Dynamical Mean-Field Theory: from classical statistical mechanics to disordered systems

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Let's start applying a coarse-grained description.

Goal: ending up with the Langevin equations for the dynamics of the single degree of freedom (spin, density, species abundance, etc.)

What is the starting point?

Let's consider the simple example of a 1dim. particle coupled to an external bath (at temperature T).

GOAL: obtaining the Langevin equations, namely a self-consistent stochastic description for the single degree of freedom.

Coupling of a massive particle (1d) to the environment

Single particle of mass M and potential energy $V(x)$ in 1 dim, coupled to an ensemble of HARMONIC OSCILLAT.

SYSTEM + ENVIRONMENT = CLOSED

SYSTEM = OPEN

"thermal bath"

$$H_{tot} = H_{syst} + H_{env} + H_{int} + \underline{H_{counter}} =$$

to avoid a negative harmonic potential (unstable dynamics)

$$= H_{syst} + \tilde{H}_{env}$$

$$\frac{p^2}{2M} + V(x)$$

• $H_{env} = \sum_{a=1}^{N} \frac{\pi_a^2}{2m_a} + \frac{m_a \omega_a^2}{2} q_a^2$

momenta; positions

independent harmonic oscillators ("phonons")
(Normal modes of a generic Hamiltonian to quadratic order around the GS).

• $H_{int} = x \sum_{a=1}^{N} c_a q_a$

coupling constant (system – environ.)

To obtain the counter-term, Z_{red}:

$$Z_{red} = \sum_{\{syst.\}} \sum_{\{env\}} e^{-\beta H_{tot}} = \sum_{\{p,x\}} e^{-\beta H_{syst}} \sum_{\{\pi_a, q_a\}} e^{-\beta(H_{env} + H_{int})}$$

Coupling of a massive particle (1d) to the environment

$$\frac{1}{N!}\int \frac{d^3\pi_a\, d^3 q_a}{h^3}\, e^{-\beta \pi_a^2/2m_a}\, e^{-\beta \frac{m_a w_a^2 q_a^2}{2}}\, e^{-\beta x \sum_a c_a q_a}$$

$$\downarrow$$

$$\left(\frac{\sqrt{2\pi m_a K_B T}}{h}\right)^3 \left(\sqrt{\frac{2\pi K_B T}{m_a w_a^2}}\right)^3 e^{\frac{1}{2}\sum_a \beta \frac{x^2 c_a^2}{m_a w_a^2}}$$

$$Z_{red} \propto \sum_{\substack{\{syst\} \\ p,x}} e^{-\beta\left(H_{syst.} - \frac{1}{2}\sum_a \frac{c_a^2 x^2}{m_a w_a^2}\right)} \Rightarrow Z_{red} \propto Z_{syst}$$

*) $c_a \sim o(1/N^2)$

*) counterterm $H_{counter} = \frac{1}{2}\sum_a \frac{c_a^2 x^2}{m_a w_a^2}$

to avoid renormalization of $V(x)$
and possible instabilities.

$$H_{env} + H_{int} + H_{counter} = \tilde{H}$$

$$\sum_a \frac{\pi_a^2}{2m_a} + \sum_a \left(\frac{1}{2} m_a w_a^2 q_a^2 + q_a c_a x + \frac{1}{2}\frac{c_a^2 x^2}{m_a w_a^2}\right)$$

Hamiltonian equations: particle + thermal bath

(1) $\dot{x}(t) = \frac{\partial H}{\partial p} = \frac{p(t)}{M}$

(*)

$$\dot{p}(t) = -\frac{\partial H}{\partial x} = \vec{F} = -V'[x(t)] - \sum_{a=1}^{N} c_a q_a(t) - \sum_{a=1}^{N} \frac{c_a^2 x(t)}{m_a w_a^2}$$

$$\boxed{\ddot{x}(t) = -V'(x) - \sum_a c_a q_a - \sum_a \frac{c_a^2 x}{m_a W_a^2}}$$

Hamiltonian equations: particle + thermal bath

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② $\dot{q}_a(t) = \frac{\partial \tilde{H}}{\partial \Pi_a} = \frac{\Pi_a(t)}{m_a}$

GENERAL SOLUTION

$\dot{\Pi}_a(t) = -\frac{\partial \tilde{H}}{\partial q_a} = -m_a \omega_a^2 q_a(t) - c_a x(t)$

$$\boxed{m_a \ddot{q}_a = -m_a \omega_a^2 q_a - x c_a}$$

Hamiltonian equations: particle + thermal bath

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$\dot{p}(t) = -\frac{\partial H}{\partial x} = \vec{F} = -V'[x(t)] - \sum_{\alpha=1}^{N} c_\alpha q_\alpha(t) - \sum_{\alpha=1}^{N} \frac{c_\alpha^2 x(t)}{m_\alpha w_\alpha^2}$ ⊛

$$\boxed{\ddot{x}(t) = -V'(x) - \sum_\alpha c_\alpha q_\alpha - \sum_\alpha \frac{c_\alpha^2 x}{m_\alpha w_\alpha^2}}$$

② $\dot{q}_\alpha(t) = \frac{\partial \tilde{H}}{\partial \Pi_\alpha} = \frac{\Pi_\alpha(t)}{m_\alpha}$ GENERAL SOLUTION

$\dot{\Pi}_\alpha(t) = -\frac{\partial \tilde{H}}{\partial q_\alpha} = -m_\alpha w_\alpha^2 q_\alpha(t) - c_\alpha x(t)$

$$\boxed{m_\alpha \ddot{q}_\alpha = -m_\alpha w_\alpha^2 q_\alpha - x c_\alpha}$$

③ $q_\alpha(t) = a\cos(w_\alpha t) + b\sin(w_\alpha t)$

$a = q_\alpha(0)$

$b = \dot{q}_\alpha(0)/w_\alpha = \frac{\Pi_\alpha(0)}{m_\alpha w_\alpha}$

PARTICULAR SOLUTION

$q_\alpha(t) = q_\alpha(0)\cos(w_\alpha t) + \frac{\Pi_\alpha(0)}{m_\alpha w_\alpha}\sin(w_\alpha t) + \dots$

$$-\frac{c_\alpha}{m_\alpha w_\alpha}\underbrace{\int_0^t dt' \sin[w_\alpha(t-t')]\, x(t')}_{\text{to be integrated by parts}}$$

④ Show that we end up with:

$$\boxed{\dot{p}(t) = -V'[x(t)] + \xi(t) - \int_0^t dt'\, \Gamma(t-t')\, \dot{x}(t')}$$

friction force making the equ non-Markovien

To prove this, let's consider:

$$\dot{p} = -V' - \sum_a c_a q_a - \sum_a \frac{c_a^2 x}{m_a w_a^2}$$

$$\dot{p} = -V' - \sum_a c_a \Big[q_a(0)\cos(w_a t) + \frac{\Pi_a(0)}{m_a w_a}\sin(w_a t)$$
$$- \underbrace{\frac{c_a}{m_a w_a}\int_0^t \sin(w_a(t-t'))\, x(t')\, dt'}\Big] - \sum_a \frac{c_a^2 x}{m_a w_a^2}$$

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$$\dot{p} = -V' - \sum_a c_a \Big[q_a(0)\cos(w_a t) + \frac{\Pi_a(0)}{m_a w_a}\sin(w_a t) - \frac{c_a}{m_a w_a} \underbrace{\int_0^t \sin(w_a(t-t'))\, x(t')\, dt'} \Big] - \sum_a \frac{c_a^2 x}{m_a w_a^2}$$

Integration by parts:

$$\int_0^t \sin[w_a(t-t')]\, x(t')\, dt' = \frac{\cos[w_a(t-t')]\, x(t')}{w_a} \Big|_0^t - \frac{1}{w_a} \int_0^t \cos[w_a(t-t')]\, \dot{x}(t')\, dt'$$

$$= \frac{x(t)}{w_a} - \frac{\cos(w_a t)\, x(0)}{w_a} - \frac{1}{w_a} \int_0^t \cos[w_a(t-t')]\, \dot{x}(t')\, dt'$$

Langevin equation for the effective particle

$$\dot{p} = -V' + \xi(t) - \int_0^t dt' \, \Gamma(t-t') \, \dot{x}(t')$$

where $\Gamma(t-t') = \sum_a \frac{c_a^2}{m_a w_a^2} \cos[w_a(t-t')]$

Friction term $\Rightarrow$ can be also written as an integral running up to a tot. time $\tau > \max(t,t')$

$$\gamma(t,t') = \Gamma(t-t') \, \theta(t-t')$$

$$\xi(t) = -\sum_a \Big[c_a q_a(0) \cos(w_a t) + \frac{c_a \pi_a(0)}{m_a w_a} \sin(w_a t) - \cancel{\frac{c_a^2 x(t)}{m_a w_a^2}} + \cancel{\frac{c_a^2 x(t)}{m_a w_a^2}} + \frac{c_a^2 \cos(w_a t) \, x(0)}{m_a w_a^2} \Big]$$

$$\xi(t) = -\sum_a \left[c_a \left(q_a(0) + \frac{c_a x(0)}{m_a w_a^2} \right) \cos(w_a t) + \frac{c_a \pi_a(0)}{m_a w_a} \sin(w_a t) \right]$$

$\Downarrow$

$\xi(t)$ is a PSEUDO-RANDOM NUMBER

(We will prove in the following that it corresponds to a Gaussian white noise).

From ordered models to disordered (highly heterogeneous) systems

Let's consider a Hamiltonian of the form:

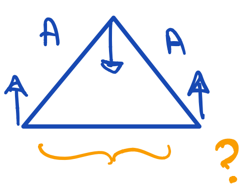
$$\mathcal{H} = -\sum_{(ij)} J_{ij}\,\sigma_i \sigma_j - h\sum_i \sigma_i \qquad \sigma_i = \pm 1 \quad \text{Ising spins}$$

where $\{J_{ij}\}$ are RANDOM VARIABLES extracted from a given probability distribution. Consider, for instance:

$$P(J_{ij}) = \frac{1}{\sqrt{2\pi J^2}} \exp\left(-\frac{J_{ij}^{\,2}}{2J^2}\right)$$

DISORDERED INTERACTIONS

"frustration" / non convex optimization



Two classes of complex landscapes

In mean-field glassy models precise meaning of the free-energy landscape via the TAP formalism, or more generally through the analysis of critical points of given index.

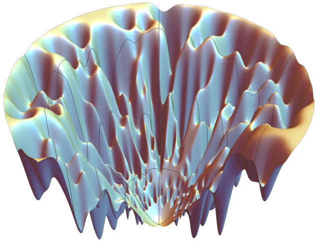
- **SPIN GLASS LANDSCAPES**
  - Sub-exponential number of free-energy minima;
  - Sub-extensive barriers;
  - Convergence of one-time observables to their equilibrium thermodynamic limit;
  - Thermodynamic and dynamical transition temperature generally coincide.

$T_d = T_s$

- **STRUCTURAL GLASS LANDSCAPES**

p-spin model with $p \geq 3$

$H = -\sum_{i_1 \dots i_p} J_{i_1 \dots i_p} \sigma_{i_1} \dots \sigma_{i_p}$



  - Exponential number of free-energy minima;
  - Extensive barriers among stationary points;
  - No convergence of one-time observables to the equilibrium value;
  - Nonetheless, long-time dynamics explores configurations with the largest & marginally stable basins (*threshold states*).
  - Different dynamical and thermodynamic glass transition temperatures.

Analysis of the dynamics

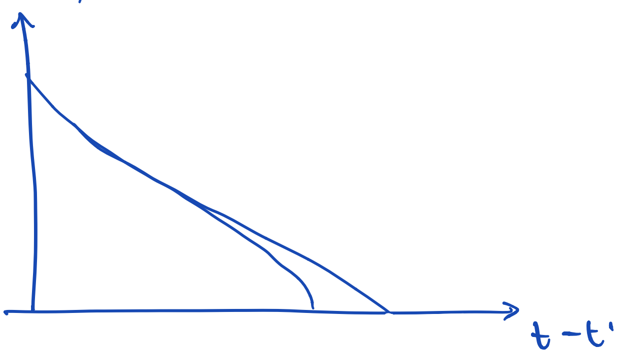
$$C(t,t') = \frac{1}{N}\sum_i \overline{\langle s_i(t)\, s_i(t')\rangle}$$

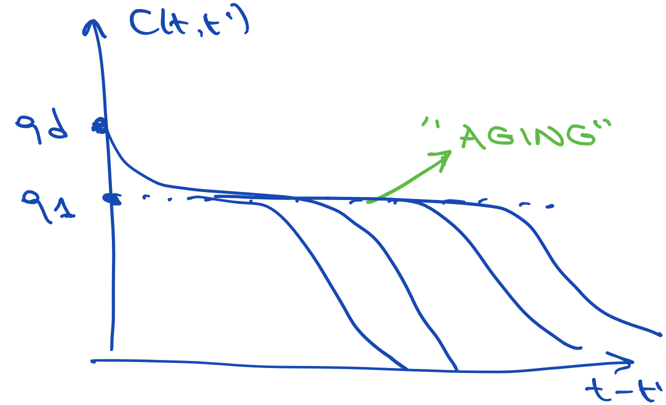
→ disorder average

↳ thermal average

$$R(t,t') = \frac{1}{N}\sum_i \overline{\left\langle \frac{\partial s_i(t)}{\partial h(t')}\right\rangle}\Bigg|_{h\to 0}$$

$C(t,t') = C(t-t')$





Equilibrium

$t - t' \sim O(1)$

$$R_{TTI}(t,t') = R_{TTI}(t-t') = \frac{1}{T}\frac{d}{dt'}C(t-t')$$

Out-of-equilibrium

$t/t' \sim O(1)$.

- Violation of FDT.
- No TTI.

Basics of Dynamical Mean-Fiel Theory (DMFT)

DMFT traces its origins in condensed matter and strongly correlated electron systems with potential applications to high-T superconductors and cluster extensions.

More recently:

Aging of spin glasses, Rheology and amorphous materials, Ecology & Evolution, Inference.

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1) Identify the correct degrees of freedom and treat the rest of the system as a bath

$$\mathcal{H} = H_{\text{syst}} + \underbrace{H_{\text{env}}}_{\text{Thermal bath}} + H_{\text{int}}$$

2) The bath is statistically equivalent to singled-out degree of freedom.

- Equilibrium dynamics: response and correlation related by FDT;
- **Aging**: extremely slow functions. The thermal bath is aging with the rest of the system. Then**?**

Why a self-consistent dynamical formalism?

Closed-form equations can be recovered only for a narrowed class of solvable models:

- spherical p-spin model with $p > 2$ [Cugliandolo, Kurchan (1993); Crisanti, Horner, Sommers (1993); Barrat (1997)]
- truncated SK model (“soft-spin" version) [Sompolinsky, Zippelius (1982); Bouchaud, Cugliandolo, Kurchan, Mézard (1997); Kennett, Chamon (2001); Cugliandolo-Kurchan (2007); Chamon, Cugliandolo (2007)]
- high-dimensional random manifold [Franz, Mézard (1994); Cugliandolo, Kurchan, Le Doussal (1996); Cugliandolo, Le Doussal (1996)]

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Goal: determining the (local) asymptotic behaviour of dynamical systems **without explicitly solving the equations**.

The dynamics we are focusing on is induced by the following Langevin equations:

$$\frac{ds_i(t)}{dt} = -\frac{\partial V}{\partial s_i} + \frac{1}{(p-1)!} \sum_{i_2 \dots i_p} \underbrace{J_{i,i_2 \dots i_p} s_{i_2} \dots s_{i_p}}_{\textit{Multi-spin interactions}} + \underbrace{\eta_i(t)}_{\textit{White noise}} \qquad P(J_{i_1 \dots i_p}) = \sqrt{\frac{N^{p-1}}{\pi p!}} \exp\left(-\frac{J^2_{i_1 \dots i_p} N^{p-1}}{p!}\right)$$

Disentanglement of fast and slow timescales

A mean-field treatment allows us to write

$$\frac{ds(t)}{dt} = -\frac{\partial V(s(t))}{\partial s} + \frac{p(p-1)}{2}\int_0^t dt' R(t,t') C^{p-2}(t,t') s(t') + \xi(t)$$

$$\langle \xi(t)\xi(t')\rangle = 2T\delta(t-t') + \frac{p}{2}C^{p-1}(t,t')$$

the correlation and the response functions being: $C(t,t') \equiv \langle s(t)s(t')\rangle$, $R(t,t') \equiv \left.\frac{\partial \langle s(t)\rangle}{\partial h(t')}\right|_{h=0}$

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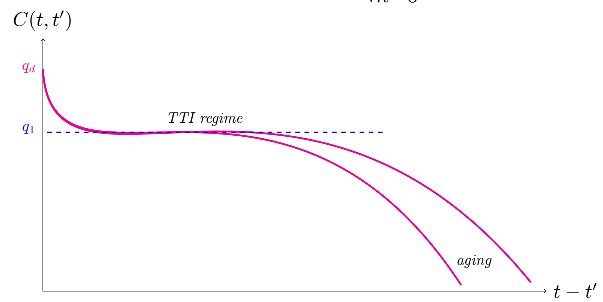
$$C(t,t') = (q_d - q_1)\boxed{f_1(t-t')} + q_1\boxed{f_2\left(\frac{t}{t'}\right)}$$

TTI regime $t-t' \sim O(1)$ **out-of-equilibrium** $t/t' \sim O(1)$



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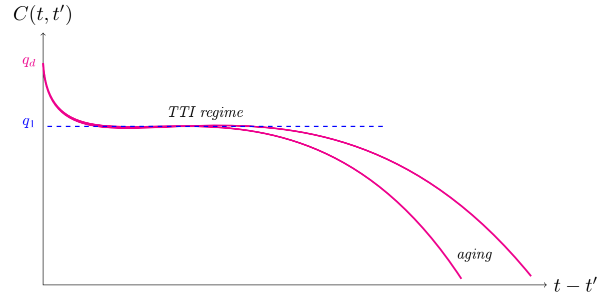

- $\int_0^t dt'' R(t,t'') C^{p-2}(t,t'') s(t'') \simeq \int_{t-\tau}^{t} R_{\mathrm{TTI}}(t-t') \left[C_{\mathrm{TTI}}(t-t') + q_1\right]^{p-2} s(t') dt' + \int_0^{t-\tau} R_A(t,t') C_A^{p-2}(t,t') s(t') dt'$

Disentanglement of fast and slow timescales

A mean-field treatment allows us to write

$$\frac{ds(t)}{dt} = -\frac{\partial V(s(t))}{\partial s} + \frac{p(p-1)}{2}\boxed{\int_0^t dt' R(t,t') C^{p-2}(t,t') s(t')} + \xi(t)$$

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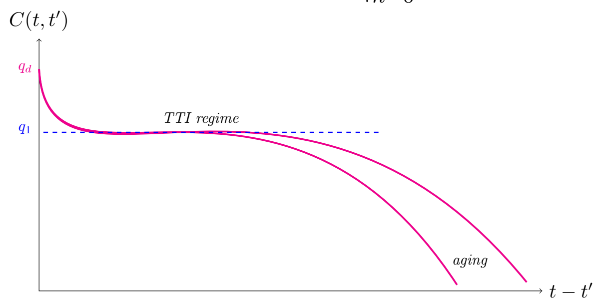

- $$\int_0^t dt'' R(t,t'') C^{p-2}(t,t'') s(t'') \simeq \boxed{\int_{t-\tau}^t R_{\mathrm{TTI}}(t-t') \left[C_{\mathrm{TTI}}(t-t') + q_1\right]^{p-2} s(t') dt'} + \int_0^{t-\tau} R_A(t,t') C_A^{p-2}(t,t') s(t') dt'$$

$$\simeq \frac{1}{T(p-1)}\left(q_d^{p-1} - q_1^{p-1}\right) s(t) - \frac{2}{p(p-1)} \int_{t-\tau}^t \nu_{\mathrm{TTI}}(t-t') \dot{s}(t') dt'$$

$$+ \int_0^{t-\tau} R_A(t,t') C_A^{p-2}(t,t') s(t') dt'$$

- Similarly, for the noise-noise correlation function

$$\langle \xi_{\mathrm{TTI}}(t)\xi_{\mathrm{TTI}}(t')\rangle = \left\{ [C_{\mathrm{TTI}}(t-t') + q_1]^{p-1} - q_1^{p-1} \right\} \frac{p}{2} + 2T\delta(t-t') = T\left[\nu_{\mathrm{TTI}}(t-t') + 2\delta(t-t')\right]$$

$$\langle \xi_A(t)\xi_A(t')\rangle = \frac{p}{2} C_A^{p-1}(t,t')$$

A. Altieri, G. Biroli, C. Cammarota, J. Phys. A.: Math. Theor. **53** (2020).

Effective process for the single variable

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The original equation can thus be rewritten as

$$\dot{s}(t) \simeq -\frac{\partial V(s)}{\partial s} + \frac{p}{2T}(q_d^{p-1} - q_1^{p-1})s(t) - \int_{t-\tau}^{t} \nu_{\mathrm{TTI}}(t-t')\dot{s}(t')dt' + \xi_{\mathrm{TTI}}(t) + h(t)$$

$$h(t) \equiv \frac{p(p-1)}{2} \int_0^{t-\tau} R_A(t,t') C_A^{p-2}(t,t') s(t') dt' + \xi_A(t)$$

Slowly evolving *effective field*

A. Altieri, G. Biroli, C. Cammarota, J. Phys. A.: Math. Theor. **53** (2020).

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Slowly evolving *effective field*

Mapping into a stochastic frictional process for the single spin subject to a quasi-stationary effective potential

$$P(s|h(t)) = \frac{1}{Z(h)} \exp\left[-\frac{\mathcal{V}(s,h(t))}{T} \right]$$

The Boltzmann-Gibbs distribution at a given temperature.

$$\mathcal{V}(s,h(t)) = V(s) - \frac{p}{4T}(q_d^{p-1} - q_1^{p-1})s^2 - h(t)s$$

Similar results obtained for a particle in a random potential
*[Cugliandolo, Kurchan, J. Phys. Soc. Japan (2000)]

A. Altieri, G. Biroli, C. Cammarota, J. Phys. A.: Math. Theor. **53** (2020).

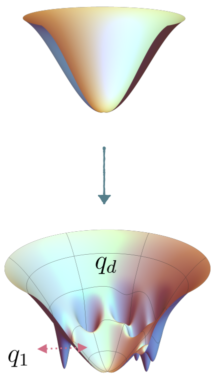
Effective process for the single variable

Mapping between a stochastic frictional process for the single spin and a quasi-stationary probability distribution

$$P(s|h(t)) = \frac{1}{Z(h)} \exp\left[-\frac{\mathcal{V}(s,h(t))}{T}\right]$$

Boltzmann-Gibbs distribution at a fixed external temperature.

$$\mathcal{V}(s,h(t)) = V(s) - \frac{p}{4T}(q_d^{p-1} - q_1^{p-1})s^2 - h(t)s$$

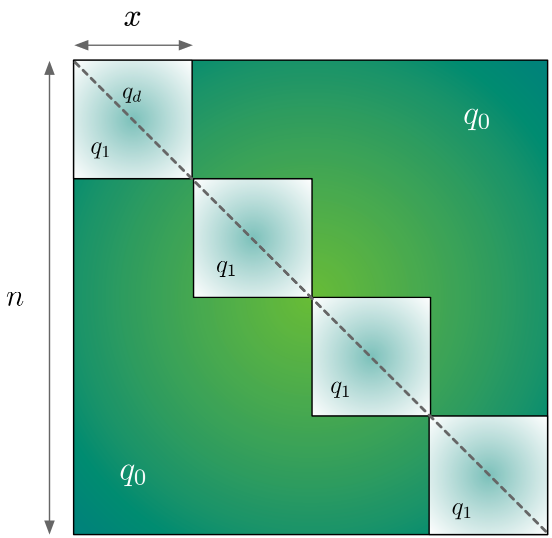


What do the overlap parameters represent?

What is their role in the statics?

A. Altieri, G. Biroli, C. Cammarota, J. Phys. A.: Math. Theor. **53** (2020).

Mapping with the statics



The mutual correlation functions at the initial and final times can be related to the "OVERLAP PARAMETERS" $\{q_d, q_1 \dots\}$ of the Parisi solution.

$q_d \rightarrow$ self - overlap

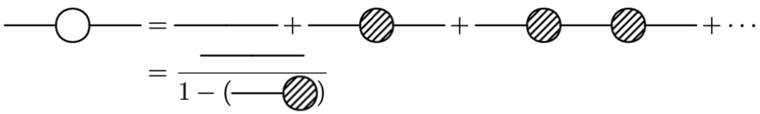
$q_1 \rightarrow$ innermost block overlap ("1RSB" parametrization)

$q_0 \rightarrow$ off - diagonal value.

Aging regime: one slow timescale

Physical requirement for aging: the dynamics in the TTI sector has to be marginal, *i.e.* the relaxation to the plateau occurs via a power-law.

Write an equation the inverse response function (Dyson): $$\int_{-\infty}^{t} R_0^{-1}(t-t')s(t')dt' = -\frac{\partial V(s)}{\partial s} + \xi_{\mathrm{TTI}}(t) + h(t)$$



$$R_{\mathrm{TTI}}(\omega) = \frac{1}{R_0^{-1}(\omega) - \Sigma(\omega)}$$

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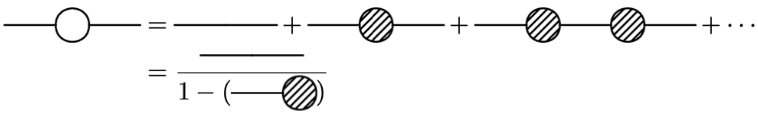

$$R_{\rm TTI}(\omega) = \frac{1}{R_0^{-1}(\omega) - \Sigma(\omega)}$$

The condition accounting for the existence of a marginal dynamics is

$$\lim_{\omega\to 0} \frac{\partial R_{\rm TTI}^{-1}(\omega)}{\partial \omega} = \infty \quad \longrightarrow \quad \frac{\partial R_{\rm TTI}^{-1}(\omega)}{\partial \omega} = \frac{-i - \ \partial\Sigma(\omega)/\partial\omega}{1 - \frac{p(p-1)}{2} q_1^{p-2} R_{\rm TTI}^2(\omega)}$$

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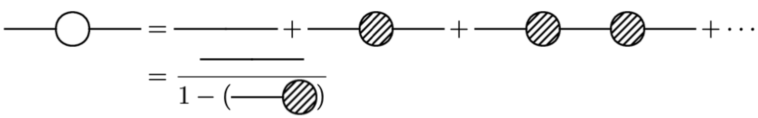

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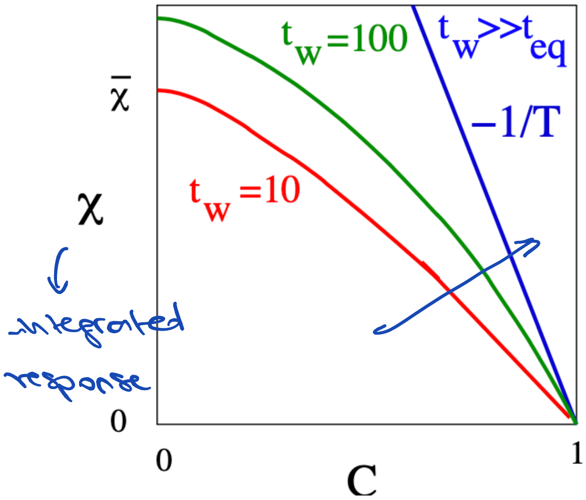
The denominator is singular for $1 = \frac{p(p-1)}{2} q_1^{p-2} \overline{\left(\frac{\langle s^2\rangle - \langle s\rangle^2}{T}\right)^2}$

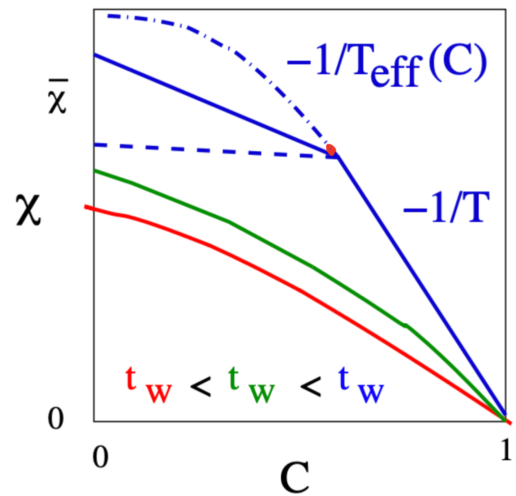
corresponding to a vanishing eigenvalue of the stability matrix.

Able to recover the missing equation on the **effective temperature / onset of non-ergodicity.**

A. Altieri, G. Biroli, C. Cammarota, J. Phys. A.: Math. Theor. **53** (2020).

Equilibrium versus non-equilibrium





The continuous line with a breaking point corresponds to a systems, like the p-spin spherical model (only one effective T).

The dot-dashed line corresponds to a case with an infinite number of timescales (aka infinite effective temperatures).

Interdisciplinary applications

Inference problems

In inference, it is usually important to identify the condition for which the gradient flow has a positive correlation with the signal.

Agoritsas, Biroli, Urbani, Zamponi ***J. Phys. A*** **(2018)**
Mannelli, Biroli, Cammarota, Krzakala, Urbani, Zdeborová, ***PRX*** **(2020)**
Mignacco, Krzakala, Urbani, Zdeborová, ***J. Stat. Mech.*** **(2021)**

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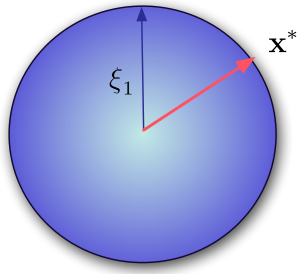
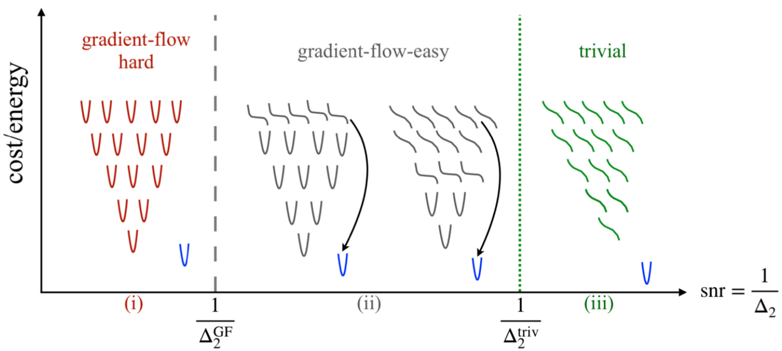


Loss function: $$L_1(\mathbf{x}^*) = \frac{1}{M}\sum_{\mu=1}^{M}\phi(\xi^\mu \cdot \mathbf{x}^*)$$

Evidence of very rough energy landscapes with many spurious minima, which can trap the dynamics.

However, gradient descent/SGD are often observed to work well even far away from the "easy" region of the landscapes.



Goal:

deriving the algorithmic threshold of the gradient flow.

Agoritsas, Biroli, Urbani, Zamponi *J. Phys. A* (2018)
Mannelli, Biroli, Cammarota, Krzakala, Urbani, Zdeborová, *PRX* (2020)
Mignacco, Krzakala, Urbani, Zdeborová, *J. Stat. Mech.* (2021)

The random Lotka-Volterra equations for species-rich ecosystems

Dynamical equations for the relative species abundances $N_i \geq 0$, with $i = 1, ..., S$

$$\frac{dN_i}{dt} = -N_i \left[\nabla_{N_i} V_i(N_i) + \sum_{j(j\neq i)} \alpha_{ij} N_j \right] + \underbrace{\sqrt{N_i}\eta_i(t)}_{\text{noise}} + \lambda_i$$

immigration rate

$$V_i(N_i) = -\rho_i \left(K_i N_i - \frac{N_i^2}{2} \right)$$ with $\rho_i = r_i / K_i$ (growth rate/carrying capacity).

Main assumptions:

- Demographic fluctuations modelled by Gaussian white noise with $\langle \eta_i(t) \rangle = 0$ and $\langle \eta_i(t)\eta_j(t') \rangle = 2T\delta_{ij}\delta(t-t')$
- Complex behaviour described by random interactions α_{ij} with $\langle \alpha_{ij} \rangle = \mu/S$, $\langle \alpha_{ij}^2 \rangle_c = \sigma^2/S$, $\langle \alpha_{ij}\alpha_{ji} \rangle_c = \gamma \langle \alpha_{ij}^2 \rangle_c$

DMFT formalism: $$\dot{N} = N\left\{ 1 - N - \mu\langle N(t)\rangle - \sigma\eta(t) + \gamma\sigma^2 \int_0^t ds\, \langle \chi(t,s)\rangle N(s) + H(t) \right\}$$

M. Barbier, J.F. Arnoldi, G. Bunin, M. Loreau, PNAS 115 (2018)
G. Bunin, Phys. Rev. E 95 (2017)
A. Altieri, F. Roy, C. Cammarota, G. Biroli, Phys. Rev. Lett. 126 (2021)

Conclusions and Perspectives

New approach proposed to deal with cases whose resulting integro-differential equations on the correlation and response functions DO NOT simplify.

- Definition of **self-consistent stochastic process for the single degree of freedom in a thermal bath**;
- Determination of a **marginal stability criterion** in models with **one slow timescale**;
- **Equation for the slow degrees of freedom with an infinite # of timescales**;
- **Mapping with the statics** & determination of the main quantities of interest (violation parameter/effective temperature, overlaps, distribution of the effective fields in the F-RSB ansatz).

Thank you for your attention!