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Beyond MHD -
A Closed Fluid Description

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Beyond MHD — A Closed Fluid Description

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Abstract

With the constraints of the Lorentz covariance as a guide, we construct the most general energy-momentum tensor for a magnetized (to be defined in the text) plasma subject to a dominant electromagnetic force. A consistent scheme is developed to derive a closed set of fluid equations determining all the unknowns in the energy-momentum tensor. Since a complete knowledge of the energy-momentum tensor is sufficient to close the Plasma-Maxwell system, our set of equations represents a relatively complete description of a collisionless magnetized plasma. The new theory subsumes the standard magnetohydrodynamics (MHD); in fact, it takes the original MHD program (i.e., a theory of magnetized plasma) to its logical limit. Relativistic as well as the Nonrelativistic (directed as well as thermal speed much smaller than the speed of light) manifestations of the system are displayed.

I. INTRODUCTION

Magnetic fields are the universal and principal instruments for confining plasmas barring a few very special cases,like the stellar interiors, where the intense gravitational fields do the job. The study of magnetized plasmas (an appropriate definition will be provided later), therefore, is a study of most plasmas that are relatively long lived. Within the framework of classical physics the most detailed descriptions of plasma dynamics are based on kinetic theory in which each component of the plasma is viewed as a fluid in the six-dimensional phase space. The kinetic approach is relatively complete but highly complicated especially when applied to plasmas which are spatially inhomogeneous, that is, most plasmas of interest. A less ambitious but often more manageable formulation of plasma dynamics emerges when we are content to treat it as an ordinary fluid in the three-dimensional configuration space. The program, then, consists of deriving the evolution equations of various physically meaningful quantities (velocity moments of the kinetic distribution function) like the mass and charge densities, the velocity (momentum) field, and pressure ; these equations are the equivalent of the conservation laws one encounters in ordinary hydrodynamics but with electromagnetic forces dominating the show.

All fluid theories have to cross a serious generic hurdle to be taken seriously; it is the problem of closure. The evolution of each successive moment depends on a higher order moment leading to an infinite set of equations which are of little use unless we can find a prescription to truncate the system, i.e., we can express the nth order moment fully in terms of the the lower $n-1$ moments. The most successful and widely used fluid theory of magnetized plasmas, the magnetohydrodynamics (MHD) [1], affects this closure by assuming a plasma stress tensor (energy-momentum tensor) dictated fully by a local thermodynamic equilibrium. We remind the reader that a knowledge of the stress tensor is sufficient to calculate the charge and current densities needed to complete the Plasma-Maxwell system.

Although MHD does capture several key features of a magnetized plasma (the electromagnetic nature of its flow ($E \times B$ drift), for example), its reliance on the thermodynamic closure fails to do justice to the dominant determinant of plasma dynamics, the electromagnetic force. A more consistent treatment should allow the stress tensor to be determined by electrodynamics just as the plasma flow velocity is. The "gyrotropic" CGL [2] tensor introduced by Chew, Goldberger and Low was a step in this direction. Reflecting the presence of a strong magnetic field, the CGL tensor displays the characteristic anisotropy between directions parallel and perpendicular to the magnetic field. The CGL theory , however, was seriously flawed because the ("double-adiabatic") laws used to advance the stress tensor are not obviously physical, especially since heat flow along the field lines of a low collisionality plasma can be rapid. In any case, we shall soon show that this tensor (and its relativistic generalization) is not the most general one consistent with a dominant electromagnetic force. It is fortunate that the most general energy-momentum tensor we find in this paper is also physically reasonable (and warranted) and leads to a consistent and relatively clean fluid description.

In this paper we develop a closed, Lorentz invariant, fluid theory of magnetized plasmas. The Lorentz invariance of the theory extends the domain of validity of this theory to include relativistic plasmas [3–7] : the plasmas in which either the thermal speed (the *rms* speed of individual particles) measured in the fluid rest frame, or the local bulk flow measured in some convenient frame, can approach the speed of light. Many astrophysical and some laboratory plasmas fall in this category although for a majority of laboratory plasmas a non-relativistic limit (a low velocity and a low-temperature limit of the general theory) will generally suffice.

The demands of special relativity (Lorentz invariance) have a rather profound effect on the very formulation of the theory. In the modern theories of elementary particles, the symmetries are often the only guide to determine the form of interactions. We find that Lorentz invariance does precisely that in this purely classical context; the form of the energy momentum tensor, the centerpiece of the theory, is dictated almost entirely by the considerations of Lorentz invariance. We are tempted to suggest that perhaps the best way to derive even purely nonrelativistic theories is to first set up a Lorentz invariant formalism and then take the appropriate limit.

Starting from the exact relativistic fluid equations, obtained by taking the moments of the kinetic equation, we will first work out in Sec. 2 the schematics of a closure program for a magnetized plasma. Quite predictably, the plasma magnetization alone proves to be an insufficient assumption for a full closure: the scalar functions appearing in the theory — such as enthalpy density, and the perpendicular and parallel pressures — outnumber the field equations. To achieve closure we must resort to a representative distribution function for each plasma species, chosen consistently with relativity, magnetization, anisotropy and heat flow, in fact, chosen to reproduce the unique form of the energy momentum tensor found independently by solving the appropriate moment equation with the constraints of Lorentz invariance.

For many applications of the magnetized fluid set we can avoid the calculational complications of the relativistic equations (the compact tensor notation often hides their complexity) and deal directly with their non-relativistic limit. Since the original equations are relativistic both in the directed and the thermal speed, the derivation of the nonrelativistic set will require taking two simultaneous limits $v/c \ll 1, v_{th}/c \ll 1$. Derivation and display of the nonrelativistic set and comparing it with the standard MHD and the CGL theory makes up the content of Sec. 3.

In Sec. 4 we carry out a token illustrative calculation and derive a dispersion relation for the low-frequency waves in a homogenous plasma. By comparing it with similar calculations for MHD and the CGL systems, we show that the inclusion of heat-flow (an essential element of the new theory) brings the physics closer to the predictions of the drift-kinetic theory.

It is worth emphasizing that the covariant analysis of fluid equations for magnetized plasmas is simpler and more transparent (primarily due to the use of compact tensor notation) than the nonrelativistic version found in many textbooks. In particular the relativistic derivation suggests straightforward means for the inclusion of finite gyroradius physics. Such generalization will be the subject of future work.

II. FLUID CLOSURE

General

Our search for a closed fluid description begins with the following three exact (collisionless) conservation laws for each species of the plasma,

$$\frac{\partial \Gamma^{\nu}}{\partial x^{\nu}} = 0 \tag{1}$$

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} - eF^{\mu\nu}\Gamma_{\nu} = 0 \tag{2}$$

$$\frac{\partial M^{\mu\alpha\beta}}{\partial x^{\mu}} - e(F^{\alpha\nu}T_{\nu}^{\ \beta} + F^{\beta\nu}T_{\nu}^{\alpha}) = 0. \tag{3}$$

Here Γ^{μ} is the four-vector measure of the fluid particle-flux density, $T^{\mu\nu}$ is the energy-momentum tensor, and $M^{\mu\alpha\beta}$ is the third-rank moment which we will call the "stress-flow" tensor. The flux

$$\Gamma^{\mu} = n_R U^{\mu} \tag{4}$$

where n_R is the plasma density in the rest-frame, and $U^{\mu} = (\gamma, \gamma \boldsymbol{V})$ is the local four-velocity of the fluid, with

$$\gamma^2 = (1 - V^2)^{-1} \tag{5}$$

the relativistic dilation factor. Notice that in an arbitrary frame the density $n = \gamma n_R$.

These moments are defined in terms of the (Lorentz-scalar) distribution function $f(x, p)$, where p represents the four-momentum p^{μ}:

$$\Gamma^{\alpha} \equiv \int \frac{d^3p}{p^0} f p^{\alpha} \tag{6}$$

$$T^{\alpha\beta} \equiv \int \frac{d^3p}{p^0} f p^{\alpha} p^{\beta} \tag{7}$$

$$M^{\alpha\beta\gamma} \equiv \int \frac{d^3p}{p^0} f p^{\alpha} p^{\beta} p^{\gamma} \tag{8}$$

where d^3p/p^0 is the invariant momentum-space volume element. Recall that for physical particles, the four-momentum p^{μ} satisfies the mass-shell condition

$$p^0 = \sqrt{m^2 + \boldsymbol{p}^2}. \tag{9}$$

where m is the particle rest-mass. The distribution function obeys

$$\frac{p^{\mu}}{m}\frac{\partial f}{\partial x^{\mu}} + g^{\mu}\frac{\partial f}{\partial p^{\mu}} = 0 \tag{10}$$

where $g^{\mu} = eF^{\mu\nu}p_{\nu}$ is the electromagnetic four-force experienced by the particle of charge e. The faraday tensor $F^{\mu\nu}$ will have the standard form; the reader is requested to consult

Ref. [8] for various properties of the electromagnetic (e.m.) field tensor and other tensors derived from it.

There are two unique Lorentz scalars (relativistic invariants) associated with the e.m.field; these are the contractions of the the tensor F with itself, and with its dual

$$\mathcal{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}$$

where $\epsilon^{\mu\nu\kappa\lambda}$ is the completely antisymmetric tensor:

$$\frac{1}{2}F_{\kappa\lambda}F^{\kappa\lambda} = B^2 - E^2 \equiv W$$

and

$$\frac{1}{2}\mathcal{F}^{\mu\kappa}F_{\kappa\mu} = \boldsymbol{E}\cdot\boldsymbol{B} \equiv \lambda W \tag{11}$$

The scalars W and λ play an essential role in defining the ordering used in this paper.

Magnetized plasma

A plasma system is considered to be magnetized if two criteria are satisfied:

1. The two electromagnetic field invariants, W and λ, satisfy

$$W > 0 \tag{12}$$

$$\lambda \ll 1. \tag{13}$$

2. The thermal gyroradius (gyrotime) is small compared to any gradient scale length (time scale of variation), that is, their ratio

$$\delta \ll 1 \tag{14}$$

The first condition implies that the magnetic field is larger than the electric field (much larger than its parallel component; parallel in this paper means parallel to the magnetic field) while the second condition is a statement that the magnetic field is strong (in some appropriate sense). Although the first statement is fully covariant, we have deliberately written the second statement in a more familiar form; a covariant definition is not very transparent. In this paper we shall use the ordering $\lambda \sim \delta$.

Before embarking on the fluid closure we must recall the central importance of the energy -momentum tensor in affecting the charged fluid-Maxwell closure. The coupling of the plasma to the e.m. field first enters a fluid system through the second moment equation, the conservation law for energy-momentum. If the total (summed over all plasma species) energy-momentum tensor for the plasma is denoted by $\mathcal{T}$, then (2) implies

$$\frac{\partial \mathcal{T}^{\mu\nu}}{\partial x^{\nu}} - F^{\mu\nu} J_{\nu} = 0 \tag{15}$$

where J_{ν} is the four-vector current density.

All fluid descriptions of *magnetized* plasma evolution use this second moment as a constitutive relation, determining the four current in terms of the fields, and thus closing Maxwell's equations:

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = J^{\mu} \tag{16}$$

$$\frac{\partial F_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial F_{\beta\gamma}}{\partial x^{\alpha}} + \frac{\partial F_{\gamma\alpha}}{\partial x^{\beta}} = 0. \tag{17}$$

Explicit expressions for the current density are give by the following set of equations (see Ref. [8] for details):

$$e^{\mu\nu} J_{\nu} = -\frac{F^{\mu}_{\kappa}}{W} \frac{\partial \mathcal{T}^{\kappa\nu}}{\partial x^{\nu}}, \tag{18}$$

$$\frac{\partial J^{\nu}}{\partial x^{\nu}} = 0, \tag{19}$$

$$J^{\nu} U_{\nu} = 0, \tag{20}$$

where the last two equations are the covariant expressions of the charge conservation and quasineutrality yielding respectively the parallel component of the current density and the charge density. In a relativistic plasma the charge density can be presumed to vanish only in the instantaneous rest-frame. The first equation is the main equation of the trio and determines the perpendicular components of J in terms of T. The symmetric tensor ($\eta^{\mu\nu}$ is the standard Minkowsky metric tensor)

$$e_{\mu}^{\nu} \equiv -F_{\mu\kappa}F^{\kappa\nu}/W, \tag{21}$$

and its compliment

$$b_{\mu}^{\nu} \equiv \eta_{\mu}^{\nu} - e_{\mu}^{\nu}, \tag{22}$$

constructed from the antisymmetric field tensor, are very important tensors of the theory; In the magnetized limit, $\lambda \sim \delta \to 0$, these tensors become approximate projection operators.

Energy-Momentum Tensor

The preceding discussion clearly shows that once the plasma stress tensor is known the four-current density is determined fully closing Maxwell's equations. The principal task ahead, then, is to compute T. Recall that conventional MHD (including relativistic MHD) avoids the challenge by assuming the stress tensor to have the thermodynamic form,

$$\mathcal{T}^{\mu\nu} = p\eta^{\mu\nu} + hU^{\mu}U^{\nu} \tag{23}$$

in terms of the pressure, p, the enthalpy density, h, and the fluid velocity four-vector U^{μ}. This form would pertain if thermal relaxation due to collisions occurred more rapidly than any other process of interest. Hence, the present work can be described as an extension of MHD into regimes of much lower collisionality. In fact we ignore collisions altogether, and find the form of the stress tensor for a plasma subject to the electromagnetic force alone.

The magnetized plasma assumption (MPA), coupled with Lorentz covariance allows us to write a unique general form for the energy-momentum tensor T. Operationally, to the leading order, MPA reduces the two moment equations (2) and(3) to

$$F^{\mu\nu}\Gamma_{\mu} = 0, \tag{24}$$

$$F^{\alpha\nu}T_{\nu}^{\ \beta} + F^{\beta\nu}T_{\nu}^{\ \alpha} = 0 \tag{25}$$

where we have suppressed any additional suffixes for notational simplicity. The solution of the first is the well-known MHD law $\Gamma^0 \boldsymbol{E} + \boldsymbol{\Gamma} \times \boldsymbol{B} = 0$ giving the general form of the four velocity

$$U^\mu = \gamma(1, \boldsymbol{V}_{\|} + \boldsymbol{V}_E), \tag{26}$$

where $\boldsymbol{V}_E \equiv \boldsymbol{E} \times \boldsymbol{b}/B$ and $\boldsymbol{V}_{\|} = \boldsymbol{b}\boldsymbol{b}\cdot\boldsymbol{V}$. Note in particular that all factors of γ are evaluated at the lowest-order flow $\boldsymbol{V} = \boldsymbol{V}_{\|} + \boldsymbol{V}_E$. The perpendicular velocity is already expressed in terms of the e.m fields but $\boldsymbol{V}_{\|}$ is still to be calculated. This brings us to the very heart of the problem- the solution of (25) to find the energy momentum tensor T. Lorentz invariance proves to be an invaluable guide. We have to construct a second order symmetric tensor which solves (25). The only second order tensors of this description in the theory are $e^{\mu\nu}$, $b^{\mu\nu}$ from the e.m.field, and $U^\mu U^\nu$ and $q^\mu U^\nu + U^\mu q^\nu$ representing the fluid momentum and energy flow. We shall soon identify q. Lorentz invariance demands that when T is written as a linear combination of these tensors , the combining coefficients must be scalars. The identification of these scalars is done by the standard techniques of looking at T in the rest frame of the fluid in which e and b become exact perpendicular and parallel projectors. We find

$$T^{\mu\nu} = b^{\mu\nu} p_{\|} + e^{\mu\nu} p_{\perp} + hU^\mu U^\nu + q^\mu U^\nu + U^\mu q^\nu \tag{27}$$

where $p_{\|}$, $p_\perp$ and h are Lorentz scalars corresponding respectively to parallel pressure, perpendicular pressure and enthalpy density in the rest-frame, and where the four-vector q^μ must satisfy

$$e_{\alpha\beta} q^\alpha = 0 \tag{28}$$

in order to satisfy force-balance, and

$$U_\alpha q^\alpha = 0 \tag{29}$$

in order to preserve the significance of $p_{\|}$ and $p_\perp$. Thus there is only one independent component in q^μ; this represents parallel heat flow in the rest-frame and is denoted by $q_{\|}$.

It is then convenient to introduce the dimensionless four-vector k^α such that

$$q^\alpha \equiv q_{||} k^\alpha$$

We emphasize that (27) represents the unique, general form of the stress tensor in a plasma dominated by the electromagnetic force. It is instructively compared to the special case (23) used in MHD; evidently collisional dissipation has been allowed to remove stress anisotropy in the latter. Compared to the CGL stress tensor, (27) differs in allowing heat flow. The fact that the electromagnetic field appears in the stress tensor only through quasi-projectors $b^{\mu\nu}$ and $e^{\mu\nu}$ is a reflection of gauge-invariance and the indicial symmetry of $T^{\mu\nu}$; recall (7).

The stress tensor contains eight unknown scalar functions: $n_R(x,t)$, $p_{||}(x,t)$, $p_\perp(x,t)$, $h(x,t)$, the three independent components of $\Gamma^\mu(x,t)$, and the single independent component of $q^\mu(x,t)$. Since (1) provides the evolution of the density, and (24) determines the two perpendicular components of the flow, closure of our system requires five additional equations for each plasma species. This task of finding the additional equations was accomplished in Ref. [8]. Here we simply write down the closed system in its compact tensor form:

$$\mathcal{F}_{\mu\kappa} \frac{\partial T^{\kappa\nu}}{\partial x^\nu} = eE_{||} B \Gamma_\mu, \tag{30}$$

obtained by multiplying (2) by the dual tensor $\mathcal{F}$, provides two independent equations. The exact consequences,

$$e_{\alpha\beta} \frac{\partial M^{\kappa\beta\alpha}}{\partial x^\kappa} = 0, \tag{31}$$

$$T_{\alpha\beta} \frac{\partial M^{\kappa\alpha\beta}}{\partial x^\kappa} = 0, \tag{32}$$

or equivalently

$$(U_\alpha k_\beta + U_\beta k_\alpha) \frac{\partial M^{\kappa\alpha\beta}}{\partial x^\kappa} = -2eE_{||} h, \tag{33}$$

derived from the third moment (3) are the source of the additional two. For the final relation that relates enthalpy h to the the pressure, density and temperature (not all independent) we have to resort to a representative distribution function which faithfully reproduces the general form of T. In the process we also manage to evaluate the scalars m_1, m_2, and m_3 appearing in the general form of the third rank symmetric tensor M. We will not display here the detailed expression for M but simply state that one of the crucial steps in the the closure program was to express M in terms of the third rank symmetric tensors constructed from the lower rank tensors of the theory which solve the leading order, electromagnetically dominated, fourth moment equation (not displayed here); no new unknown tensors were introduced. We believe that this procedure is eminently sensible.

We end this section by listing expressions for the above-mentioned scalars:

$$h = \frac{mn_R K_3}{K_2}\left(1 - \frac{2\Delta}{\zeta}\mathcal{F}\right) \tag{34}$$

$$m_1 = \frac{m}{K_2}\left[K_3 p_{\|} + (p_{\|} - p_{\perp})\left(K_3 - 2\frac{K_4}{K_3}K_2\right)\right] \tag{35}$$

$$m_2 = m(p_{\|} - p_{\perp})\frac{K_4}{K_3} \tag{36}$$

$$m_3 = q_{\|}\frac{m\mathcal{K}}{1+\zeta\mathcal{K}} \tag{37}$$

with

$$\mathcal{K} \equiv \frac{K_3}{K_2} - \frac{K_4}{K_3}, \tag{38}$$

$$p_{\|} - p_{\perp} = -2n_R T\Delta\frac{K_3}{\zeta K_2}, \tag{39}$$

where $K_n(\zeta)$ are the MacDonald functions associated with the momentum integrals of relativistic Maxwellians, $\zeta = m/T$ is the inverse of the temperature measured in units of the rest-mass, and Δ is a measure of the pressure anisotropy. For considerations of the next section, it will be useful to remember that $p_{\|} = n_R T$ and $m_1 + m_2 = Th$.

We have thus derived a closed system of fluid equations valid for a magnetized plasma with arbitrary directed speed and arbitrary temperature. All dynamical variables of the system (the pressures, the parallel velocity and heat flow etc.) needed to construct the energy momentum tensor, have appropriate equations for their temporal advancement. The equations are coupled and highly nonlinear; they represent, to the leading order, a complete description of the macroscopic low- frequency motions of a magnetized plasma.

III. NONRELATIVISTIC LIMIT — NR

Most of the laboratory magnetized plasmas are not relativistic, neither the directed velocity nor the thermal speed are anywhere near the speed of light; the former is often much smaller than the latter for hot thermonuclear plasmas. Thus a nonrelativistic limit of the theory is extremely important for applications to familiar plasma systems. One could ask if it was necessary to spend the labour of first deriving a fully covariant relativistic theory and then go through the cumbersome process (it certainly is tedious but interestingly enough turns out to be less tedious than conventional nonrelativistic in many ways) of taking the double limit. We believe that this effort is fully justifiable — not only because we have a very general theory applicable to a vastly larger set of physical systems, but also because the dictates of space-time symmetries (Lorentz invariance) were crucial in the determination of the unique form of the electromagnetically dominated energy-momentum tensor T — the centerpiece of the problem and of the theory. Without the constraints of Lorentz covariance, it is quite difficult, if not impossible, to derive the correct general expression for T.

There is also another non-trivial advantage inherent in this procedure. Once the approximations are spelled out, the theory can be worked out to a given order by a straightforward mechanical prescription; there is almost no fear of missing terms of equal magnitude in a specific equation.

On our way to the NR limit, we first write down the three-vector forms of various equations. It will require the three-vector forms of various constituents of T and M: These include the vector k^α,

$$k^\alpha = (k^0, \boldsymbol{k}) = \gamma\sqrt{\frac{W}{B^2}}\left(\frac{B^2}{W}V_\|, \boldsymbol{b} + \frac{B^2}{W}V_\|\boldsymbol{V}_E\right) \tag{40}$$

obtained by using its defining relations (28)-(29) or by Lorentz boosting the rest-frame $k^\alpha = \{0,\ \boldsymbol{b}\}$, and the four-curvature

$$\frac{\partial b_\mu{}^\nu}{\partial x^\nu} = \frac{F_{\mu\nu}}{W}J^\nu + \left(\frac{1}{2}\eta_\mu{}^\nu - b_\mu{}^\nu\right)\frac{\partial \log W}{\partial x^\nu}, \tag{41}$$

Notice that $\partial \nu e^\nu_\mu = -\partial \nu b^\nu_\mu$. It will be convenient to use the identity $U^\nu \partial \nu = \gamma(\partial_t + \boldsymbol{V} \cdot \nabla) \equiv \gamma\, d/dt$, and the notation $k^\nu \partial \nu \equiv d/ds$. Using all this, we obtain

$$\frac{d}{dt}(\gamma n_R) + \gamma n_R \boldsymbol{\nabla} \cdot \boldsymbol{V} = 0, \tag{42}$$

$$\begin{aligned}&\sqrt{W}\nabla_{\|}\left(\frac{p_{\|}}{\sqrt{W}}\right) + \frac{p_\perp}{2}\nabla_{\|}\log W + \gamma n_R \boldsymbol{b} \cdot \frac{d}{dt}\left(\frac{h\gamma \boldsymbol{V} + \boldsymbol{q}}{n_R}\right)\\ &\quad + q_{\|}\boldsymbol{b} \cdot \frac{d\gamma \boldsymbol{V}}{ds} + \gamma V_{\|}\left(\frac{\partial q^0}{\partial t} + \nabla \cdot \boldsymbol{q}\right) = \gamma e n_R E_{\|},\end{aligned} \tag{43}$$

$$\sqrt{W}\frac{d}{dt}\frac{p_{\|}}{\sqrt{W}} + \frac{p_\perp}{2}\frac{d\log W}{dt} - n_R \frac{d}{dt}\frac{h}{n_R} - \gamma \boldsymbol{q} \cdot \frac{d\boldsymbol{V}}{dt} - \frac{1}{\gamma}\left(\frac{\partial q^0}{\partial t} + \nabla \cdot \boldsymbol{q}\right) = 0 \tag{44}$$

$$\partial_\mu G^\mu = 0, \tag{45}$$

with the definitions

$$W^{1/2}G^0 = \gamma\left(m_1 + m_3\sqrt{\frac{B^2}{W}}V_{\|}\right)$$

$$W^{1/2}\boldsymbol{G} = \gamma\left[\boldsymbol{b}\left(m_1 V_{\|} + m_3\sqrt{\frac{W}{B^2}}\right) + \boldsymbol{V}_E\left(m_1 + m_3\sqrt{\frac{B^2}{W}}V_{\|}\right)\right],$$

and

$$\begin{aligned}&5m_3\gamma\frac{d\log(m_3 n_R^{-6/5})}{dt} + (m^2 n_R + 5Th - 2m_3)\gamma^2 \boldsymbol{k} \cdot \frac{d\boldsymbol{V}}{dt}\\ &\quad + \frac{dTh}{ds} - m_2\frac{d\log\sqrt{W}}{ds} + 7m_3\gamma \boldsymbol{k} \cdot \frac{d\boldsymbol{V}}{ds} = ehE_{\|}\end{aligned} \tag{46}$$

as the three-vector expressions of (1), (30) (two equations), (31) and (33). We must augment this set with the three-vector version of (18) [(19) and (20) are trivial]:

$$\boldsymbol{J}_\perp = \frac{B^2}{W}J_{\|}V_{\|}\boldsymbol{V}_E - \frac{1}{W}(\boldsymbol{G} \times \boldsymbol{B} + G^0 \boldsymbol{E}) \tag{47}$$

with$\boldsymbol{G}, and G^0$ given by

$$\begin{aligned}\left(1 - \frac{P_{\|} - P_\perp}{w}\right)G^0 &= \gamma n\frac{d}{dt}\left(\frac{Q^0 + \gamma H}{n}\right) + \partial_\nu(\gamma Q^\nu) - \partial_t P_\perp\\ &\quad - (P_{\|} - P_\perp)\left[\partial_t \log\sqrt{W} + \frac{B^2}{W}D_E\left(\log\frac{(P_{\|} - P_\perp)}{W}\right)\right]\end{aligned} \tag{48}$$

and

$$\begin{aligned}\left(1 - \frac{P_{\|} - P_\perp}{w}\right)\boldsymbol{G} &= \gamma n\frac{d}{dt}\left(\frac{\boldsymbol{q} + \gamma h\boldsymbol{V}}{n}\right) + \gamma \boldsymbol{V}\partial_\nu q^\nu + q\frac{d(\gamma \boldsymbol{V})}{ds} + \nabla P_\perp\\ &\quad + (P_{\|} - P_\perp)\left[\nabla \log\sqrt{W} - \frac{B^2}{W}\boldsymbol{V}_E D_E\left(\log\frac{(P_{\|} - P_\perp)}{W}\right)\right].\end{aligned} \tag{49}$$

It would seem that we have taken a giant step backward; from lofty heights of the explicit elegance and compactness of the covariant equations we have descended into a complicated mess. This is, however, an intermediate step; our aim is to derive the low-temperature, low-velocity limit of these exact equations. We do not expect to recapture their earlier beauty, but we will have a set of equations which we can compare and contrast with known systems and find what essential new physics we have incorporated in the present system.

The nonrelativistic limit has to be taken carefully; there are occasions when all leading-order terms cancel in the equation and one has to resort to higher orders in expansion, especially in $\zeta = m/T$. However, the program is straightforward and mechanical and the results are unique for a prescribed ordering; No intuition on the relative importance of terms is needed to arrive at the desired consistent set. The inequalities $V, \zeta^{-1/2} \ll 1$ capture the essentials of the NR limit. In the procedure followed here, we also take $\partial/\partial t \sim d/dt \sim V \ll 1$. We just remind the reader that various physical quantities appearing in the equations are relativistically inequivalent; their appearance in the same equation, therefore, implies that they must be 'multiplied' by factors that are also inequivalent in the expansion so that the terms in which they appear are of the same order.

Since $V \sim E/B$, in the NR limit $W = B^2 - E^2 \sim B^2$. Similarly we find that $d/ds \rightarrow \nabla_{\|}$. Putting it all together and using (the large argument limits of the MacDonald functions K),

$$h = \zeta p_{\|} + \frac{3}{2} p_{\|} + p_{\perp} + \mathcal{O}(\infty/\zeta), \tag{50}$$

$$m_1 = \frac{m}{K_2}\left[K_3 p_{\|} + (p_{\|} - p_{\perp})\left(K_3 - 2\frac{K_4}{K_3}K_2\right)\right], \tag{51}$$

$$m_2 = m(p_{\|} - p_{\perp})\left(1 + \frac{7}{2\zeta}\right), \tag{52}$$

$$m_3 = \frac{2}{5} m q_{\|}\left(1 + \frac{5}{2\zeta}\right), \tag{53}$$

we arrive finally at the following set of NR equations:

The parallel equation of motion

$$mn\boldsymbol{b} \cdot \frac{d\boldsymbol{V}}{dt} + \nabla_{\|} p_{\|} + (p_{\perp} - p_{\|})\nabla_{\|} \log B = enE_{\|} - \nu_1 q_{\|} - \frac{\nu_0}{e} J_{\|}, \tag{54}$$

two equations for the evolution of parallel and perpendicular pressures

$$\frac{d}{dt}\log\left(\frac{p_{\|}B^2}{n^3}\right)+\frac{6}{5}\frac{q_{\|}}{p_{\|}}\nabla_{\|}\log\left(\frac{q_{\|}}{B^{1/3}}\right)=0, \tag{55}$$

$$\frac{d}{dt}\log\left(\frac{p_{\perp}}{Bn}\right)+\frac{2}{5}\frac{q_{\|}}{p_{\perp}}\nabla_{\|}\log\left(\frac{q_{\|}}{B^2}\right)=\nu_2\frac{p_{\|}-p_{\perp}}{p_{\perp}}, \tag{56}$$

and an equation for the evolution of parallel heat flow,

$$\begin{aligned}&\frac{dq_{\|}}{dt}+q_{\|}\left[\frac{9}{5}\boldsymbol{b}\cdot(\nabla_{\|}\boldsymbol{V})-\frac{7}{5}\frac{d\log n}{dt}\right]+\left(\frac{7}{2}p_{\|}-p_{\perp}\right)\boldsymbol{b}\cdot\frac{d\boldsymbol{V}}{dt}\\ &-\frac{1}{2}enE_{\|}V_{\|}^2+\frac{7T}{2m}(p_{\perp}-p_{\|})\nabla_{\|}\log B+\nabla_{\|}\left[\frac{T}{m}\left(\frac{3}{2}p_{\|}+p_{\perp}\right)\right]\\ &=\frac{e}{m}E_{\|}\left(\frac{3}{2}p_{\|}+p_{\perp}\right)-\frac{3}{2}\nu_3q_{\|},\end{aligned} \tag{57}$$

and finally the equation for the perpendicular current

$$\boldsymbol{J}_{\perp}=\left(B^2-(P_{\|}-P_{\perp})\right)^{-1}\boldsymbol{B}\times\left[\nabla P_{\perp}+(P_{\|}-P_{\perp})\left(\nabla\ell n\,B+\boldsymbol{b}\nabla_{\|}\ell n\frac{P_{\|}-P_{\perp}}{B^2}\right)+Mn\frac{d\boldsymbol{V}}{dt}\right]. \tag{58}$$

In the process of taking the nonrelativistic limit we have also added collisions (as reflected by the presence of the collision frequencies (νs) on the right-hand side of (54), (56), and (57). All equations except (58) hold for each individual species (species index suppressed) while in (58), the upper case P's denote the total plasma pressure.

We incorporated the collisions with two ends in view: the first was to present a more or less complete theory of magnetized plasmas valid for arbitrary collision frequency ((54)-(58) with equations of continuity for each species) and the second was to show how the current theory subsumes less general and less encompassing systems like MHD.

There are several important features which distinguish the current theory from its predecessors:

The parallel heat flow ($q_{\|}$) has attained the status of a dynamic variable of the system on an equal a priori footing with density, parallel velocity or pressure; it is not given by an additionally assumed diffusive transport equation but by a a perfectly well-defined (rather complicated) evolution equation (57) which relates it to other dynamic variables. This factor adds both to the complexity and to the richness of the self-consistent dynamics of a magnetized plasma.

For plasma motions for which the heat flow along the field line cannot be ignored, there are no "equations of state" for either the parallel or the perpendicular pressure. It is only when $\langle q_{\|}\rangle$ is negligible that the pressure evolution equations (55)-(56) reduce to the familiar form for the equation of state with the difference that the magnetic field strength has worked its way into the scheme of things. The equation of state for a magnetized plasma in the limit of vanishing parallel heat flow, relates not just the particle pressure but some combination of the particle and the magnetic field pressure(B^2) with the plasma density. The two "adiabatic laws" obtained from the $q_{\|} = 0$ limit of (55)-(56),

$$\frac{d}{dt}\log\left(\frac{p_{\|}B^2}{n^3}\right) = 0, \tag{59}$$

$$\frac{d}{dt}\log\left(\frac{p_{\perp}}{Bn}\right) = 0, \tag{60}$$

are precisely the double adiabatic laws of the CGL theory.

This is perhaps as good a juncture as any to comment on the conditions when $q_{\|} = 0$ may be a reasonable assumption. For this we will have to introduce another ordering parameter, $\mu = (d/dt)/\nu$, which measures the ratio of the inverse time scales of interest and any of the collision frequencies occurring on the right hand-side of the above system of equations. For large collision frequencies ($\mu \to 0$), it is easy to infer that the leading order solution of the evolution equation (57) must be $q_{\|} = 0$. If this ordering were applied to the rest of the set, we are forced to conclude from (56) that $p_{\|} - p_{\perp} \to 0$. High collisionality, therefore, does not permit either the parallel heat flow or the pressure anisotropy. This raises doubts about the justification of the CGL theory (pressure anisotropy but no neat flow) in any regime of interest unless one can devise a yet unknown ordering. The general statement (in the nonrelativistic magnetized plasmas) that MHD is collisional while CGL is applicable for collisionless plasmas is definitely contradicted by our general formulation. CGL requires high collisionality to make $q_{\|} = 0$; but when this is done the resulting theory is MHD and not CGL; the latter does not constitute a consistent approximation to the physical system.

When terms proportional to $p_\| - p_\perp$ and $q_\|$ are neglected in our general energy momentum tensor (27) it reduces precisely to the MHD energy- momentum tensor (23) as it must. This shows the consistency of the formalism and also displays the fact the energy momentum tensor must reflect all the information on the momentum and energy flows.

IV. LINEAR THEORY — SOUND WAVES

We now carry out an extremely simple calculation to show how the current theory may be a closer approximation to the detailed kinetic theory than MHD. For this purpose we choose the problem of deriving the dispersion relation for the low-frequency MHD waves in a homogeneous collisionless plasma. This problem can be exactly solved in the fluid as well as equivalent kinetic models.

Let the density n_0, the pressure p_0, the temperature T_0, and the magnetic field B_0 be uniform, and $q_\|^0 = 0$, $p_\|^0 = p_\perp^0$, $\boldsymbol{V}_0 = 0$ in the equilibrium state. Let us also allow the ion and electron temperatures to be different (in the $T_i^0 \to 0$ limit, the dispersion relations are the simplest) with $\tau = T_i^0/T_e^0 = p_i^0/p_e^0$. ¿From the linearization of (54) and (55), we have

$$\frac{\tilde{q}_\|}{p^0} = \frac{5}{6}\frac{\omega}{k_\|}\left[\frac{\tilde{p}_\|}{p^0} + 2\frac{b_\|}{B_0} - 3\frac{\tilde{n}}{n_0}\right], \tag{61}$$

$$\frac{\tilde{q}_\|}{p^0} = \frac{5}{2}\frac{\omega}{k_\|}\left[\frac{\tilde{p}_\perp}{B} - \frac{b_\|}{B_0} - \frac{\tilde{n}}{n_0}\right], \tag{62}$$

which yield the relation

$$\frac{\tilde{p}_\|}{p^0} = 3\frac{\tilde{p}_\perp}{p^0} - 5\frac{b_\|}{B_0} \tag{63}$$

valid for each species. The linearized equations (58) and (57) lead to ($\widetilde{\boldsymbol{V}} = \boldsymbol{V}_E + \widetilde{V}_\|\widehat{b}_0$)

$$\frac{\tilde{q}_\|}{p^0} + \frac{5}{2}\left[\widetilde{V}_\| + \frac{ie\,\widetilde{E}_\|}{\omega m}\right] = \frac{k_\| T^0}{\omega m}\left[4\frac{\tilde{p}_\|}{p^0} + \frac{\tilde{p}_\perp}{p^0} - \frac{5}{2}\frac{\tilde{n}}{n_0}\right], \tag{64}$$

$$V_\| - \frac{ie\widetilde{E}_\|}{\omega m} = \frac{k_\|}{m\omega}\frac{\tilde{p}_\|}{n_0} = \frac{k_\|}{\omega}\frac{T_0}{m}\frac{\tilde{p}_\|}{p^0}, \tag{65}$$

and may be combined (eliminating $E_{\|}$) to relate the heat-flow perturbation to the others,

$$\frac{\widetilde{q}_{\|}}{p^0} = \frac{k_{\|}}{m\omega}\frac{T_0}{m}\left[\frac{3}{2}\frac{\widetilde{p}_{\|}}{p^0} + \frac{\widetilde{p}_{\perp}}{p^0} - \frac{5}{2}\frac{\widetilde{n}}{n_0}\right]. \tag{66}$$

Equations (62)-(65) may be readily solved to derive ($z = (T_0/m)k_{\|}^2/\omega^2$)

$$\frac{\widetilde{p}_{\perp}}{p^0} = \frac{\frac{b_{\|}}{B_0}(1-3z) + \frac{\widetilde{n}}{n_0}(1-z)}{1-\frac{11}{5}z} \tag{67}$$

$$\frac{\widetilde{p}_{\|}}{p^0} = \frac{3(1-z)}{1-\frac{11}{5}z}\left[\frac{\widetilde{n}}{n_0} - \frac{2}{13}\frac{b_{\|}}{B_0}\right] \tag{68}$$

determining $\widetilde{p}_{\perp}$ and $\widetilde{p}_{\|}$ in terms of $b_{\|}$ and $\widetilde{n}$. Let us consider the standard case of a hot plasma for which the Alfvén speed, $v_A \ll v_{the}$, the electron thermal speed. Then for electrons ($z \to \infty$), (67) simplifies to [$P_0 = p_e^0 + p_i^0$ is the total pressure]

$$\frac{\widetilde{p}_{\|e}}{P_0} = \frac{p_{\|e}}{(1+\tau){p^0}_e} = \frac{15}{11}\left[\frac{\widetilde{n}}{n_0} - \frac{2}{3}\frac{b_{\|}}{B_0}\right]\frac{1}{1+\tau}. \tag{69}$$

For the ions, however, z must be kept finite

$$\frac{\widetilde{p}_{\|i}}{P_0} = \frac{3(1-z)}{(1-\frac{11}{5}z)}\left[\frac{\widetilde{n}}{n_0} - \frac{3}{3}\frac{b_{\|}}{B_0}\right]\frac{\tau}{1+\tau} \tag{70}$$

Adding (69) and (70) yields the total pressure perturbation $\widetilde{p}_{\|}$

$$\begin{aligned}\frac{\widetilde{p}_{\|}}{P_0} &= \frac{1}{1+\tau}\left[\frac{\widetilde{n}}{n_0} - \frac{2}{3}\frac{b_{\|}}{B_0}\right]\left[\frac{15}{11} + \frac{3(1-z)\tau}{1-\frac{11}{5}z}\right] \\ &\equiv \alpha(\omega)\left[\frac{\widetilde{n}}{n_0} - \frac{2}{3}\frac{b_{\|}}{B_0}\right],\end{aligned} \tag{71}$$

where

$$\alpha(\omega) = \frac{1}{1+\tau}\left[\frac{15}{11} + \frac{3(1-z)\tau}{1-\frac{11}{5}z}\right] \xrightarrow[\tau \to 0]{} \frac{15}{11} \tag{72}$$

Using the linearized continuity equation

$$\frac{\widetilde{n}}{n_0} = \frac{b_{\|}}{B_0} + \frac{k_{\|}^2 C_s^2}{\omega^2}(1+\tau)\frac{\widetilde{p}_{\|}}{P_0} = \frac{b_{\|}}{B_0} + \widehat{z}(1+\tau)\frac{\widetilde{p}_{\|}}{P_0} \tag{73}$$

where $\widehat{z} = \tau z$, we express the pressure perturbation fully in terms of $b_{\|}$, the parallel magnetic perturbation

$$\frac{\widetilde{P}_{\|}}{P_0} = \frac{\frac{1}{3}\alpha(\omega)}{1-\alpha(\omega)\widehat{z}(1+\tau)}\frac{b_{\|}}{B_0}. \tag{74}$$

Since in the force balance equation we will need the perturbed perpendicular pressure, we may calculate it from (63) and (74),

$$\frac{\widetilde{P}_\perp}{P_0} = \frac{1}{3}\left[\frac{1}{3}\frac{\alpha(\omega)}{1-\alpha(\omega)\widehat{z}(1+\tau)} + 5\right]\frac{b_\|}{B_0} \equiv \lambda(\omega)\frac{b_\|}{B_0}. \tag{75}$$

The perpendiclar component of the force balance equation (58) yields on linearization and some manipulation

$$k_\|\boldsymbol{b} - \boldsymbol{k}_\perp b_\| = \beta\lambda(\omega)b_\|\boldsymbol{k} - \frac{\omega}{v_A^2}(\boldsymbol{E}_\perp \times \boldsymbol{b}_0) \tag{76}$$

where $\beta = 8\pi P_0/B_0^2$ is the plasma beta. Equation (76) allows two consequences:

(i) Dotting it with $\boldsymbol{k}_\perp$ gives

$$\left(-k_\|^2 - k_\perp^2\right) b_\| = \beta\lambda(\omega)(\boldsymbol{E}_\perp \times \boldsymbol{b}_0) - \frac{\omega}{v_A^2}(\boldsymbol{k}_\perp \cdot \boldsymbol{E}_\perp)b_0, \tag{77}$$

(ii) Crossing it with $\boldsymbol{k}_\perp$ yields

$$k_\|\boldsymbol{b}_0 \cdot (\boldsymbol{k}_\perp \times \boldsymbol{b}_\perp) = \frac{\omega}{v_A^2}\boldsymbol{k}_\perp \cdot \boldsymbol{E}_\perp. \tag{78}$$

The use of Faraday's law $\omega\boldsymbol{b} = \boldsymbol{k} \times \boldsymbol{E}$, translating as

$$\omega b_\| = \boldsymbol{k}_\perp \cdot (\boldsymbol{E}_\perp \times \boldsymbol{b}_0), \tag{79}$$

and

$$\omega(\boldsymbol{b}_0 \times \boldsymbol{b}_\perp) = -k\boldsymbol{E}_\perp \tag{80}$$

completes the system. From (78)-(80) it is trivial to derive

$$\omega^2 = k_\|^2 v_A^2 \tag{81}$$

implying that the shear Alfvén wave stilll remains unchanged in this simple geometry. The description relation for the other two coupled modes (the compressional and the sound wave) is naturally severly altered; it is obtained from (77) and (79) to be

$$\frac{\omega^2}{v_A^2} - k_\|^2 - k_\perp^2 = \beta\lambda(\omega)k_\perp^2. \tag{82}$$

In the limit of zero ion temperature and $\beta \to 0$, the modes decouple and become simplified, yielding,

$$\omega^2 \simeq (k_{\|}^2 + k_{\perp}^2) v_A^2 \tag{83}$$

$$\tag{84}$$

$$\omega^2 = \frac{15}{11} k_{\|}^2 c_s^2. \tag{85}$$

We find that, for this very primitive case, we have obtained quite an interesting result: the effective sound speed $v_s = \sqrt{15/11}\ C_s$ which lies between the kinetic value C_s and the MHD value $\sqrt{5/3}\ C_s$. This is a very encouraging result for the new theory because the influence of the parallel heat flow, by not allowing the standard adiabatic law with $\gamma = 5/3$, has affected the rate of sound propagation along the field line. We expect many more changes in more substantial and nontrivial cases. Incidentally, the CGL model would give a sound speed $\sqrt{(3)} C_s$ which is much worse than the MHD result.

V. SUMMARY

A relatively complete closed fluid model of magnetized plasmas moving with arbitrary thermal and directed speeds is derived from the exact moments of the kinetic equation by appealing to space-time symmetries(Lorentz invariance) and the fact that the electromagnetic force is the principal determinant of plasma dynamics. The electrodynamically determined energy momentum tensor has a built in pressure anisotropy and nonzero parallel heat flow distinguishing it from MHD which is subsumed in the new theory. The system derived in this paper may be viewed as the logical culmination of the intended MHD program (theory of magnetized plasmas); it is obtained essentially by replacing the thermodynamic stress tensor of MHD by the more relevant and general tensor dictated by the electrmanetic nature of the dominant interaction. We expect both the relativstic and the nonrelativistic manifestations to find widespread applications in problems ranging from the structure of intergalactic jets to low frequency motions of hot confined laboratory plasmas.

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