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Astrophysical Plasmas

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Astrophysical plasmas

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1. – Introduction

The term *plasma* was first used by Tonks and Langmuir in a paper in the *Physical Review* in 1929 referring to a gas with a sufficient ionization degree and noticing that the presence of free charges modifies the behavior of matter with respect to the hydrodynamic limit by enforcing a collective behavior by long-range electromagnetic forces. Crookes had already analyzed this effect in experiments on gas discharges since 1879 and called the configuration a *fourth state of matter*, although the passage from the neutral gas to the ionized gas state is not a sudden phase transition.

The plasma state forms when a substantial population of free charges is present, and this happens:

1. for systems in thermodynamical equilibrium at $T \geq 10^4\,\mathrm{K}$;

2. for non-equilibrium systems when typical particle energies are above an ionization limit, $E \geq 10^{-2}\,\mathrm{eV}$.

These values clearly show that the objects in the Universe are almost everywhere in the plasma state, the only exceptions being in fact cold planetary surfaces. Stars are dominated by gravitation, but their surface activity is due to electromagnetic forces. Similarly the interstellar and the intergalactic matter are shaped by plasma forces; and galaxies also show a plasma collective behavior where long-range forces are gravitational forces acting on a gas of stars. Active stars (pulsars, X-ray binaries, transient sources, etc.) and active galactic nuclei appear also to be dominated by plasma effects.

1'1. *Characteristic parameters.* – The characterization of a plasma state requires the definition of a few basic parameters.

1. *Debye length* λ_{D}, the distance over which electromagnetic long-range forces prevail over short-range electrostatic effects and yield the collective behavior:

$$V = \frac{q}{r} e^{-(r/\lambda_{\mathrm{D}})(1+Z)^{1/2}} \quad (Z \text{ average ionic charge}).$$

For a thermal plasma the Debye length is obtained by balancing electrostatic and thermal pressures:

$$\lambda_{\mathrm{D}} = \sqrt{\frac{kT}{4\pi q^2 n}},$$

which for electrons, the high-mobility component in a plasma, becomes (n_{e} is the free-electron density)

$$\lambda_{\mathrm{D}} = 740 \sqrt{\frac{E_{\mathrm{eV}}}{n_{\mathrm{e}}}} \,\mathrm{cm}.$$

2. *Plasma parameter* Λ, a measure of the number of free charges in a Debye volume, that must be necessarily large to allow the collective forces to be dominant:

$$\Lambda = 3N_{\mathrm{D}} = 3\frac{4\pi}{3} n \lambda_{\mathrm{D}}^3 \gg 1.$$

3. *Quasi-neutrality*, astrophysical plasmas are globally quasi-neutral:

$$\left| n_{\mathrm{e}} - \sum Z_i n_i \right| \ll n_{\mathrm{e}}.$$

4. *Plasma frequency* ω_{p}, the characteristic frequency of oscillation of a system of charges connected by collective electrostatic forces; the link is due to the field over a Debye length and the typical velocity of moving particles is their thermal velocity:

$$\omega_{\mathrm{p}} = \frac{v_{\mathrm{th}}}{\lambda_{\mathrm{D}}} = \sqrt{\frac{4\pi q^2 n}{m}},$$

$$\omega_{\mathrm{pe}} = 5.6 \times 10^4 \sqrt{n_{\mathrm{e}}} \,\mathrm{rad\ s^{-1}} \quad (\text{for electrons}).$$

5. *Electric conductivity* σ, defined by the short-range collision frequency ν_{c}

$$\sigma = \frac{ne^2}{m\nu_{\mathrm{c}}}.$$

TABLE I. – *Astrophysical plasmas parameters.*

	L (cm)	n_e (cm^{-3})	T (K)	λ_{De} (cm)	σ (s^{-1})	H (gauss)	ν_g (Hz)	ν_{pe} (Hz)	ν_c (Hz)
Ionosphere	10^7	10^3–10^6	10^2–10^3	7×10^{-2}–7	6×10^9–10^{11}	0.1	3×10^5	3×10^5–10^7	10–10^3
Solar wind	10^{13}–10^{15}	1–10^4	10^2–10^3	0.7–2×10^2	6×10^9–10^{11}	10^{-6}–10^{-5}	3–30	10^4–10^6	2×10^{-2}–6
Solar corona	6×10^9–10^{11}	10^8–10^{12}	10^6–10^7	10^{-2}–2	7×10^{15}	10^{-5}–1	30–3×10^6	10^8–10^{10}	8
Stellar interiors	10^{10}–10^{12}	10^{27}	4×10^7	10^{-9}	7×10^{18}	—	—	3×10^{17}	2×10^{16}
Neutron stras	10^6	10^{42}	10^6–10^9	10^{-17}–10^{-16}	10^{17}–3×10^{21}	10^{12}	3×10^{18}	10^{25}–3×10^{26}	10^{23}–10^{28}
Interstellar gas	10^1–10^{22}	10^{-3}–10	10^2	20–2×10^3	6×10^{12}	10^{-6}	3	3×10^2–3×10^4	9×10^{-5}
Intergalactic gas	$\geq 10^{24}$	$\leq 10^{-5}$	10^5–10^6	$\geq 2 \times 10^6$	10^{14}	$\leq 10^{-8}$	$\leq 3 \times 10^{-2}$	≤ 30	$\leq 10^{-11}$
Galactic nuclei	$\leq 10^{15}$	$\leq 10^{12}$	$\geq 10^8$	7×10^{-2}	$\geq 4 \times 10^{18}$	$\geq 10^5$	$\geq 3 \times 10^{11}$	$\leq 10^{10}$	30
Thermonucl. plasma	10^2	10^{16}	10^8	7×10^{-4}	6×10^{18}	10^5	3×10^{11}	10^{12}	2×10^5

6. *Cyclotron frequency* Ω, in the presence of an external or induced magnetic field:

$$\Omega = \left|\frac{qB}{\gamma mc}\right|,$$

$$\Omega_{\mathrm{e}} = \left|\frac{e^- B}{m^- c}\right| = 1.8\times 10^7 B_G \,\mathrm{rad\ s^{-1}} \quad \text{(for electrons)}.$$

Characteristic values of the above quantities in astrophysical plasmas are given in table I.

2. – Plasma models

The treatment of plasmas has been developed following three basic descriptions:

1. *kinetic equations*, that provide a detailed study of multibody systems in terms of phase-space distribution functions, and apply also to non-equilibrium, anisotropic conditions;
2. *fluid equations*, that take the moments of the distribution functions of kinetic equations over momentum space and therefore apply to equilibrium situations which can be described by average macroscopic quantities as density, pressure, temperature, etc.; in this category are included the *two-fluid model*, in which negative and positive charges are dealt with separately and coupled through energy and momentum exchanges, the *magnetohydrodynamic* (MHD) *model*, in which a further averaging is done on the two charge components in the assumption that they are strongly coupled by collisions, and the *cold plasma model*, in which thermal dispersion is neglected and the particles interact only through long-range electromagnetic forces;
3. *orbit theory*, that applies to the dynamics of systems dominated by external fields so that all particles follow the same trajectories.

2·1. *Kinetic equations.* – The distribution function in phase space of a system of particles of the same type is defined as

$$\text{number of particles in } \mathrm{d}^3q\,\mathrm{d}^3p = f(\mathbf{q},\mathbf{p},t)\,\mathrm{d}^3q\,\mathrm{d}^3p$$

normalized through the space number density

$$\int f\,\mathrm{d}^3v = n.$$

This function is not directly measurable, but its macroscopic measurable moments in velocity space are well-known measurable quantities:

– space mass density $\to \rho = \int mf\,\mathrm{d}^3v$,

- flow velocity $\to \mathbf{V} = \int \mathbf{v} f \, \mathrm{d}^3 v$,
- thermal velocity $\to \mathbf{u} = \mathbf{v} - \mathbf{V}$,
- pressure tensor $\to \Pi_{ij} = \int m v_i v_j f \, \mathrm{d}^3 = \int m u_i u_j \, \mathrm{d}^3 u + \rho V_i V_j$, thermal + kinetic pressure,
- thermal energy density $\to \varepsilon = \int \frac{1}{2} m u^2 f \, \mathrm{d}^3 v$,
- thermal flux $\to \mathbf{q} = \int \frac{1}{2} m u^2 \mathbf{u} f \, \mathrm{d}^3 v$,
- force density $\to \mathcal{F} = \int \mathbf{F} f \, \mathrm{d}^3 v$.

The evolution of the distribution function is governed by the *Liouville equation*, a continuity equation in phase space:

$$\frac{\partial f}{\partial t} + \nabla \cdot (f \mathbf{U}) = 0$$

or

$$\frac{\mathrm{d} f}{\mathrm{d} t} + f \nabla \cdot \mathbf{U} = 0,$$

where

$$\mathbf{U} = \left(\frac{\mathrm{d} q_i}{\mathrm{d} t}, \frac{\mathrm{d} p_i}{\mathrm{d} t} \right), \quad i = 1, 2, 3,$$

$$\nabla \equiv \left(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i} \right), \qquad \frac{\mathrm{d}}{\mathrm{d} t} \equiv \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla.$$

When short-range collisions are present, the possibility of discontinuous trajectories in the phase space must be taken into account:

$$\frac{\partial f}{\partial t} + f \nabla \cdot \mathbf{U} = \left(\frac{\partial f}{\partial t} \right)_{\mathrm{coll}}.$$

When the forces acting upon charges of charge q and mass m are only electromagnetic, with potentials $\mathbf{A}$ and ϕ, the conjugate momenta are

$$q_i = x_i, \qquad p_i = m v_i + \frac{e}{c} A_i$$

with Hamiltonian

$$\mathcal{H} = \frac{1}{2m} \left(p_i - \frac{e}{c} A_i \right)^2 + e \phi.$$

The Liouville equation becomes the *Vlasov equation*

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + \frac{F_i}{m}\frac{\partial f}{\partial v_i} = 0,$$

where

$$F_i = -e\frac{\partial \phi}{\partial x_i} - \frac{e}{c}\frac{\partial A_i}{\partial t} + \frac{e}{c}\big[\mathbf{v} \times (\nabla \times \mathbf{A})\big],$$

$$\mathbf{F} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B},$$

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \qquad \mathbf{B} = \nabla \times \mathbf{A},$$

and with short-range collisions the *Boltzmann equation*

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + \frac{F_i}{m}\frac{\partial f}{\partial v_i} = \left(\frac{\partial f}{\partial t}\right)_{\rm coll}.$$

The system of kinetic equations in the electromagnetic case is very complex as it must be coupled with the full system of Maxwell equations to define fields through the distribution of charges and currents. In fact solutions can be obtained only in some relatively simplified cases.

2·2. *Macroscopic fluid equations.* – From the system of kinetic equations we can derive moment equations averaging over the velocity distribution and obtain the evolution of macroscopic measurable parameters. We calculate the first three moments of the Liouville equations multiplying by

$$\psi_0(\mathbf{v}) = mv^0, \qquad \psi_1(\mathbf{v}) = mv^1, \qquad \psi_2(\mathbf{v}) = \frac{1}{2}mv^2$$

and integrating over the velocity space from 0 to ∞. Taking into account that electrostatic forces are independent of velocity and the Lorenzt force has the form $\mathbf{v} \times \mathbf{B}$, and assuming that short-range collisions are binary and elastic, one obtains

$$\int \psi(\mathbf{v})\frac{\partial f}{\partial t}\,\mathrm{d}^3v = \frac{\partial}{\partial t}\int \psi f\,\mathrm{d}^3v,$$

$$\int \psi(\mathbf{v})v_i\frac{\partial f}{\partial x_i}\,\mathrm{d}^3v = \frac{\partial}{\partial x_i}\int \psi f v_i\,\mathrm{d}^3v,$$

$$\int \psi(\mathbf{v})\frac{F_i}{m}\frac{\partial f}{\partial v_i}\,\mathrm{d}^3v = \frac{F_i}{m}\int \psi\frac{\partial f}{\partial v_i}\,\mathrm{d}^3v = -\frac{F_i}{m}\int \frac{\partial \psi}{\partial v_i}f\,\mathrm{d}^3v,$$

$$\int m\left(\frac{\partial f}{\partial t}\right)_{\rm coll}\mathrm{d}^3v \simeq \int m\mathbf{v}\left(\frac{\partial f}{\partial t}\right)_{\rm coll}\mathrm{d}^3v \simeq \int \frac{1}{2}mv^2\left(\frac{\partial f}{\partial t}\right)_{\rm coll}\mathrm{d}^3v \simeq 0.$$

Correspondingly the moment equations are:

1. *Continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0;$$

2. *Linear momentum equation or motion equation*

$$\rho \frac{D\mathbf{V}}{Dt} + \nabla \cdot \tilde{\mathbf{\Pi}} = \mathcal{F},$$

where $D/Dt = (\partial/\partial t + \mathbf{V} \cdot \nabla)$ and $\tilde{\mathbf{\Pi}}$ is the pressure tensor; for homogeneous and isotropic plasmas $\nabla \cdot \tilde{\mathbf{\Pi}} \to \nabla p$; $\mathcal{F}$ are electromagnetic and non-electromagnetic forces;

3. *Energy equation*

$$\frac{D\varepsilon}{Dt} + \nabla \cdot \mathbf{t} + \varepsilon \nabla \cdot \mathbf{V} + \tilde{\mathbf{\Pi}} * \nabla \mathbf{V} = 0.$$

The system is not closed; closure assumptions are typically:

- by an equation of state $p = p(\rho)$,
- by the adiabaticity condition $\mathbf{t} = 0$, or $(D/Dt)(p\rho^{-5/3}) = 0$.

The only condition for applying these equations to a real plasma is that the averaging procedure corresponds to the measurable quantities. This requires that the plasma be close to an equilibrium state. Further assumptions must be used to make the fluid equations more tractable analytically.

2.2.1. Two-fluid model. The typical particle species in a plasma are: i) neutrals, f^0, ii) electrons, f^-, iii) ions, f_i^+, and for each of them a set of fluid equations must be solved. In astrophysics most plasmas are fully ionized and this allows to use two-fluid models with f^- and f^+ only.

We discuss the model equations for a plasma in thermodynamic equilibrium, so that the energy density can be written for the two species as $\varepsilon^\mp = c_v^\mp T^\mp$.

The system of equations for the two distributions with momentum and energy exchange terms $\Delta \mathbf{p}^{\mp\pm}$, $\Delta \mathcal{E}^{\mp\pm}$ is

$$\frac{\partial \rho^\mp}{\partial t} + \nabla \cdot (\rho^\mp \mathbf{V}^\mp) = 0,$$

$$\rho^\mp \left(\frac{\partial}{\partial t} + \mathbf{V}^\mp \cdot \nabla\right) \mathbf{V}^\mp + \nabla \cdot \tilde{\mathbf{\Pi}}^\mp - n^\mp e^\mp \left(\mathbf{E} + \frac{\mathbf{V}^\mp}{c} \times \mathbf{B}\right) = \Delta \mathbf{p}^{\mp\pm},$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}^\mp \cdot \nabla\right) c_v^\mp T^\mp + c_v^\mp T^\mp \nabla \cdot \mathbf{V}^\mp + \nabla \cdot \mathbf{t}^\mp + \tilde{\mathbf{\Pi}} * \nabla \mathbf{V}^\mp = \Delta \mathcal{E}^{\mp\pm}.$$

Averaging over the two distributions with

- mass density $\rightarrow \rho = \rho^+ + \rho^- = n^+m^+ + n^-m^-$,
- average flow $\rightarrow \mathbf{V} = (\rho^+\mathbf{V}^+ + \rho^-\mathbf{V}^-)/(\rho^+ + \rho^-)$,
- charge density $\rightarrow Q = n^+e^+ + n^-e^-$,
- current densities $\rightarrow \mathbf{J} = n^+e^+\mathbf{V}^+ + n^-e^-\mathbf{V}^-$,
- $\rightarrow \mathbf{J}_{\mathrm{conv}} = Q\mathbf{V} = (n^+e^+ + n^-e^-)\mathbf{V}$,
- $\rightarrow \mathbf{j} = \mathbf{J} - \mathbf{J}_{\mathrm{conv}} = n^+e^+(\mathbf{V}^+ - \mathbf{V} + n^-e^-(\mathbf{V}^- - \mathbf{V})$,
- total pressure tensor $\rightarrow \tilde{\mathbf{\Pi}} = \tilde{\mathbf{\Pi}}^+ + \tilde{\mathbf{\Pi}}^-$

and assuming elastic binary collisions

$$\triangle\mathbf{p}^{-+} + \triangle\mathbf{p}^{+-} \simeq 0, \qquad \triangle\mathcal{E}^{-+} + \triangle\mathcal{E}^{+-} \simeq 0$$

yields the following set of equations:

1. *Mass continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0;$$

2. *Charge continuity equation*

$$\frac{\partial Q}{\partial t} + \nabla \cdot \mathbf{J} = 0;$$

3. *Equation of motion*

$$\rho \frac{D\mathbf{V}}{Dt} + \nabla \cdot \tilde{\mathbf{\Pi}} - Q\mathbf{E} - \frac{1}{c}\mathbf{J} \times \mathbf{B} = 0;$$

4. *Generalized Ohm's law* (in the limit $m^-/m^+ \ll 1$, isotropic scalar pressure p, no thermoelectric currents $\nabla V_i^{\pm} \approx 0$, and $\triangle\mathbf{p}^{-+} \simeq -m^-\nu_c\mathbf{j}/e^-$)

$$\frac{\partial \mathbf{J}}{\partial t} + \Omega_e \mathbf{J} \times \frac{\mathbf{B}}{B} - \frac{e^-}{2m^-}\nabla p + \nu_c \mathbf{j} = \frac{n^- e^{-2}}{m^-}\left(\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B}\right);$$

5. *Energy equation* ($c_vT = c_v^-T^- + c_v^+T^+$)

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)c_vT + c_vT\nabla\cdot\mathbf{V} + \nabla\cdot\mathbf{t} + \tilde{\mathbf{\Pi}} * \nabla\mathbf{V} = \mathbf{j}\cdot\left(\mathbf{E} + \frac{\mathbf{V}}{c}\times\mathbf{B}\right);$$

6. *Maxwell's equations*

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \cdot \mathbf{E} = 4\pi Q, \qquad \nabla \cdot \mathbf{B} = 0$$

(the last two equations are in fact boundary conditions).

With the closure condition $\mathbf{t} = 0$, the two-fluid model has 15 scalar equations in 15 scalar unknowns ρ, $\mathbf{V}$, Q, $\mathbf{J}$, p, $\mathbf{E}$, $\mathbf{B}$.

2˙2.2. MHD equations. Plasmas dominated by short-range collisions and with large electric conductivity are quasi-neutral and Maxwellian:

$$n^+ \approx n^-, \qquad Q \approx 0, \qquad p = p^+ + p^- \approx 2p^+ \approx 2p^-,$$
$$\mathbf{J} \approx \mathbf{j}, \qquad \mathbf{V}^+ \approx \mathbf{V}^-, \qquad p = \frac{2}{3}c_v T.$$

Scaling over the characteristic time $\omega^{-1} = L/V$ (L plasma extension, V flow velocity) one derives the MHD system with 14 scalar equations in 14 scalar unknowns ρ, $\mathbf{V}$, $\mathbf{J}$, p, $\mathbf{E}$, $\mathbf{H}$:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \mathbf{V}),\\
\rho \frac{D\rho}{Dt} &= -\nabla p + \frac{1}{c}\mathbf{J} \times \mathbf{B},\\
\mathbf{J} &= \sigma\left(\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B}\right),\\
\frac{D}{Dt}\left(\frac{p}{\rho^{5/3}}\right) &= \frac{2}{3}\rho^{-5/3}\mathbf{J} \cdot \left(\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B}\right),\\
\nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{J},\\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}.
\end{aligned}$$

2˙2.3. Ideal MHD equations. In the limit of infinite conductivity $\sigma \to \infty$

$$\left(\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B}\right) = 0$$

and the electric field can be explicitly eliminated; the ideal MHD system is then formed by 11 scalar equations in 11 scalar unknowns ρ, $\mathbf{V}$, $\mathbf{J}$, p, $\mathbf{H}$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0,$$
$$\rho \frac{D\rho}{Dt} = -\nabla p + \frac{1}{c} \mathbf{J} \times \mathbf{B},$$
$$\frac{D}{Dt}\left(\frac{p}{\rho^{5/3}}\right) = 0,$$
$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}, \qquad \nabla \times (\mathbf{V} \times \mathbf{B}) = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}.$$

This is the plasma model most commonly used in astrophysical applications.

2·2.4. Applicability of a fluid treatment. Consider a plasma with spatial and temporal scales λ and τ, and short-range collision mean free path and frequency λ_{c} and τ_{c}^{-1}, respectively. In a gas the collective behavior is maintained by short-range collisions

$$\lambda \gg \lambda_{\mathrm{c}}, \qquad \tau = \frac{\lambda}{V} \gg \tau_{\mathrm{c}}.$$

In a plasma electromagnetic long-range collisions contribute:

$$\tau_{\mathrm{c,lr}} = \frac{2\pi n \lambda_{\mathrm{D}}^3}{\omega_{\mathrm{p}} \log(\lambda_{\mathrm{D}}/b_0)} \approx \frac{\tau_{\mathrm{c}}}{\log \Lambda},$$

where $\log \Lambda = 8 \log(\lambda_{\mathrm{D}}/b_0) \geq 10$. Correspondingly, an MHD treatment is allowed for:

$$\tau \gg \tau_{\mathrm{c,lr}}, \qquad \lambda \gg \lambda_{\mathrm{c,lr}} = V\tau_{\mathrm{c,lr}}.$$

2·2.5. Cold plasma equations. Long-range electromagnetic forces allow a coherent behavior even for plasmas with very low thermal velocities, $u \ll V$, *i.e.* low collision frequency. Equations for a cold plasma are obtained from the two-fluid model neglecting pressure, temperature and transport terms, yielding a system of 14 scalar equations in 14 scalar unknowns ρ, Q, $\mathbf{V}$, $\mathbf{J}$, $\mathbf{E}$, $\mathbf{B}$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \qquad \frac{\partial Q}{\partial t} + \nabla \cdot (\rho \mathbf{J}) = 0,$$
$$\rho \frac{D\mathbf{V}}{Dt} = Q\mathbf{E} + \frac{1}{c}\mathbf{J} \times \mathbf{B}, \qquad \frac{m^+ m^-}{\rho e^2}\frac{\partial \mathbf{J}}{\partial t} = \mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{B},$$
$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}.$$

2.3. *Orbit theory: the strong magnetic field limit.* – In non-collisional plasmas, $\lambda_c \gg \lambda$, a coherent behavior can be insured by a strong (external) magnetic field:

$$\lambda_c \gg r_{gyr} = \frac{mvc}{eB} \quad (r_{gyr} \text{ Larmor radius}).$$

The orbit theory is used to describe these plasmas as all particles of the same type follow the same paths. We shall not discuss here these plasma model equations, but simply mention that one can derive fluid equations in the assumption $\lambda_\perp \gg r_{gyr}$.

However a fluid behavior is guaranteed in the two space coordinates perpendicular to the magnetic field only. For the third coordinate, along the magnetic field, the cold plasma approximation is used. Such a plasma is essentially anisotropic with $p_\perp \neq p_\parallel$, $T_\perp \neq T_\parallel$, etc.

The fluid theory of anisotropic plasmas has been developed by Chew, Goldberger and Low; the energy equations assume adiabaticity in the perpendicular and parallel coordinates that can be written in terms of adiabatic invariants in the orbit theory:

$$\frac{D}{Dt}\left(\frac{p_\perp}{\rho B}\right) = 0, \qquad \frac{D}{Dt}\left(\frac{p_\parallel B^2}{\rho^3}\right) = 0.$$

2.4. *The plasma–magnetic-field interaction.* – Maxwell'equations and Ohm's law provide the so-called MHD equation

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} + \nabla \times (\mathbf{V} \times \mathbf{B}),$$

where $\eta = c^2/(4\pi\sigma)$ is the electric resistivity. Two limiting solutions are:

1. Plasma at rest, $\mathbf{V} = 0$:

   $$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B},$$

   $\mathbf{B}$ varies by diffusion on a time scale

   $$\tau_{\text{diff}} = \frac{L^2}{\eta} = \frac{4\pi\sigma L^2}{c^2}.$$

   In table II examples of diffusion time scales for astrophysical situations are given.

2. Plasma in motion, $\mathbf{V} \neq 0$, with negligible resistivity, $\eta \to 0$ (infinite conductivity):

   $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}).$$

   The form of this equation is similar to the equation for vorticity in turbulent fluids and, as shown by Alfvén, can be interpreted as plasma tying to magnetic lines, or conversely magnetic lines freezing in the plasma.

TABLE II. – *Diffusion time scales for astrophysical plasmas.*

	L (cm)	τ_{diff} (s)
Gas discharges	10	10^{-3}
Earth's nucleus	10^{8}	10^{12}
Sunspots	10^{9}	10^{14}
Solar corona	10^{11}	10^{18}
Interplanetary space	10^{13}	10^{20}

The applicability of the two solutions can be defined in terms of the magnetic Reynolds number $\mathcal{R}_{\mathcal{M}}$:

$$|\nabla \times (\mathbf{V} \times \mathbf{B})| \gg |\eta \nabla^2 \mathbf{B}|, \qquad \frac{VB}{L} \gg \eta \frac{B}{L^2}, \qquad \mathcal{R}_{\mathcal{M}} \equiv \frac{LV}{\eta}.$$

For $\mathcal{R}_{\mathcal{M}} \gg 1$ the freezing-in condition applies.

3. – Waves in plasmas

The collective behavior of plasmas supports many types of waves and oscillations characterized by complex gas and electromagnetic field couplings. Waves provide energy transport and radiation emission; at the same time they represent a diagnostic tool for measuring plasma parameters.

The analysis of dispersion relations of waves is an important chapter in the study of plasmas. We here summarize some of the basic elements.

3·1. MHD *waves.* – Using the ideal MHD equations in the incompressible limit

$$\begin{aligned}
\nabla \cdot \mathbf{V} &= 0, \\
\rho \frac{D\mathbf{V}}{Dt} &= -\nabla p + \frac{1}{c} \mathbf{J} \times \mathbf{B}, \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{V} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{V} - (\mathbf{V} \cdot \nabla)\mathbf{B}, \\
\nabla \times \mathbf{B} &= \frac{4\pi}{c} \mathbf{J},
\end{aligned}$$

one derives

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p - \frac{1}{4\pi} \mathbf{B} \times (\nabla \times \mathbf{B}) = -\nabla \left(p + \frac{B^2}{8\pi} \right) + \frac{1}{4\pi} (\mathbf{B} \cdot \nabla)\mathbf{B}.$$

Starting from an initial equilibrium state with $\mathbf{V}_0 = 0$ and p_0, ρ_0, $\mathbf{B}_0$ spatially uniform and applying perturbations of the form

$$\mathbf{V} = 0 + \mathbf{V}', \qquad p = p_0 + p', \qquad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}',$$

one obtains the following linearly perturbed equations:

$$\rho \frac{\partial^2 \mathbf{V}'}{\partial t^2} = \frac{1}{4\pi}(\mathbf{B}_0 \cdot \nabla)^2 \mathbf{V}', \qquad \rho \frac{\partial^2 \mathbf{B}'}{\partial t^2} = \frac{1}{4\pi}(\mathbf{B}_0 \cdot \nabla)^2 \mathbf{B}'.$$

Choosing $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ one gets the typical wave equations

$$\frac{\partial^2 \mathbf{V}'}{\partial t^2} = \frac{B_0^2}{4\pi\rho} \frac{\partial^2 \mathbf{V}'}{\partial z^2}, \qquad \frac{\partial^2 \mathbf{B}'}{\partial t^2} = \frac{B_0^2}{4\pi\rho} \frac{\partial^2 \mathbf{B}'}{\partial z^2},$$

corresponding to velocity and magnetic field oscillations transverse to $\mathbf{B}_0$ with phase velocity:

$$\mathbf{V}_{\mathrm{A}} = \frac{\mathbf{B}_0}{\sqrt{4\pi\rho}}.$$

Fourier-analyzing the perturbations $\mathbf{V}'$, $\mathbf{B}' \propto \exp[i(\omega t - kz)]$ one gets the dispersion relation for the Alfvén waves

$$\omega^2 = V_{\mathrm{A}}^2 k^2,$$

that are clearly non-dispersive.

For compressible plasmas one obtains three types of MHD waves (θ angle to $\mathbf{B}_0$):

1. fast magnetosonic $\Longrightarrow \omega^2 \simeq \left(V_{\mathrm{s}}^2 + V_{\mathrm{A}}^2\right) k^2$,
2. slow magnetosonic $\Longrightarrow \omega^2 \simeq \left(\frac{V_{\mathrm{s}}^2 V_{\mathrm{A}}^2}{V_{\mathrm{s}}^2 + V_{\mathrm{A}}^2}\right) k^2 \cos^2\theta$,
3. Alfvénic $\Longrightarrow \omega^2 = V_{\mathrm{A}}^2 k^2 \cos^2\theta$.

Non-ideal MHD waves are dispersive:

$$\omega = -\frac{1}{2} i k^2 \eta \pm \frac{1}{2}\sqrt{4k^2 V_{\mathrm{A}}^2 - k^4 \eta^2}.$$

3·2. *Waves in the two-fluid model.* – A general discussion of waves in plasmas can be done using the two-fluid equations. One starts from an equilibrium state with $n^+ = n^-$, $Q = 0$, $\mathbf{V}^+ = \mathbf{V}^- = 0$ and uniform $\mathbf{B}_0$. Further assume thermal equilibrium between positive ions and electrons $T^+ = T^-$, $p^+ = n^+kT$, $p^- = n^-kT$. With linear perturbations δn^-, δn^+, $\delta\mathbf{V}^-$, $\delta\mathbf{V}^+$, etc. one obtains the linearized equations

$$\nabla\times\delta\mathbf{E} = -\frac{1}{c}\frac{\partial\delta\mathbf{B}}{\partial t}, \qquad \nabla\times\delta\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\delta\mathbf{E}}{\partial t},$$

$$\nabla\cdot\delta\mathbf{E} = 4\pi\delta Q, \qquad \nabla\cdot\delta\mathbf{B} = 0,$$

$$\frac{\partial\delta n^\pm}{\partial t} + \nabla\cdot(n^\pm\delta\mathbf{V}^\pm) = 0,$$

$$n^\pm m^\pm\frac{\partial\delta\mathbf{V}^\pm}{\partial t} = n^\pm e^\pm\left(\delta\mathbf{E} + \frac{1}{c}\delta\mathbf{V}^\pm\times\mathbf{B}_0\right) - \nabla p^\pm - n^\pm m^\pm\nu_c(\delta\mathbf{V}^\pm - \delta\mathbf{V}),$$

$$\delta Q = \delta n^- e^- + \delta n^+ e^+,$$

$$\mathbf{V} = n^- e^-\delta\mathbf{V}^- + n^+ e^+\delta\mathbf{V}^+,$$

$$\delta p^\pm = m^\pm\left(V_T^\pm\right)^2\delta n^\pm,$$

$$V_\mathrm{T}^\pm = \sqrt{\frac{3kT}{m^\pm}}.$$

A plane wave analysis for propagation vector along $\hat{\mathbf{z}}$ with $\delta f = f_0\exp[-i(\omega t + kz)]$ yields a fourth-order homogeneous algebraic dispersion relation in k^2, $F(\omega, k) = 0$. In principle ω, k can be both complex: usually the choice is k real and ω complex to study the local amplification or decay of waves of given wave number. The solution of the dispersion relation provides *four modes propagating in opposite directions:* 1 *electronic mode,* 1 *ionic mode,* 2 *electromagnetic modes (ordinary and extraordinary).*

We first analyze the solutions in the limit of positive ions at rest, $m_\mathrm{e}/m_\mathrm{i} \ll 1$. In this case the dispersion relations becomes of third order, without ion modes.

3·2.1. No external magnetic field, $\Omega_\mathrm{e} = 0\,(\beta_\mathrm{T} = V_\mathrm{T}/c)$. Two basic types of waves exist (V_T thermal velocity, V_f phase velocity, V_g group velocity, $n^2 = \omega^2/k^2$ refractive index):

1. Longitudinal electrostatic oscillations ($\mathbf{k} \parallel \mathbf{E}$) or electron plasma waves:

$$\omega^2 = \omega_\mathrm{pe}^2 + V_\mathrm{T}^2 k^2, \qquad n^2 = \beta_\mathrm{T}^{-2}\left(1 - \frac{\omega_\mathrm{pe}^2}{\omega^2}\right),$$

$$\lambda_\mathrm{pe} = 2\pi\frac{V_\mathrm{T}}{\omega_\mathrm{pe}} = 2\pi\sqrt{3}\lambda_\mathrm{D} \approx 10\lambda_\mathrm{D},$$

$$V_\mathrm{f} = \frac{\omega}{k} = V_\mathrm{T}\left(1 - \frac{\omega_\mathrm{pe}^2}{\omega^2}\right)^{-1/2} > V_\mathrm{T},$$

$$V_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = V_{\mathrm{T}}\left(1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right)^{1/2} < V_{\mathrm{T}};$$

2. Transverse electromagnetic waves ($\mathbf{k} \perp \mathbf{E} \perp \mathbf{B}$):

$$\omega^2 = \omega_{\mathrm{pe}}^2 + c^2 k^2, \qquad n^2 = \left(1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right) < 1,$$

$$V_{\mathrm{f}} = \frac{\omega}{k} = c\left(1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right)^{-1/2} > c,$$

$$V_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = c\left(1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right)^{1/2} < c.$$

3·2.2. Longitudinal magnetic field $\mathbf{k} \parallel \mathbf{B}_0$, $\theta = 0$. The dispersion relation splits into two branches:

1. Longitudinal electrostatic oscillations or electron plasma waves:

$$\omega^2\left(1 - \beta_{\mathrm{T}}^2 n^2\right) - \omega_{\mathrm{pe}}^2 = 0;$$

2. Transverse electromagnetic modes:

$$\left[\omega^2\left(1 - n^2\right) - \omega_{\mathrm{pe}}^2\right]^2 \omega^2 \Omega_{\mathrm{e}}^2 \left(1 - n^2\right)^2 = 0,$$

- ordinary waves, left circularly polarized, gyrating with electrons:

$$n^2 = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega(\omega + \Omega_{\mathrm{e}})};$$

- extraordinary waves, right circularly polarized, gyrating opposite to electrons:

$$n^2 = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega(\omega - \Omega_{\mathrm{e}})};$$

- in the limit of strong magnetic fields the two waves reduce to well-known results $\omega^2 = c^2 k^2$ electromagnetic, $\omega^2 = \Omega_{\mathrm{e}}^2$ electron cyclotron;
- cut-off frequencies ($n \to 0$): $\omega_{\mathrm{min}} = (\omega_{\mathrm{pe}}^2 + \Omega_{\mathrm{e}}^2/4) - (\Omega_{\mathrm{e}}/2)$ for ordinary modes and $\omega_{\mathrm{min}} = (\omega_{\mathrm{pe}}^2 + \Omega_{\mathrm{e}}^2/4) + (\Omega_{\mathrm{e}}/2)$ for extraordinary modes.

3·2.3. Transverse magnetic field $\mathbf{k} \perp \mathbf{B}_0$, $\theta = \pi/2$. Again we have two branches:

1. Electromagnetic transverse ordinary waves:

$$\omega^2 = \omega_{\text{pe}}^2 + c^2k^2;$$

2. Hybrid electromagnetic modes:

$$n^2 = 1 - \frac{\omega_{\text{pe}}^2/\omega^2}{1 - \Omega_{\text{e}}^2/(\omega^2 - \omega_{\text{pe}}^2)} = \frac{1}{\omega^2}\frac{(\omega^2 - \omega_{\text{pe}}^2) - \omega^2\Omega_{\text{e}}^2}{\omega^2 - (\omega_{\text{pe}}^2 + \Omega_{\text{e}}^2)},$$

– upper hybrid resonance ($n \to \infty$) for $\omega_{\text{res}} = (\omega_{\text{pe}}^2 + \Omega_{\text{e}}^2)^{1/2}$;

– cut-off ($n \to 0$) for $\omega_{\text{c}} = (\pm\Omega_{\text{e}}/2) + (\omega_{\text{pe}}^2 + \Omega_{\text{e}}^2/4)$.

3·2.4. Generic field orientation, cold plasma. The dispersion relation in this case is known as the Appleton-Hartree equation:

$$n^2 = 1 - \frac{2\omega_{\text{pe}}^2/\omega^2(1 - \omega_{\text{pe}}^2/\omega^2)}{2(1 - \omega_{\text{pe}}^2/\omega^2) - \mathrm{B}^2\sin^2\theta \pm \Gamma},$$

$$\mathrm{B} = \frac{\Omega_{\text{e}}}{\omega},$$

$$\Gamma = \sqrt{\mathrm{B}^4\sin^4\theta + 4\mathrm{B}^2(1 - \omega_{\text{pe}}^2/\omega^2)^2\cos^2\theta}.$$

3·2.5. Ion modes. If one takes into account the mobility of ions, in addition to some modifications in the above dispersion relations, ion modes are also found.

1. Alfvén transverse shear waves ($\mathbf{k} \parallel \mathbf{B}_0$):

$$(\omega_{\text{pe}}^2 + \omega_{\text{pi}}^2) = \Omega_{\text{e}}\Omega_{\text{i}}\left(1 - \frac{k^2c^2}{\omega^2}\right),$$

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{c^2}{V_{\text{A}}^2}, \qquad V_{\text{A}} = \sqrt{\frac{B_0^2}{4\pi m^+ n^-}};$$

2. Ion cyclotron waves ($\mathbf{k} \parallel \mathbf{B}_0$) gyrating with ions:

$$\frac{k^2c^2}{\omega^2} = \frac{2\omega_{\text{pi}}^2}{\Omega_{\text{i}}^2 - \omega^2};$$

3. Compressible Alfvén waves ($\mathbf{k} \perp \mathbf{B}_0$):

$$\left[\Omega_e\Omega_i - (\omega_{pe}^2 + \omega_{pi}^2)\right]\left[(\omega_{pe}^2 + \omega_{pi}^2)\frac{\omega^2}{\omega^2 - k^2c^2} - \Omega_e\Omega_i\right] = 0,$$

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{c^2}{V_A^2};$$

4. Ion sound waves (ion plasma oscillations):

$$\omega^2 = k^2\frac{\omega_{pe}^2 V_{Ti}^2 + \omega_{pi}^2 V_{Te}^2}{\omega_{pe}^2},$$

$$\omega^2 \approx k^2\left(\gamma^+\frac{kT^+}{m^+} + \gamma^-\frac{kT^-}{m^-}\right) \quad \text{for } k^2 \ll \frac{\omega_{pe}^2}{V_{Te}^2},$$

$$\omega^2 \approx \omega_{pi}^2 + k^2V_{Ti}^2 \quad \text{for } k^2 \simeq \frac{\omega_{pi}^2}{V_{Ti}^2} \gg \frac{\omega_{pe}^2}{V_{Te}^2}.$$

In fig. 1 we present the dispersion relations of plasma waves for a specific choice of the plasma parameters.

4. – Plasma instabilities

The study of the time evolution of a plasma starting from an equilibrium configuration is important in astrophysical applications to understand the stability of configurations and the excitation of transient phases. An analytic linear approach based on perturbations developed in orthonormal Fourier modes $f \propto \exp[i\omega t]$ allows to define the following situations:

- perturbations grow monotonically, $\mathrm{Im}\,\omega < 0$, $\mathrm{Re}\,\omega \to 0$: the plasma configuration is unstable;

- perturbations perform oscillations of increasing amplitude, $\mathrm{Re}\,\omega \neq 0$, $\mathrm{Im}\,\omega < 0$: the plasma configuration is overstable;

- perturbations perform oscillations of decreasing amplitude, $\mathrm{Re}\,\omega \neq 0$, $\mathrm{Im}\,\omega > 0$: the plasma configuration is stable.

The linear analysis of plasma instabilities, that corresponds at some level to deriving the mode dispersion relation, is limited because it is cumbersome to prove the orthonormality and completeness conditions. In addition, given the large number of oscillations and waves allowed in plasmas, non-linear effects of mode coupling and saturation are crucial in defining the condition of instability beyond its initial development stage.

As an alternative to mode analysis, global variational models are often used in plasma theory.

4·1. MHD *instabilities*. – In this brief review we shall only comment upon the study of instabilities in the framework of the MHD theory and discuss two instabilities of wide astrophysical application.

In MHD theory one starts from an equilibrium configuration as

$$p_0 = \frac{1}{c}\,\mathbf{J}_0 \times \mathbf{B}_0 + \rho_0 \mathbf{g}, \qquad \nabla \times \mathbf{B}_0 = \frac{4\pi}{c}\,\mathbf{J}_0 .$$

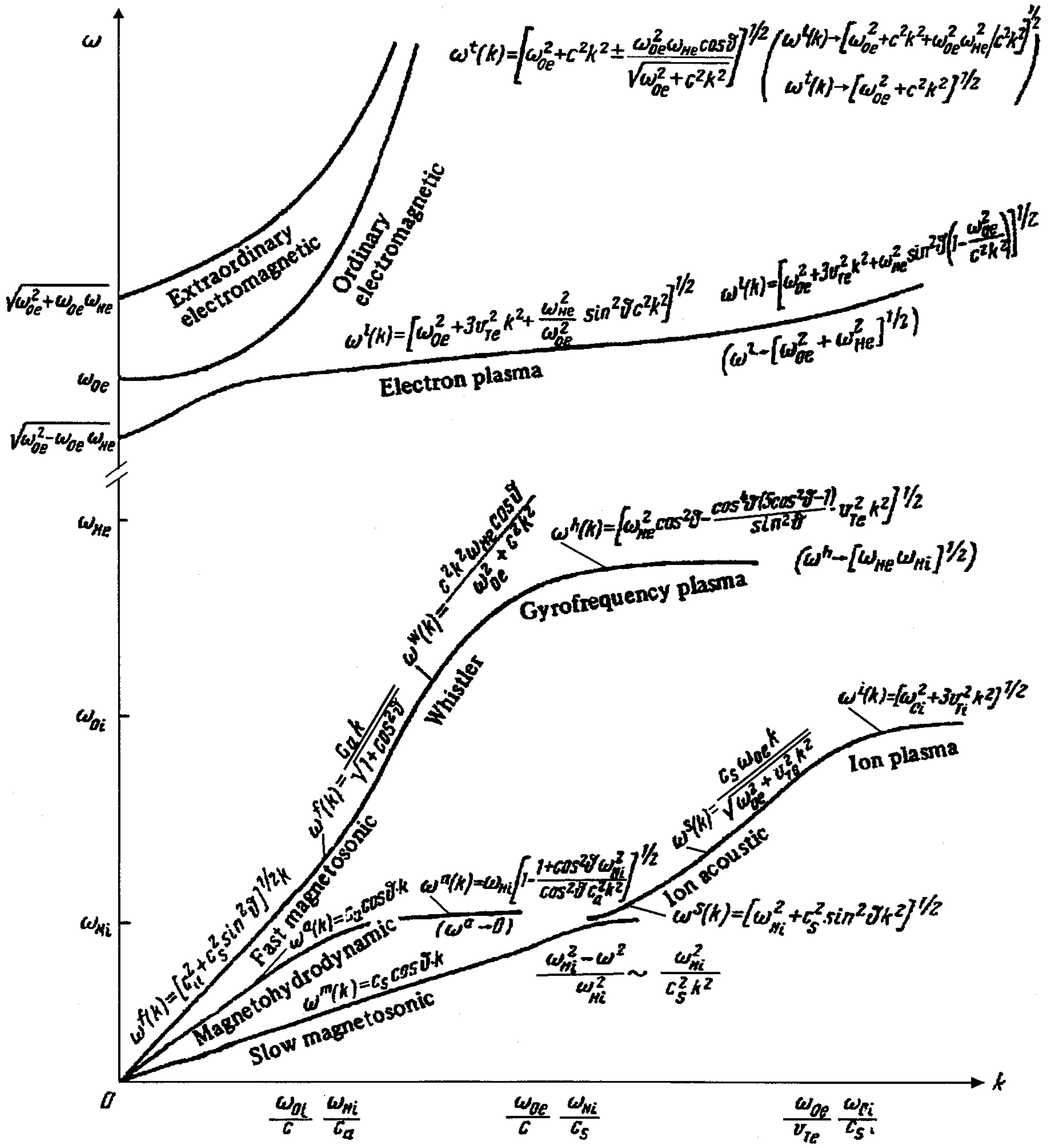

Fig. 1. – Dispersion relations of waves in magnetized plasmas.

Assuming for instance an adiabatic equation of state $p\rho^{-\gamma} = \text{const}$, and applying a linear mode analysis, one obtains the perturbed equations (perturbed quantities are indicated with an apex)

$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{V}'),$$
$$\rho_0 \frac{\partial \mathbf{V}'}{\partial t} = -\nabla p + \frac{1}{c}(\mathbf{J}' \times \mathbf{B}_0 + \mathbf{J}_0 \times \mathbf{B}'),$$

$$p' \rho_0 = \gamma \rho' p_0,$$
$$\frac{\partial \mathbf{B}'}{\partial t} = \nabla \times (\mathbf{V}' \times \mathbf{B}_0),$$
$$\nabla \times \mathbf{B}' = \frac{4\pi}{c}\mathbf{J}'.$$

It is customary to introduce the Lagrangian variable $\boldsymbol{\xi}(\mathbf{r}_0, t) = \mathbf{r} - \mathbf{r}_0$, so that $\mathbf{V}' = D\mathbf{r}/Dt = \partial\boldsymbol{\xi}/\partial t$. With this variable the above equations are combined in the single expression

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \nabla(\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi}) + \frac{1}{4\pi}\left[(\nabla \times \nabla \times [\boldsymbol{\xi} \times \mathbf{B}_0]) \times \mathbf{B}_0\right] + \\ + \frac{1}{4\pi}\left[(\nabla \times \mathbf{B}_0) \times (\nabla \times [\xi \times \mathbf{B}_0])\right] \equiv \mathbf{Q}\left[\xi(\mathbf{r}, \omega_n)\right].$$

The normal mode analysis with $\boldsymbol{\xi}(\mathbf{r}, t) = \sum_n \boldsymbol{\xi}(\mathbf{r}, \omega_n) e^{i\omega_n t}$ leads to the equation

$$-\rho_0 \omega_n^2 \boldsymbol{\xi}(\mathbf{r}, \omega_n) = \mathbf{Q}\left[\boldsymbol{\xi}(\mathbf{r}, \omega_n)\right]$$

with appropriate boundary conditions. The mathematical conditions to satisfy by eigenfunctions are: orthonormality, completeness and limitation in amplitude. Singularities in $\mathbf{Q}$ can also arise that give rise to localized unphysical unstable modes.

4'1.1. Rayleigh-Taylor instability. This instability gives rise to mixing when heavy fluids are located above light fluids in a gravity field or in the presence of non-inertial effects yielding an effective gravity.

We assume (fig. 2) a plasma with a density gradient along the x-axis and a homogeneous gravitational field along the negative x-axis; a magnetic field is present in the (y, z)-plane. The equilibrium condition for an inviscid, incompressible fluid is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p_0 + \frac{B_0^2}{8\pi}\right) = -\rho_0 g$$

that is perturbed in the form of plane waves:

$$\boldsymbol{\xi}(\mathbf{r}, t) = \boldsymbol{\xi}(x) e^{i\omega t + i(k_y y + k_z z)}.$$

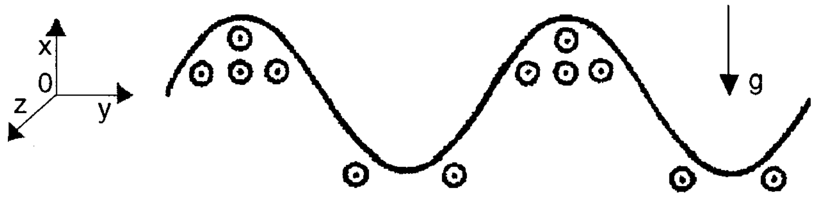

Fig. 2. – Rayleigh-Taylor instability in magnetized plasmas.

The local analysis with ω, $\boldsymbol{\xi}$ complex and $\mathbf{k}$ real provides the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left[\omega^2\rho_0 - \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{4\pi}\right]\frac{\mathrm{d}\xi_x}{\mathrm{d}x}\right\} + \left(k_y^2 + k_z^2\right)\left[-\rho_0\omega^2 + \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{4\pi} - g\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right]\xi_x = 0.$$

Applying the condition $\xi_x \to 0$ at the physical domain boundaries, and assuming further $B_{0,y} = 0$, $k_z = 0$, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho_0\frac{\mathrm{d}\xi_x}{\mathrm{d}x}\right] + \left[-\rho_0 k_y^2 + \left(-\frac{k_y^2}{\omega^2}g\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right)\right]\xi_x = 0$$

with $\xi_x = 0$ at $x = \pm\infty$. This is a typical Sturm-Liouville problem, allowing non-trivial solutions if

$$-\frac{k_y^2}{\omega^2}g\frac{\mathrm{d}\rho_0}{\mathrm{d}x} > 0,$$

requiring

- $\omega^2 < 0$, where $\mathrm{d}\rho_0/\mathrm{d}x > 0$,
- $\omega^2 > 0$, where $\mathrm{d}\rho_0/\mathrm{d}x < 0$.

Therefore the condition for stability is that $\mathrm{d}\rho_0/\mathrm{d}x < 0$ everywhere. The growth rate for incompressible fluids separated by a discontinuity at $x = 0$ (ρ_1 above, ρ_2 below) is

$$\omega^2 = -gk\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}.$$

More generally with $\mathbf{k}\cdot\mathbf{B}_0 \neq 0$ the growth rate is

$$\omega^2 = -gk\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} + \frac{B^2 k_\parallel^2}{2\pi(\rho_1 + \rho_2)gk}.$$

For compressible fluids with an equation of state $p\rho^{-\gamma} = \text{const}$,

$$\omega^2 = g\frac{\mathrm{d}}{\mathrm{d}x}\left(\log\frac{\rho_0}{p_0^{1/\gamma}}\right).$$

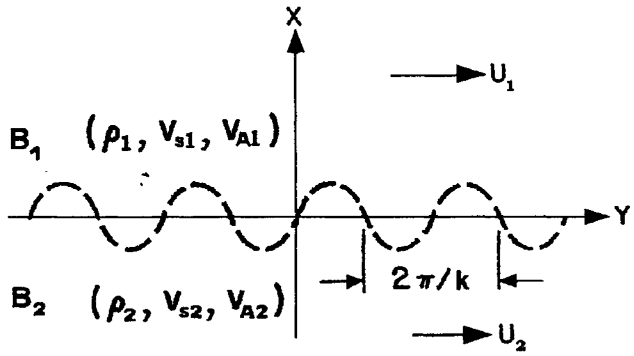

Fig. 3. – Kelvin-Helmholtz instability.

This last result corresponds to the *Schwarzschild criterion* for convective instability: instability arises ($\omega^2 > 0$) when $\log \rho_0$ increases more rapidly against gravity than $(1/\gamma)\log p_0$. Magnetic fields do not change the instability criterion, they simply contribute to the total pressure.

Effective gravity can be substituted by inertial forces: for instance, supernova shells expanding into the circumstellar gas decelerate and become Rayleigh-Taylor unstable to mixing with external gas.

4˙1.2. Kelvin-Helmholtz instability. This instability leads to mixing between fluids in relative motion along a contact discontinuity with formation of shear (turbulent) layers.

Incompressible hydrodynamic case. The classical case is that of the contact layer between two incompressible, non-magnetized fluids in pressure equilibrium and with absolute velocities along the x-axis $U_{1,2}$ (fig. 3). We write the perturbed equations in the two fluids in terms of the perturbations $(\mathbf{v}_{1,2}, p_{1,2})$ in the (y,z)-plane:

$$\nabla \cdot \mathbf{v}_{1,2} = 0,$$

$$\rho_{1,2}\left(\frac{\partial}{\partial t} + U_{1,2}\frac{\partial}{\partial y}\right)\mathbf{v}_{1,2} = -\nabla p_{1,2}$$

and develop the perturbations in plane waves $\propto \xi_x \exp[i(k_y y + k_z z + \omega t)]$. The dispersion relation is

$$\omega = -k_y\big(\alpha_1 U_1 +_2 U_2\big) \pm i\Big[k_y^2 \alpha_1 \alpha_2 \big(U_1 - U_2\big)^2\Big]^{1/2},$$

where $\alpha_{1,2} = \rho_{1,2}/(\rho_1 + \rho_2)$. Unstable solutions always exist for wave vectors with a non-zero component along the relative velocity.

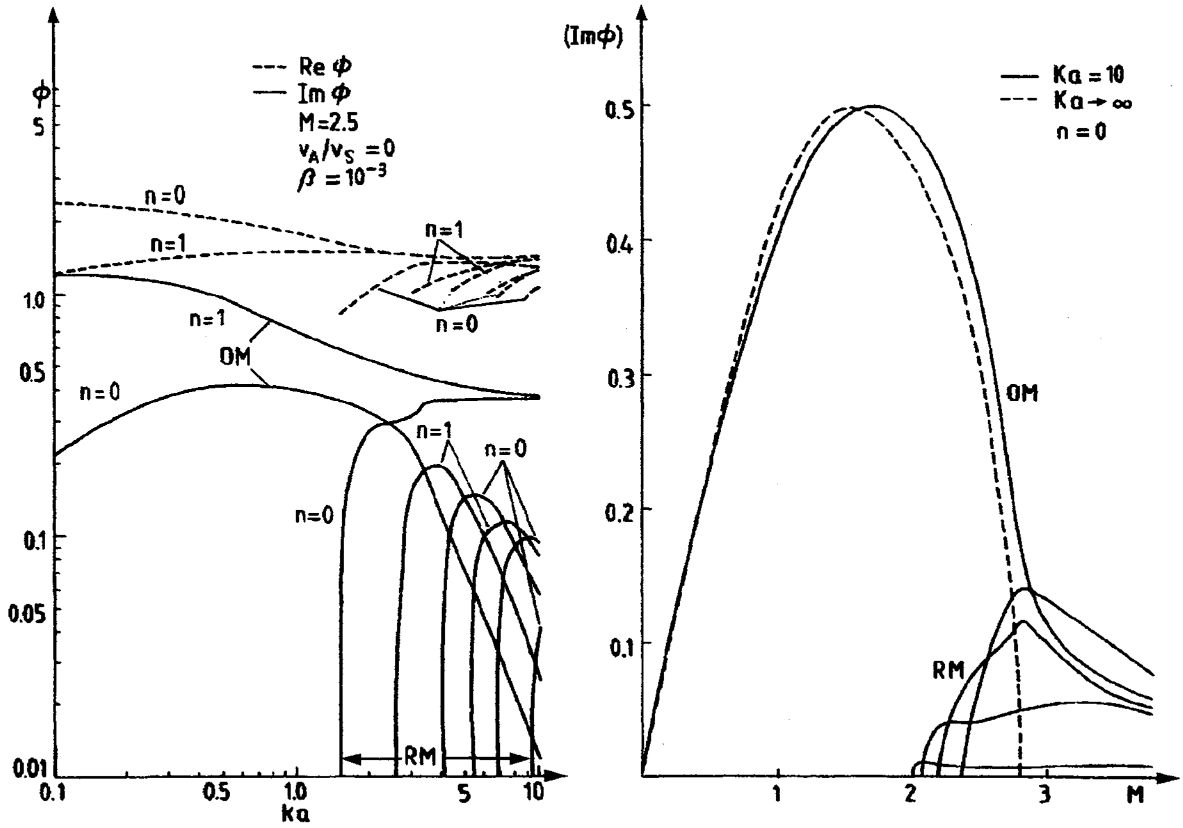

Fig. 4. – Growth rates of Kelvin-Helmholtz modes. Left panel: frequencies (Re Φ) and growth rates (Im Φ) *vs.* wavelengths; right panel: growth rates against flow Mach number (OM = ordinary modes, RM = reflected modes).

Compressible hydrodynamic case. Introducing compressibility with $p\rho^{-\gamma}$ = const and choosing the relative velocities as $U_1 = 0$ e $U_2 = U \neq 0$, we write the perturbations in the form $p_{1,2} \propto \exp[\pm q_{1,2}x]$ and obtain the dispersion relation

$$\omega^2 q_2 = -\left(\omega + Uk_y\right)^2 q_1$$

with

$$q_1 = \left[k^2 - \frac{\omega^2}{V_{s1}^2}\right]^{1/2}, \qquad q_2 = \left[k^2 - \frac{(\omega + Uk_y)^2}{V_{s2}^2}\right]^{1/2}.$$

In adimensional form:

$$\Phi \equiv \frac{\omega}{kV_{s2}}, \qquad M \equiv \frac{Uk_y}{kV_{s2}} = \frac{U\cos\theta}{V_{s2}},$$

the dispersion relation is

$$F(\Phi) = \Phi^2\left[(\Phi - M)^2 - 1\right]^{1/2} - (\Phi - M)^2(\Phi^2 - 1)^{1/2} = 0.$$

MHD *relativistic case with plane interface.* Introducing magnetic fields and using relativistic dynamics as applicable to some astrophysical applications (relativistic jets with $\Gamma = (1 - V^2/c^2)^{1/2}$), the initial equilibrium state must satisfy the following system of equations:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}),$$

$$\Gamma^2\left(\rho + \frac{p}{c^2}\right)\frac{D\mathbf{V}}{Dt} = -\nabla p - \frac{\mathbf{V}}{c^2}\frac{\partial p}{\partial t} + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B},$$

$$\Gamma\frac{D\rho}{Dt} = -\left(\rho + \frac{p}{c^2}\right)\left[\frac{\partial \Gamma}{\partial t} + \nabla \cdot (\Gamma \mathbf{V})\right],$$

$$\frac{D}{Dt}(p\rho^{-\gamma}) = 0.$$

We then assume

- plasma at rest in $x < 0$ (plasma 2) and plasma in motion at velocity V in $x > 0$ (plasma 1),
- (y, z) is the plane contact interface,
- relative motion along z,
- magnetic field parallel to the interface,
- perturbation $f \propto \exp[i(k_y y + k_z z - \omega t) \pm q(\omega)x]$.

The final dispersion relation is again in the form

$$F\left(\Phi, M = \frac{V}{V_{s1}}, \nu = \frac{\rho_1}{\rho_2}, \frac{B_1}{B_2}, R = \frac{k_y}{k_z}, \mathbf{V} \cdot \mathbf{B}_{1,2} = \cos\phi_{1,2}, \frac{V_{A1,2}}{V_{s1}}\right) = 0$$

and must be treated numerically.

The main results are the following:

- frequency of unstable oscillations are in the range $\omega_A \leq \mathrm{Re}\,\omega \leq \omega_f$ between Alfvén and fast modes;
- magnetic fields reduce instability ($M = V/V_{s1}$ flow Mach number); the range of instability is

$$2\frac{V_A}{V_s} \leq M \leq \sqrt{8(1 + R^2)}, \quad \frac{V_A}{V_s} \ll 1,$$

$$2\frac{V_A}{V_s} \leq M \leq \frac{V_A}{V_s}\sqrt{8(1 + R^2)}, \quad \frac{V_A}{V_s} \gg 1;$$

– relativistic effects

$$\operatorname{Re}\Phi \approx \frac{M}{2} \quad \text{non-relativistic,}$$
$$\operatorname{Re}\Phi \approx M\frac{c}{V}\left(1-\frac{1}{\Gamma}\right) \quad \text{relativistic.}$$

MHD *case with cylindrical interface (jets).* Using cylindrical perturbations

$$f(r,\theta,z,t) = g(r)\exp\left[i(kz+n\theta-\omega t)\right]$$

the dispersion relation becomes

$$\nu\frac{\Delta_\mathrm{e}}{\Delta_\mathrm{i}}\frac{\Gamma^2(M-\Phi)^2-(V_\mathrm{Ai}/V_\mathrm{si})^2}{\Phi^2-(V_\mathrm{Ae}/V_\mathrm{si})^2} = \frac{J_n'(ka\Delta_\mathrm{i})H_n^{(1)}(ka\Delta_\mathrm{e})}{J_n(ka\Delta_\mathrm{i})H_n'^{(1)}(ka\Delta_\mathrm{e})}$$

yielding various types of unstable modes: pinches ($n=0$), kinks or helices ($n=1,2,...$), flutes ($n\geq 2$, $k=0$). In addition two types of unstable perturbations are found (fig. 4):

– ordinary surface modes ($ka\sim 1$) rapidly decaying away from the contact layer,

– global reflected modes ($ka\gg 1$) with finite amplitude inside the cylinder.

Non-linear evolution. The long-term evolution of instabilities, including mode interactions and transition to turbulent mixing, must be studied numerically. In particular supersonic and relativistic jets in 2D and 3D, with different density ratios with respect to the surrounding medium, without and with magnetic fields have been studied for astrophysical applications up to maximum time scales $t\sim 40a/V_\mathrm{s}$. Different phases of evolution have been found:

1. linear phase, up to the formation of internal shocks,
2. acoustic phase, with formation of shocks at the contact interface leading to energy dissipation from jets to ambient; light jets transfer more energy to ambient,
3. mixing phase, heavy jets become very turbulent,
4. quasi-stationary phase, highly mixed and turbulent end result.

In general magnetic fields tend to slow down the instability, while 3D effects accelerate the transition to the final turbulent state.