Monday: Overview ; linear systems + control (freq.)

- What is control th? Why? Apps?
- dynamics and dynamical systems
- linearization + linear systems
- freq. domain, transfer functions, block diagrams
- PID control, delays and instabilities

Tuesday: Control of linear systems (time)

- controllability, observability, duality
- full-state control
- observers + output control

Wednesday: Optimal control

- parametric → Lagrangian (calc. variations)
- Bellman + dynamic prog., LQR
- constraints

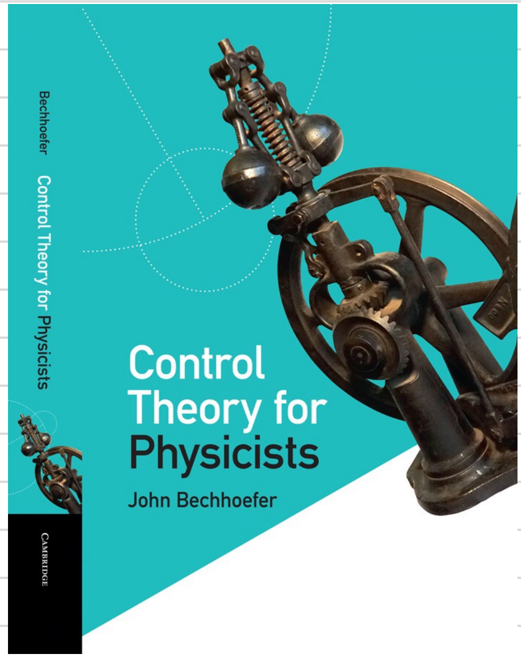
Thursday: Stochastic Control

- state estimation
- Kalman filter
- hidden Markov Models
- Bayesian methods
- nonlinear extensions (EKF, UKF, part.)

Friday: Bayesian State estimation / Noisy Maxwell Demon (seminar)

- redo Kalman filter

- Overall reference



- Cambridge Univ. Press
  2021

- www.sfu.ca/chaos
  → book → .CUP

→ for exercises +
  solutions

and also Math Appendix

- "Any sufficiently advanced technology is indistinguishable from magic."

  — Arthur C. Clarke, 1973

- dynamics with a purpose
  - profound implications
  - not just math! (a way of thinking)

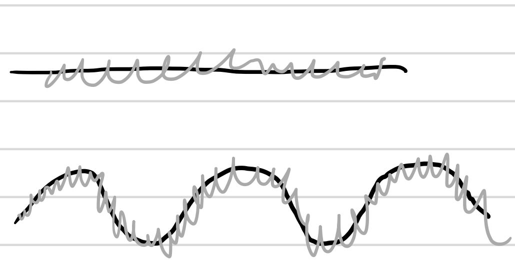
  purpose: - "intelligent design"
  - evolution (biology, technological)

Applications

- Technology ↔ better experiments
- Physics ↔ control can change physical dynamics (stabilize unstable state / Paul trap)
  - fundamentals of thermo.
  - quantum systems
  - control of complex networks
- Biology ↔
  - large scale (physiology)
  - small scale (genetic regulation)
  - single-mol. control
  - evolutionary time scales (pop. control)

Goals of control

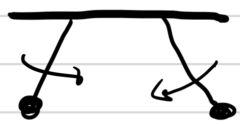
- Regulation
- Tracking



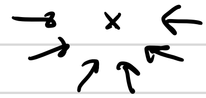
Our focus

- Changing the attractor
  - or potential, or...

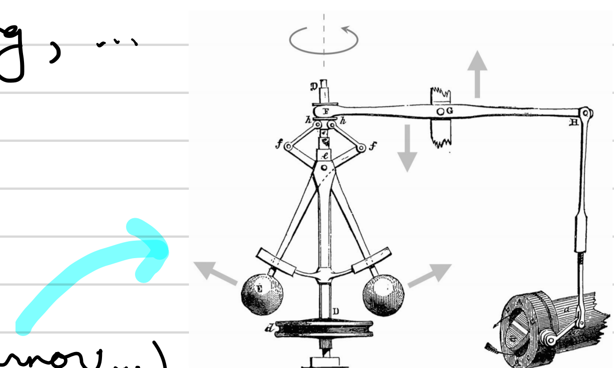
- Collective motion



or



synchronization, dancing, swarming, ...



Feedback loops

- Gadgets (flush toilet, governor...)
- Natural sys (climate feedbacks ...)
- Analogue control: circuit → amp → sys., sensor → circuit
- Digital control: computer → D/A → ▷ → sys., sys. → ▭ → A/D → computer

$$\boxed{x} \rightleftarrows \boxed{y}$$

$\dot{x} = f(x, y)$ $\qquad$ $\dot{y} = g(x, y)$

Autonomy

Causality

vs. $\dot{\vec{x}} = \vec{f}(\vec{x})$

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \vec{f} = \begin{pmatrix} f \\ g \end{pmatrix}$

Open loop

Cont. $\rightarrow$ Sys.

Closed loop

Cont. $\rightleftarrows$ Sys.

Autonomous $\leftrightarrow$ **Non-autonomous**

$\dot{x} = f(x)$ $\qquad$ $\dot{x} = f(x,t)$

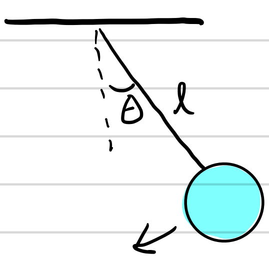
Complex controller — Universal Turing machine (can do anything)

Simple controller — all others

Review of Dynamical Systems (Ch. 2)

goal: set notation, introduce point of view

Pendulum as a dynamical sys.



torques: $m l^2 \ddot{\theta} + m g l \sin\theta = 0$

$\ddot{\theta} + \omega_0^2 \sin\theta = 0 \qquad \omega_0 = \sqrt{g/l}$

Rescale time $\quad \bar{t} = \omega_0 t \Rightarrow \frac{d^2\theta}{dt^2} \to \omega_0^2 \frac{d^2\theta}{d\bar{t}^2} = \omega_0^2 \theta''$

$\Rightarrow \quad \theta''(\bar{t}) + \sin\theta = 0$

Rescaling simplifies problem by reducing # pars. $(1 \to 0)$
- usually will be lazy and use same symbol after scaling

$\Rightarrow \quad \ddot{\theta} + \sin\theta = 0$

2^{nd}-order nonlinear ODE → 2 coupled, 1^{st}-order nonlin. ODEs.

$x_1 = \theta,\ x_2 = \dot\theta \quad \Rightarrow \quad \frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_2\\-\sin x_1\end{pmatrix} \quad \to \quad \dot x = f(x)$

The Systems Point of View

- Control: dynamical systems $\dot x = f(x)$

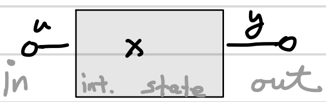
  \+ inputs u and outputs y

Control System

x → internal states — abstract

u → inputs — what we do

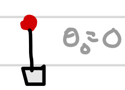
y → outputs — what we see



$$\dot x = f(x,u) \qquad y = h(x,u) \text{ or } h(x)$$

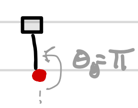
Linearization: small deviations from equilibrium

e.g. $\theta_0 = \dot\theta_0 = 0$ $\quad x_0 = \begin{pmatrix}0\\0\end{pmatrix}$ $\quad$ for small θ, $\sin\theta \approx \theta$



$\Rightarrow \quad \ddot\theta + \theta = 0 \qquad \frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \qquad \dot x = Ax$

Q: Linearize about $\theta_0 = \pi$, $\dot\theta_0 = 0$?

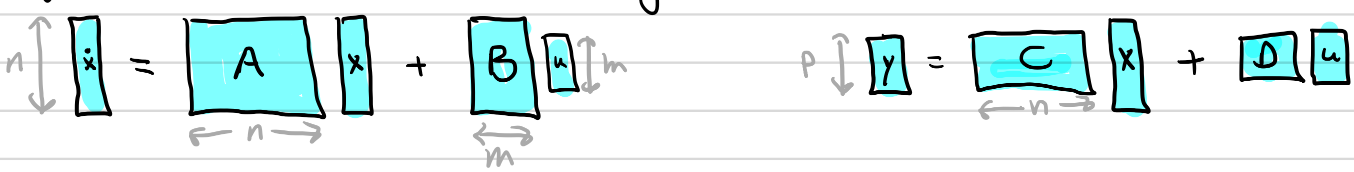


Add inputs, outputs: $\ddot\theta + \sin\theta = u(t)$ — direct torque $\qquad \dot x = f(x,u)$

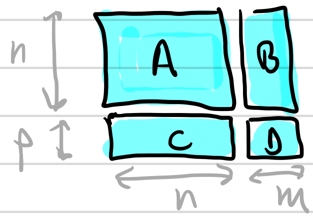
$y = (1\ 0)\begin{pmatrix}x_1\\x_2\end{pmatrix} = x_1$ (for ex.) $\qquad y = h(x,u)$

↳ observe angle, not ang. velocity

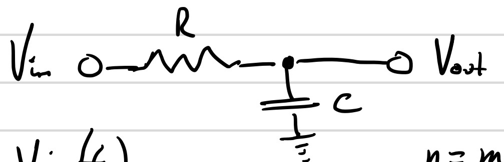
Often deal w/ linear systems $\dot{x} = Ax + Bu$ $\quad y = Cx + Du$



Nice packing of matrices:



Ex 1: Low-pass filter



$$\dot{V}_{out} = -\frac{1}{RC} V_{out}(t) + \frac{1}{RC} V_{in}(t)$$

$n = m = p = 1$, $x = y = V_{out}$, $u = V_{in}$

scale time $\quad \tau = RC \quad \Rightarrow \quad \dot{x} = -x + u, \quad y = x$

$A = -1, \quad B = 1, \quad C = 1, \quad D = 0$

Ex. 2: Harmonic osc.

$$\ddot{q} + 2\zeta\dot{q} + q = u \quad \rightarrow \quad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \overset{A}{\begin{pmatrix} 0 & 1 \\ -1 & -2\zeta \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overset{B}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} u \qquad y = \overset{C}{(1\ 0)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overset{D}{0}$$

Recall: scaling $\Rightarrow$ $\underset{\text{dimensional}}{3} \rightarrow \underbrace{1 \text{ parameter}}_{\text{dimensionless}}$

Why study linear systems?

1) easy; nice general methods
2) real eq., deviations obey linear dyn.
3) use nonlin. tracking $\Rightarrow$ deviations small enough so lin. is OK (maybe)

Nonlin to lin. (linearization) $\dot{x} = f(x,u)$ $y = h(x,u)$

fixed pt: $f(x_0, u_0) = 0$ $h(x_0, u_0) = y_0$

$x \equiv x_0 + x_1(t)$ $u \equiv u_0 + u_1(t)$ $y \equiv y_0 + y_1(t)$ $\{x_1, u_1, y_1\}$ small

Taylor: $\dot{x}_1 = f(x_0 + x_1, u_0 + u_1) \approx \cancel{f(x_0, u_0)}^{0} + \frac{\partial f}{\partial x}\Big|_{x_0,u_0} x_1 + \frac{\partial f}{\partial u}\Big|_{x_0,u_0} u_1 + \cdots$

$\cancel{y_0} + y_1 = h(x_0 + x_1, u_0 + u_1) \approx \cancel{h(x_0, u_0)} + \frac{\partial h}{\partial x}\Big|_{x_0,u_0} x_1 + \frac{\partial h}{\partial u}\Big|_{x_0,u_0} u_1$

$A = \frac{\partial f}{\partial x}$ $B = \frac{\partial f}{\partial u}$ $C = \frac{\partial h}{\partial x}$ $D = \frac{\partial h}{\partial u}$ all at (x_0, u_0)

Frequency-domain analysis

Fourier transform $f(\omega) = \int_{-\infty}^{\infty} dt \cdot f(t)\, e^{-i\omega t} = \mathcal{F}[f]$

→ Laplace transform $f(s) = \int_{0}^{\infty} dt \cdot f(t)\, e^{-st} = \mathcal{L}[f]$

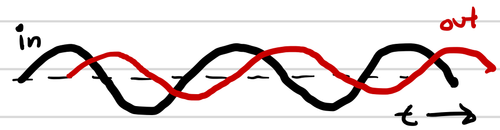
↳ complex

Transfer functions (dynamical response)

- in Laplace domain, $G(s) = \frac{y(s)}{u(s)}$ $y(s) = \int_0^\infty dt \cdot y(t)\, e^{-st}$

$G(s = i\omega)$ ~ Fourier transform

mag $|G(i\omega)|$ phase: $\tan^{-1} \frac{\mathrm{Im}\, G}{\mathrm{Re}\, G}$



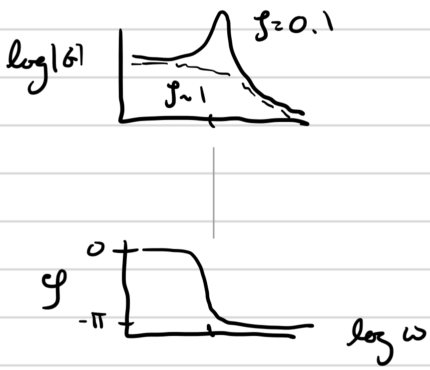
Ex: 2^{nd} order sys. $\ddot{y} + 2\zeta \dot{y} + y = u(t)$ (ignore init cond.)

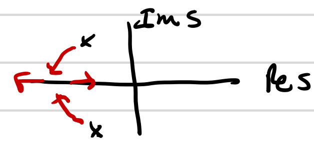
→ $G(s) = \frac{1}{1 + 2\zeta s + s^2}$

Graphical tools:

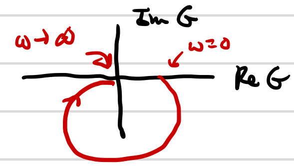
Complex s-plane

Bode plots





Pole-zero



Nyquist

- n^{th} - order ODE $\qquad y^{(n)} + \cdots \quad y = 0$

$\Rightarrow \quad G(s) \sim s^{-n} \quad \Rightarrow \quad G(i\omega) \sim (-i)^n \omega^n$

$-i \Rightarrow$ phase lag of $\pi/2$ (90°) $\Rightarrow (-i)^n \to$ phase lag of $n\pi/2$

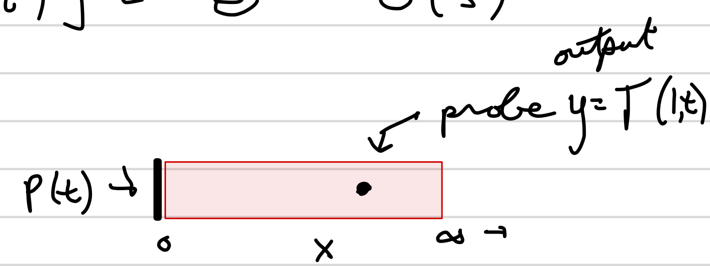
if Bode amplitude has asymptotic slope $= -n$ $\qquad$ Bode phase at least $-\frac{n\pi}{2}$

- More generally $\quad G(s) = \dfrac{b(s)}{a(s)} \begin{matrix} \sim s^k \\ \sim s^n \end{matrix}$ (rational poly.)

n = order of $G(s)$ $\qquad n-k$ = relative degree $\qquad$ (a, b coprime)

- Systems w/ non-rational $G(s)$
  - delay $\qquad \mathcal{L}[g(t-\tau)] = e^{-s\tau}\, G(s)$
  - spatially extended

Ex: 1d rod:



$\partial_t T = \partial_{xx} T \qquad T \to 0$ at $x \to \infty \qquad -\partial_x T = u(t)$ input

$sT = \partial_{xx} T \quad \Rightarrow \quad T = \cancel{A e^{\sqrt{s}x}} + B e^{-\sqrt{s}x}$

$\Rightarrow \cdots \qquad \boxed{G(s) = e^{-\sqrt{s}}/\sqrt{s}}$

"irrational"

can approx $G(s)$ by "rational" poly ...

Convolution Thm. and Block Diagrams

given $g(t)$ and $h(t)$ $\quad f = g * h = \int_0^\infty dt' \, g(t') \, h(t'-t)$

$$\mathcal{L}[g * h] = g(s) \, h(s)$$

arises when output of one sys. = input to the next ...

Ex: **Sensor dynamics**

$$\ddot{\nu} + 2\zeta\dot{\nu} + \nu = u(t) \qquad \dot{y} + y = \nu(t)$$

ie u drives osc. position ν, which is "measured" by y

Laplace $\Rightarrow$

$$\nu(s) = G(s)\, u(s) = \left(\frac{1}{1+2\zeta s+s^2}\right) u(s)$$

$$y(s) = H(s)\, \nu(s) = \left(\frac{1}{1+s}\right)\left(\frac{1}{1+2\zeta s+s^2}\right) u(s)$$

Depicted as

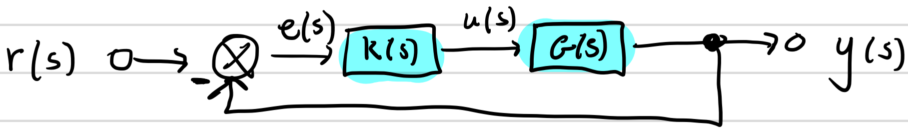
$$u(s) \circ\!\!\longrightarrow \boxed{G(s)} \longrightarrow \boxed{H(s)} \longrightarrow\!\!\circ\; y(s)$$

Series connection

- Frequency (Laplace) domain is intuitive but mostly limited to linear, time-invariant (LTI) systems.

- Developed (mostly) in US 1920-1960 (Bell Labs)
  - Nyquist, Bode, ...

Elementary SISO Feedback loops



$G(s)$ — system transfer function ("plant" in engineering jargon)

$K(s)$ — controller " "

$y(s)$ — output of dynamical system

$u(s)$ — input " " (output of controller)

$r(s)$ — reference (command)

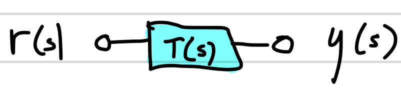
$e(s)$ $=$ $r-y$ (negative) error signal

Closed-loop dynamics

$$u = Ke = K(r-y)$$

$$y = Gu = GK(r-y)$$

$$y(1+GK) = GK\,r \qquad y = \frac{GK}{1+GK}\,r \equiv T(s)\,r$$



$$T = \frac{GK}{1+GK}$$ → transfer func. $r(s) \longrightarrow y(s)$ (in → out)

loosely: $K \to \infty \Rightarrow T \to 1$

but usually goes unstable for large K ...

Proportional control

$K(s) = K_p$

- $u(s) = K_p\, e(s) = -K_p\,(y(s) - r(s))$

  – negative feedback $\propto$ minus the deviation from the setup

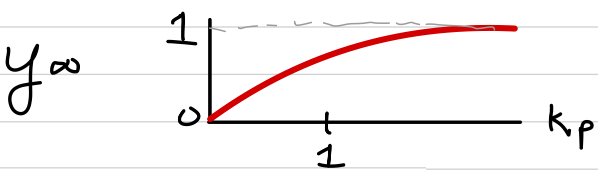
- eg $G(s) = \frac{1}{1+s}$

Q: Check in time domain

$$T(s) = \frac{GK}{1+GK} = \frac{1}{1+(GK)^{-1}} = \frac{1}{1+(1+s)/K_p} = \frac{1}{(1+1/K_p) + s/K_p}$$

$\Rightarrow$ new 1^{st}-order dynamics, – $y_\infty = \frac{K_p}{K_p+1}$

– relaxes at a rate K_p times faster.



– "proportional droop"

– intuition w/ heater

Integral control

$$u(t) = K_i \int_{-\infty}^{t} dt'\, e(t')$$

$$\dot{y} = -y + K_i \int_{-\infty}^{t} dt'\, e(t') \qquad e = r_\infty - y(t)$$

$$\ddot{y} = -\dot{y} + K_i\,(r_\infty - y) \quad \Rightarrow \quad y(t) \to r_\infty \;!!$$

In freq. space $K(s) = \frac{K_i}{s}$ "freq-dep gain"

$K \to \infty$ at $\omega = 0$ $\Rightarrow$ at zero freq., ∞ gain

Ex: $G(s) = \frac{1}{1+s}$ $\qquad T(s) = \frac{KG}{1+KG} = \frac{1}{1+(KG)^{-1}} = \frac{1}{1+s\left(\frac{1}{K_i}\right)(1+s)}$

$$= \frac{1}{1+\frac{1}{K_i}s+\frac{1}{K_i}s^2}$$

$T(s\to 0) = 1 \quad \Rightarrow$ unit step response!

As K_i increases, go from over $\to$ underdamped

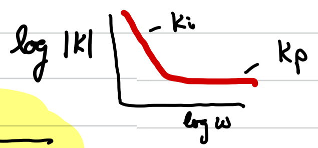
poles: $s = -\frac{1}{2} \pm \sqrt{\frac{1}{4}-K_i}$

Notice that the feedback has turned a 1^{st}-order sys. into a second-order sys. (closed loop)

- Robustness of integral control: $y \to r_\infty$ whatever the value of K_i, etc. only the structure of the loop (and int. cont.) count

- famous application: Robustness of chemotaxis response. (Barkai - Leibler, Nature 97)
  - connection to integral control, Yi et al (John Doyle) PNAS 2000

PI control: $K(s) = K_p + \frac{K_i}{s}$



$$T(s) = \frac{1}{1+\frac{s}{K_p s+K_i}(1+s)} = \frac{K_p s+K_i}{s^2+(1+K_p)s+K_i}$$

still have $T(s\to 0) = 1$ but $K_p \gg 1 \Rightarrow$ no osc. response

PID control: Proportional – Integral – Derivative

- "D" anticipates the future (crudely)
- also: adds damping $\ddot{x} + 2\zeta\dot{x} + x = -K_d\dot{x}$

$$u(t) = \underbrace{K_p\, e(t)}_{\text{present}} + \underbrace{K_i \int dt'\, e(t')}_{\text{past}} + \underbrace{K_d \frac{de}{dt}}_{\text{future}}$$

$$K(s) = K_p + \frac{K_i}{s} + K_d s$$

How to choose the parameters?

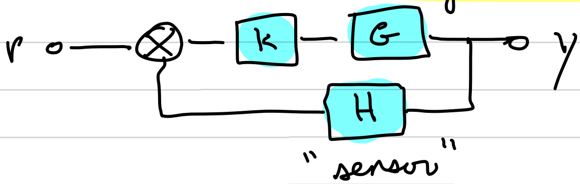
PID is simple; heuristic rules work

- e.g. increase K_p ("P") until osc.; decrease a bit
- More complex sys. need better methods; eg optimal cont.

→ Wed.

Feedback and Instabilities

Ex: 1st - order sys. with short delay



$$T = \frac{KG}{1+KGH}$$

$G = \frac{1}{s+1}$ — first-order system

$K = K_p$ — prop. gain

$H = e^{-s\tau}$ — delay $(e^{-i\omega\tau})$

Condition for instability $L(s) = KGH = -1$

- short delay $\Rightarrow$ $\tau \ll 1$; at high freq. $G \approx \frac{1}{s}$

$$\Rightarrow \quad L(s) = K_p \frac{1}{s} \cdot e^{-s\tau}\Big|_{s=i\omega} = -1$$

$$K_p \cdot \frac{1}{i\omega} \cdot e^{-i\omega\tau} = -1$$

$$\Rightarrow \quad K_p = \omega \qquad e^{-i\omega\tau} = -i = e^{-i\pi/2}$$

$$\omega^*\tau = \pi/2 \quad \text{or} \quad \boxed{K_p^* = \omega^* = \frac{\pi}{2}\left(\frac{1}{\tau}\right)} \qquad \left(f^* = \frac{1}{4\tau}\right)$$

Max gain $K_p^* = \omega^* \sim \tau_0/\tau = \frac{\text{sys}}{\text{delay}}$ $\left(G(s) = \frac{1}{1+s\tau_0}\right)$

Turn up gain on typical closed-loop control

$\longrightarrow$ get a Hopf bif. w/

critical gain K^*, osc. freq. ω^*

WHY ??

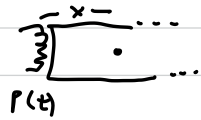
- Increasing prop. gain $\Rightarrow$ shorter time const.
  (higher feedback bandwidth)
  $\rightarrow$ ω_c' increases
- lumped element behavior $\rightarrow$ field behavior
- fields + input-output separation $\approx$ delay
- delay + fb $\Rightarrow$ Hopf

Ex: 1d rod $\quad \omega \to 0 \quad G(s) \sim 1/s$ (capacitor)
eventually semi-∞ $\quad G(s) \sim e^{-\sqrt{s}}/\sqrt{s}$
$\Rightarrow$ Hopf (see book text, problem)

Or: Microphone + speaker (freq $\sim \frac{c_{sound}}{l}$)
$\vdash\!\!\underset{l}{\text{———}}\!\!\dashv$

Things to do:

$T(x(t)) = y(t)$

1) For 1d semi-∞ rod, show 

$$G(s) = \frac{y(s)}{P(s)} = e^{-\sqrt{s}}/\sqrt{s}$$

See p. 9, bottom

2) Show that a proportional temperature controller for 1d rod has inst. (like delay ex.) for finite gain.

See book problem 3.9
(+ solution, too ...)

For Tuesday: linear control in time dom.