# Introduction to Digital Design

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# Outline

- Digital CMOS design
  - Boolean algebra
  - Basic digital CMOS gates
  - Combinational and sequential circuits
  - Coding - Representation of numbers

# Boolean Algebra

- English mathematician 1815 - 1864

  1854 : *An Investigation of the Laws of Thought*

# Boolean Algebra

- Let $B = \{ 0 , 1 \}$ — $B$ is the Boolean set
  0 and 1 are the Boolean constants
- Let $x \in B$ — $x$ is a Boolean variable

# Boolean Algebra

- Unary functions : $B \longrightarrow B$

Unary function *0* : $\forall x \in B, \quad x \mapsto 0$

Unary function *1* : $\forall x \in B, \quad x \mapsto 1$

Unary function *Identity* : $\forall x \in B, \quad x \mapsto x$

Unary function *Not* :

$$0 \mapsto 1$$
$$1 \mapsto 0$$

$Not(x)$ is denoted $\bar{x}$

# Boolean Algebra

- Binary functions : $B^2 \longrightarrow B$

function *And* :

$\forall\, x, y \in B,\ And(x, y) = 1$ if and only if $x = 1$ and $y = 1$

| |
|---|
| $And$ is also called $Min$ and denoted $x \cdot y$ |

function *Or* :

$\forall\, x, y \in B,\ Or(x, y) = 0$ if and only if $x = 0$ and $y = 0$

| |
|---|
| $Or$ is also called $Max$ and denoted $x + y$ |

# Boolean Algebra

- Other binary functions can be defined combining *And*, *Or* and *Not*

function *Nand* : $Nand(x,y) = Not\big(And(x,y)\big)$

function *Nor* : $Nor\ (x,y)\ = Not\big(Or\ (x,y)\big)$

function *Xor* : $Xor\ (x,y)\ = Not(x) \cdot y\ +\ x \cdot Not(y)$

$Xor(x,y)$ is denoted $x \oplus y$

# Boolean Algebra

- Noticeable properties

$$Not\left(Not\left(x\right)\right) = x \qquad \bar{\bar{x}} = x$$

$$x \cdot x = x \qquad x + x = x \qquad x \oplus x = 0$$

$$x \cdot \bar{x} = 0 \qquad x + \bar{x} = 1 \qquad x \oplus \bar{x} = 1$$

$$x \cdot 0 = 0 \qquad x + 0 = x \qquad x \oplus 0 = x$$

$$x \cdot 1 = x \qquad x + 1 = 1 \qquad x \oplus 1 = \bar{x}$$

# Boolean Algebra

- Noticeable properties

Commutativity

$$x \cdot y = y \cdot x$$
$$x + y = y + x$$
$$x \oplus y = y \oplus x$$

Associativity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$x + (y + z) = (x + y) + z$$
$$x \oplus (y \oplus z) = (x \oplus y) \oplus z$$

# Boolean Algebra

- Noticeable properties

Distributivity

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

$$x \cdot (y \oplus z) = (x \cdot y) \oplus (x \cdot z)$$

$$x + (y \cdot z) = (x + y) \cdot (x + z)$$

De Morgan

$$\overline{x \cdot y} = \bar{x} + \bar{y}$$

$$\overline{x + y} = \bar{x} \cdot \bar{y}$$

Absorption

$$x + (\bar{x} \cdot y) = x + y$$

# Boolean Algebra

- Let $B = \{0, 1\}$ — $B$ is the Boolean set
  0 and 1 are the Boolean constants
- Let $x \in B$ — $x$ is a Boolean variable
- Let $V \in B^n$ — $V$ is a Boolean vector

# Boolean Algebra

- $V \in B^n, V = (v_1, \cdots, v_i, \cdots, v_n)$
  $U \in B^n, U = (u_1, \cdots, u_i, \cdots, u_n)$

  The number of Boolean variables that are different between $V$ and $U$ is called the Hamming Distance $(V, U)$

  Example :

  $$HD\big((0,0,0,1),(1,0,1,0)\big) = 3$$

# Boolean Algebra

To vectors are said adjacent when their Hamming distance = 1

Example :

$$HD\left((0,0,0,1),(1,0,0,1)\right) = 1$$

# Boolean Algebra

- Let $B = \{0, 1\}$ — $B$ is the Boolean set; 0 and 1 are the Boolean constants
- Let $x \in B$ — $x$ is a Boolean variable
- Let $V \in B^n$ — $V$ is a Boolean vector
- Let $f : B^n \longrightarrow B$ — $f$ is a Boolean function (dimension n)
- Let $\mathcal{B}_n$ — the set of Boolean functions of dimension n

$$Card\ (\mathcal{B}_n) = 2^{2^n}$$

# Boolean Algebra

- $Card\ (B^n)$ is finite

A Boolean function $f$ may be defined by its Truth Table :

for each Boolean vector $V$ give the value $f\ (V)$

| $x$ | $y$ | $z$ | $f$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

# Boolean Algebra

- Unary functions : $\mathcal{B}_n \longrightarrow \mathcal{B}_n$

  function $Not$ : $\big(Not(f)\big)\,(V) = Not\,\big(f\,(V)\big)$

- Binary functions : $\mathcal{B}_n{}^2 \longrightarrow \mathcal{B}_n$

  function $And$ : $\big(And(f,g)\big)\,(V) = And\,\big(f\,(V), g\,(V)\big)$

  function $Or$ : $\big(Or\,(f,g)\big)\,(V) = Or\,\big(f\,(V), g\,(V)\big)$

# Boolean Algebra

- Let $V \in B^n, V = (v_1, \cdots, v_i, \cdots, v_n)$

The Boolean function

$f \in \mathcal{B}_n \,/\, f(V) = v_i$

is denoted $v_i$

| $x$ | $y$ | $z$ | $f = y$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

# Boolean Algebra

- A Boolean function $f$ may be defined by giving a Boolean expression

$$f = \bar{x} \cdot y \cdot z + x \cdot \bar{y} \cdot \bar{z} + x \cdot z$$

$$f = x \cdot \bar{y} + y \cdot z$$

| $x$ | $y$ | $z$ | $f$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

The expression is not unique

# Boolean Algebra

- Let $f \in \mathcal{B}_n$

$$f = \bar{x}yz + x\bar{y}\bar{z} + x\bar{y}z + xyz$$

| $x$ | $y$ | $z$ | $f$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

min-term

$$f = \sum\left(\alpha_j \cdot \prod(\tilde{v}_i)\right)$$

$$\alpha_j = \begin{cases} 0 \\ 1 \end{cases} \qquad \tilde{v}_i = \begin{cases} v_i \\ \overline{v_i} \end{cases}$$

disjunctive normal form (DNF)

# Boolean Algebra

- Let $f \in \mathcal{B}_n$

$$f = (x+y+z)\cdot(x+y+\bar{z})\cdot(x+\bar{y}+z)\cdot(\bar{x}+\bar{y}+\bar{z})$$

| $x$ | $y$ | $z$ | $f$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

max-term

$$f = \prod\left(\beta_j + \sum(\tilde{v}_i)\right)$$

conjunctive normal form (CNF)

# Boolean Algebra

- Let $f \in \mathcal{B}_n$

  $f$ is independent from the variable $v_i$ iff :

  $$\forall\ V \in B^n, V = (v_1, \cdots, v_i, \cdots, v_n)$$

  $$f\big((v_1, \cdots, v_i, \cdots, v_n)\big) = f\big((v_1, \cdots, \overline{v_i}, \cdots, v_n)\big)$$

# Boolean Algebra

- Let $f \in \mathcal{B}_n$

  $\exists!\ f_{i0}\ , f_{i0}$ independent from the variable $v_i$ /

  $$f = v_i \cdot f_{i1} + \overline{v_i} \cdot f_{i0}$$

**Shannon decomposition**

# Boolean Algebra

- Let $f \in \mathcal{B}_n$

$$f = v_i \cdot f_{i1} + \overline{v_i} \cdot f_{i0}$$

Example :

$$f = x \cdot (\bar{y} + z) + \bar{x} \cdot (yz)$$

Shannon decomposition of $f$ regarding $x$

| $x$ | $y$ | $z$ | $f$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

normal form (given an order of variables)

# Boolean Algebra

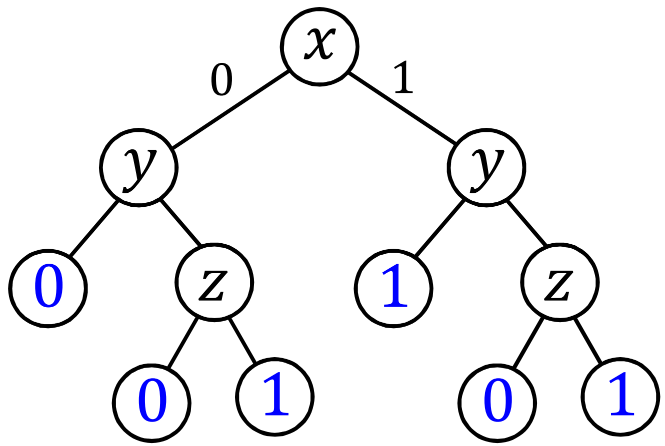
$$f = \bar{x}yz + x\bar{y}\bar{z} + x\bar{y}z + xyz$$

$$f = (x + y + z) \cdot (x + y + \bar{z}) \cdot (x + \bar{y} + z) \cdot (\bar{x} + \bar{y} + \bar{z})$$

list of terms

$$f = x \cdot (\bar{y} + z) + \bar{x} \cdot (yz)$$

binary tree




binary decision diagram (BDD)

# Boolean Algebra

- Let $f = v_i \cdot f_{i1} + \overline{v_i} \cdot f_{i0}$

  $f$ is independent from the variable $v_i$ iff :

$$f = f_{i1} = f_{i0}$$

$$f_{i1} \oplus f_{i0} = 0$$

$f_{i1} \oplus f_{i0} = 0$ : $f$ is insensitive to $v_i$

notion of derivative

# Boolean Algebra

- Let $f = v_i \cdot f_{i1} + \overline{v_i} \cdot f_{i0}$

$$\frac{\partial f}{\partial v_i} = f_{i1} \oplus f_{i0}$$

# Boolean Algebra

- Let $f = v_i \cdot f_{i1} + \overline{v_i} \cdot f_{i0}$

$$\frac{\partial f}{\partial v_i} = f_{i1} \oplus f_{i0} = f_{i1} \cdot \overline{f_{i0}} + f_{i0} \cdot \overline{f_{i1}}$$

$f_{i1} \cdot \overline{f_{i0}}$ and $f_{i0} \cdot \overline{f_{i1}}$ cannot be 1 for the same vector

if $f_{i1} \oplus f_{i0} \neq 0$

$f$ may be sensitive to $v_i$ in two ways

# Boolean Algebra

- Let $f = v_i \cdot f_{i1} + \overline{v_i} \cdot f_{i0}$

$$\frac{\partial f}{\partial v_i} = f_{i1} \oplus f_{i0} = f_{i1} \cdot \overline{f_{i0}} + f_{i0} \cdot \overline{f_{i1}}$$

Let $f_{i1}(V) = 1$ and $f_{i0}(V) = 0$ then $f = v_i \cdot 1 + \overline{v_i} \cdot 0$

Let $f_{i1}(V) = 0$ and $f_{i0}(V) = 1$ then $f = v_i \cdot 0 + \overline{v_i} \cdot 1$

# Boolean Algebra

- Let $f = v_i \cdot f_{i1} + \overline{v_i} \cdot f_{i0}$ $$\frac{\partial f}{\partial v_i} = f_{i1} \cdot \overline{f_{i0}} + f_{i0} \cdot \overline{f_{i1}}$$

if $(f_{i1}\overline{f_{i0}})(V) = 1,$ $f$ varies in direct way with $v_i$
$f$ is a positive function of $v_i$

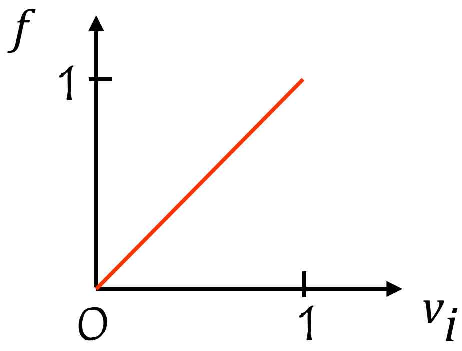
if $(f_{i0}\overline{f_{i1}})(V) = 1,$ $f$ varies in opposite way with $v_i$
$f$ is a negative function of $v_i$

$$\frac{\partial f^+}{\partial v_i} = f_{i1} \cdot \overline{f_{i0}}$$

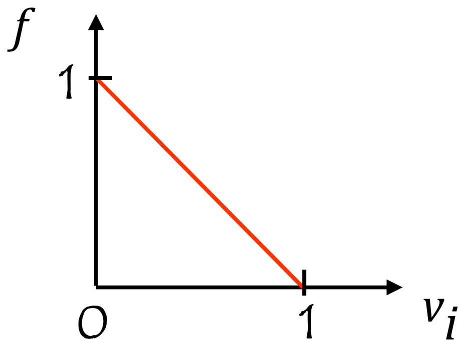
$$\frac{\partial f^-}{\partial v_i} = f_{i0} \cdot \overline{f_{i1}}$$

# Boolean Algebra

- $$\frac{\partial f^+}{\partial v_i} = f_{i1} \cdot \overline{f_{i0}}$$




- $$\frac{\partial f^-}{\partial v_i} = f_{i0} \cdot \overline{f_{i1}}$$



# Boolean Algebra

Example :

$$f = y \cdot (z) + \bar{y} \cdot (x)$$

$$\frac{\partial f}{\partial y} = (z) \oplus (x)$$

$$\frac{\partial f^{+}}{\partial y} = z\bar{x}$$

$$\frac{\partial f^{-}}{\partial y} = \bar{z}x$$

| $x$ | $y$ | $z$ | $f$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |