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Introduction to Nonlinear Gyrokinetic Equation

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Intro to Nonlinear Gyrokinetic Equation

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I. Microinstabilities & MHD instabilities often driven by "bad curvature" & ∇p.

$\Rightarrow$ intuitive picture of this

II. Properties of the Gyrokinetic Eq.

& GK ordering

compare with Kulsrud's Collisionless MHD & drift-kinetic ordering.

THEORY-BASED MODELS OF TURBULENCE AND ANOMALOUS TRANSPORT IN FUSION PLASMAS

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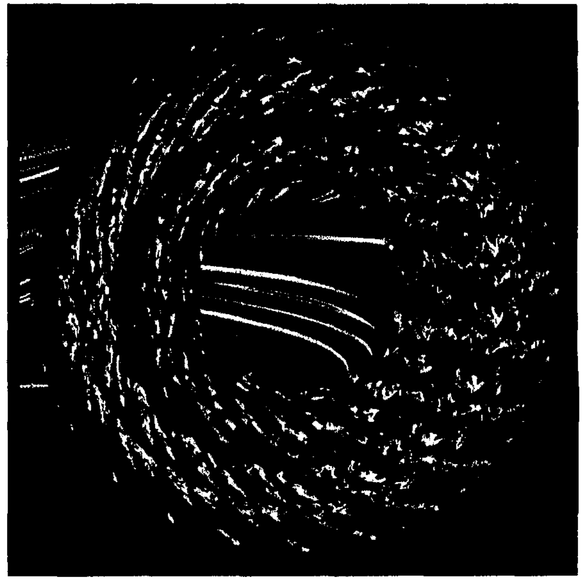
APS Centennial, Atlanta, March, 1999

Close collaboration with M.A. Beer (PPPL), W. Dorland (Univ. of Maryland), M. Kotschenreuther (Univ. of Texas), R.E. Waltz (General Atomics).



Acknowledgments: A. Dimits, G.D. Kerbel (LLNL), T.S. Hahm, Z. Lin, P.B. Snyder (PPPL), S.E. Parker (U. Colorado), and many others.

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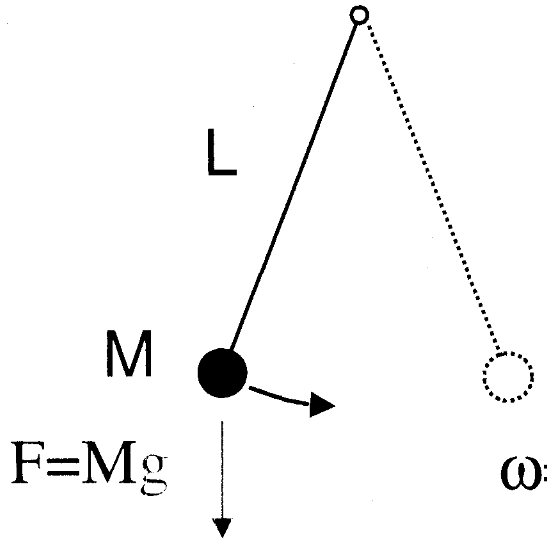
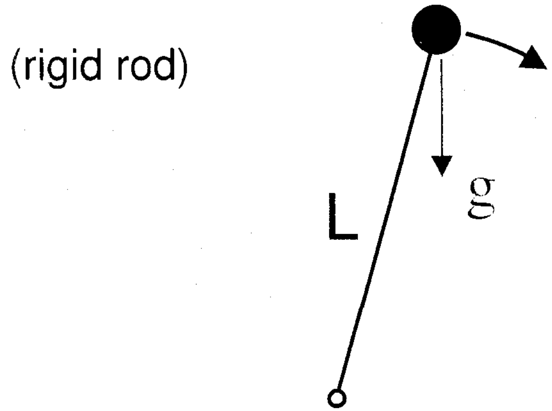
Stable Pendulum



$\omega=(g/L)^{1/2}$

Unstable Inverted Pendulum

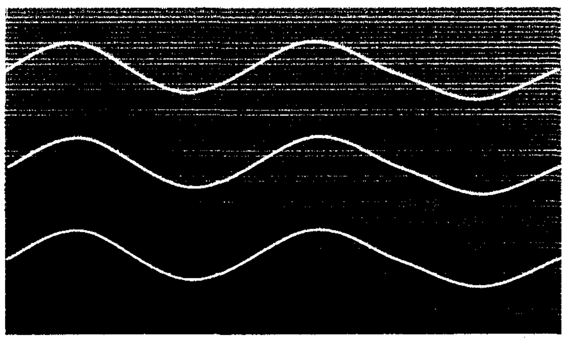


$\omega = (-g/|L|)^{1/2} = i(g/|L|)^{1/2} = i\gamma$

Instability

Density-stratified Fluid

$\rho=\exp(-y/L)$

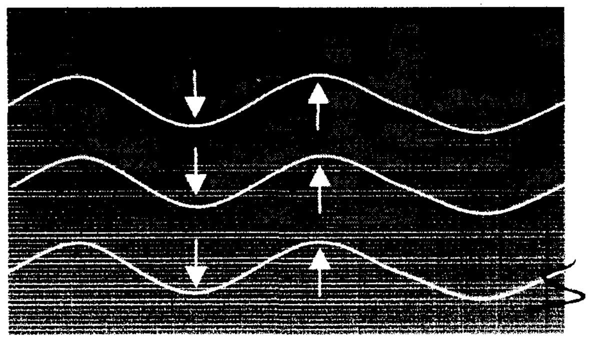


stable $\omega=(g/L)^{1/2}$

Inverted-density fluid

$\Rightarrow$Rayleigh-Taylor Instability

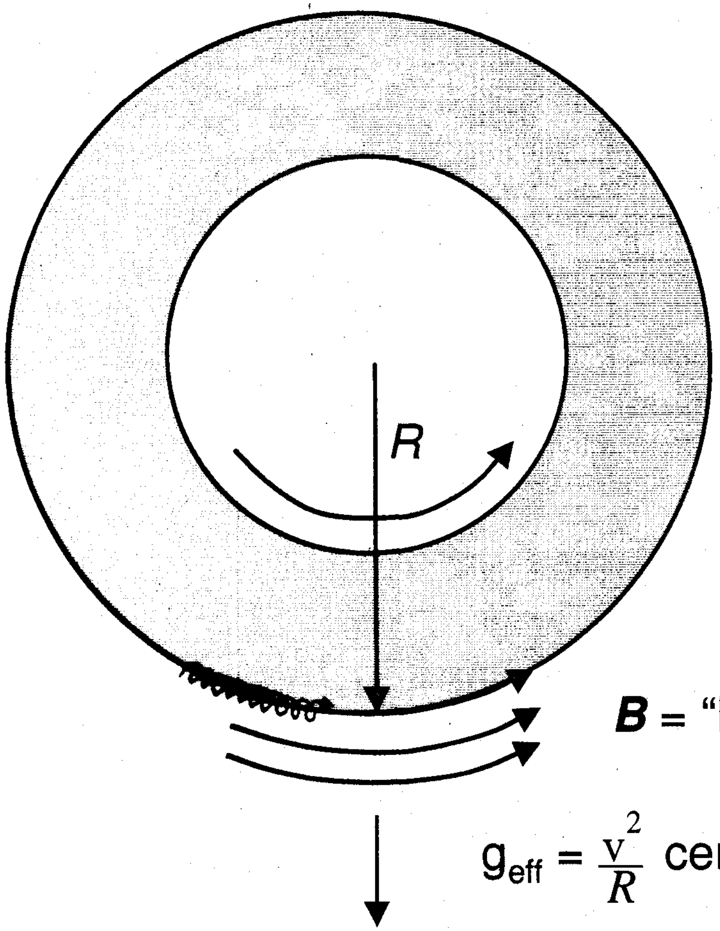
$\rho=\exp(y/L)$



Max growth rate $\gamma=(g/L)^{1/2}$

"Bad Curvature" instability in plasmas ≈ Inverted Pendulum / Rayleigh-Taylor Instability

Top view of toroidal plasma:



plasma = heavy fluid

B = "light fluid"

$g_{eff} = \frac{v^2}{R}$ centrifugal force

Growth rate:

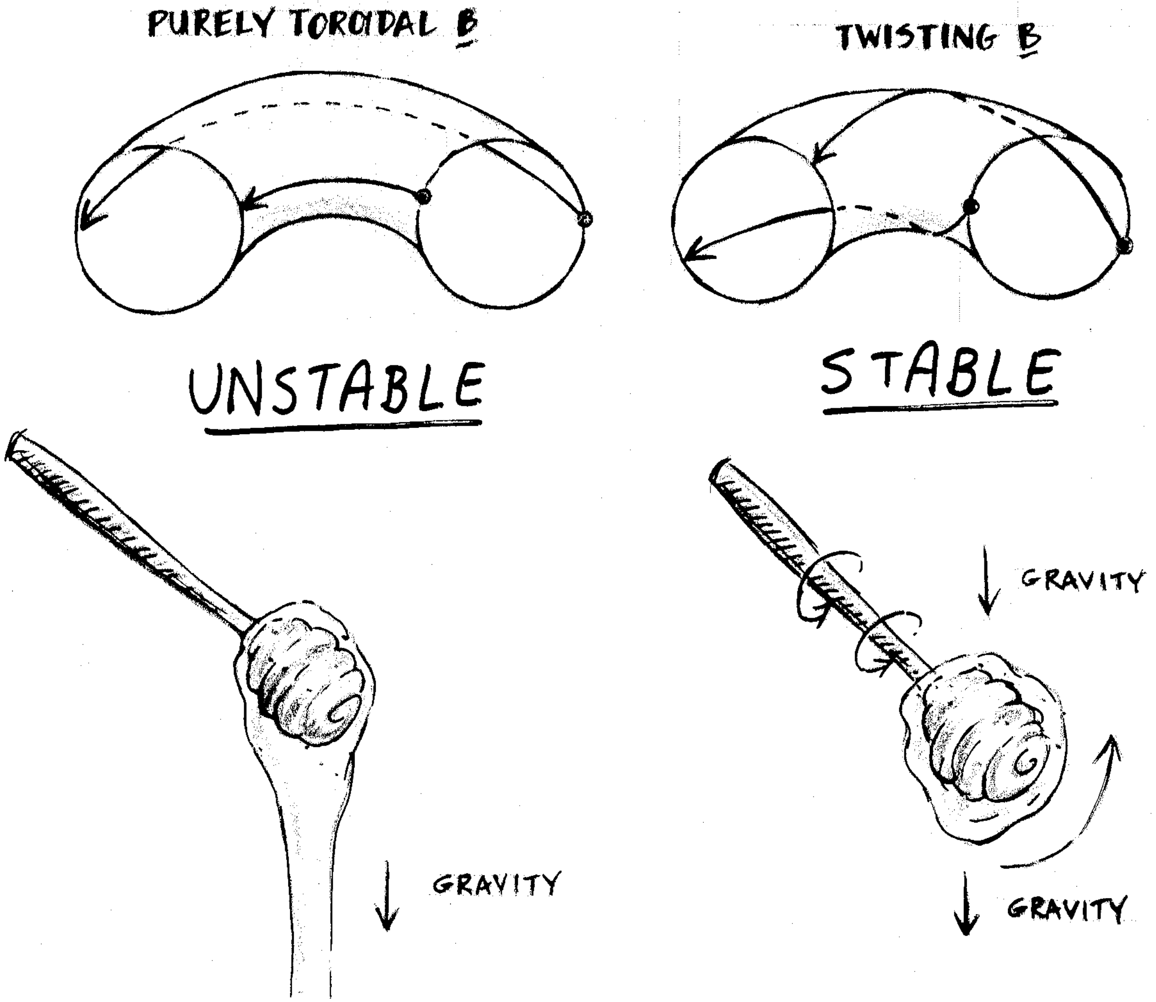
$$\gamma = \sqrt{\frac{g_{eff}}{L}} = \sqrt{\frac{v_t^2}{RL}} = \frac{v_t}{\sqrt{RL}}$$

Similar instability mechanism in MHD & drift/microinstabilities

$1/L = \nabla p/p$ in MHD,
$\propto$ combination of ∇n & ∇T in microinstabilities.

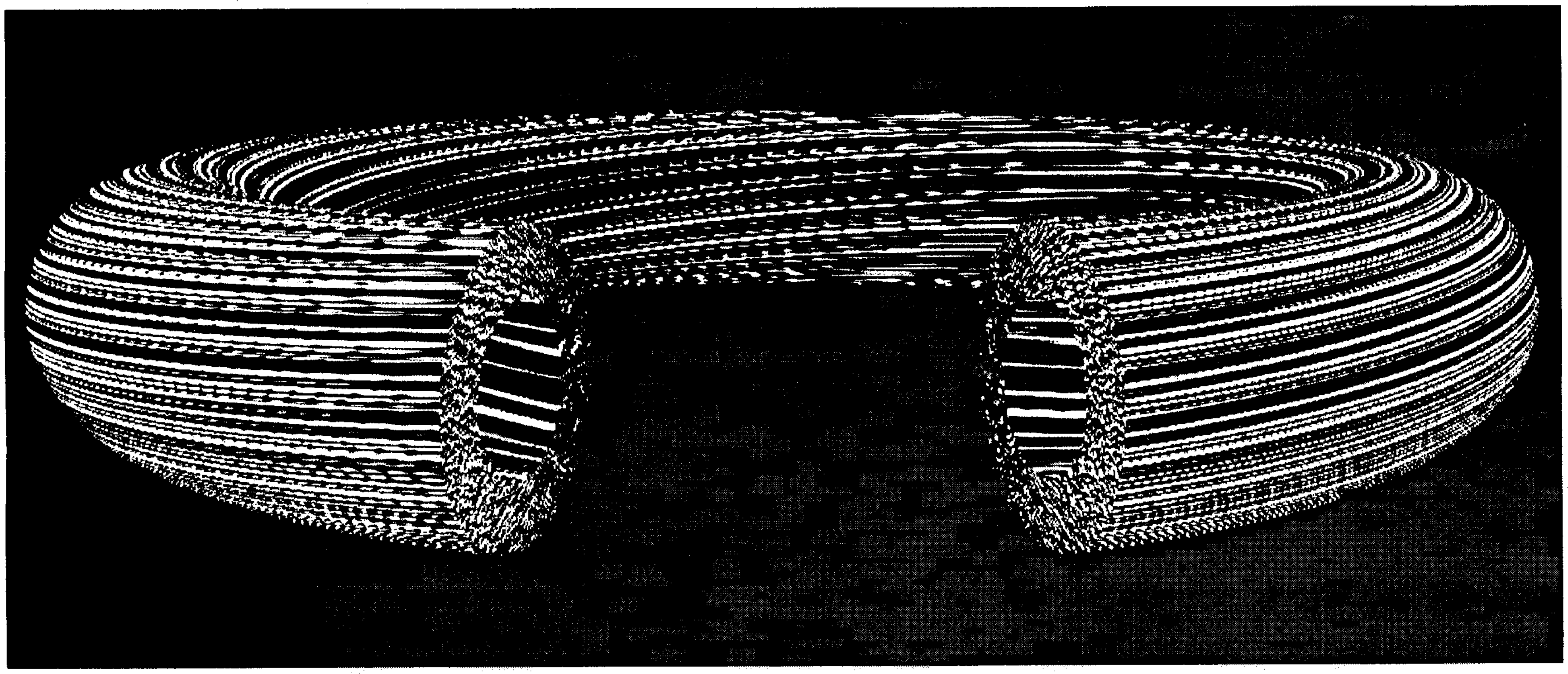
The Secret for Stabilizing Bad-Curvature Instabilities

Twist in $\underset{\sim}{B}$ carries plasma from bad curvature region to good curvature region.



Similar to how a twirling honey dipper can prevent honey from dripping.

Cut-away view of tokamak turbulence simulation



Waltz (General Atomics), Kerbel (LLNL), et.al., gyrofluid simulations. Similar pictures from gyrokinetc particle simulations.

Lots more pictures at www.acl.lanl.gov/GrandChal/Tok/gallery.html.

Waltz + Kerbel Gyrofluid

The Simplest Drift Wave

(oversimplified...)

Classic drift wave ordering:

$$v_{ti} << \frac{\omega}{k_\parallel} << v_{te}$$

Fluid ions (ions don't move much along field line)

Wave phase speed

Kinetic electrons $\Rightarrow$ adiabatic (Boltzmann) response

$+ i\delta$ small non-adiabatic part (collisions, trapped particles can't flow along field line...)

$$\boxed{\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i \underset{\sim}{v_E}) = 0} \quad + \quad \boxed{\text{Quasineutrality} \quad \tilde{n}_i = \tilde{n}_e = n_{e0}\frac{e\tilde{\Phi}}{T_e}(1+i\delta)}$$

$$\underset{\sim}{v_E} = \frac{c}{B^2}\underset{\sim}{E}\times\underset{\sim}{B} = \frac{c}{B}\hat{b}\times\nabla\Phi$$

$$n_i = n_0 + \tilde{n}_i$$

$$\frac{\partial \tilde{n}_i}{\partial t} + \underset{\sim}{v_E}\cdot\nabla(n_0+\tilde{n}_i) + (n_0+\tilde{n}_i)\underbrace{\nabla\cdot\underset{\sim}{v_E}}_{\frac{c}{B}\nabla\cdot[\hat{b}\times\nabla\Phi] = 0} = 0$$

in slab ($\hat{b} = \hat{z}$)

$$\underset{\sim}{v_E}\cdot\nabla n_0 = \frac{c}{B}\hat{b}\times\nabla\tilde{\Phi}\cdot\left(-\hat{x}\frac{n_0}{L_n}\right)$$

$$= i\left(\frac{cT_e}{eB}\frac{k_y}{L_n}\right)\frac{e\tilde{\Phi}}{T_e}n_0$$

$\omega_{*e} \equiv \frac{cT_e}{eB}\frac{k_y}{L_n}$

$$-i\omega(1+i\delta)\frac{e\tilde{\Phi}}{T_e} + i\omega_{*e}\frac{e\tilde{\Phi}}{T_e} + \underset{\sim}{V_E}\cdot\nabla\left[\frac{e\tilde{\Phi}}{T_e}(1+i\delta)\right]=0$$

Linear $\quad \omega = \dfrac{\omega_{*e}}{1+i\delta} \approx \omega_{*e}(1 - i\delta)$

Unstable if $\quad \omega_{*e}\delta < 0$

$\delta = \delta_0 k_y$ in some models.

$\omega_{*e}\delta \propto k_y^2$ independent of Sign(k_y).

Stability depends on Sign(δ_0).

$\gamma \propto k_y^2 \Rightarrow$ Fastest growing mode at highest k_y (Usually cut off by FLR at $k_y\rho \sim 1$)

Nonlinear term:

$$\underset{\sim}{V_E}\cdot\nabla\left[(1+i\delta)\frac{e\tilde{\Phi}}{T_e}\right]$$

$$= \underset{\sim}{V_E}\cdot\nabla\left[\left(1+\delta_0\frac{\partial}{\partial y}\right)\frac{e\tilde{\Phi}}{T_e}\right]$$

$$= \frac{c}{B}\hat{b}\times\underbrace{\nabla\tilde{\Phi}\cdot\nabla\left[\frac{e\tilde{\Phi}}{T_e}\right.}_{=0}\left. + \delta_0\frac{\partial}{\partial y}\frac{e\tilde{\Phi}}{T_e}\right]$$

Only surviving nonlinearity is from $i\delta$, unless FLR corrections kept...

Gyrokinetic Ordering for Nonlinear Plasma Turbulence

Frieman & Chen
Phys. Fluids $\underline{25}$, 502 (1982)

$$\boxed{\frac{\omega}{\Omega} \sim \frac{\rho}{L} \sim \frac{k_\parallel v_t}{\Omega} \sim \frac{\tilde{F}_1}{F_0} \sim \frac{e\tilde{\Phi}}{T} \sim \frac{|\delta \underset{\sim}{B}|}{B} \sim \varepsilon << 1}$$

$k_\perp \rho \sim 1$ "Maximal ordering" (small $k_\perp \rho$ sometimes)
FLR effects important

$L \rightarrow$ equilibrium scale length, $\omega, k_\parallel, k_\perp \Rightarrow$ time & space scales of perturbations

If perturbations are small, how can there be nonlinearities?

$$\frac{\partial F}{\partial t} + (v_\parallel \hat{b} + \underset{\sim}{v_E}) \cdot \nabla F + \frac{e}{m} E_\parallel \frac{\partial F}{\partial v_\parallel} = 0$$

$$\underset{\sim}{v_E} = \frac{c}{B}(\underset{\sim}{E} \times \hat{b})$$

$$F = F_0(\underset{\sim}{x}, \underset{\sim}{v}) + \tilde{F}_1(\underset{\sim}{x}, \underset{\sim}{v}, t)$$

equilibrium + perturbation

$$\frac{\partial F_0}{\partial t} = 0 \quad \hat{b} \cdot \nabla F_0 = 0$$

Nonlinear

$$\underset{\sim \omega F_1}{\frac{\partial \tilde{F}_1}{\partial t}} + \underset{\sim k_\parallel v_t F_1 \sim \omega F_1}{v_\parallel \hat{b} \cdot \nabla \tilde{F}_1} + \underset{\sim}{v_E} \cdot \nabla (F_0 + \tilde{F}_1) + \frac{e}{m} E_\parallel \frac{\partial}{\partial v_\parallel} (F_0 + \tilde{F}_1) = 0$$

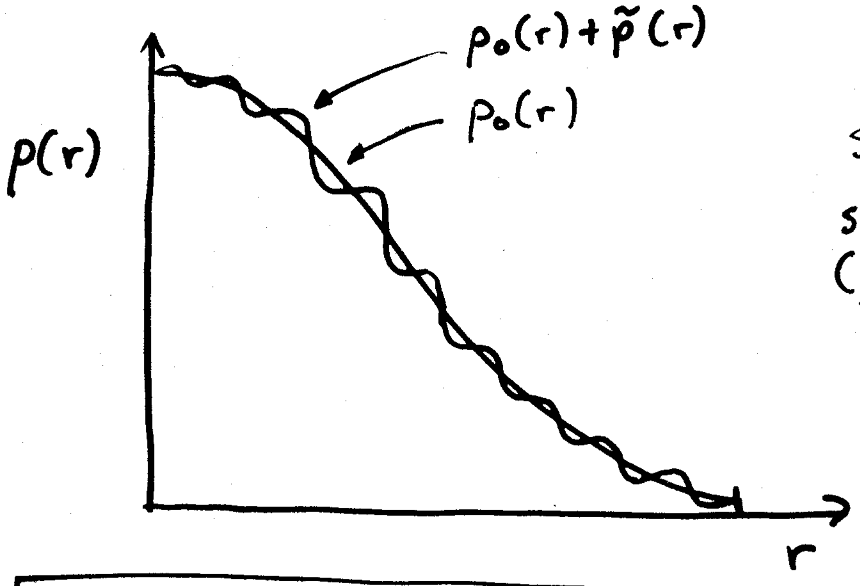
Can we ignore this $\tilde{F}_1$ compared to F_0?

(No...) —

Aside: The ordering above simplifies equations (fluid &/or kinetic) for many types of plasma turbulence. Not just used to derive the Gyrokinetic Eq. Ordering motivated by experiments & years of theoretical analysis of various instabilities.

Though $\frac{F_1}{F_0} \sim \epsilon << 1$

$$\frac{\nabla F_1}{\nabla F_0} \sim \frac{k_\perp F_1}{\frac{1}{L} F_0} \sim k_\perp L \frac{F_1}{F_0} \sim \underbrace{k_\perp \rho}_{\sim 1} \underbrace{\frac{L}{\rho}}_{\sim \frac{1}{\epsilon}} \underbrace{\frac{F_1}{F_0}}_{\sim \epsilon} \sim 1$$



Perturbations have Small amplitude, but short wavelength (sharp gradients).

$\boxed{\text{E x B nonlinearity is the same order as linear terms.}} \leftrightarrow \boxed{\nabla_\perp F_1 \sim \nabla_\perp F_0}$

linear E x B term: $\underset{\sim}{V_E} \cdot \nabla F_0 \sim \frac{c}{B} E_\perp \frac{F_0}{L_n} \sim \frac{c\Phi}{B} \frac{k_\perp}{L_n} F_0$

$$\sim \frac{e\Phi}{T} \frac{cT}{eB} \frac{k_\perp}{L_n} F_0 \sim \epsilon \omega_* F_0$$
$$\sim \omega_* F_1$$
$$(\Rightarrow \omega \sim \omega_*)$$

"Parallel" nonlinearity usually ignored assuming $\frac{\partial \tilde{F}_1}{\partial v_{\parallel}} << \frac{\partial \tilde{F}_0}{\partial v_{\parallel}}$

Here "$\sim$" doesn't mean "close", but merely the same order in an ϵ expansion. I.e.,
$10\, \epsilon \sim \frac{1}{2} \epsilon$

Large $k_\perp L \gg 1$ also means magnetic nonlinearities are included in the gyrokinetic ordering:

$$\hat{b} = \frac{\underset{\sim}{B}}{|\underset{\sim}{B}|} = \frac{\underset{\sim}{B}_0 + \delta \underset{\sim}{B}_1}{|\underset{\sim}{B}|} \qquad |\hat{b}_1| \sim \varepsilon$$

$$= \hat{b}_0 + \hat{b}_1$$

$$v_\parallel \hat{b}_1 \cdot \nabla \tilde{F}_1 \sim v_\parallel \varepsilon k_\perp \tilde{F}_1 \sim v_\parallel k_\parallel \tilde{F}_1 \sim \omega \tilde{F}_1$$

(Other magnetic nonlinearities also...)

The gyrokinetic ordering includes the "Reduced MHD" ordering (MHD minus the fast compressional Alfvén wave, while retaining the shear Alfvén wave).

$\omega \sim k_\parallel v_A$ Shear Alfvén

$$\frac{\omega}{\Omega} \sim \frac{k_\parallel v_A}{\Omega} \sim \frac{k_\parallel v_{ti}}{\Omega} \frac{v_A}{v_{ti}}$$

Homework ⇒ $$\sim \varepsilon \frac{1}{\sqrt{\beta_i/2}}$$

Satisfies gyrokinetics (β isn't necessarily an expansion parameter...)

$\omega \sim |\underset{\sim}{k}| v_A$ Compressional Alfvén

$$\frac{\omega}{\Omega} \sim \frac{k_\perp v_A}{\Omega} \sim 1 \qquad \left(\frac{k_\perp}{k_\parallel} \sim \frac{1}{\varepsilon} \gg\right.$$

A modified form of gyrokinetics might be able to keep long wavelength $k_\perp \rho \ll 1$ compressional Alfvén waves while still allowing $k_\perp \rho \sim 1$ for drift-like modes?

Compare with MHD / drift Ordering:

$$\frac{\omega}{\Omega} \sim \frac{\rho}{L} \sim \rho k \sim \mathcal{O}(\varepsilon)$$

$$\frac{F_1}{F_0} \sim \frac{V_{E\times B}}{V_t} \sim \frac{\delta B}{B} \sim \mathcal{O}(1)$$

$$\frac{\partial F}{\partial t} \sim \omega F \qquad \nabla F \sim \frac{F}{L}$$

But this ordering misses drift waves:

$$\frac{\omega_*}{\Omega} \sim \frac{\frac{cT}{eB}\frac{k_y}{L}}{\Omega} \sim \frac{V_t^2}{\Omega^2}\frac{1}{L^2} \sim \frac{\rho^2}{L^2} \sim \mathcal{O}(\varepsilon^2)$$

Would have to work to higher order in ε
(but could only get FLR $k_\perp^2 \rho^2$ effects perturbatively)

MHD waves okay (of course):

$$\frac{\omega}{\Omega} \sim \frac{k_\perp V_A}{\Omega} \sim \frac{k_\perp V_t}{\Omega\sqrt{\beta}} \sim \frac{k_\perp \rho}{\sqrt{\beta}} \sim \mathcal{O}(\varepsilon)$$

Summary of Kulsrud's Collisionless MHD + Drift Kinetic Eq.
Ref #2 & 3 Varenna '63

In this ordering, the Vlasov equation reduces to a condition on the zeroth-order parallel (relative to the magnetic field) electric field $E_{\parallel_0}=0$, and the following kinetic equation for the zeroth-order distribution function of each species $f_{0_s}(v_\parallel,\mu,\mathbf{r},t)$:

$$\frac{\partial f_{0_s}}{\partial t}+(v_\parallel\hat{\mathbf{b}}+\mathbf{v}_E)\cdot\nabla f_{0_s}+\left(-\hat{\mathbf{b}}\cdot\frac{D\mathbf{v}_E}{Dt}-\mu\hat{\mathbf{b}}\cdot\nabla B+\frac{e_s}{m_s}E_\parallel\right)$$
$$\times\frac{\partial f_{0_s}}{\partial v_\parallel}=0, \tag{1}$$

where e_s is the charge on species s, $\hat{\mathbf{b}}$ is a unit vector in the magnetic field direction $\hat{\mathbf{b}}=\mathbf{B}/B$, $\mathbf{v}_E\doteq c(\mathbf{E}\times\mathbf{B})/B^2$, $\mu\doteq v_\perp^2/2B$, and $D/Dt\doteq\partial/\partial t+(v_\parallel\hat{\mathbf{b}}+\mathbf{v}_E)\cdot\nabla$.

Combining moments of this kinetic equation with Maxwell's equations and taking the usual low Alfvén speed limit $v_A^2\ll c^2$ yields Kulsrud's set of collisionless MHD equations:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{U})=0, \tag{2}$$

$$\rho\left(\frac{\partial\mathbf{U}}{\partial t}+\mathbf{U}\cdot\nabla\mathbf{U}\right)=\frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi}-\nabla\cdot\mathbf{P}, \tag{3}$$

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{U}\times\mathbf{B}), \tag{4}$$

$$\mathbf{P}=p_\perp\mathbf{I}+(p_\perp-p_\parallel)\hat{\mathbf{b}}\hat{\mathbf{b}}, \tag{5}$$

$$p_\perp=\sum_s\frac{m_s}{2}\int f_{0_s}v_\perp^2d^3v, \tag{6}$$

$$p_\parallel=\sum_s m_s\int f_{0_s}(v_\parallel-\mathbf{U}\cdot\hat{\mathbf{b}})^2d^3v, \tag{7}$$

$$\sum_s e_s\int f_{0_s}\,d^3v=0, \tag{8}$$

where ρ is the total mass density, $\mathbf{U}=\mathbf{v}_E+u_\parallel\hat{\mathbf{b}}$ is the fluid velocity, and $\mathbf{P}$ is the pressure tensor.

The above set of equations is exact to zeroth order in the expansion parameter, but the kinetic equation itself, Eq. (1), must be used to evaluate $p_\parallel$ and $p_\perp$ to close the system. Because Eq. (1) is difficult to solve directly, this system is rarely employed without further simplification.

lem, it is necessary to introduce an additional ordering which removes the compressional Alfvén time scale.

Another complication is the evaluation of the $|k_{\|}|/k_{\|}$ terms found in the Landau closures. As pointed out by Finn and Gerwin,[18] the Landau damping must be evaluated along perturbed field lines. Hence, for nonlinear calculations, transforming the closure to real space requires an integral along the perturbed field line. The numerical evaluation of these nonlinear closures may be burdensome in some cases, as discussed in Sec. VII.

It is anticipated that the model will be useful for nonlinear numerical simulations. Some of the caveats involved in using Landau closures in nonlinear simulations have been extensively discussed in the gyrofluid literature,[11,12,23,24,26–30] but these caveats are an area of ongoing research. There are some regimes where certain nonlinear kinetic effects are not well modeled by Landau-fluid closures.[30] But we generally believe[12,24,27,28] these closures will be adequate for stronger turbulence regimes where rapid decorrelation is occurring and the velocity space details of the distribution function are not critically important.

It is hoped that the model will prove useful for simulating both laboratory and astrophysical plasmas in the collisionless MHD regime. The model should be able to predict the onset and structure of instabilities, as well as the heat and particle transport caused by the instabilities.

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Kulsrud Refs.

Caution: Gyrokinetic ordering may break down (or may need extension) near the plasma edge where

$$\frac{e\tilde{\Phi}}{T} \sim 1$$

$$\frac{\rho}{L} \sim \rho \frac{\nabla n}{n} \sim 1$$

(Perhaps can include $\frac{e\Phi}{T} \sim 1$ as long as $k_\perp \rho \ll 1$?

i.e., $\underset{\sim}{V}_E \propto \nabla_\perp \Phi$ is the relevant quantity...)

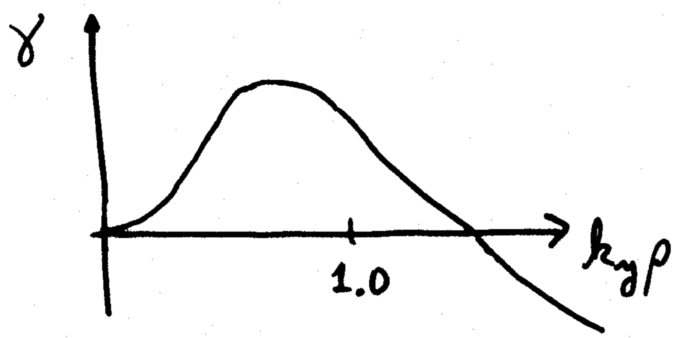
FLR effects

FLR = Finite Larmor (gyro) Radius $\Rightarrow$ $k_\perp \rho > 0$

*→ Popular Forms...

$$\omega \sim \omega_{*e} = \frac{cT_e}{eB}\frac{k_y}{L_n} \sim \frac{m_i c}{eB}\frac{T_e}{m_i}\frac{k_y}{L_n} \sim \frac{c_s}{L_n} k_y \rho_s$$

Often, $\gamma \propto \omega_{*e}$, so the fastest growing mode has the largest k_y. FLR provides cutoff.



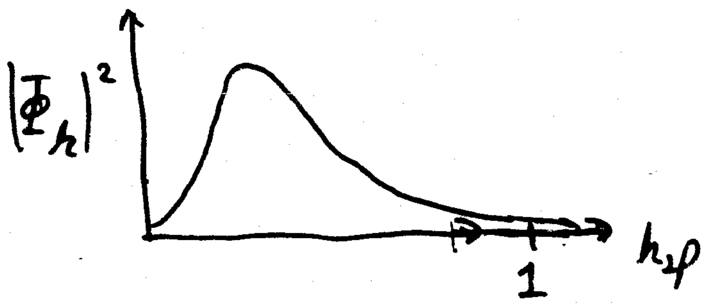
* Fastest growing linear mode is not necessarily going to dominate in the nonlinear state.
* Indeed, BES + μwave scattering on TFTR indicate $k_\perp \rho \sim .1 - .3 \ll 1$

Speculations...

* Even at $k_\perp \rho \ll 1$, residual FLR plays important roles in driving instabilities + nonlinearities.

* Also, damped modes at $k_\perp \rho > 1$ may be an important sink of energy in nonlinear simulations, even if not dominant in the spectrum?

(Actually, lots of ion Landau damping at high $k_\parallel$ ($k_\parallel v_t > \omega_*$), not just high $k_\perp$)



The Gyrokinetic Eq.

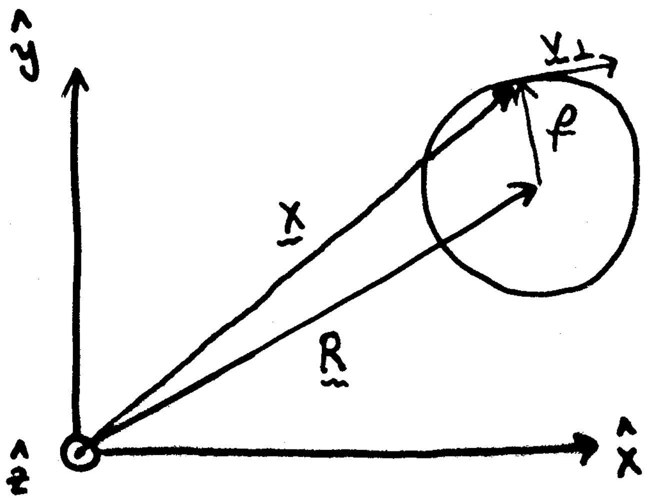
(Not as mysterious as you might think...)

$$\boxed{\frac{\partial F_{gc}}{\partial t} + \nabla\cdot\left[F_{gc}\left(v_\parallel \hat{b} + \langle \underset{\sim}{v_E}\rangle\right)\right] - \frac{\partial}{\partial v_\parallel}\left[F_{gc}\frac{e}{m}\hat{b}\cdot\nabla\langle\Phi\rangle\right] = 0}$$

Parallel flow | $\underset{\sim}{E}\times\underset{\sim}{B}$ (Gyroaveraged) | Parallel Acceleration (Gyro averaged)

Conservation Eq. for $F_{gc}(\underset{\sim}{R}, v_\parallel, v_\perp, t)$, the density of particles with <u>guiding centers</u> at position $\underset{\sim}{R}$, with velocities $v_\parallel$ & $v_\perp$.

$$n_{gc}(\underset{\sim}{R}) = \int d^3v\, F_{gc} \neq \text{particle density } n$$



Particle position

$$\underset{\sim}{X} = \underset{\sim}{R} + \rho$$

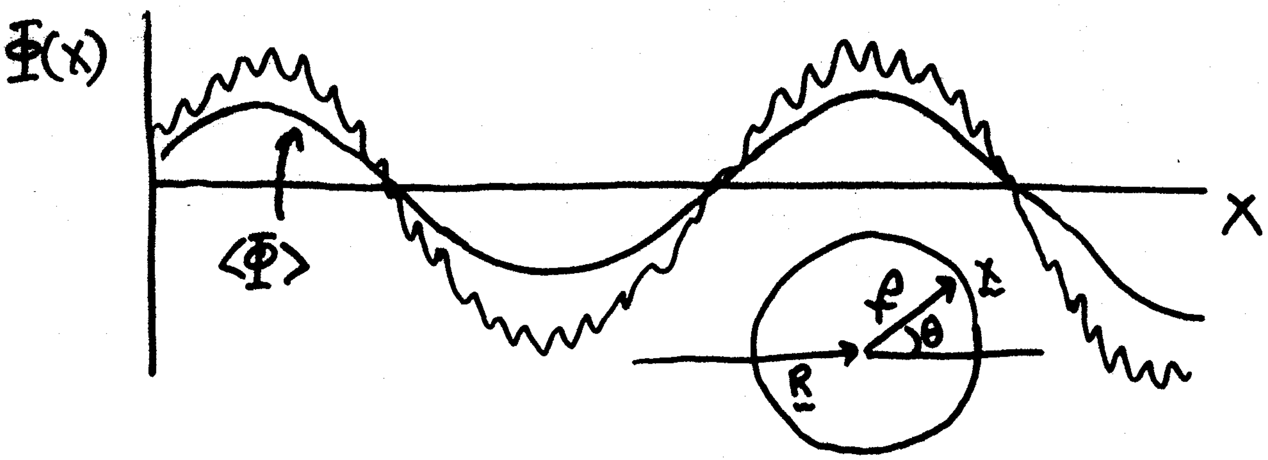
The above GKE is for slab geometry ($\underset{\sim}{B}$ straight, may be sheared) for electrostatic perturbations only. General geometry + $\delta \underset{\sim}{B}_\perp$ GKE exists. Tomorrow: + toroidal ITG...

Gyroaveraging

Particles respond to the gyro-orbit-averaged

$\langle \underset{\sim}{V}_E \rangle = \frac{c}{B} \hat{b} \times \nabla \langle \Phi \rangle$.

$\Rightarrow$ Smooths out high $k_\perp \rho$ components:



$$\Phi(\underset{\sim}{x}) = \sum_{\underset{\sim}{k}} \Phi_k e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}} \qquad \underset{\sim}{x} = \underset{\sim}{R} + \underset{\sim}{\rho}(\theta)$$

Gyro-average: $\langle \Phi \rangle = \frac{1}{2\pi} \int d\theta \, \Phi(\underset{\sim}{R} + \underset{\sim}{\rho}(\theta))$

$$= \sum_{\underset{\sim}{k}} \Phi_{\underset{\sim}{k}} \frac{1}{2\pi} \int d\theta \, e^{i \underset{\sim}{k} \cdot (\underset{\sim}{R} + \underset{\sim}{\rho}(\theta))}$$

$$= \underbrace{\sum_{\underset{\sim}{k}} \Phi_{\underset{\sim}{k}} e^{i \underset{\sim}{k} \cdot \underset{\sim}{R}}}_{= \Phi(\underset{\sim}{R})} \underbrace{\frac{1}{2\pi} \int d\theta \, e^{i k_\perp \rho \cos\theta}}_{\equiv J_0(k_\perp \rho)}$$

Bessel Function

$$J_0(k_\perp \rho) = \frac{1}{2\pi}\int d\theta\, e^{i k_\perp \rho \cos\theta}$$

$$\approx \frac{1}{2\pi}\int d\theta \left[1 + i k_\perp \rho \cos\theta - \frac{k_\perp^2 \rho^2}{2}\cos^2\theta + \cdots\right]$$

$$= \quad 1 + 0 \quad - \frac{k_\perp^2 \rho^2}{4}$$

$$\Rightarrow 1 + \frac{\rho^2}{4}\nabla_\perp^2$$

$$\langle \Phi \rangle \approx \left(1 + \frac{\rho^2}{4}\nabla_\perp^2\right)\Phi$$

$$\approx \left(1 - \frac{\rho^2}{4}k_\perp^2\right)\Phi$$

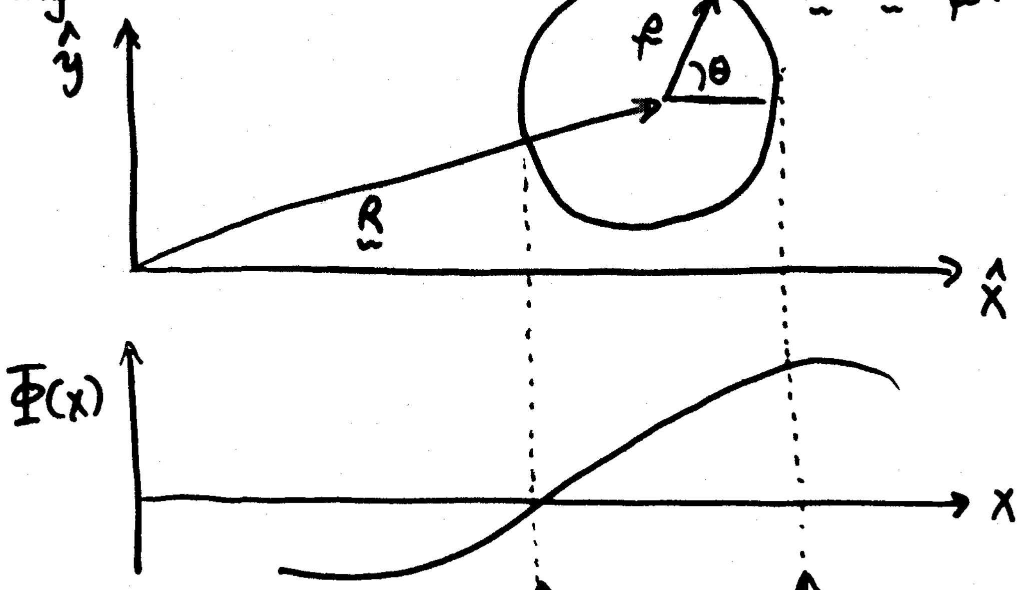
$$\equiv J_0 \Phi$$

[Think of J_0 as an operator, like in quantum mechanics, which happens to have a simple form in k space.

Surprise: Particles are not uniformly distributed around the gyro-orbit.

There is a correction due to the variation of Φ around the orbit. Since $\omega \ll \Omega$, the particles set up an adiabatic (Boltzmann) response around the gyro-ring:

$$\underset{\sim}{x} = \underset{\sim}{R} + \underset{\sim}{\rho}(\theta)$$



There are slightly more particles here at the bottom of the potential well.

Slightly fewer particles here.

$$F_{TOT}(\underset{\sim}{R}, v_\parallel, v_\perp, \theta, t) = F_{gc}(\underset{\sim}{R}, v_\parallel, v_\perp, t)$$

$$- F_0(\underset{\sim}{R}, v_\parallel, v_\perp)\left[\frac{e\Phi}{T} - \frac{e\langle\Phi\rangle}{T}\right]$$

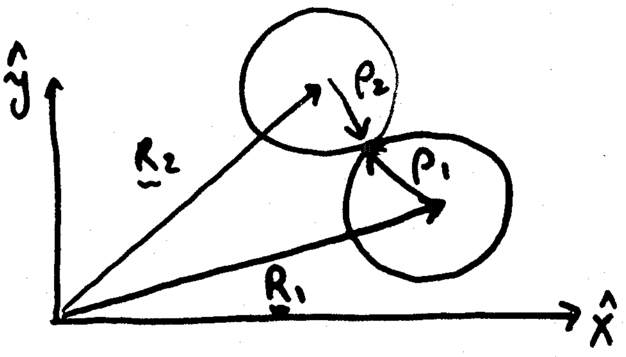
Evaluate at $\underset{\sim}{x} = \underset{\sim}{R} + \underset{\sim}{\rho}(\theta)$

Averaged over θ at fixed $\underset{\sim}{R}$.

Guiding-center density ≠ Particle density.

$$\underset{\sim}{x} = \underset{\sim}{R} + \underset{\sim}{\rho}(\theta)$$

$$\underset{\sim}{R} = \underset{\sim}{x} - \underset{\sim}{\rho}(\theta)$$



Two particles at the same $\underset{\sim}{x}$, but different $\underset{\sim}{R}$.

$f(\underset{\sim}{x}, \underset{\sim}{v})$ = particle distribution function

$$= F_{TOT}(\underbrace{\underset{\sim}{x} - \underset{\sim}{\rho}(\theta)}_{\underset{\sim}{R}}, v_\parallel, v_\perp, \theta)$$

$$= F_{gc}(\underset{\sim}{x} - \underset{\sim}{\rho}(\theta), v_\parallel, v_\perp) - F_0(\underset{\sim}{x} - \underset{\sim}{\rho}(\theta), v_\parallel, v_\perp)\left[\frac{e\Phi(\underset{\sim}{x})}{T} - \frac{e\langle\Phi\rangle}{T}\right]$$

evaluate at $\underset{\sim}{R} = \underset{\sim}{x} - \underset{\sim}{\rho}$

$$n(\underset{\sim}{x}) = \int d^3v\Big|_{\underset{\sim}{x}} f = \bar{n} + n_{pol}$$

$$\bar{n} = \int dv_\parallel \int dv_\perp v_\perp \int d\theta \sum_{\underset{\sim}{k}}{}' F_{gc\underset{\sim}{k}}(v_\parallel, v_\perp)\, e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x} - \underset{\sim}{\rho}(\theta))}$$

$$= \underbrace{\int dv_\parallel \int dv_\perp v_\perp 2\pi}_{\int d^3v} \sum_{\underset{\sim}{k}} F_{gc\underset{\sim}{k}}(v_\parallel, v_\perp)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \underbrace{\frac{1}{2\pi}\int d\theta\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\rho}(\theta)}}_{= J_0(k_\perp \rho)!}$$

$$\bar{n} = \int d^3v \, J_0 \, F_{gc}(\underset{\sim}{x}, v_\parallel, v_\perp) \approx \int d^3v \left(1 + \frac{\rho^2}{4}\nabla_\perp^2\right) F_{gc}$$

$$n_{pol} = -\int d^3v \, \underbrace{F_0(\underset{\sim}{x} - \rho(\theta), v_\parallel, v_\perp)}_{\approx F_0(\underset{\sim}{x}, v_\parallel, v_\perp) - \rho \cdot \frac{\partial F_0}{\partial \underset{\sim}{x}}} \left[\frac{e\Phi(\underset{\sim}{x})}{T} - \frac{e\langle\Phi\rangle}{T}\bigg|_{\underset{\sim}{R} = \underset{\sim}{x} - \rho} \right]$$

$$\sim \frac{\rho}{L_n} F_0 \qquad \text{ignore}$$

$$\langle\Phi\rangle\Big|_{\underset{\sim}{R} = \underset{\sim}{x} - \rho} = \sum_{\underset{\sim}{k}} \Phi_{\underset{\sim}{k}} \, e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x} - \rho(\theta))} J_0(k_\perp \rho)$$

$$n_{pol} = -n_0 \frac{e\Phi(\underset{\sim}{x})}{T} + \sum_{\underset{\sim}{k}} \Phi_{\underset{\sim}{k}} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \underbrace{\int d^3v \, F_0 \, e^{-i\underset{\sim}{k}\cdot\rho(\theta)} J_0(k_\perp \rho)}$$

$$\int d^3v \, F_0 \, J_0 \frac{1}{2\pi} \int d\theta \, e^{-ik_\perp \rho \cos\theta}$$

$$\int d^3v \, F_0 \, J_0^2$$

$$n_0 \, \Gamma_0(k_\perp^2 \rho^2)$$

1 Bessel Function from
$\Phi(\underset{\sim}{x}) \rightarrow \langle\Phi\rangle(\underset{\sim}{R})$

*1 Bessel Function from
$F(\underset{\sim}{R}) \rightarrow \langle F\rangle(\underset{\sim}{x})$

$n =$ particle density

$= \bar{n} + n_{pol}$

$$\boxed{n = \int d^3v\, J_0 F_{gc} - n_0 \left(1 - \Gamma_0\right) \frac{e\Phi}{T}}$$

Useful I.D.'s: $b = k_\perp^2 \rho_t^2 = k_\perp^2 \dfrac{T_i}{m_i \Omega_{ci}^2}$

$$\Gamma_0(b) = e^{-b} I_0(b) = e^{-b} J_0(i b)$$

$$\approx 1 - b \approx 1 - k_\perp^2 \rho_t^2 = 1 + \rho_t^2 \nabla_\perp^2$$

Ion particle density in the cold ion limit $(T_i \to 0,\ k_\perp^2 \rho_i^2 \to 0)$:

$$n_i = \int d^3v\, F_{gc} - n_0 \left[1 - \left(1 - k_\perp^2 \rho_i^2\right)\right] \frac{e\Phi}{T_i}$$

$$= n_{gc} - n_0 \underbrace{k_\perp^2 \rho_i^2 \frac{e\Phi}{T_i}}$$

independent of T_i!
Survives the $T_i \to 0$ limit.

$$n_i = n_{gc} - n_0 k_\perp^2 \rho_s^2 \frac{e\Phi}{T_e}$$

$$\rho_s^2 = \frac{c_s^2}{\Omega_{ci}^2} = \frac{T_e / m_i}{\Omega_{ci}^2}$$

Combine with adiabatic electron response in quasineutrality condition:

$$\tilde{n}_e = \tilde{n}_i$$

$$n_{e0} \frac{e\tilde{\Phi}}{T_e}(1+i\delta) = \tilde{n}_{gc} - n_0 k_\perp^2 \rho_s^2 \frac{e\tilde{\Phi}}{T_e}$$

$$\boxed{\frac{e\tilde{\Phi}}{T_e} = \frac{\tilde{n}_{gc}/n_0}{1+i\delta + k_\perp^2\rho_s^2}}$$

Provides "filtering" at high $k_\perp^2 \rho_s^2$, even with $k_\perp^2 \rho_i^2 \to 0$.

(Most tokamaks have $T_i \sim T_e$, so $k_\perp^2 \rho_i^2$ effects similar in size to $k_\perp^2 \rho_s^2$. Supershots + hot ion modes have $T_i/T_e \sim 3-4$.)

The Toroidal Gyrokinetic Eq.

parallel flow: Gyro-avg. $E \times B$: ∇B & Curvature Drifts:

$$\frac{\partial}{\partial t}(FB) + \nabla \cdot \left[FB \left(v_\parallel \hat{b} + J_0 \underset{\sim}{V_E} + \underset{\sim}{V_d} \right) \right]$$

$$+ \frac{\partial}{\partial v_\parallel} \left[FB \left(-\frac{e}{m} \hat{b} \cdot \nabla J_0 \Phi - \mu \hat{b} \cdot \nabla B + v_\parallel (\hat{b} \cdot \nabla \hat{b}) \cdot J_0 \underset{\sim}{V_E} \right) \right] = 0$$

$E_\parallel$ force (properly gyro-avg.)

Magnetic Mirroring Force

Acceleration to conserve toroidal angular momentum

($m v_\parallel R$ = constant during toroidal adiabatic compression)

$$J_0 \underset{\sim}{V_E} \approx \frac{c}{B} \hat{b} \times \nabla J_0 \Phi \qquad \underset{\sim}{V_d} \approx \frac{\hat{b} \times \nabla B}{B} \frac{\frac{1}{2} v_\perp^2 + v_\parallel^2}{\Omega}$$

$$F = F_{gc}(\underset{\sim}{R}, v_\parallel, \mu, t) \qquad \mu = \frac{1}{2} \frac{v_\perp^2}{B}$$

$$n_{gc} = \int d^3v \, F_{gc} = 2\pi \int dv_\parallel \int d\mu \, B \, F_{gc}$$

$$F_{TOT} = F_{gc} + F_{pol}$$

Quasineutrality condition: $\tilde{n}_e \approx \tilde{n}_i$

$$\tilde{n}_i = \int d^3v \, J_0 F_{gc} + \underbrace{n_0 (\Gamma_0 - 1) \frac{e \tilde{\Phi}}{T_i}}_{n_{pol}}$$

$$J_0 \approx 1 - \frac{k_\perp^2 \rho^2}{4}$$

$$\Gamma_0 \approx 1 - k_\perp^2 \rho_{ti}^2 \approx \frac{1}{1 + k_\perp^2 \rho_{ti}^2} \text{ (better)}$$

From component of F_{TOT} which has an adiabatic variation around the gyro-orbit

Nonlinear Gyrokinetic Eq. Refs.

Frieman & Chen, Phys. Fluids 25, 502 (1982)

First Nonlinear GK Eq. It's all there, but somewhat complicated.

Others:

W. W. Lee, Phys. Fluids 26, 556 (1983)
" " , J. Comput. Phys. 72, 243 (1987)

Dubin, Krommes, et. al. Phys. Fluids 26, 3524 (1983).

More modern Approach:

Littlejohn, "Variational principles of guiding center motion", J. Plasma Physics 29, 111 (1983)

T. S. Hahm, Phys. Fluids 31, 2670 (1988)

1989 –
1990's papers by Hahm, Brizard, + by Hong Qin.

Note: some of 1980's papers went to higher order in ε expansion than needed.