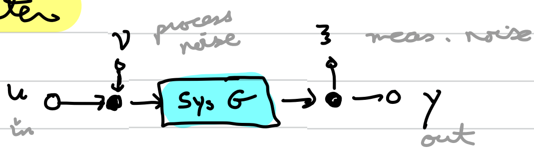
L4, Thursday

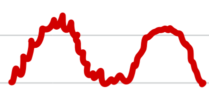
Ch. 8 Stochastic Systems

- The only reason to use feedback is uncertainty
  ↔ otherwise, feedforward is better
- no "reaction delay", even "acausal" (anticipation)
- Natural language for uncertainty ~
  probability, statistics, stochastic processes

8.1 Kalman filter

Setting:



Ex: $G = \frac{1}{1+s^2}$ (undamped osc.) $y(t) \sim$ 

- output noise at all freqs.
- input noise filtered by sys.

$$\dot{x} \approx \frac{x_{k+1} - x_k}{\Delta t}$$

↔ naive differencing to estimate $\dot{x}$ amplifies meas. noise
Kalman filter accounts for + takes advantage of idea that low-freq. var. more likely sys.
high-freq. variations more likely meas. noise.

1d ex: **Tracking a diffusing particle**

ν = thermal noise, ξ ~ microscope res., photon stats.

$$\dot{x} = Ax + B'u + B\nu, \qquad y = Cx + \xi$$
$$\dot{\hat{x}} = A\hat{x} + B'u + L(y - C\hat{x})$$

(for input noise, $B' = B$. In general can differ.)
or: $B \to$ matrix

Discretize, scale $\Rightarrow$ $x_{k+1} = x_k + \nu_k$, $y_k = x_k + \xi_k$

- x_k = actual pos. $x(t = k\,\Delta t)$
- only force is stoch. force from thermal motions

Noise statistics

$$\langle \nu_k \rangle = \langle \xi_k \rangle = 0$$

$$\langle \nu_k \xi_{k'} \rangle = \langle \nu_k x_{k'} \rangle = \langle \xi_k x_{k'} \rangle = 0 \qquad \forall k, k'$$

$$\langle \nu_k, \nu_{k'} \rangle = \nu^2 \delta_{kk'} \qquad \langle \xi_k \xi_{k'} \rangle = \xi^2 \delta_{kk'}$$

$\langle \cdots \rangle \equiv$ **ensemble averages**; eg $\langle \nu_k \rangle = \int_{-\infty}^{\infty} d\nu_k \cdot \nu_k \cdot p(\nu_k)$

Here, we will often assume $p(\nu_k)$ is **Gaussian**

eg $$p(\nu_k) = \frac{1}{\sqrt{2\pi \nu^2}} e^{-\nu_k^2/2\nu^2}$$

Another notation: $\nu_k \sim \underbrace{N}_{\text{normal dist.}}(\underbrace{0}_{\text{mean}}, \underbrace{\nu^2}_{\text{variance}})$

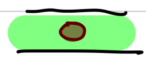
An aside on the physics

Can show $\nu^2 = 2DT_s = 2\left(\frac{k_BT}{\gamma}\right)T_s$

D = diffusion coef. $\overset{\text{un}}{=} \ell^2/t$

Sphere of radius R in fluid of viscosity η

$\Rightarrow$ $\gamma = 6\pi R\eta$ "Stokes - Einstein" reln.

if confined  γ increases

Observer

use "current obs." structure (use y_k at k) → timing ...

Prediction : $\hat{x}^-_{k+1}$ Using estimate $\hat{x}_k$, predict $k+1$

Estimate : $\hat{x}_{k+1}$ Acquires y_{k+1}, update prediction

Here: $\hat{x}^-_{k+1} = \hat{x}_k$ $\hat{y}_{k+1} = \hat{x}^-_{k+1}$

$$\hat{x}_{k+1} = \hat{x}^-_{k+1} + L(y_{k+1} - \hat{y}_{k+1}) = (1-L)\hat{x}_k + L y_{k+1}$$

Cost function

want to choose "best" observer gain L.

error $e_k = x_k - \hat{x}_k$ use $\langle e_k^2 \rangle$ as cost funct. "J"

P is standard in control th. Σ another common notation

Change notation slightly: $P_{k+1} = \langle e_{k+1}^2 \rangle \equiv \langle (x_{k+1} - \hat{x}_{k+1})^2 \rangle$

and also $P_{k+1}^- = \langle e_{k+1}^{-2} \rangle \equiv \langle (x_{k+1} - \hat{x}_{k+1}^-)^2 \rangle$

note: $e_{k+1}^- = x_{k+1} - \hat{x}_{k+1}^- = (x_k + \nu_k) - \hat{x}_k = e_k + \nu_k$

$$\Rightarrow \quad P_{k+1}^- = \langle (e_k + \nu_k)^2 \rangle = P_k + \nu^2$$

$\langle e_k \cdot \nu_k \rangle = 0$

After measuring y_{k+1}:

$$e_{k+1} = x_{k+1} - [\hat{x}_{k+1}^- + L(y_{k+1} - \hat{y}_{k+1})]$$

$$= e_{k+1}^- - L(x_{k+1} + \xi_{k+1} - \hat{x}_{k+1}^-)$$

$$= (1-L)\, e_{k+1}^- - L\, \xi_{k+1}$$

$$\Rightarrow \quad P_{k+1} = \langle e_{k+1}^2 \rangle = (1-L)^2 P_{k+1}^- + L^2 \xi^2$$

Choose L to minimize P_{k+1}:

$$\frac{dP_{k+1}}{dL} = -2(1-L)\, P_{k+1}^- + 2L\xi^2 = 0$$

$$\Rightarrow \quad L_{k+1}^* = \frac{P_{k+1}^-}{P_{k+1}^- + \xi^2} = \frac{P_k + \nu^2}{P_k + \nu^2 + \xi^2}$$

L^* = optimum value of observer gain.

$$\frac{d^2 P_{k+1}}{dL^2} = 2(P_{k+1}^- + \xi^2) > 0$$

$\Rightarrow$ minimum

Then: $P^*_{k+1} = \langle e^2_{k+1} \rangle = (1 - L^*_{k+1})^2 P^-_{k+1} + (L^*_{k+1})^2 \xi^2$

with $L^*_{k+1} = \dfrac{P^-_{k+1}}{P^-_{k+1} + \xi^2}$ and $1 - L^*_{k+1} = \dfrac{\xi^2}{P^-_k + \xi^2}$

$$\Rightarrow \quad P^*_{k+1} = \frac{\xi^4}{(P^-_k + \xi^2)^2} \cdot P^-_{k+1} + \frac{(P^-_{k+1})^2}{(P^-_{k+1} + \xi^2)^2} \cdot \xi^2$$

$$= \xi^2 L^*_{k+1} \left[\frac{\xi^2}{(P^-_{k+1} + \xi^2)} + \frac{P^-_{k+1}}{(P^-_{k+1} + \xi^2)} \right]$$

$$= \xi^2 L^*_{k+1}$$

Stationary state $\Leftrightarrow$ ξ^2, ν^2 indep. of k

$\Leftrightarrow$ $L^*_k \to L^*$, $P^*_k \to P^*$

$$\Rightarrow \quad L^* = \frac{P^* + \nu^2}{P^* + \nu^2 + \xi^2} \qquad P^* = \xi^2 L^*$$

$$\Rightarrow \quad L^{*2} + \alpha L^* - \alpha = 0 \qquad \alpha = \nu^2 / \xi^2 \sim SNR^2$$

need $L > 0$

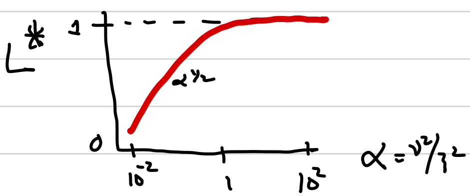
$$L^* = \frac{1}{2}\left[-\alpha + \sqrt{\alpha^2 + 4\alpha} \right]$$

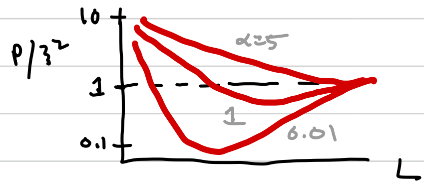
$\alpha \gg 1$: $L^* \to 1$, $P^* \approx \xi^2$ trust the measurements

$\alpha \ll 1$: $L^* \to \sqrt{\alpha}$, $P^* \approx \nu\xi$ trust the model

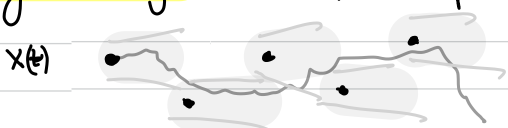
$\hookrightarrow \delta x \approx \sqrt{\nu \xi} = (2DT_s)^{1/4}\, \xi^{1/2}$

The very weak scaling is nice: increase D by 10^4 (eg $R \to R/10^4$)
$\Rightarrow$ increase δx by only 10!





Hybrid Dyn. $\partial_t p = \partial_{xx} p \Rightarrow p(x,t) \sim N\left(\hat{x}_k, P_k + 2D(t-t_k)\right)$



spreading Gaussian
- variance spreads between obs.
- "collapses" after each y_k

Estimating a constant (w/ meas. errors)

$$x_{k+1} = x_k \qquad y_{k+1} = x_{k+1} + \xi_{k+1} \qquad \langle \xi_k \xi_{k'} \rangle = \xi^2 \delta_{kk'}$$

- same as before w/ $\nu^2 \to 0 \quad \Rightarrow \quad \alpha \to 0$

$$\Rightarrow \quad L^*_{k+1} = \frac{P^*_k}{P^*_k + \xi^2}, \qquad P^*_{k+1} = \xi^2 L^*_{k+1}$$

$$P_0 = \xi^2 \quad \Rightarrow \quad L^*_{k+1} = \frac{1}{k+1}, \qquad P^*_{k+1} = \frac{\xi^2}{k+1}$$

$$\hat{x}_{k+1} = (1 - L^*_{k+1})\,\hat{x}_k + L^*_{k+1}\, y_{k+1}$$

$$= \left(\frac{k}{k+1}\right)\hat{x}_k + \left(\frac{1}{k+1}\right) y_{k+1}$$

Compare:

$$\hat{x}_{k+1} = \frac{1}{k+1} \sum_{i=1}^{k+1} x_i \qquad \text{batch alg.}$$

$$= \frac{1}{k+1} \sum_{i=1}^{k} x_i + \left(\frac{1}{k+1}\right) x_{k+1}$$

$$= \left(\frac{k}{k+1}\right) \frac{1}{k} \sum_{i=1}^{k} x_i + \left(\frac{1}{k+1}\right) x_{k+1}$$

$$= \left(\frac{k}{k+1}\right) \hat{x}_k + \left(\frac{1}{k+1}\right) x_{k+1} \qquad \text{recursive alg}$$

$$= \hat{x}_k + \left(\frac{1}{k+1}\right)(x_{k+1} - \hat{x}_k) \qquad \text{Kalman gain}$$

Note: $x_k \leftrightarrow y_k$ ... also: running avg ...

Kalman filter, general case

$$x_{k+1} = A x_k + B u_k + v_k \,, \qquad y_k = C x_k + \xi_k$$

$$\langle v_k \rangle = \langle \xi_k \rangle = \langle \xi_k v_\ell^T \rangle = 0$$

$$\langle v_k v_\ell^T \rangle = Q_v \cdot \delta_{k\ell} \,, \qquad \langle \xi_k \xi_\ell^T \rangle = Q_\xi \cdot \delta_{k\ell}$$

Note that we include coupling in covariance matrices.
eg, if $\vec{v}_k = \vec{B} \nu_k$, then $Q_v = B B^T \nu^2$

$(\nu^2 = \langle \nu_k^2 \rangle)$

Predicted state: $\hat{x}_{k+1}^- = A \hat{x}_k + B u_k$

$$\hat{x}_{k+1} = \hat{x}_{k+1}^- + L \left(y_{k+1} - \underbrace{\hat{y}_{k+1}}_{C \hat{x}_{k+1}^-} \right)$$

Covariance matrix for state estimation errors

$$P_k = \langle e_k e_k^T \rangle \qquad e_k = x_k - \hat{x}_k$$

also: $P_{k+1}^- = \langle e_{k+1}^- e_{k+1}^{-T} \rangle$

with $e_{k+1}^- = x_{k+1} - \hat{x}_{k+1}^- = A x_k + B u_k + v_k - A \hat{x}_k - B u_k$

$$= A e_k + v_k$$

$$\Rightarrow \quad P_{k+1}^- = \langle (A e_k + v_k)(e_k^T A^T + v_k^T) \rangle$$

$$= A P_k A^T + Q_v$$

After the observation y_{k+1},

$$
\begin{aligned}
e_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\
&= x_{k+1} - \hat{x}^-_{k+1} - L\left(y_{k+1} - \hat{y}_{k+1}\right) \\
&= e^-_{k+1} - L\,\varepsilon_{k+1}
\end{aligned}
$$

where $\varepsilon_{k+1} \equiv y_{k+1} - \hat{y}_{k+1}$ innovations

Thus,
$$
\begin{aligned}
P_{k+1} &= \langle e_{k+1}\, e^T_{k+1} \rangle \\
&= \langle (e^-_{k+1} - L\,\varepsilon_{k+1})(e^-_{k+1} - L\,\varepsilon_{k+1})^T \rangle \\
&= P^-_{k+1} - L P^{xy\,T}_{k+1} - P^{xy}_{k+1} L^T + L P^y_{k+1} L^T
\end{aligned}
$$

where
$$
\begin{aligned}
P^y_{k+1} &= \langle \varepsilon_{k+1}\, \varepsilon^T_{k+1} \rangle \qquad \text{covariance matrix of innovations} \\
&= \langle (C x_{k+1} + \zeta_{k+1} - C\hat{x}^-_{k+1})(\cdots)^T \rangle \\
&= \langle (C e^-_{k+1} + \zeta_{k+1})(\cdots)^T \rangle \\
&= C P^-_{k+1} C^T + Q_\zeta
\end{aligned}
$$

and
$$
\begin{aligned}
P^{xy}_{k+1} &= \langle e^-_{k+1} \cdot \varepsilon^T_{k+1} \rangle = \langle e^-_{k+1} (C e^-_{k+1} + \zeta_{k+1})^T \rangle \\
&= P^-_{k+1} C^T
\end{aligned}
$$

Pick L to min $\overbrace{\langle e^T_k e_k \rangle = \operatorname{Tr}\langle e_k e^T_k \rangle = \operatorname{Tr} P_k}^{\text{cost function}}$

So $\dfrac{d}{dL} \operatorname{Tr} P_{k+1} = -2 P^{xy\,T}_{k+1} + 2 P^y_{k+1} L^T = 0$

$$\Rightarrow \quad L^*_{k+1} = P^{xy}_{k+1} \left(P^y_{k+1}\right)^{-1}$$

$$\Rightarrow \quad P^{xy}_{k+1} = L^*_{k+1} P^y_{k+1} \quad \Rightarrow \quad P^*_{k+1} = P^-_{k+1} - L^*_{k+1} P^y_{k+1} L^{*T}$$

using optimal observer gain L^*

To summarize:

$$\left.\begin{aligned}
\hat{x}^-_{k+1} &= A x_k + B u_k && \text{state mean.}\\
\hat{y}_{k+1} &= C \hat{x}^-_{k+1} && \text{observation mean.}\\
P^-_{k+1} &= A P_k A^T + Q_v && \text{state cov.}\\
P^y_{k+1} &= C P^-_{k+1} C^T + Q_3 && \text{innovation cov.}\\
P^{xy}_{k+1} &= P^-_{k+1} C^T && \text{state - obs. cov.}
\end{aligned}\right\}\text{predict}$$

$$\left.\begin{aligned}
L^*_{k+1} &= P^{xy}_{k+1} \left(P^y_{k+1}\right)^{-1} && \text{obs. gain.}\\
\hat{x}_{k+1} &= \hat{x}^-_{k+1} + L^*_{k+1}\left(y_{k+1} - \hat{y}_{k+1}\right) && \text{state mean}\\
P^*_{k+1} &= P^-_{k+1} - L^*_{k+1} P^y_{k+1} L^{*T}_{k+1} && \text{state cov.}
\end{aligned}\right\}\text{update}$$

$$P^*_{k+1} = + \underbrace{A P^*_k A^T}_{\text{dynamics}} + \underbrace{Q_v}_{\text{disturb.}} - \underbrace{L^*_{k+1} P^y_{k+1} L^{*T}_{k+1}}_{\text{observations}}$$

The dynamics and disturb increase P_k; obs. decrease P_k.

Steady-state eqs. iterate w/ stationary stats until L^*, P^* converge.

The structure of such eqs. is clearer for prediction observn.

$\longrightarrow$ Riccati eq.

again duality w/control: $A \to A^T$, $B \to C^T$, $K \to L^T$

Continuous / Hybrid Dynamics

$$\dot{x} = A_c x + B_c u + \nu_c \qquad y = Cx + \xi_c$$

$$\langle \nu_c(t)\, \nu_c^T(t') \rangle = Q_\nu^c\, \delta(t-t') \,, \qquad \langle \xi_c(t)\, \xi_c^T(t') \rangle = Q_\xi^c\, \delta(t-t')$$

$$Q_\nu = \frac{1}{T_s} Q_\nu^c \qquad Q_\xi = \frac{1}{T_s} Q_\xi^c$$

integrate white noise to bandwidth T_s^{-1}

$$\dot{\hat{x}} = A_c \hat{x} + B_c u + L(y - \hat{y}) \qquad \hat{y} = C\hat{x}$$

Similar analysis $\Rightarrow$ $L^* = P^{xy}(P^y)^{-1} = P^* C^T (Q_\xi^c)^{-1}$

compare: discrete case has smaller $L^* = P^- C^T (CP^-C^T + Q_\xi)^{-1}$

extra variance accumulates during T_s^{-1}

$$\dot{P}^* = A_c P^* + P^* A_c^T + Q_\nu^c - P^* C^T (Q_\xi^c)^{-1} C P^*$$

For hybrid, we integrate continuous eqs from $t_k \to t_{k+1}$.

8.2 Linear Quadratic Gaussian (LQG) control

- Combine LQR, Kalman

1d ex: Trapping a diffusing particle

$$x_{k+1} = x_k + u_k + \nu_k, \qquad y_k = x_k + \xi_k$$

$$\nu_k \sim N(0, \nu^2), \qquad \xi_k \sim N(0, \xi^2) \qquad (\nu^2 = 2DT_s)$$

- Stochastic generalization of cost function

$$J = \sum_{k=0}^{N} \left\langle x_k^2 + R u_k^2 \right\rangle = \sum_k \langle x_k^2 \rangle \qquad (\text{for } R = 0)$$

min. variance control

Try 3 strategies

1. Perfect info. $u_k = -K x_k$
2. Noisy obs. $u_k = -K y_k$
3. Observer $u_k = -K \hat{x}_k$

Here we can solve for these directly because the problem is simple.

Expect

1. Perfect info $\quad u_k = -K x_k$

$$x_{k+1} = (1-K)\, x_k + \nu_k$$

$$\langle x_{k+1}^2 \rangle = (1-K)^2 \langle x_k^2 \rangle + \nu^2$$

$$\langle x^2 \rangle = (1-K)^2 \langle x^2 \rangle + \nu^2 \qquad \text{stationary state}$$

$$\langle x^2 \rangle = \frac{\nu^2}{1-(1-K)^2} \qquad \text{so } \frac{d\langle x^2 \rangle}{dK} = 0 \;\Rightarrow\; K^* = 1,\; \Delta J^* = \nu^2$$

increment

2. Naive obs: $\quad u_k = -K y_k = -K(x_k + \xi_k)$

$$x_{k+1} = (1-K)\, x_k - K\xi_k + \nu_k$$

$$\Rightarrow \; \cdots \quad \langle x^2 \rangle = \frac{K^2\xi^2 + \nu^2}{1-(1-K)^2} \;\Rightarrow\; K^* = \tfrac{1}{2}(\sqrt{5}-1) \approx 0.62$$

$$\Delta J^* = \tfrac{1}{2}(\sqrt{5}+1) \approx 1.62$$

3. Observer fb $\quad u_k = -K \hat{x}_k \;\Rightarrow\; \hat{x}^-_{k+1} = \hat{x}_k + u_k = (1-K)\hat{x}_k$

$$\hat{x}_{k+1} = (1-L)\,\hat{x}^-_{k+1} + L y_{k+1}$$

$$\Rightarrow \; \cdots \quad \langle x^2 \rangle = L^* \xi^2 + \frac{\nu^2}{K(2-K)} \;\Rightarrow\; K^* = 1$$

$$\Delta J^* = \tfrac{1}{2}(\sqrt{5}+1) \approx 1.62$$

$K^* = 1$ is same as "perfect info" $\leftrightarrow$ separation principle

2,3 same but need to "tune" K to right value ...

General LQG

continuous case

$$\dot{J} = \langle x^T Q x \rangle + \langle u^T R u \rangle$$

$$\dot{x} = Ax + Bu + \nu, \qquad y = Cx + \xi$$

$$\langle \nu(t)\, \nu^T(t') \rangle = Q^c_\nu\, \delta(t-t'), \qquad \langle \xi(t) \xi^T(t') \rangle = Q^c_\xi\, \delta(t-t')$$

$$u = -K\hat{x}, \qquad K = R^{-1} B^T S, \qquad S B R^{-1} B^T S - SA - A^T S - Q = 0$$

$$0 = AP + PA^T + Q^c_\nu - PC^T (Q^c_\xi)^{-1} CP, \qquad L = PC^T (Q^c_\xi)^{-1}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \qquad \hat{y} = C\hat{x}$$

Solve for steady-state S, P

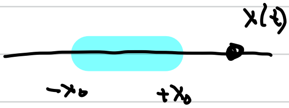
$$\Rightarrow \qquad \dot{J} = \underbrace{\mathrm{Tr}\, S Q^c_\nu}_{\text{disturb.}} + \underbrace{\mathrm{Tr}\, P K^T R K}_{\text{state-est. error}}$$

Similar for discrete case..

Separation Principle:

i) Use Kalman gain L to est. $\hat{x}$
ii) Use optimal fb matrix K, assuming knowl. of x
iii) Combine in $u = -K\hat{x}_k$

Limitations of Separation Principle



eg $\quad \dot{x} = -ax + u + v \qquad y = x, \; |x| > x_0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; 0, \; x$ outside

- The best strategy is to do nothing while particle is "hidden", control when "visible".

- Main point is that this is a nonlinear control law (varies in character w/ state of system) and ability to estimate depends on its value, which depends on the control, etc.

  $\Rightarrow$ problem of estimating state + control are coupled.

8.3 Stochastic Optimal Control

How do we control a nonlinear, stoch. system?

eg $\dot{x} = f(x,u) + \nu(t)$ $\quad \langle \nu(t) \rangle = 0, \quad \langle \nu(t)\nu(t') \rangle = \nu^2 \delta(t-t')$

— let all quantities be scalar, for simplicity

Cost-to-go over interval $[t, \tau]$

$$J(x,u,t) = \left\langle \varphi[x(\tau)] + \int_t^\tau dt' \, L(x,u) \right\rangle$$

$$J^*(x,t) = \inf_u J(x,u,t)$$ infimum is over paths u_t^τ

Repeat earlier derivation. The only difference in the stochastic case is a new, diffusive term

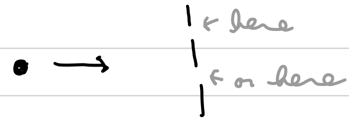
Since diffusion $\Rightarrow \langle \Delta x^2 \rangle \sim \nu^2 \Delta t$

we need to expand $\langle J(x+\Delta x) \rangle$ to 2nd order in Δx

$$\rightarrow \quad \partial_t J^*(x,t) + \inf_u \left[L(x,u) + (\partial_x J^*) f(x,u) + \tfrac{1}{2}\nu^2 \partial_{xx} J^* \right] = 0$$

Stochastic HJB

Ex: **Delayed Choice**



Particle drifts at const velocity but has a transverse position $x(t)$ affected by noise

$x(t=0)=0$; Must pass through slits at $x=\pm 1$ at time τ

$\Rightarrow \quad \dot{x} = u + \nu \qquad x(0)=0 \qquad x(\tau) = \pm 1$

$\hookrightarrow \quad \langle \nu(t)\, \nu(t') \rangle = 2D\, \delta(t-t')$

Also: running cost is $\quad L = \frac{1}{2} R u^2$

Cost-to-go $\quad J(x,u,t) = \int_t^{\tau} dt'\, L(u(t'))$

stoch. HJB $\Rightarrow \quad \partial_t J^*(x,t) + \inf_u \left[\frac{1}{2} R u^2 + (\partial_x J^*) u + D\, \partial_{xx} J^* \right] = 0$

Minimize $u(t)$ by $\partial_u = 0 \quad \Rightarrow u^* = -R^{-1} (\partial_x J^*)$

$$\Rightarrow \quad \partial_t J^*(x,t) - \frac{1}{2R} (\partial_x J^*)^2 + D\, \partial_{xx} J^* = 0$$

Change of variable (Cole-Hopf)

$$J^*(x,t) \equiv -\lambda \log \Psi(x,t)$$

$$\Rightarrow \quad -\partial_t \Psi = D\, \partial_{xx} \Psi \qquad \text{linearizes!}$$

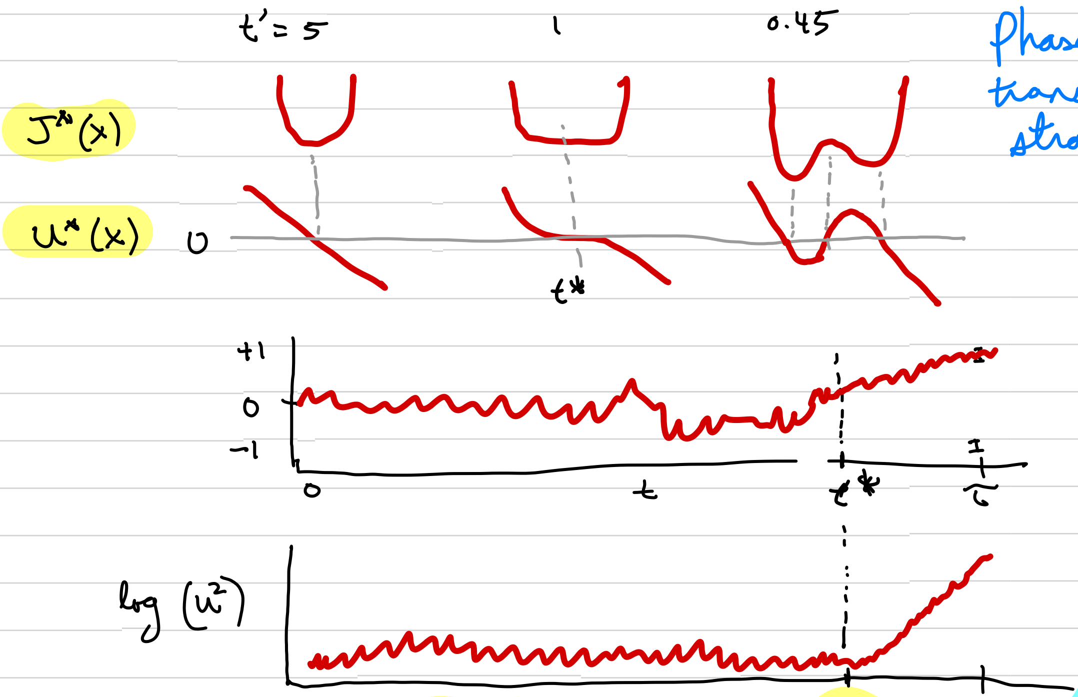
Diff. eq. in negative time. $\quad x(\tau) = \pm 1 \longrightarrow$ back to t

final cond. $\Rightarrow$ $\psi(x,\tau) = \frac{1}{2}\left[\delta(x-1) + \delta(x+1)\right]$

$$J^*(x,t') = 2RD\left[\frac{x^2}{4Dt'} - \ln\cosh\left(\frac{x}{2Dt'}\right)\right] + f(t')$$

where $t' = \tau - t$ and $f(t')$ is a const. (diverges as $t' \to 0$)

$u^* = -R^{-1}\partial_x J^*$ independent of $f(t')$



Phase trans. in strategy!

Wait ...

Act now!

$t^* = \dfrac{x_0^2}{2D}$