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SMR 1331/2

AUTUMN COLLEGE ON PLASMA PHYSICS

8 October - 2 November 2001

Relativistic Kinetic Theory

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These are preliminary lecture notes, intended only for distribution to participants.

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October 1, 2001

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1.1 Notation

We use the metric tensor $\eta_{\alpha\beta}$ with $\eta_{00} = -1$, $\eta_{ii} = 1$; Greek indices vary from 0 to 3, Roman indices vary from 1 to 3. Boldface indicates a 3-vector, so that

$$V^\mu = (V^0, \mathbf{V})$$

and

$$V^\mu V_\mu = \mathbf{V}^2 - (V^0)^2$$

We consider two reference frames, the "lab frame" S and a frame S' that moves at velocity $\mathbf{v}$ relative to S (that is, the velocity of S', measured in S, is $\mathbf{v}$). Coordinates of a particular event, as measured in S and S', are denoted by x and x' respectively.

The Lorentz transformation matrix is denoted by Λ, so that a 4-vector transforms according to

$$V^{\mu\prime} = \Lambda^\mu{}_\nu V_\mu$$

We recall that

$$\Lambda = \begin{matrix} \gamma & -\gamma v_1 & -\gamma v_2 & -\gamma v_3 \\ -\gamma v_1 & 1+\hat{v}_1^2(\gamma-1) & \hat{v}_1\hat{v}_2(\gamma-1) & \hat{v}_1\hat{v}_3(\gamma-1) \\ -\gamma v_2 & \hat{v}_1\hat{v}_2(\gamma-1) & 1+\hat{v}_2^2(\gamma-1) & \hat{v}_2\hat{v}_3(\gamma-1) \\ -\gamma v_3 & \hat{v}_1\hat{v}_3(\gamma-1) & \hat{v}_2\hat{v}_3(\gamma-1) & 1+\hat{v}_3^2(\gamma-1) \end{matrix} \tag{1}$$

where $\hat{\mathbf{v}} \equiv \mathbf{v}/v$. An alternative expression for Λ is

$$\Lambda^0{}_0 = \gamma \tag{2}$$

$$\Lambda^i{}_j = \delta^i{}_j + \hat{v}_i\hat{v}_j(\gamma - 1) \tag{3}$$

$$\Lambda^0{}_j = -\gamma v_j \tag{4}$$

It is useful to consider the special case in which $\mathbf{v}$ is aligned with a coordinate axis. We choose $\mathbf{v} = (0, v, 0)$ and find

$$\Lambda = \begin{matrix} \gamma & 0 & -\gamma v & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma v & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \tag{5}$$

The speed of light is equated to unity, $c \to 1$, so that the 4-momentum of a particle of mass m and 3-momentum $\mathbf{p}$ is given by

$$p^\mu = (p^0, \mathbf{p})$$

with $p^0 = \sqrt{\mathbf{p}^2 + m^2}$.

1.2 Invariance

Suppose that (x, p) are the coordinates, measured in a frame S, of some point in phase space (x and p are 4-vectors). The same point as measured in a different Lorentz frame, S', has coodinates ($x' = \Lambda x, p' = \Lambda p$) and a function $F(x, p)$ is a *Lorentz scalar* if its value as measured by an observer in S' is given by

$$F'(x', p') = F(x, p). \tag{6}$$

That is, F has the same value to both observers when measured at same phase-space point. Similarly a 4-component object V is a Lorentz vector or 4-vector if its components in the two frames are related by

$$V'(x', p') = \Lambda V(x, p),$$

and so on. Tensors (of whatever rank) that transform according to these Lorentz rules, with the number of Λ-factors corresponding to the rank, are Lorentz tensors.

Notice that being a Lorentz tensor puts no constraint on the function form of the tensor components; it only specifies the transformed components. There is a distinct property, however, that does constrain the functional dependence. We say that a function $G(x, p)$ is *invariant* under Lorentz transformation if

$$G'(x', p') = G(x', p')$$

In other words the functional form is preserved. Notice that this definition is meaningful only when a rule for determining G' from G is given. For example, if G is a Lorentz scalar, then the invariance property becomes $G(x', p') = G(x, p)$ or

$$G(\Lambda x, \Lambda p) = G(x, p),$$

which does indeed constrain the form of G. An example of an invariant (scalar) function is

$$p \cdot x = p^\mu x_\mu$$

In this sense the 4-dimensional volume element is invariant,

$$d^4x = d^4x'$$

because volume elements transform with the Jacobian, and the Jacobian of a (proper) Lorentz matrix is unity:

$$|\Lambda| = 1$$

For the same reason,

1. d^4p, where p denotes 4-momentum, is invariant;

2. the 4-dimensional Dirac delta functions,

$$\delta^4(x - \bar{x}) = \delta(x^1 - \bar{x}^1)\delta(x^2 - \bar{x}^2)\delta(x^3 - \bar{x}^3)\delta(x^4 - \bar{x}^4)$$

   and $\delta^4(p - \bar{p})$ are invariant.

Here $\bar{x}^\mu$ and $\bar{p}^\mu$ are specified four vectors.

The mass-shell restriction

$$(p^0)^2 - \mathbf{p}^2 = m^2$$

involves only the invariant function $p^{mu}p_{mu}$, so that the $\delta-$function

$$\delta((p^0)^2 - \mathbf{p}^2 - m^2) = \frac{1}{|2p^0|}\delta(p^0 - \sqrt{\mathbf{p}^2 + m^2}) \tag{7}$$

is scalar. (Here we have discarded the negative energy root.)

1.3 Equations of motion

The relativistic Hamiltonian for a single particle having position $\mathbf{x}$ and momentum $\mathbf{p}$ is expressed in terms of the 4-vector potential

$$A^\mu = (\phi, \mathbf{A})$$

where $\mathbf{A}$ is the vector potential and ϕ the electrostatic potential. We introduce the canonical momentum

$$P^\mu = p^\mu + qA^\mu$$

which is obviously a 4-vector, in order to write the Hamiltonian as

$$H(\mathbf{x}, \mathbf{P}) = P^0 = \sqrt{m^2 + (\mathbf{P} - q\mathbf{A})^2} + \Phi \tag{8}$$

where $\Phi = q\phi$ is the potential energy. Notice that

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{P}} = \frac{\mathbf{P} - q\mathbf{A}}{\sqrt{m^2 + (\mathbf{P} - q\mathbf{A})^2}}$$

so that the relation between velocity $\dot{\mathbf{x}}$ and momentum $\mathbf{p}$ is independent of the magnetic field:

$$\dot{\mathbf{x}}\sqrt{m^2 + \mathbf{p}^2} = \mathbf{p}$$

After solving this relation for $\mathbf{p}$, one finds that

$$\mathbf{p} = m\gamma\dot{\mathbf{x}} \tag{9}$$

where

$$\gamma \equiv (1 - \dot{\mathbf{x}}^2)^{-1/2}$$

Notice that γ has the equivalent expression

$$\gamma = \sqrt{1 + \mathbf{p}^2/m^2};$$

which implies

$$p^0 = m\gamma. \tag{10}$$

The other set of Hamilton's equations

$$\dot{\mathbf{P}} = -\frac{\partial H}{\partial \mathbf{x}}$$

reproduce the Lorentz force law, as follows:

$$\nabla_i(\sqrt{(\mathbf{P} - q\mathbf{A})^2 + m^2} + q\phi) = \frac{q}{p^0}(P_j - qA_j)\nabla_i A_j - q\nabla_i\phi.$$

Hence we have

$$\dot{p^i} + q\dot{A}_i = \frac{q}{p^0}\nabla_i(P_j A_j) - \frac{q^2}{p^0}A_j\nabla_i A_j - q\nabla_i\phi$$

We next use a standard vector identity for $\nabla(\mathbf{P}\cdot\mathbf{A})$, and then eliminate $\mathbf{P} = \mathbf{p} + q\mathbf{A}$ in favor of $\mathbf{p}$. The resulting quadratic terms in $\mathbf{A}$ precisely cancel,

$$\mathbf{A}\times\nabla\times\mathbf{A} + \mathbf{A}\cdot\nabla\mathbf{A} - \nabla(\mathbf{A}^2/2) = 0,$$

leaving

$$\dot{\mathbf{p}} + q\dot{\mathbf{A}} = \frac{q}{p^0}(\mathbf{p}\cdot\nabla\mathbf{A} + \mathbf{p}\times\mathbf{B}) - q\nabla\phi$$

or, in view of (10 and (9),

$$\dot{\mathbf{p}} = q(\dot{\mathbf{x}}\times\mathbf{B} - \nabla\phi) - q(\dot{\mathbf{A}} - \dot{\mathbf{x}}\cdot\nabla\mathbf{A})$$

In the last term here, $\dot{\mathbf{A}}$ is the total time derivative

$$\dot{\mathbf{A}} = \frac{\partial\mathbf{A}}{\partial t} + \dot{\mathbf{x}}\cdot\nabla\mathbf{A}$$

Hence we have

$$\dot{\mathbf{p}} = q\left(\dot{\mathbf{x}} \times \mathbf{B} - \nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\right) \equiv \mathbf{F}_L \tag{11}$$

where $\mathbf{F}_L = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is the usual Lorentz force.

Next we write the equations of motion in covariant form. This requires the "4-force" F^μ satisfying

$$\frac{dp^\mu}{d\tau} = F^\mu$$

where τ is the proper time. Since

$$dt = \gamma d\tau$$

we see that the spatial components of the 4-force are

$$F^i = \gamma F_L^i \tag{12}$$

(Thus the ordinary 3-vector force is not part of a 4-vector.) For the remaining component, we equate dp^0, the energy change, to the work done by $\mathbf{F}_L$:

$$dp^0 = \mathbf{F}_L \cdot \mathbf{v} dt.$$

But (10) implies that $\mathbf{v}dt = m^{-1}\mathbf{p}d\tau$, so

$$F^0 = \frac{dp^0}{d\tau} = \frac{\mathbf{F}_L \cdot \mathbf{p}}{m} = \frac{\mathbf{F} \cdot \mathbf{p}}{E}, \tag{13}$$

or $F^0 = \mathbf{F} \cdot \mathbf{v}$, (Note that $\mathbf{F}$ without the 'L' subscript refers to the spatial components of the 4-force.)

It is helpful to recall here that

$$\left(\frac{\partial}{\partial \mathbf{p}}\right) \cdot \mathbf{F}_L = 0$$

It follows from (12) that the spatial components of the 4-force satisfy a slightly different condition:

$$\left(\frac{\partial}{\partial \mathbf{p}}\right) \cdot \left(\frac{\mathbf{F}}{E}\right) = 0 \tag{14}$$

since $E = m\gamma$.

1.4 Field strength tensor

It is helpful to express the electromagnetic force in terms of the field strength tensor,

$$F^{\mu\nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu$$

or, in Schutz' language, $\tilde{F} = \tilde{d}\tilde{A}$. Here

$$\nabla^\mu = \eta^{\mu\nu} \frac{\partial}{\partial x^{nu}} = (-\frac{\partial}{\partial t}, \nabla)$$

Explicitly,

$$F^{\mu\nu} = \begin{matrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{matrix} \tag{15}$$

Since lowering the first index reverses the sign of the 0^{th} row, and lowering the second reverses the sign of the 0^{th} column, we have

$$F_{\mu\nu} = \begin{matrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{matrix} \tag{16}$$

The action of F on any four-vector K can be seen from direct multiplication:

$$F^{\mu\kappa} K_\kappa = (\mathbf{E} \cdot \mathbf{K}, \mathbf{E} K_0 + \mathbf{B} \times \mathbf{K}) \tag{17}$$

In particular we consider the product

$$F^{\mu\nu} p_\nu = -F^{\mu 0} p^0 + \nabla^\mu (\mathbf{p} \cdot \mathbf{A}) - (\mathbf{p} \cdot \nabla) A^\mu$$

¿From (17) we see that

$$F^{\mu\nu} p_\nu = \frac{m}{q} F^\mu. \tag{18}$$

1.5 Energy-momentum tensor

Landau and Lifschitz show from symmetry considerations that the energy-momentum tensor of an ideal fluid, when measured in the rest-frame of the fluid, will have the form

$$T(0) = \begin{matrix} u & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{matrix} \tag{19}$$

where u is the energy density, p is the pressure and the 0-argument refers to the state of rest. Notice that this definition makes p and u Lorentz scalars. The fact that the symbol 'p' has more than one meaning will rarely cause confusion.

When the energy-momentum tensor of the moving fluid is measured in the lab frame, its components will be boosted by $-\mathbf{v}$, the velocity of S as measured in S'. Denoting the tensor of the moving fluid, measured in the rest frame, by $T(\mathbf{v})$, we have

$$T(\mathbf{v}) = \Lambda(-\mathbf{v}) \cdot \Lambda(-\mathbf{v}) \cdot T(0)$$

where $\Lambda(-\mathbf{v}) = \Lambda(\mathbf{v})^{-1}$ is the Lorentz boost inverse to the $\Lambda(\mathbf{v})$ of (1); its components are denoted by

$$\bar{\Lambda}^{\mu}{}_{\nu} \equiv (\Lambda^{-1})^{\mu}{}_{\nu}$$

Straightforward, albeit lengthy, calculation yields

$$T^{00}(\mathbf{v}) = \bar{\Lambda}^{0}{}_{\alpha}\bar{\Lambda}^{0}{}_{\beta}T^{\alpha\beta}(0) = \gamma^2(u + v^2 p);$$

$$T^{0i}(\mathbf{v}) = \bar{\Lambda}^{0}{}_{\alpha}\bar{\Lambda}^{i}{}_{\beta}T^{\alpha\beta}(0) = \gamma^2 v_i(u + p);$$

and

$$T^{ij}(\mathbf{v}) = \bar{\Lambda}^{i}{}_{\alpha}\bar{\Lambda}^{j}{}_{\beta}T^{\alpha\beta}(0) = \delta_{ij}p + \gamma^2(u + p)v_i v_j.$$

Recall that the combination $U + pV$, where V is the volume, is a thermodynamic potential called enthalpy. Therefore the quantity appearing in the energy-momentum tensor

$$u + p \equiv h$$

is enthalpy per unit volume, or enthalpy density.

A simpler expression for $T(\mathbf{v})$ is

$$T^{\alpha\beta}(\mathbf{v}) = \eta^{\alpha\beta}p + (u + p)U^{\alpha}U^{\beta} \tag{20}$$

where

$$U = (\gamma, \gamma\mathbf{v})$$

is the obvious 4-vector for fluid flow. It is not hard to verify the two expressions for the energy-momentum tensor agree.

2 Kinetic theory

2.1 Scalar distribution function

The distribution function $f(\mathbf{x}, \mathbf{p}, t)$ is defined such that

$$n(\mathbf{x}, t) = \int d^3 p f$$

is the density of particles at $\mathbf{x}$ at time t. Here we show that f is a Lorentz scalar.

We begin with the particle trajectories given by $\mathbf{x}_i(t)$ and $\mathbf{p}_i(t)$. Then the distribution function can be expressed as the ensemble average of the micro-distribution (Klimontovich-Dupree)

$$f_* \equiv \sum_i \delta^3(\mathbf{x} - \mathbf{x}_i(t))\delta^3(\mathbf{p} - \mathbf{p}_i(t))$$

We next introduce a separate time variable t_i for each i in order to write

$$f_* = \sum_i \int dt_i \delta(t - t_i)\delta^3(\mathbf{x} - \mathbf{x}_i(t_i))\delta^3(\mathbf{p} - \mathbf{p}_i(t_i))$$

or, choosing $x_i^0 = t_i$,

$$f_* = \sum_i \int dt_i \delta^4(\mathbf{x} - \mathbf{x}_i(t_i))\delta^3(\mathbf{p} - \mathbf{p}_i(t_i))$$

Now the proper time interval for the *ith* particle is

$$d\tau_i = \frac{dt_i}{\gamma_i}$$

where γ_i is the relativistic factor for the *ith* particle. We recall (10) in order to write

$$dt_i = \frac{p_i^0}{m} d\tau_i \tag{21}$$

whence

$$f_* = \sum_i \int d\tau_i \frac{p_i^0}{m} \delta^4(x - x_i(\tau_i))\delta^3(\mathbf{p} - \mathbf{p}_i(\tau_i))$$

We multiply this function by the Lorentz scalar appearing in (7):

$$F(\mathbf{x}, \mathbf{p}, t) \equiv \frac{1}{p^0}\delta(p^0 - \sqrt{\mathbf{p}^2 + m^2}) f_*(\mathbf{x}, \mathbf{p}, t)$$

and note that the mass-shell factor can be put inside the sum and then evaluated at each $\mathbf{p}_i$:

$$F = \sum_i \int d\tau_i \frac{p_i^0}{mp^0} \delta^4(x - x_i)\delta^3(\mathbf{p} - \mathbf{p}_i)\delta(p^0 - p_i^0)$$

or

$$F = \frac{1}{m} \sum_i \int d\tau_i \delta^4(x - x_i)\delta^4(p - p_i)$$

This quantity is manifestly a scalar. Since it differs from f_* by a scalar factor, and since the physical distribution $f(\mathbf{x}, \mathbf{p}, t)$ is the ensemble average of f_*, we conclude that $f(\mathbf{x}, \mathbf{p}, t)$ is a Lorentz scalar.

It is worthwhile to recall the significance of this fact. Suppose that the distribution function of some fluid is known in the fluid rest frame, R: $f'(x', p') = f_R(x', p')$. Then, if the local fluid velocity (as measured in the "lab frame" S) is $\mathbf{V}$, we have $x' = \Lambda(\mathbf{V})x$, $p' = \Lambda(\mathbf{V})p$ and the distribution observed in the lab frame is, according to (6),

$$f(x, p) = f_R(\Lambda x, \Lambda p). \tag{22}$$

In the presence of an electromagnetic field, f_R can depend on position both directly (through its, density, for example), and also through the (rest-frame) field variables $A'^\mu(x')$. It is natural to express the lab-frame distribution in terms of the lab-frame fields $A^\mu(x)$, and therefore to write the transformation law in the form

$$f(x, p, A(x)) = f_R(\Lambda x, \Lambda p, \Lambda A(\Lambda x)) \tag{23}$$

2.2 Momentum-space volume

We have found that the quantity d^3p/E is an invariant. An alternative derivation of this fact is instructive, albeit tedious. The idea is to compute the metric tensor g_{ij} measuring distance on the mass shell. The volume element on the mass shell is then given by the Jacobian $\sqrt{|g|}$, and it will turn out that

$$\sqrt{|g|} = m/E. \tag{24}$$

Thus we recall the four velocity $u^\mu = dx^\mu/d\tau$ where τ is proper time. It follows that

$$\begin{aligned} u^0 &= \gamma, \\ \mathbf{u} &= \gamma \mathbf{v} \end{aligned}$$

Note that the 3-vector $\mathbf{v}$ determines a 4-vector $u^\mu = (\gamma, \gamma\mathbf{v})$ essentially because of the implicit mass-shell restriction. That is, if we define

$$p^\mu \equiv m u^\mu$$

then we find that

$$p^\mu p_\mu = \mathbf{p}\cdot\mathbf{p} - m^2\gamma^2$$

or

$$m^2\gamma^2 \equiv E^2 = m^2 + \mathbf{p}^2 \tag{25}$$

Next consider the infinitesimal length $dp^\mu dp_\mu = d\mathbf{p}^2 - (dp^0)^2$, constrained by (25). Thus we make the substitution

$$dp^0 = \frac{\mathbf{p}\cdot d\mathbf{p}}{E}$$

and find, after some algebra,

$$dp^\mu dp_\mu = g_{ij} dp^i dp^j$$

with

$$g = \begin{matrix} 1-\hat{p}_x^2 & -\hat{p}_x\hat{p}_y & -\hat{p}_x\hat{p}_z \\ -\hat{p}_x\hat{p}_y & 1-\hat{p}_y^2 & -\hat{p}_y\hat{p}_z \\ -\hat{p}_x\hat{p}_z & -\hat{p}_y\hat{p}_z & 1-\hat{p}_z^2 \end{matrix} \tag{26}$$

and $\hat{p}_i = p_i/E$. Now a straightforward calculation gives

$$|g| = \det(g) = 1 - \frac{p^2}{E^2} = \frac{m^2}{E^2}$$

as was to be demonstrated.

2.3 Kinetic equation

An invariant kinetic equation with the correct nonrelativistic limit is easily written down:

$$\frac{p^\mu}{m}\frac{\partial f}{\partial x^\mu} + F^\mu \frac{\partial f}{\partial p^\mu} = C \tag{27}$$

where C is a collision operator, and F^μ is the 4-vector force constructed in Sec. 1. The form of (27) seems obvious when one notices that it can be written as

$$\frac{dx^\mu}{d\tau}\frac{\partial f}{\partial x^\mu} + \frac{dp^\mu}{d\tau}\frac{\partial f}{\partial p^\mu} = C$$

To see how this equation reduces to the familiar version, we write it more explicitly as

$$\frac{p^0}{m}\frac{\partial f}{\partial t}+\gamma\dot{x}^i\frac{\partial f}{\partial x^i}+\frac{\mathbf{p}\cdot\mathbf{F}}{p^0}\frac{\partial f}{\partial p^0}+\mathbf{F}\cdot\frac{\partial f}{\partial \mathbf{p}}=C$$

and then observe that the mass-shell restriction

$$p^0=\sqrt{m^2+p^2}$$

allows us to write

$$\frac{\partial f(p^0(\mathbf{p}),\mathbf{p})}{\partial \mathbf{p}}=\frac{dp^0}{d\mathbf{p}}\frac{\partial f}{\partial p^0}+\frac{\partial f}{\partial \mathbf{p}}$$

by the chain rule. Hence, since

$$\frac{dp^0}{d\mathbf{p}}=\frac{\mathbf{p}}{p^0}$$

the p^0-derivative in the kinetic equation can be considered part of the $\mathbf{p}$-derivative and suppressed, giving an obvious relativistic version of the usual kinetic equation,

$$\frac{p^\mu}{m}\frac{\partial f}{\partial x^\mu}+F^i\frac{\partial f}{\partial p^i}=C \tag{28}$$

3 Moments of kinetic equation

3.1 Tensor moments

It is clear that the density $n(\mathbf{x},t)$ is not a scalar, since d^3p is not. To construct moments of f that are Lorentz tensors, we use the scalar momentum-sapce volume element d^3p/E, where, as before,

$$E(\mathbf{p})\equiv\sqrt{\mathbf{p}^2+m^2}=p^0(\mathbf{p})=m\gamma$$

Thus the moment

$$M^{\alpha\beta\ldots\nu}\equiv\int\frac{d^3p}{E(p)}p^\alpha p^\beta\ldots p^\nu f(\mathbf{x},\mathbf{p},t) \tag{29}$$

is a Lorentz tensor—the general tensor moment of the distribution f.

Next consider the quantity (not a 4-vector)

$$\frac{p^\alpha}{E}=\left(1,\frac{\mathbf{p}}{m\gamma}\right).$$

Recalling (21) we see that

$$\frac{p^\alpha}{E} = \frac{dx^\alpha}{dt} \tag{30}$$

Hence our tensor can be written as

$$M^{\alpha\beta\ldots\nu} \equiv \int d^3p \frac{dx^\alpha}{dt} p^\beta \ldots p^\nu f(\mathbf{x}, \mathbf{p}, t)$$

showing that it measures the flow of $p^\beta \ldots p^\nu$.

When necessary we indicate the rank of our moment tensor with a subscript: M_r is the tensor with r-factors of 4-momentum in the integrand. Examples are

1. $M_0 = \rho$ is the scalar mass density,

$$\rho = \int \frac{d^3p}{E(p)} f$$

2. $M_1^\alpha = \Gamma^\alpha$ is the flow 4-vector

$$\Gamma^\alpha = \int \frac{d^3p}{E(p)} p^\alpha f$$

whose time-component is the density,

$$\Gamma^0 = \int d^3p f = n, \tag{31}$$

and whose spatial components give the fluid mean-flow vector

$$\Gamma^k = nV^k = \int d^3p v^k f \tag{32}$$

3. $M_2^{\alpha\beta} = T^{\alpha\beta}$ is the energy-momentum tensor,

$$T^{\alpha\beta} = \int \frac{d^3p}{E(p)} p^\alpha p^\beta f,$$

which we have already considered.

3.2 Lorentz transformation of moments

Here we confirm that the scalar property of f is consistent with the tensor character of its momentum-moments. Thus consider the moment $M^{\alpha\cdots\gamma}$ measured by an observer in the lab frame:

$$M^{\alpha\ldots\gamma} = \int \frac{d^3p}{E} p^\alpha \cdots p^\gamma f(x,p)$$

Usually the simplest function will be f_R, so it make sense to express the integral as

$$M^{\alpha\ldots\gamma} = \int \frac{d^3p}{E} p^\alpha \cdots p^\gamma f_R(\Lambda x, \Lambda p)$$

and then to change the integration variable $p \to p'$. Since $p = \bar{\Lambda} p'$, and since d^3p/E is invariant, the result of this change is

$$M^{\alpha\cdots\gamma} = \bar{\Lambda}^\alpha{}_\mu \cdots \bar{\Lambda}^\gamma{}_\nu \int \frac{d^3p'}{E'} p'^\alpha \cdots p'^\gamma f_R(x',p')$$

or

$$M^{\alpha\cdots\gamma} = \bar{\Lambda}^\alpha{}_\mu \cdots \bar{\Lambda}^\gamma{}_\nu M'^{\mu\cdots\nu}$$

so that the r^{th} moment is indeed a tensor of rank r.

3.3 General moment

We return to the kinetic equation (28)

$$\frac{p^\mu}{m}\frac{\partial f}{\partial x^\mu} + F^i \frac{\partial f}{\partial p^i} = C \tag{33}$$

and recall the tensor moment

$$M^{\alpha\cdots\gamma} \equiv \int \frac{d^3p}{E} f p^\alpha \cdots p^\gamma$$

Here we operate of (33) with

$$\int \frac{d^3p}{E} p^{\alpha\cdots p^\gamma}$$

to obtain a sequence of equations for the moments M.

The moment of the first, convective term in (33) is simply

$$\int \frac{d^3p}{E} p^\alpha \cdots p^\gamma \frac{p^\mu}{m}\frac{\partial f}{\partial x^\mu} = \frac{1}{m}\frac{\partial M^{\mu\alpha\cdots\gamma}}{\partial x^\mu}$$

In the second term

$$\mathcal{F}^{\alpha\cdots\gamma} \equiv \int \frac{d^3p}{E} p^\alpha \cdots p^\gamma F^i \frac{\partial f}{\partial p^i}$$

we integrate by parts and use (14) to obtain

$$\mathcal{F}^{\alpha\cdots\gamma} = -\int \frac{d^3p}{E} f F^i \frac{\partial}{\partial p^i}(p^\alpha \cdots p^\gamma)$$

or, in view of (18),

$$\mathcal{F}^{\alpha\cdots\gamma} = -\frac{q}{m} F^{i\nu} \int \frac{d^3p}{E} f p_\nu \frac{\partial}{\partial p^i}(p^\alpha \cdots p^\gamma)$$

After performing the derivative we find that

$$\mathcal{F}^{\alpha\cdots\gamma} = -\frac{q}{m} F^{\{\alpha\nu} M_\nu{}^{\beta\cdots\gamma\}} \tag{34}$$

Here

$$M_\nu{}^{\beta\cdots\gamma} \equiv \int \frac{d^3p}{E} f p_\nu p^\beta \cdots p^\gamma$$

and the curly brackets instruct us to symmetrize by exchanging the superscript α with each of the superscript indices on M:

$$F^{\{\alpha\nu} M_\nu{}^{\beta\cdots\gamma\}} = F^{\alpha\nu} M_\nu{}^{\beta\cdots\gamma} + F^{\beta\nu} M_\nu{}^{\alpha\cdots\gamma} + \ldots$$

It is clear that M transforms like a mixed tensor.

We denote the corresponding moment of the collision operator by

$$\mathcal{C}^{\alpha\cdots\gamma} = m \int \frac{d^3p}{E} p^\alpha \cdots p^\gamma C$$

Then the general moment of the kinetic equation can be expressed as

$$\frac{\partial M^{\mu\alpha\cdots\gamma}}{\partial x^\mu} - q F^{\{\alpha\nu} M_\nu{}^{\beta\cdots\gamma\}} = \mathcal{C}^{\alpha\cdots\gamma} \tag{35}$$

3.4 Examples

1. The zeroth moment describes particle conservation:

$$\frac{\partial \Gamma^\mu}{\partial x^\mu} = 0 \tag{36}$$

Here we have noted that $M_\nu = 0$ when the rank of M is zero, and that the zeroth moment of the collision operator vanishes for any operator that conserves particles.

2. The first moment describes conversion of particle momentum and energy into electromagnetic field energy-momentum and collisional dissipation (friction),

$$\frac{\partial T^{\mu\alpha}}{\partial x^\mu} - qF^{\alpha\nu}\Gamma_\nu = \mathcal{C}^\alpha \tag{37}$$

4 Maxwellian case

4.1 Maxwellian distribution

A local Maxwellian distribution function is given by

$$f_M(x,p) \equiv N_M e^{-H(\mathbf{x},\mathbf{p})/T} \tag{38}$$

where N_M is the normalization, T measures temperature and

$$H(\mathbf{x},\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2} + \Phi(\mathbf{x})$$

is the energy (Hamiltonian) associated with the point $(\mathbf{x},\mathbf{p})$. Here Φ is the potential energy; for electromagnetic interaction it can be written in terms of the electrostatic potential $\phi(x)$ as

$$\Phi = q\phi$$

where q is the particle charge.

It is significant that the Hamiltonian is the sum of the time-components of two 4-vectors, $p^\mu = (\mathbf{p}, \sqrt{\mathbf{p}^2 + m^2})$ and $A^\mu = (\mathbf{A}, \phi)$. Thus, in terms of the canonical momentum $P^\mu = p^\mu + qA^\mu$ we can write

$$f_M(x,p) = N_M e^{-P^0(x,p)/T} \tag{39}$$

Because f_M must be scalar, $f'_M(x',p') = f_M(x,p)$, the constants N_M and T are scalars. And because H is a constant of the motion, f_M obviously satisfies the kinetic equation.

4.2 Maxwellian moments

Using (29) we can compute the Maxwellian tensors

$$\begin{aligned} M_M^{\alpha\beta\ldots\nu}(\mathbf{x}) &\equiv \int \frac{d^3p}{E(p)} p^\alpha p^\beta \ldots p^\nu f_M(\mathbf{x},\mathbf{p}) \\ &= N_M m^{r+2} e^{-\Phi/T} \int \frac{d^3s}{\sqrt{1+s^2}} s^\alpha \cdots s^\nu e^{-z\sqrt{1+s^2}} \end{aligned}$$

where $s^\alpha = p^\alpha/m$,

$$z \equiv m/T \ (= mc^2/T)$$

and r is the rank of the tensor. The isotropy of f_M implies that odd-rank tensors vanish and that the second-rank tensor is diagonal.

The scalar mass density $\rho(x) = mT_M^{(0)}$ is given by

$$\begin{aligned}
\rho &= N_M m^4 e^{-\Phi/T} \int \frac{d^3 s}{\sqrt{1+s^2}} e^{-z\sqrt{1+s^2}} \\
&= 4\pi N_M m^4 e^{-\Phi/T} \int \frac{ds\, s^2}{\sqrt{1+s^2}} e^{-z\sqrt{1+s^2}}
\end{aligned}$$

In addition one is interested in the stress tensor, or energy-momentum tensor ($r = 2$). For a general ideal fluid at rest, we know from (19) that this tensor has only diagonal components, the mean energy

$$T^{00} = u = \int d^3 p E(p) f,$$

and the pressure

$$T^{ij} = p\delta_{ij} = \int \frac{d^3 p}{E(p)} p^i p^j f.$$

In the Maxwellian case we have

$$\begin{aligned}
u &= N_M m^4 e^{-\Phi/T} \int \frac{d^3 s}{\sqrt{1+s^2}} (1+s^2) e^{-z\sqrt{1+s^2}} \\
&= 4\pi N_M m^4 e^{-\Phi/T} \int \frac{ds}{\sqrt{1+s^2}} (s^2+s^4) e^{-z\sqrt{1+s^2}}
\end{aligned}$$

and

$$\begin{aligned}
\delta_{ij} p &= N_M e^{-\Phi/T} m^4 \int \frac{d^3 s}{\sqrt{1+s^2}} s^i s^j e^{-z\sqrt{1+s^2}} \\
&= \frac{4\pi}{3} \delta_{ij} N_M e^{-\Phi/T} m^4 \int_0^\infty \frac{ds}{\sqrt{1+s^2}} s^4 e^{-z\sqrt{1+s^2}}
\end{aligned}$$

The integrals are performed using the formula

$$\int_0^\infty \frac{ds}{\sqrt{1+s^2}} s^{2n} e^{-z\sqrt{1+s^2}} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}} \left(\frac{2}{z}\right)^n K_n(z)$$

where K_n is the MacDonald function. Notice that

$$\frac{\Gamma(n+1/2)}{\sqrt{\pi}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}$$

Therefore

$$\int_0^\infty \frac{ds}{\sqrt{1+s^2}} s^{2n} e^{-z\sqrt{1+s^2}} = \frac{1\cdot 3\cdots(2n-1)K_n(z)}{z^n} \tag{40}$$

As Bessel functions the K_n satisfy

$$\begin{aligned} \frac{d}{dz}(z^n K_n(z)) &= z^n K_{n-1}(z) && (41)\\ \frac{d}{dz}(z^{-n} K_n(z)) &= -z^{-n} K_{n+1}(z) && (42) \end{aligned}$$

Their limiting forms are given by

$$K_\nu(z) \sim \frac{\Gamma(\nu)}{2}\left(\frac{2}{z}\right)^\nu \tag{43}$$

for $z \to 0$, and

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}\left[1 + \frac{4\nu^2-1}{8z} + O(z^{-2})\right] \tag{44}$$

for $z \to \infty$.

Our formulae have become

$$\begin{aligned} u &= 4\pi N_M m^4 e^{-\Phi/T}\left(\frac{K_1}{z} + \frac{3K_2}{z^2}\right) && (45)\\ p &= 4\pi N_M e^{-\Phi/T} m^4 \frac{K_2}{z^2} && (46) \end{aligned}$$

4.3 Normalization

The normalization is conveniently expressed in term of the density (not a scalar and not an element of T_M)

$$n(x) = 4\pi m^3 N_M e^{-\Phi/T} I_0$$

where

$$I_0 \equiv \int ds s^2 e^{-z\sqrt{1+s^2}}$$

It can be seen that

$$I_0 = -\frac{d}{dz}\left(\frac{K_1}{z}\right)$$

so (42) implies $I_0 = K_2/z$ and we have

$$N_M = \frac{ne^{\Phi/T}}{4\pi m^2 T K_2(z)} \tag{47}$$

Thus our Maxwellian can be expressed in terms of its density as

$$f_M = \frac{ne^{-E/T}}{4\pi m^2 T K_2(z)} \tag{48}$$

with $E = p^0 = \sqrt{p^2 + m^2}$.

The fact that p and n are both proportional to K_2 may seem surprising, so we digress to verify it. The point is that the two integrals

$$\begin{aligned} I_0 &\equiv \int ds s^2 e^{-z\sqrt{1+s^2}}, \\ I_0' &\equiv \frac{z}{3}\int ds \frac{s^4}{\sqrt{1+s^2}} e^{-z\sqrt{1+s^2}} \end{aligned}$$

are both equal to K_2/z. To see why, we note that

$$\frac{d}{ds}e^{-z\sqrt{1+s^2}} = -\frac{zs}{\sqrt{1+s^2}}e^{-z\sqrt{1+s^2}}$$

whence

$$I_0' = -\frac{1}{3}\int ds s^3 \frac{d}{ds} e^{-z\sqrt{1+s^2}}$$

and partial integration shows that, indeed, $I_0' = I_0$.

Finally we substitute (47) into our expression for the moments and find

$$\begin{aligned} u &= mn\left(\frac{K_1(z)}{K_2(z)} + \frac{3}{z}\right) && (49) \\ p &= nT && (50) \\ \rho &= mn\frac{K_1(z)}{K_2(z)} && (51) \end{aligned}$$

4.4 Nonrelativistic limit

The nonrelativistic limit has $z \to \infty$ and is therefore characterized by (44). Since

$$\frac{K_1}{K_2} \to \frac{1 + (3/8z)}{1 + (15/8z)} \to 1 - \frac{3}{2z}$$

we see that

$$\begin{aligned} u &\to mn\left(\frac{K_1}{K_2} + \frac{3T}{m}\right) = mn\left(1 + \frac{3T}{2m}\right) \\ \rho &\to mn\left(1 - \frac{3T}{2m}\right) \end{aligned}$$

Note here that, in dimensional units,

$$\frac{T}{m} = \frac{v_t^2}{2c^2}$$

where $v_t \equiv \sqrt{2T/m}$ is the usual thermal speed and c the speed of light. However neglecting the corrections would be the same as neglecting the $(1/2)mv^2$ energy correction to rest mass. Indeed replacing m by mc^2 yields

$$\begin{aligned} u &\rightarrow mnc^2 + \frac{3}{2}nT \\ \rho &\rightarrow mnc^2 - \frac{3}{2}nT \end{aligned}$$

4.5 Moving Maxwellian

Again S' is a frame moving at the local fluid velocity $\mathbf{V}$ and S is the lab frame. If the distribution of the fluid as measured in its rest frame is f_R, then, according to (23), the lab-frame distribution is

$$f(x, p, A) = f_R(x', p', A').$$

In the Maxwellian case,

$$f_R(x', p', A') = f_M(P'^0(x', p', A')),$$

can be expressed in terms of $P'^0 = p'^0 + qA'^0$ where

$$P'^0 = \Lambda^0{}_\mu P^\mu = \gamma(P^0 - \mathbf{P} \cdot \mathbf{V}).$$

[It can be seen that this rule coincides with the non-relativistic version

$$\frac{1}{2}mv^2 \rightarrow \frac{1}{2}m(\mathbf{v} - \mathbf{V})^2$$

through terms quadratic in V, but not beyond the quadratic terms.] The moving Maxwellian is therefore described in the lab-frame by

$$f(x, p, A) = f_M(\gamma(P^0 - \mathbf{P} \cdot \mathbf{V}))$$

or

$$f(x, p) = N_M e^{-\gamma q(\phi - \mathbf{V}\cdot\mathbf{A})/T} e^{-\gamma(E - \mathbf{V}\cdot\mathbf{p})/T} \tag{52}$$

Note that in this formula $\gamma \equiv \gamma(\mathbf{V})$ is associated with the fluid velocity, not the phase-space point; recall (22).