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SMR 1331/22

AUTUMN COLLEGE ON PLASMA PHYSICS

8 October - 2 November 2001

Topology and Nonlinear Theory - Mathematical Concepts to Explore the Complexity of Plasmas

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These are preliminary lecture notes, intended only for distribution to participants.

Topology and Nonlinear Theory
– mathematical concepts to explore the complexity of plasmas –

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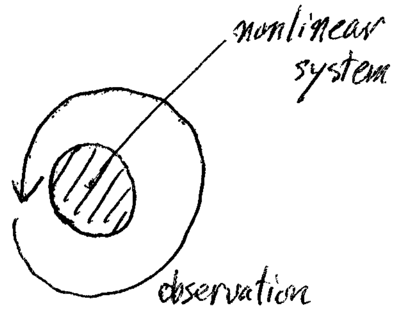
This lecture is addressed to a basic question: How can we construct a "rigorous" theory for complex system (like a plasma) with a small number of exact information? We study some mathematical concepts that may provide us a methodology to explore fundamental nature of plasmas. The theory of "topology" is the main theme of this lecture.

I. TOPOLOGY (GEOMETRIC THEORY)

How can we construct a rigorous theory with a small number of exact data? Since many complex systems are not "integrable" (see Sec. II.A), we cannot expect that we could obtain a complete understanding of the system. If we have some exact information that can be measured or estimated, what rigorous assertion can we derive? This question may be answered by studying the concept of "topology", a general notion of characteristics that is insensitive to details.



A. Degree theory

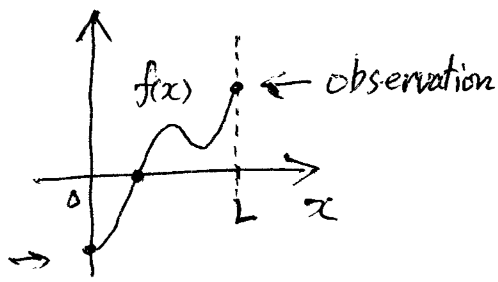
The degree theory is to characterize what we can know about "nonlinear system" from the "outside".

For example, let $f(x)$ be a continuous function on $[0, L]$ to $\mathbf{R}$. If we have "boundary data"

$$\begin{cases} f(0) < 0, \\ f(L) > 0, \end{cases}$$

then, there must be a point $x \in [0, L]$ such that $f(x) = 0$.

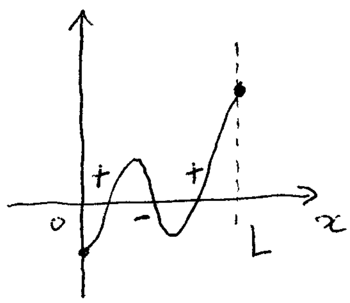
To generalize this trivial but profound fact for general maps, we must quantify what we can estimate from the observation at the boundary. That is the "degree".



- $\Omega \subset \mathbf{R}^1$:

  Let f be a smooth map (or a limit of smooth map) on Ω ($\subset \mathbf{R}$) to $\mathbf{R}$. For some y ($\in \mathbf{R}$), we find x_j ($j = 1, \cdots, m$) such that $f(x_j) = y$. Then, we define

$$\deg\,(f, \Omega, y) = \sum_{j=1}^{m} \operatorname{sgn} f'(x_j). \qquad (1)$$



- $\Omega \subset \mathbf{R}^N$:

  Let $\boldsymbol{f}$ be a smooth map (or a limit of smooth map) on Ω ($\subset \mathbf{R}^N$) to $\mathbf{R}^N$. We denote by $J(\boldsymbol{x}) = \det \partial_x \boldsymbol{f}(\boldsymbol{x})$ the Jacobian. For some $\boldsymbol{y}$ ($\in \mathbf{R}^N$), we find $\boldsymbol{x}_j$ ($j = 1, \cdots, m$) such that $\boldsymbol{f}(\boldsymbol{x}_j) = \boldsymbol{y}$. We define

$$\deg\,(\boldsymbol{f}, \Omega, \boldsymbol{y}) = \sum_{j=1}^{m} \operatorname{sgn}\, J(\boldsymbol{x}_j). \qquad (2)$$

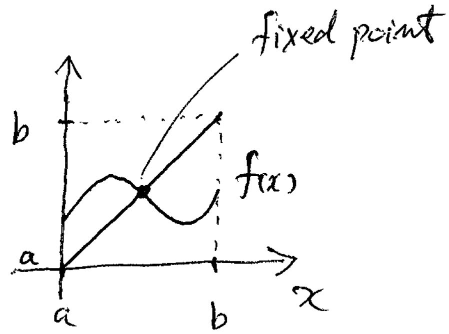
It is shown that deg is uniquely determined by the boundary value $\boldsymbol{f}|_\Gamma$ and $\boldsymbol{y}$, and independent to any continuous deformation of $\boldsymbol{f}$ in Ω^o (**homotopy invariant**).

B. Fixed point

We can apply the degree theory to prove the "fixed point theorem". For a map $f(x)$, we call x_0 a fixed point, if

$$f(x_0) = x_0.$$

In the one-dimension case, it is obvious that a continuous function f that maps $[a, b]$ into $[a, b]$ has at least one fixed point.



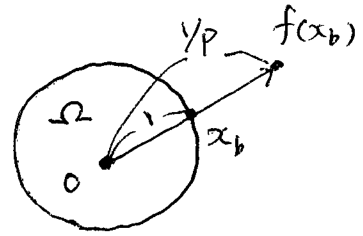
In $\mathbf{R}^N$, we have

Theorem 1. [Brouwer] *Let Ω be a bounded convex set in $\mathbf{R}^N$, and $\boldsymbol{f}$ be a continuous map on Ω into Ω. Then, $\boldsymbol{f}$ has at least one fixed point.*

(proof) Let the origin 0 be included in Ω (after appropriate coordinate transform). We define

$$\boldsymbol{g}_p(\boldsymbol{x}) = \boldsymbol{x} - p\boldsymbol{f}(\boldsymbol{x}) \quad (0 \le p \le 1). \qquad (3)$$

It suffices to show that $\mathbf{0} \in \boldsymbol{g}_1(\Omega)$. We denote the boundary of Ω by Γ. If $\mathbf{0} \in \boldsymbol{g}_1(\Gamma)$, then the theorem is proved. So, we assume otherwise; $\mathbf{0} \notin \boldsymbol{g}_1(\Gamma)$. Then, we find that $\mathbf{0} \notin \boldsymbol{g}_p(\Gamma)$ $(0 \le \forall p < 1)$ (see figure). Hence, we find $\mathbf{0} \notin \boldsymbol{g}_p(\Gamma)$ $(0 \le \forall p \le 1)$.



If $0 \in g_p(\partial\Omega)$
$\Leftrightarrow \exists x_b \in \partial\Omega$
$x_b = pf(x_b)$

For $\boldsymbol{g}_0(\boldsymbol{x}) = \boldsymbol{x}$, it is obvious that $\deg\,(\boldsymbol{g}_0, \Omega, \mathbf{0}) = 1$. When we change p, the "boundary value" $\boldsymbol{g}_p(\Gamma)$ does not meet $\mathbf{0}$ for all p $(0 \le p \le 1)$. Hence, $\deg\,(\boldsymbol{g}_p, \Omega, \mathbf{0})$ is conserved against the change of p from 0 to 1;

$$\deg\,(\boldsymbol{g}_1, \Omega, \mathbf{0}) = \deg\,(\boldsymbol{g}_p, \Omega, \mathbf{0}) = \deg\,(\boldsymbol{g}_0, \Omega, \mathbf{0}) = 1.$$

We thus conclude that $\boldsymbol{g}_1(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x}) = 0$ must occur at least one point $\boldsymbol{x}$ $(\in \Omega)$.

(QED)

We note that the existence of a solution to a nonlinear problem $\boldsymbol{g}_1(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x}) = 0$ is concluded by a linear problem $\boldsymbol{g}_0(\boldsymbol{x}) = \boldsymbol{x} = 0$. The used fact is $\mathbf{0} \notin \boldsymbol{g}_p(\Gamma)$ $(0 \le \forall p \le 1)$, that is an observation of the "boundary vale".

This theorem is generalized to infinite-dimension topological vector spaces.

Application 1. Consider an evolution equation of dissipative type:

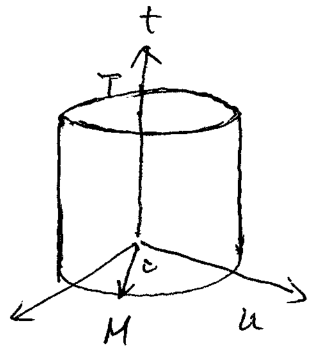
$$\begin{cases} \dfrac{d}{dt}u = -u + J(u), \\ \\ u(0) = u_0, \end{cases} \tag{4}$$

where the initial value satisfies $||u_0|| \leq M$ with some M (> 0). We assume

$$||J(u)|| \leq ||u|| \quad \text{(contraction map)}.$$

Condier a convex domain of functions $u(t)$;

$$\Omega = \{u(t);\ t \in [0,T],\ \sup_{t\in[0,T]} ||u(t)|| \leq M\}.$$



Then, (4) has a solution in Ω.

To prove this, we write (4) in an equivalent form of

$$u = e^{-t}u_0 + \int_0^t e^{s-t} J(u(s))\, ds. \tag{5}$$

We define a map

$$f(u(t)) = e^{-t}u_0 + \int_0^t e^{s-t} J(u(s))\, ds,$$

and show that $f(u) = u$ has a solution (fixed point). Since f is continuous map, it suffices to show that $\Omega \subseteq f(\Omega)$. Let

$$\langle\langle x(t) \rangle\rangle = \sup_{t\in[0,T]} ||x(t)||.$$

We observe

$$\begin{aligned} \langle\langle f(u) \rangle\rangle &\leq \sup_{t\in[0,T]} \left[e^{-t}||u_0|| + \int_0^t e^{s-t} ||J(u(s))||\, ds \right] \\ &\leq \sup_{t\in[0,T]} \left[e^{-t}||u_0|| + (1-e^{-t}) \sup_s ||J(u(s))|| \right] \\ &= \sup_{t\in[0,T]} \left[\sup_s ||J(u(s))|| (e^{-t} + (1-e^{-t})) \right] \\ &\leq \langle\langle u \rangle\rangle. \end{aligned}$$

Hence, we find $\Omega \subseteq f(\Omega)$.

- Using another version of fixed point theorem, that is the contraction map theorem, we can prove that the solution is unique.

- It is easy to generalize (4) to

$$\frac{d}{dt}u = \frac{1}{\lambda}[-u + J(u)]. \tag{6}$$

- The evolution equation (6) is the "Yosida approximation" of

$$\frac{d}{dt}u = A(u). \tag{7}$$

The Yosida approximation of the operator A is

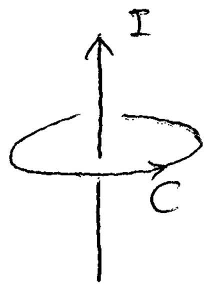
$$A_\lambda = A(1+\lambda A)^{-1}. \qquad (8)$$

If we write

$$J_\lambda = (1+\lambda A)^{-1},$$

which is called the "resolvent operator", we may write $A_\lambda = \lambda^{-1}(I - J_\lambda)$, and hence, the Yosida approximation of (7) reads as (6). We study the limit of $\lambda \to 0$ for a general class of evolution equation.

C. Linkage



Study of magnetic fields leads to an interesting topological concept of "linkage", which, introduced by Gauss, was the start of the degree theory.

We consider a stationary magnetic field that obey

$$\nabla \times \boldsymbol{B} = \boldsymbol{j}, \qquad (9)$$

where $\boldsymbol{j}$ corresponds to the electric current density. Let us consider a wire that carries a total current I. Let C be a loop circulating around the current, and S be a surface span by C. We have

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{x} = \int_S \boldsymbol{j} \cdot \boldsymbol{n}\ ds = I. \qquad (10)$$

If C does not link the current, the integral (10) vanishes.

We can invert (9) by using the Biot-Savart integral;

$$\boldsymbol{B}(\boldsymbol{x}) = \mathrm{curl}^{-1}\boldsymbol{j} = \int \boldsymbol{K}(\boldsymbol{x},\boldsymbol{y}) \times \boldsymbol{j}(\boldsymbol{y})dy, \qquad (11)$$

where the kernel is

$$\boldsymbol{K}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi}\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}. \qquad (12)$$

Using (11), we rewrite (10) as

$$\oint_C \int \boldsymbol{K}(\boldsymbol{x},\boldsymbol{y}) \times \boldsymbol{j}(\boldsymbol{y}) \cdot d\boldsymbol{x}dy. \qquad (13)$$

Assuming the current I is unity and carried by a loop C', the integral (13) reads

$$\ell(C,C') = \oint_C \oint_{C'} \boldsymbol{K}(\boldsymbol{x},\boldsymbol{y}) \times d\boldsymbol{y} \cdot d\boldsymbol{x}, \qquad (14)$$

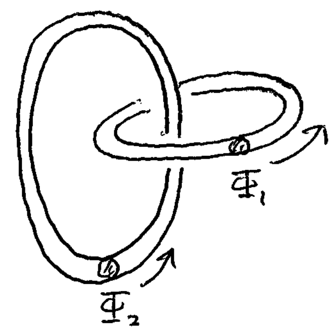
which we call the "Gauss linkage number". Following the above argument, we easily find that $\ell(C,C')$ is an integer that depends only on the linkage of two loops C and C', i.e., $\ell(C,C')$ is a homotopy invariant.

Application 2. The so-called "helicity" reduces into the constant multiple of the linkage number, when the fluxes are localized into two filaments. For a solenoidal (divergence-free) 3D vector field $\boldsymbol{u}$, we define the helicity by

$$H = \int \boldsymbol{u} \cdot (\mathrm{curl}^{-1}\boldsymbol{u})\ dx. \tag{15}$$

Suppose that $\boldsymbol{u}$ is localized in two loop filaments $\mathcal{C}$ and $\mathcal{C}'$, and they carry fluxes Φ and Φ', respectively. Then, we obtain

$$\begin{aligned} H &= \Psi\Psi' \oint_{\mathcal{C}} \left[\oint_{\mathcal{C}} \boldsymbol{K}(\boldsymbol{x},\boldsymbol{y}) \times d\boldsymbol{y} + \oint_{\mathcal{C}'} \boldsymbol{K}(\boldsymbol{x},\boldsymbol{y}) \times d\boldsymbol{y} \right] \cdot d\boldsymbol{x} \\ &\quad + \Psi\Psi' \oint_{\mathcal{C}'} \left[\oint_{\mathcal{C}} \boldsymbol{K}(\boldsymbol{x},\boldsymbol{y}) \times d\boldsymbol{y} + \oint_{\mathcal{C}'} \boldsymbol{K}(\boldsymbol{x},\boldsymbol{y}) \times d\boldsymbol{y} \right] \cdot d\boldsymbol{x} \\ &= 2\Psi\Psi'\ell(\mathcal{C},\mathcal{C}'). \end{aligned}$$



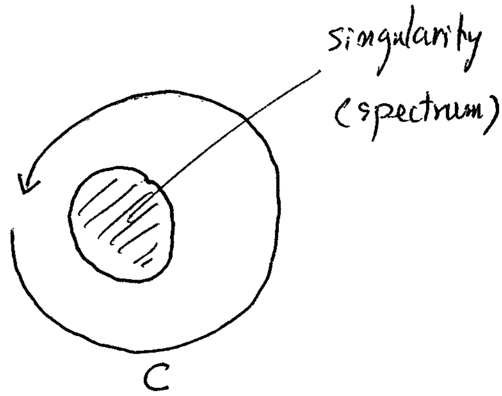
For a general (smoothly distributed) field $\boldsymbol{u}$, the helicity H gives a sum of the linkage. It also detects other geometric characteristics like twist.

D. Singularity

The Gauss linkage number (14) is defined by a singular integral. Singularities can be detected from outside by an appropriate integral. Here, we see a most beautiful example of homotopy invariant integral.

Let A be a bounded linear map. The Cauchy-Dunford integral gives an analytic function of A;

$$f(A) = \frac{1}{2\pi i} \oint_{\mathcal{C}} f(\lambda)(\lambda I - A)^{-1}\ d\lambda, \tag{16}$$



where λ is a complex number and f is a certain complex function that is analytic inside the loop $\mathcal{C}$. The loop must include whole singularities of A (which is a bounded set on the complex plane, if A is a bounded operator). We note that $(\lambda I - A)^{-1}$ is the "resolvent" operator, and its singularity is the spectrum of A.

Application 3. If we take $f(\lambda) = e^{t\lambda}$ in (16), we can define the exponential function of the operator A

$$e^{tA} = \frac{1}{2\pi i} \oint_{\mathcal{C}} e^{t\lambda}(\lambda I - A)^{-1}\ d\lambda, \tag{17}$$

which is the "inverse Laplace transform". The e^{tA} gives the solution operator to the autonomous evolution equation

$$\begin{cases} \dfrac{d}{dt}u = Au, \\ u(0) = u_0. \end{cases}$$

The solution is given by

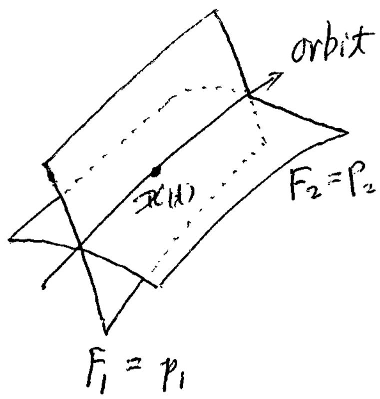
$$u(t) = e^{tA}u_0.$$

II. CONSERVATION LAW

The idea of the physics of dynamics is to find "unchanged quantities" in the system. Once one finds as many unchanged quantities (constants of motion) as the degree of freedom, the dynamics is totally understood; The system is seen as stationary. In general, however, the quest for constants of motion falls short, and in many nonlinear systems, a very small number of such quantities can be found. Even if we have a small number of such "information", they provides us a profound insight into the complex system.

A. Integrability

Dynamics of a system is represented by an orbit (curve) in the phase space, and the position on the curve is parameterized by the time t. Geometrically, a curve in any N dimensional space is defined by the intersection of surfaces.



In an N-dimensional space, we need $N' = N - 1$ independent real-number valued functions $F_j(x_1, \cdots, x_N)$ $(j = 1, \cdots, N')$ to determine a curve. The level set of each function defines an N' dimensional hyper-surface (manifold). A one-dimensional curve is formed by the intersection of these surfaces, and is obtained by solving $F_j(x_1, \cdots, x_N) = p_j$ $(j = 1, \cdots, N')$ simultaneously.

We are thus able to demonstrate that given an appropriate set of functions $F_j(x_1, \cdots, x_N)$ $(j = 1, \cdots, N')$, we can always find a one-dimensional set of points (a curve) common to them all.

These functions F_j are the constants of motion, because the orbit is included in each levelset of F_j. Hence, the analysis of the dynamics is attributed to the quest for the constants of motion.

Let us now consider the inverse problem. For a given smooth curve in an N-dimensional space, can we find an appropriate set of functions $F_j(x_1, \cdots, x_N)$ $(j = 1, \cdots, N')$ such that the intersection of their level sets coincides with the given curve? As far as we consider this problem "locally", the answer is yes. Let P be a point on the curve. We consider a neighborhood V of P where the curve looks like a straight line. Changing the coordinates to align the x_N axis to be parallel to the line, we obtain the equations $x_j = p_j$ (constant) $(j = 1, \cdots, N')$ whose simultaneous solution represents the curve locally in V.

In trying to solve the "global" problem, however, we meet the following difficulty. The global problem consists in finding the hyper-surfaces that include the curve throughout their trajectory. For a curve that moves about in a certain domain of space, such hyper-surfaces must have a highly complicated structure. In fact, for a sufficiently complex dynamics, well-defined smooth hyper-surfaces that contain the complicated streamlines may not actually exist. This thought experiment gives us a glimpse of the pathway leading to the concept of "chaos" in dynamical systems.

B. Hamiltonian system (classical mechanics)

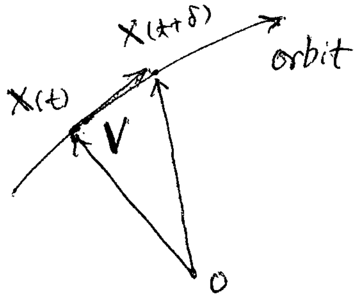
1. Hamilton's equation of motion

Let $\boldsymbol{x}$ and $\boldsymbol{p}$ denote a pair of coordinate and (canonical) momentum variables in a ν dimensional space. Although our primary interest is in the ordinary physical space ($\nu = 3$), we will develop the formalism in a more general space. We begin with the Hamilton's equations of motion which can be jointly written as

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \partial_{\boldsymbol{p}}\mathcal{H} \\ -\partial_{\boldsymbol{x}}\mathcal{H} \end{pmatrix}, \tag{18}$$

where $\mathcal{H}$ is the Hamiltonian (assumed to be a smooth real function of $\boldsymbol{x}$, $\boldsymbol{p}$ and t), and $\partial_{\boldsymbol{x}}$ and $\partial_{\boldsymbol{p}}$ are respectively the gradients with respect to $\boldsymbol{x}$ and $\boldsymbol{p}$. With the notation

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} \partial_{\boldsymbol{p}}\mathcal{H} \\ -\partial_{\boldsymbol{x}}\mathcal{H} \end{pmatrix}. \tag{19}$$



Hamilton's equation (18) reads as the streamline equation in 2ν dimensional canonical phase space,

$$\frac{d}{dt}\boldsymbol{X} = \boldsymbol{V}(\boldsymbol{X}, t). \tag{20}$$

Application 4. For a given magnetic field $\boldsymbol{B}(\boldsymbol{x})$, the field line equation is

$$\frac{d}{d\tau}\boldsymbol{x} = \boldsymbol{B}(\boldsymbol{x}), \tag{21}$$

where τ is an abstract variable that indicates the position on the streamline. This τ and the time t are totally different.

We consider the simpler case, in which $\boldsymbol{B}(\boldsymbol{x})$ is homogeneous with respect to z, one of the trio forming the Cartesian coordinates x-y-z. Because of this symmetry, and of the divergence-free property of $\boldsymbol{B}(\boldsymbol{x})$, we can write $\boldsymbol{B}(\boldsymbol{x})$ in the form

$$\boldsymbol{B}(x, y) = \nabla\psi \times \nabla z + B_z \nabla z \tag{22}$$

where $\psi(x, y)$ and $B_z(x, y)$ are two scalar functions.

From (22), we deduce

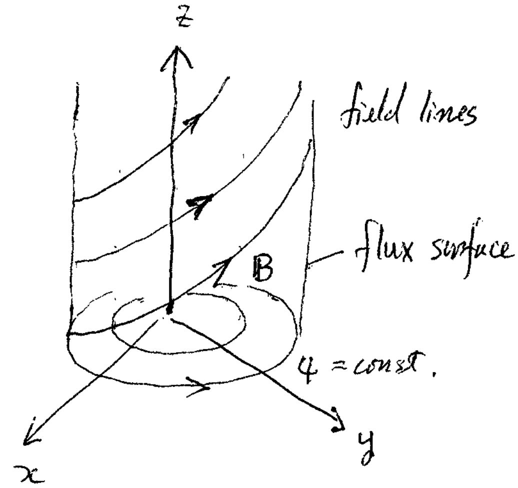
$$\boldsymbol{B} \cdot \nabla\psi = (\nabla\psi \times \nabla z) \cdot (\nabla\psi) + B_z(\nabla z) \cdot (\nabla\psi) \equiv 0, \tag{23}$$

implying that the vector field $\boldsymbol{B}(\boldsymbol{x})$ is tangential to a level set of the function $\psi(x, y)$. The level set in the x-y-z space of $\psi(x, y)$ is a column whose section by an

x-y plane is the contour curve of $\psi(x, y)$ in the plane. Hence, ψ is constant along the streamline of the flow (magnetic filed) $\boldsymbol{B}(\boldsymbol{x})$, viz., ψ is a constant of motion of the dynamics defined by $\boldsymbol{B}(\boldsymbol{x})$.

We have, thus, shown that every streamline of an incompressible stationary flow (such as a magnetic field) with an "ignorable coordinate" must be integrable. This important result can also be derived as a straight forward implication of Hamiltonian dynamics. For $\boldsymbol{B}(\boldsymbol{x})$ of (22), the x and y components of the streamline equation become

$$\begin{cases} \dfrac{dx}{dt} = \partial_y \psi, \\[2ex] \dfrac{dy}{dt} = -\partial_x \psi, \end{cases}$$

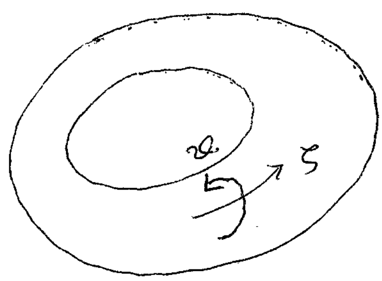


which read as Hamilton's equations of motion with the coordinate x, the momentum y, and the Hamiltonian $\psi(x, y)$. In this two dimensional phase space, one integral of motion suffices for integrability, and we have already shown that the Hamiltonian ψ is a constant of motion. The integral surfaces (curve) of this system are the level sets of ψ.

Application 5. Consider a stationary incompressible flow $\boldsymbol{B}(\boldsymbol{x})$ in a three dimensional toroidal domain Ω, which is a general three-dimensional flow (magnetic field) without any symmetry.

Such a vector field can be represented in the form [1]

$$\boldsymbol{B} = \nabla\Psi \times \nabla\zeta - \nabla\chi \times \nabla\vartheta, \tag{24}$$



where ζ and ϑ are, respectively, the appropriate toroidal and poloidal angles, and Ψ and χ are scalar functions of ζ, ϑ and ξ (a radial coordinate). Since

$$\nabla\Psi \times \nabla\zeta - \nabla\chi \times \nabla\vartheta = \nabla \times (\Psi\nabla\zeta - \chi\nabla\vartheta),$$

Ψ and χ are really nothing but the toroidal and poloidal components of the vector potential.

If we replace the radial coordinate ξ by the function χ, and assume that the Jacobian

$$\frac{D(\chi, \vartheta, \zeta)}{D(x, y, z)} = \nabla\chi \cdot (\nabla\vartheta \times \nabla\zeta) \neq 0$$

(with the implication that the streamline does not turn back in circulating the toroidal domain), the streamline equations read

$$\begin{cases} \dfrac{d\vartheta}{d\zeta} = \dfrac{\nabla\vartheta \cdot \boldsymbol{B}}{\nabla\zeta \cdot \boldsymbol{B}}, \\[2ex] \dfrac{d\chi}{d\zeta} = \dfrac{\nabla\chi \cdot \boldsymbol{B}}{\nabla\zeta \cdot \boldsymbol{B}}. \end{cases} \tag{25}$$

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[1] Z. Yoshida, Phys. Plasmas 1 (1994) 208.

After using (24) to evaluate the right-hand side of (25), we find

$$
\begin{aligned}
\nabla\vartheta\cdot\boldsymbol{B} &= \nabla\vartheta\cdot(\nabla\Psi\times\nabla\zeta) = -\nabla\chi\cdot(\nabla\vartheta\times\nabla\zeta)(\partial\Psi/\partial\chi),\\
\nabla\chi\cdot\boldsymbol{B} &= \nabla\chi\cdot(\nabla\Psi\times\nabla\zeta) = \nabla\chi\cdot(\nabla\vartheta\times\nabla\zeta)(\partial\Psi/\partial\vartheta),\\
\nabla\zeta\cdot\boldsymbol{B} &= -\nabla\chi\cdot(\nabla\vartheta\times\nabla\zeta).
\end{aligned}
$$

Plugging these relations in (25), we obtain the set of canonical equations

$$
\begin{cases}
\dfrac{d\vartheta}{d\zeta} = \partial_\chi\Psi,\\[2ex]
\dfrac{d\chi}{d\zeta} = -\partial_\vartheta\Psi,
\end{cases}
\tag{26}
$$

for which the toroidal angle ζ parallels time, the poloidal angle ϑ, is the angle coordinate, χ, mimics the canonical momentum (action variable), and $\Psi = \Psi(\chi,\vartheta,\zeta)$ plays the role of the Hamiltonian.

For a general flow (magnetic field), the Hamiltonian $\Psi(\vartheta,\chi,\zeta)$ depends on all three of its arguments, i.e, there is no ignorable coordinate and hence no constant of motion. The Hamiltonian system (26), then, is not integrable. If Ψ were independent, say, of the toroidal angle ζ, a constant of motion will emerge, and the system (26) becomes integrable. In the next section, we show an example of a non-integrable streamline.

2. Liouville equation

Through Hamiltonian dynamics, an arbitrary field $u(\boldsymbol{X},t)$ evolves as

$$
\begin{aligned}
\frac{d}{dt}u(\boldsymbol{X},t) &= \partial_t u + (\boldsymbol{V}\cdot\nabla)u\\
&= \partial_t u + (\partial_{\boldsymbol{p}}\mathcal{H})\cdot(\partial_{\boldsymbol{x}}u) - (\partial_{\boldsymbol{x}}\mathcal{H})\cdot(\partial_{\boldsymbol{p}}u)\\
&= \partial_t u + \{\mathcal{H},u\}\,,
\end{aligned}
\tag{27}
$$

where, the bilinear operator,

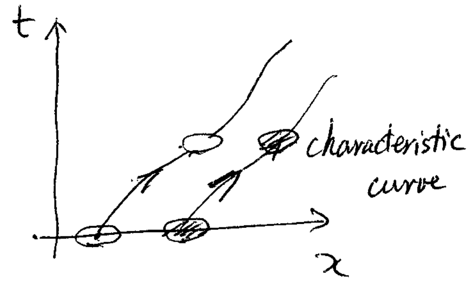
$$
\begin{aligned}
\{\mathcal{H},u\} &= (\partial_{\boldsymbol{p}}\mathcal{H})\cdot(\partial_{\boldsymbol{x}}u) - (\partial_{\boldsymbol{x}}\mathcal{H})\cdot(\partial_{\boldsymbol{p}}u)\\
&= \sum_{j=1}^{\nu}(\partial_{p_j}\mathcal{H})(\partial_{x_j}u) - \sum_{j=1}^{\nu}(\partial_{x_j}\mathcal{H})(\partial_{p_j}u).
\end{aligned}
\tag{28}
$$

is the well-known Poisson bracket. When $u(\boldsymbol{X},t)$ is constant along every streamline, it satisfies the transport equation

$$
\partial_t u + \{\mathcal{H},u\} = 0. \tag{29}
$$

This is the Liouville equation, which gives the intrinsic rate of change of a "constant of motion (first integral, constant along the streamlines)" of the Hamiltonian dynamics.

Hamilton's equation of motion (18) is said the "characteristic ordinary differential equation (ODE)" of the Liounville equation (29) which is a hyperbolic partial differential equation (PDE). Solving (18) to obtain every orbit starting from an arbitrary initial position, which is called the "characteristic curve", we can find the solution of the transport equation (29). On the contrary, if we find sufficient number of the constants of motion from (29), we can integrate the equation of motion (18). Hence, the ODE (18) and the PDE (29) are equivalent.



Application 6. In Sec. II.E.1, we will see some examples of nonlinear Liouville equations that describe transport of vortex.

3. Hamilton-Jacobi equation

We have another important PDE that is equivalent to Hamilton's equation of motion (18). Here we assume that the Hamiltonian $\mathcal{H}$ does not include t. The Hamilton-Jacobi equation is

$$\partial_t \mathcal{S} + \mathcal{H}(\boldsymbol{x}, \nabla \mathcal{S}) = 0. \tag{30}$$

This equation is derived by the following consideration. To integrate (18), we want to find a canonial transform such that $\mathcal{H} \to \hat{\mathcal{H}} \equiv 0$, which stationarize the dynamics. The transform relations are, with a generating function $\mathcal{S}(\boldsymbol{x}, \hat{\boldsymbol{p}}, t)$,

$$\hat{\mathcal{H}} = \mathcal{H} + \partial_t \mathcal{S} \ (\equiv 0), \tag{31}$$
$$\boldsymbol{p} = \partial_x \mathcal{S}, \tag{32}$$
$$\hat{\boldsymbol{x}} = -\partial_{\hat{p}} \mathcal{S}. \tag{33}$$

Using (32) in (31) yields the Hamilton-Jacobi equation (30).

Because $\mathcal{H}$ does not include t, we may write

$$\mathcal{S} = S(\boldsymbol{x}, \hat{\boldsymbol{p}}) - \omega t \quad (\omega = \text{constant}),$$

and (30) reduces into

$$\mathcal{H}(\boldsymbol{x}, \nabla \mathcal{S}) = \omega. \tag{34}$$

This is a classical-mechanical energy spectrum equation.

We note that (30) and (34) are PDEs in the space-time ($\boldsymbol{x}$-t), so that the solutions are given as functions $\mathcal{S}(\boldsymbol{x}, t)$ and $S(\boldsymbol{x}, t)$. If these solutions can be parameterized by $\hat{\boldsymbol{p}} = \boldsymbol{k}$ (constants) as

$$\mathcal{S}(\boldsymbol{x}, t; \boldsymbol{k}), \quad S(\boldsymbol{x}, t; \boldsymbol{k}),$$

these solutions are called "complete solutions". As shown in (31), they give a canonical transform to integrate Hamilton's equation of motion (18).

Because of the general nonlinearity of (30), the ODE $d\boldsymbol{x}/dt = \partial_p \mathcal{H}$ does not close (compare with the Liouville equation), because the right-hand side may still include $\boldsymbol{p} = \partial_x \mathcal{H}$. Hence, we need the ODE to determine the $\boldsymbol{p}$, that is second part of Hamilton's equation.

In optics, the $\mathcal{S}$ is the phase of a wave. The unitary transform

$$\psi = e^{i\mathcal{S}(x,t;k)} = e^{iS(x,t;k)-i\omega t} \tag{35}$$

solves the classical Schödinger equation

$$i\partial_t \psi = \mathcal{H}(\boldsymbol{x}, \boldsymbol{k})\psi. \tag{36}$$

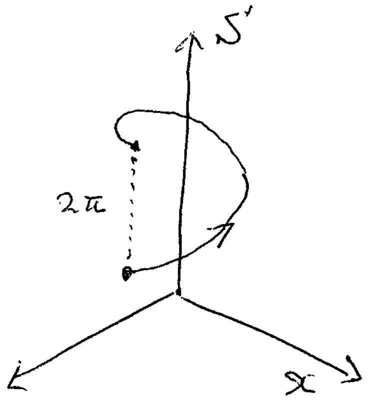
In this expression, we have a relation between ω and $\boldsymbol{k}$ (good quantum number), which is the "dispersion relation".

In general, we do not have the complete solution to (30) or (34). This "non-integrable" case is the so-called "quantum chaos".

In writing (35), the phase $\mathcal{S}$ is regarded as a 2π-modulo function, and we may consider a multiple-valued $\mathcal{S}$. Hence, a loop integral

$$\frac{1}{2\pi} \oint_{\Gamma} \nabla \mathcal{S} \cdot d\boldsymbol{x} \tag{37}$$

may take a certain integer value, which is called a "topological charge". The loop encircles the singularity of $\mathcal{S}$, where $\nabla\mathcal{S}$ is not defined. This is the phase singularity, the origin of an "angular momentum".



Let us assume

$$\mathcal{H}(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2}p^2 + \phi(\boldsymbol{x}).$$

Denoting $\nabla\mathcal{S}(\boldsymbol{x}, t)$ $(= \boldsymbol{p}) = \boldsymbol{v}$ and assuming it is a "flow velocity", the gradient of (30) reads as

$$\partial_t \boldsymbol{v} + \nabla \left(\frac{1}{2}v^2 + \phi \right) = 0, \tag{38}$$

which is the standard ideal irrotional fluid equation (the potential energy ϕ parallels the pressure). Kelvin's circulation theorem warrants the conservation of the topological charge (37) as far as the $\boldsymbol{v} = \nabla\mathcal{S}$ is well defined.

Application 7. Let us consider a nonlinear Schrödinger equation (in some generalized version)

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + f(|\psi|^2)\psi. \tag{39}$$

Writing $\psi = ae^{i\mathcal{S}}$ (a and $\mathcal{S}$ are real functions) and $|\psi|^2 = a^2 = \rho$, (39) reads

$$\partial_t \mathcal{S} + \frac{1}{2}|\nabla\mathcal{S}|^2 + f(\rho) + \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = 0, \tag{40}$$

$$\partial_t \rho + \nabla \cdot [(\nabla\mathcal{S})\rho] = 0. \tag{41}$$

The gradient of (40) resembles (38). The (41) describes the conservation of the energy density ρ.

C. Von Neumann theorem (quatum mechanics)

Let us consider an abstract Schödinger equation

$$i\partial_t \psi = \mathcal{H}\psi. \tag{42}$$

Here $\mathcal{H}$ is a Hermitian operator in a function space; compare with (36). Solving the eigenvalue problem [compare with (34)]

$$(\omega I - \mathcal{H})\varphi = 0 \tag{43}$$

we can obtain the complete orthogonal basis to span the total function space (Von Neumann's theorem). This task is equivalent to find the singularity of the resolvent operator $(\omega I - \mathcal{H})^{-1}$ [see (16)]. Unlike linear algebra of finite dimension vector space, "continuous" spectrum occurs when $(\omega I - \mathcal{H})^{-1}$ exists but is not continuous.

Denoting φ_k the eigenfunction (may be singular) belonging to the spectrum ω_k, we may write

$$u = \int_{-\infty}^{+\infty} (u, \varphi_k)\varphi_k \, dk, \tag{44}$$

$$\mathcal{H}u = \int_{-\infty}^{+\infty} \omega_k (u, \varphi_k)\varphi_k \, dk, \tag{45}$$

$$e^{-it\mathcal{H}}u = \int_{-\infty}^{+\infty} e^{-it\omega_k} (u, \varphi_k)\varphi_k \, dk. \tag{46}$$

Since the eigenfunction belonging to the a continuous spectrum is a singular function, it is more appropriate to define a "projector" E_k to write

$$(\cdot, \varphi_k)\varphi_k \, dk = dE(k). \tag{47}$$

The propagator $e^{-it\mathcal{H}}$ given in (46) gives the solution of (42); for a initial condition ψ_0, the solution is

$$\psi(t) = e^{-it\mathcal{H}}\psi_0.$$

It is remarkable here that the quantum mechanical representation of the wave (42) is "integrable", if the spectrum is "point spectrum" (eigenvalues). This is the case for "trapped (or quantized)" waves. Von Neumann's theorem warrants the existence of the complete spectrum which consists of the constants of motion ω_k. However, the continuous spectrum part does not represent a stationary mode; it describes dynamical processes such as scattering and/or phase mixing.

D. Statistical mechanics and relaxed states

1. Kinetic theory and statistical distribution

Let $f(\boldsymbol{x}, \boldsymbol{p}, t)$ be the distribution function. We assume that the evolution of f obeys the Liouvill equation (Vlasov equation)

$$\partial_t f + \{H, f\} = 0, \tag{48}$$

where H is the Hamiltonian of a test particle moving in a mean field. The mean field $\boldsymbol{A}$ and ϕ containd in H must be consistent to f through Maxewll's equations.

The steady states for (48) is given by (we consider a stationary state and assume that H is independent to t)

$$\{H, f\} = 0. \tag{49}$$

Let a_j be a constant of motion, i.e., $\{H, a_j\} = 0$. The H itself a constant. Suppose that we know N of constants of motion, and consider a distribution (F is a certain smooth function)

$$f = F(a_1, \cdots, a_N). \tag{50}$$

For an arbitrary F, we easily verify that (50) solves (49). If N is equal to the degree of freedom, the system is "integrable".

The aim of this section, however, is to find a special class of solutions that are robust (rugged) against various perturbations. We invoke only a small number of constants of motion that are robust in a sense that the ensemble averages (or the total sums) of such quantities are conserved. The most robust steady state is the Boltzmann distribution

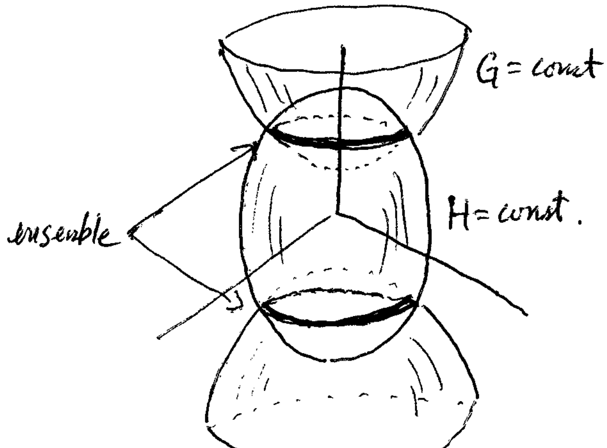
$$f = \alpha e^{-\beta H} \tag{51}$$

where α (normalization factor) and β (inverse temperature, or a Lagrange multiplier) are positive constants. We obtain this equilibrium by maximizing the entropy $-\int f(x) \log f(x) dx$ over an ensemble that is characterized by a given total energy (i.e., a constant-energy set).

If we know that another quantity G is conserved (in an ensemble average sense), we cannot maximize the entropy on a constant-energy set. With restricting the totals of H and G, we obtain

$$f = \alpha e^{-\beta H - \gamma G}, \tag{52}$$

where γ is the second Lagrange multiplier. Including some additional constants of motion, we can obtain an equilibrium (maximum entropy solution) that is slightly more restricted than the Boltzmann equilibrium, but is still robust as far as the additional constraints are valid.



2. Momentum conservation

A symmetry of the system yields a constant of motion that is the canonical momentum corresponding to the ignorable coordinate. Suppose that the Lagrangean L is independent of a coordinate x_0, as well as t. Then, $p_0 = \partial L / \partial x_0'$ is conserved ($'$ is the time derivative). With an arbitrary constant c, we define

$$\hat{H} = H - cp_0, \tag{53}$$

and consider a distribution

$$f = \alpha e^{-\beta\hat{H}} = \alpha e^{-\beta(H-cp_0)}. \tag{54}$$

As far as the collision-less dynamics is concerned, this f solves (49); see (52). Moreover, this solution has the following important meaning, and the physical meaning of the parameter c becomes clear.

When we discuss a distribution function f, we consider an ensemble of particles, which is characterized by the sum of the Hamiltonian over the all particles. We invoke the conservation of the total energy, but not the energy of each particle. We apply the same framework for the momentum p_0 in (54). When we consider a "relaxed state", we give up the conservation of individual H or p_0, while we demand the conservation of the totals of these quantities. Then, the physical meaning of $\hat{H}$ becomes essential. Indeed, we can interpret $\hat{H}$ as the Hamiltonian in a moving frame, and hence, $f = \alpha e^{-\beta\hat{H}}$ is an invariant of the collision operator (the average momentum is unchanged by collisions). This robustness of p_0 warrants the use of p_0 in determining the ensemble.

Let us first revisit the change of variables in general inhomogeneous coordinate transform. Let $\boldsymbol{U}$ be a certain temporary-constant velocity field. We write the velocity $\boldsymbol{v}$ of the laboratory frame as

$$\boldsymbol{v} = \tilde{\boldsymbol{v}} + \boldsymbol{U}, \tag{55}$$

and set $\tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{v}}$. The Lagrangean of a charged particle (q: charge, m: mass) can be written as

$$L = \frac{m}{2}\left|\tilde{\boldsymbol{v}} + \boldsymbol{U}\right|^2 + q\left(\tilde{\boldsymbol{v}} + \boldsymbol{U}\right)\cdot\boldsymbol{A} - q\phi.$$

The canonical momentum is

$$\tilde{\boldsymbol{p}} = \frac{\partial L}{\partial \tilde{\boldsymbol{v}}} = m\left(\tilde{\boldsymbol{v}} + \boldsymbol{U}\right) + q\boldsymbol{A} = m\tilde{\boldsymbol{v}} + q\tilde{\boldsymbol{A}},$$

and the Hamiltonian reads as

$$\tilde{H} = \frac{1}{2m}\left|\tilde{\boldsymbol{p}} - q\tilde{\boldsymbol{A}}\right|^2 - \frac{m}{2}U^2 + q\tilde{\phi}, \tag{56}$$

where

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} + \frac{m}{q}\boldsymbol{U}, \tag{57}$$

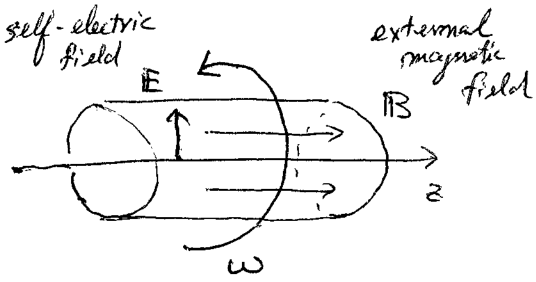
$$\tilde{\phi} = \phi - \boldsymbol{U}\cdot\boldsymbol{A}. \tag{58}$$

The effective vector potential $\tilde{\boldsymbol{A}}$ includes an additional term that yields the Coriolis force. The scalar potential $\tilde{\phi}$ has received the (nonrelativistic) Lorentz transform. In (56), $-mU^2/2$ is the centrifugal potential.

The transform of the Hamiltonian and the momentum can be written as

$$
\begin{aligned}
\tilde{H} &= H - \boldsymbol{U} \cdot (m\boldsymbol{v} + q\boldsymbol{A}) \\
&= H - \boldsymbol{U} \cdot \boldsymbol{p}, && (59) \\
\tilde{\boldsymbol{p}} &= \boldsymbol{p} \equiv m\boldsymbol{v} + q\boldsymbol{A}. && (60)
\end{aligned}
$$

Application 8. Let us consider an axisymmetric cylindrical single-species plasma confined in a homogeneous magnetic field ($\boldsymbol{B} = B\boldsymbol{e}_z$). We assume that the density is small so that the magnetic field produced by the internal current is negligibly small. Hence, we solve (49) simultaneously with the Poisson equation of the electrostatic potential. By the symmetry $\partial/\partial\theta = 0$, the canonical angular momentum $p_\theta = mrv_\theta + qrA_\theta$ is conserved. Hence, $\hat{H} = H - \omega p_\theta$ (ω = constant) is a constant of motion. This $\hat{H}$ is the Hamiltonian in a rigid rotation frame. Indeed, setting $\boldsymbol{U} = \omega r\boldsymbol{e}_\theta$ (ω is the angular velocity of the rigid rotation), (56) reads



$$
\begin{aligned}
\tilde{H} &= \frac{m}{2}\tilde{v}^2 - \frac{m}{2}U^2 - q\boldsymbol{U} \cdot \boldsymbol{A} + q\phi \\
&= \frac{m}{2}\tilde{v}^2 - \frac{m}{2}(r\omega)^2 - qr\omega A_\theta + q\phi \\
&= \hat{H}.
\end{aligned}
$$

The equilibrium $f(\hat{H}) = \alpha e^{-\beta\hat{H}}$ represents a drift Maxwellian with a constant angular velocity ω.

The potential ϕ in the Hamiltonian must be determined consistently to the field equation. Here, we seek a solution that has no spatial inhomogeneity inside the plasma, i.e.,

$$
\hat{H} = \frac{m}{2}\tilde{v}^2 = \frac{1}{2m}\left|\tilde{p} - q\tilde{A}\right|^2. \qquad (61)
$$

The vector potential $\boldsymbol{A}$ is an externally given function, because we neglect the magnetic field produced by the electron flow. Let us consider a infinitely long plasma column. The vector potential for the homogeneous longitudinal magnetic field $\boldsymbol{B} = B\boldsymbol{e}_z$ is given by $\boldsymbol{A} = (rB/2)\boldsymbol{e}_\theta$. For the distribution function with the Hamiltonian (61), the density n is constant for the radius $r < a$. Then, the potential is $\phi = -(qn/4\epsilon_0)r^2$. To satisfy (61), we demand

$$
\omega^2 + \omega_c\omega + \frac{1}{2}\omega_p^2 = 0, \qquad (62)
$$

where $\omega_c = qB/m$ and $\omega_p^2 = nq^2/m\epsilon_0$. This is the equilibrium condition for the non-neutral plasma column.

Similar treatment for electromagnetic neutral plasma reads to the "Harris sheat" (slab geometry) and "Bennet pinch" (cylindrical geometry) solutions.

E. Navier-Stokes system

1. Navier-Stokes equation and cousins

The Navier-Stokes equation is a paradigm of nonlinear nonintegrable system.

- An incompressible flow $\boldsymbol{v}$ in a three dimensional domain (we assume a bounded domain Ω with a smooth boundary Γ) obeys

$$\begin{cases} \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \epsilon \Delta \boldsymbol{v}, \\ \nabla \cdot \boldsymbol{v} = 0, \end{cases} \tag{63}$$

where p is the pressure and ϵ is the kinematic viscosity. Boundary conditions are

$$\boldsymbol{v}|_\Gamma = 0, \quad p|_\Gamma = \text{given}. \tag{64}$$

- In the ideal limit, we assume $\epsilon = 0$ in (63);

$$\begin{cases} \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p, \\ \nabla \cdot \boldsymbol{v} = 0 \end{cases} \tag{65}$$

Then, the boundary conditions are

$$\boldsymbol{n} \cdot \boldsymbol{v}|_\Gamma = 0, \quad p|_\Gamma = \text{given}. \tag{66}$$

- The ideal vortex equation is derived by taking the curl of (65). Denoting the vorticity $\nabla \times \boldsymbol{v} = \boldsymbol{W}$, we obtain

$$\partial_t \boldsymbol{W} - \nabla \times (\boldsymbol{v} \times \boldsymbol{W}) = 0. \tag{67}$$

We impose a boundary condition, as well as a circulation condition (resulting from Kelvin's theorem)

$$\boldsymbol{n} \cdot \boldsymbol{v}|_\Gamma = 0, \quad \int_\Sigma \boldsymbol{n} \cdot \boldsymbol{W} \, ds = \text{given}, \tag{68}$$

where Σ is a cross section of Ω.

- In two dimension, we denote W $(= -\Delta\Phi)$ and Φ the vorticity and the Hamiltonian (stream function) of an incompressible flow $\boldsymbol{v} = {}^t(\partial_y \Phi, -\partial_x \Phi)$. The vortisity equation (67) reduces into

$$\partial_t W + \{\Phi, W\} = 0. \tag{69}$$

The Poisson bracket is

$$\begin{aligned} \{a, b\} &= (\partial_y a)(\partial_x b) - (\partial_x a)(\partial_y b) \\ &= -(\nabla a \times \nabla b) \cdot \boldsymbol{e}_z, \end{aligned}$$

where $\boldsymbol{e}_z = \nabla x \times \nabla y$. The circulation of the flow must be conserved (Kelvin's theorem);

$$\oint_\Gamma \boldsymbol{n} \cdot \nabla\Phi \, d\gamma = K \text{ (given constant)}. \tag{70}$$

We assume that the flow $\nabla\Phi \times \boldsymbol{e}_z$ is confined in Ω, demanding

$$\Phi|_\Gamma = C \text{ (unknown constant)}. \tag{71}$$

- The Hasegawa-Mima equation is a close cousin of (69). To represent the drift wave in a plane perpendicular to a homogeneous magnetic field, we consider a generalized vorticity

$$W = \Phi - \Delta\Phi,$$

which obeys (69)-(71).

2. *Constants of motion* ($\epsilon = 0$)

The "energy" and "enstrophy" are the most important quantities in the study of fluid mechanics. They are, respectively, defined by

$$H_0 = \|\boldsymbol{v}\|^2, \quad H_1 = \|W\|^2.$$

Here we use the conventional notation

$$\|a\| = (a, a)^{1/2}, \quad (a, b) = \int_\Omega a \cdot \bar{b} \; dx.$$

The ideal incompressible flow obeying (65) conserves the energy H_0. The change of the enstrophy is given by

$$\frac{d}{dt} H_1 = -(\boldsymbol{W}, (\boldsymbol{W} \cdot \nabla)\boldsymbol{v}).$$

In two dimensional case, the right-hand side (which is called the vortex-stretching effect) vanishes, and H_1 is also conserved. This fact can be directly derived by taking the inner product of (69) with W. The constancy of H_1 is the most essential characteristic of two-dimensional flow, distinguishing three-dimensional flows.

3. *A priori estimates* ($\epsilon > 0$)

If a finite viscosity is included, the above-mentioned constants of motion receive a dissipation. In two dimension case, we observe

$$H_0(t) = H_0(0) - 2\epsilon \int_0^t \|\nabla \times \boldsymbol{v}(t')\|^2 \; dt',$$

$$H_1(t) = H_1(0) - 2\epsilon \int_0^t \|\nabla W(t')\|^2 \; dt'$$

which imply monotonic decreases of both quatities.

In three dimension case, the energy H_0 satisfies the same equation, while the enstrophy is estimated by

$$H_1(t) = H_1(0) - \int_0^t 2\epsilon \|\nabla \times \boldsymbol{W}\|^2 - (\boldsymbol{W}, (\boldsymbol{W} \cdot \nabla)\boldsymbol{u}) \; dt' \tag{72}$$

4. Existence theorem

The existence of a smooth solution to the three-dimensional Navier-Stokes equation is a long standing open problem of mathematics. The key to resolve this problem is the derivation of a bound for the enstrophy H_1. As we have seen in the previous section, the two dimension case is very different from the three dimension case, because we have a bound for H_1.

A possible approach of finding a solution is to write (63) in the form of

$$v(t) = T_t v_0 + \int_0^t T_{t-s}[-(v \cdot \nabla)v]\, ds, \qquad (73)$$

where $T_t = e^{-tA}$ with $A = -\mathcal{P}\epsilon\Delta$ (Stokes operator; the projection of $-\epsilon\Delta$ of onto the space of solenoidal fields). Denoting the right-hand side of (73) as $f(v(t))$, we seek for the fixed point of the map f; see *Application 1.* For this task, we need bounds for H_0 and H_1. For a short enough time $t < T$, we can manipulate (72) to derive a limitation for H_1, and we can conclude the existence of a temporally localized solution. It is still an open question whether the T can be extended to infinity.

F. Relaxed states in fluid mechanics

We have seen that the constants of motion (and their variations in a dissipative system) are used to show the existence of solution. We can derive from these quantities a more interesting information about the dynamics.

Let us consider a two dimensional flow with a small viscosity ϵ. As a general tendency of nonlinear fluid motion, the mixing effect yields smaller length scales in the vorticity distribution. Let L be the characteristic length scale of vortices. We estimate $\|\nabla W\|^2 \approx L^{-2}\|W\|^2$. Through the length scale reduction, we find that H_1 receives a stronger damping than H_0. This expectation leads to the concept of "selective dissipation" of a certain constant of motion, relative to the other constants, through a weakly dissipative nonlinear process.

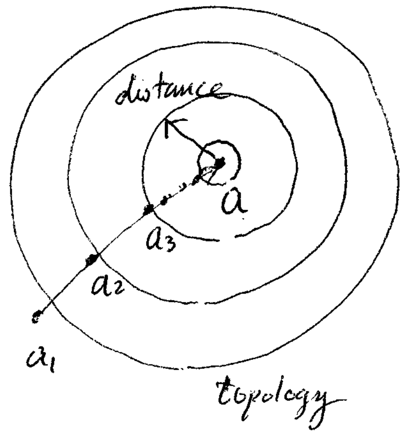
If we assume that H_1 achieves the minimum while H_0 is approximately conserved, the relaxed state may be given as the minimizer of a functional

$$F(v) = H_1(v) - \mu H_0(v). \qquad (74)$$

This problem will be solved in the next section.

III. TOPOLOGY (ANALYTIC THEORY)

The topology in analysis is the concept to quantify the distances among different points. The simplest example is the Euclidian norm that measures the distance between two points in the conventional metric. The concept can be much generalized by introducing an abstract axiom to distinguish whether two points are within a "neighborhood" or not. In many physics applications, the topology induced by a "norm" plays essential roles. In a finite dimension vector space, every norm, such as the Euclidian norm or the sup norm, defines the same topology. However, in an infinite dimension vector space, such as a function space, one can define different topologies depending on the choice of norm. This fact is the central theme of functional analysis. In this section, we study the topology of function spaces, and analyze structures and stability of rather complex nonlinear systems with the help of conservation laws.



A. Topological vector space

Let $\{a_j\}$ be a sequence of points in a vector space (possibly infinite dimension) V. To see the convergence

$$\lim_{j\to\infty} a_j = a \tag{75}$$

we need a measure to detect the distance. Here we consider a "norm" $\|\cdot\|$, and define (75) by

$$\lim_{j\to\infty} \|a_j - a\| = 0. \tag{76}$$

In many theories, we normally demand that any Cauchy sequence $\{a_j\}$ $[\lim_{j\to\infty} \|a_k - a_j\| = 0 \ (k > j)]$ must converge. If the vector space V endowed with the norm $\|\cdot\|$ satisfies this condition, we say the space V is *complete*, and call such a vector space a "Banach space".

When the norm is defined by an innerproduct

$$\|a\| = (a, a)^{1/2},$$

and if the space is complete, we call V a "Hilbert space".

Let us consider two different norms $\|\cdot\|_s$ and $\|\cdot\|_w$ such that

$$\|a\|_s \geq c\|a\|_w \quad (\forall a \in V) \tag{77}$$

with some c (> 0). Then, we say that $\|\cdot\|_s$ is coercive with respect to $\|\cdot\|_w$. Obviously, convergence with respect to the norm $\|\cdot\|_s$ warrants convergence in $\|\cdot\|_w$. In this sense, the topology induced by $\|\cdot\|_s$ is stronger than that by $\|\cdot\|_w$.

The following Hilbert spaces are frequently used.

- Lebesgue space $L^2(\Omega)$: For (vector or scalar/real or complex) Lebesgue-measurable functions on $\Omega \subseteq \mathbf{R}^N$, we define

$$(a, b) = \int_\Omega a \cdot \bar{b} \, dx$$

with the Lebesgure integral. The totality of functions such that $\|a\| = (a, a)^{1/2} < \infty$ is denoted by $L^2(\Omega)$, and called the Lebesgue space.

- Sobolev space $H^n(\Omega)$: We denote

$$D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}, \quad [\alpha] = \alpha_1 + \cdots + \alpha_N.$$

We define

$$\langle a, b \rangle_n = \sum_{[\alpha] \leq n} (D^\alpha a, D^\alpha b).$$

The totality of functions such that $\|a\|_{H^n} = \langle a, a \rangle_n^{1/2} < \infty$ is denoted by $H^n(\Omega)$, and called the Sobolev space.

Obviously $H^{n+1}(\Omega) \subset H^n(\Omega)$. The norm $\| \cdot \|_{H^n}$ is *not* coercive in $H^{n+1}(\Omega)$.

B. Variational principle

The coerciveness represents the preciseness of the sight defined by the topology. Here, we see some interesting examples where the topology of vector space plays an essential role in variational principle.

1. *Well-posed variational principle*

Let Ω be a bounded domain in $\mathbf{R}^N$ with a smooth boundary Γ. Let us consider two functionals

$$\mathcal{G}(u) = \int_\Omega |\nabla u(x)|^2 dx, \quad \mathcal{H}(u) = \int_\Omega |u(x)|^2 dx$$

with a boundary condition $u|_\Gamma = 0$.

We seek for a minimizer of $\mathcal{G}(u)$ with a constraint $\mathcal{H}(u) = 1$. This is a well-posed problem. Indeed, if Ω is bounded and if $u|_\Gamma = 0$, we have the Poincaré inequality

$$\|\nabla u\| \geq c\|u\|$$

with some positive constant c (determined by the size of Ω). Hence, $\mathcal{G}(u) \geq c'\|u\|_{H^1}^2$ (coercive). The minimization sequence of $\mathcal{G}(u)$ (denoted by $\{u_n\}$) is, thus, bounded in a space $H_0^1(\Omega) = \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma\}$. This $\{u_n\}$ is compactly embedded in $L^2(\Omega)$, and hence, there exists a subsequence of $\{u_n\}$ that converges to a minimizer in the topology of $L^2(\Omega)$.

The minimizer is found by the corresponding variational principle

$$\delta[\mathcal{G}(u) - \lambda\mathcal{H}(u)] = 0 \tag{78}$$

(λ is a Lagrange multiplier). The Euler-Lagrange equation reads as an eigenvalue problem $-\Delta u = \lambda u$, where λ is an eigenvalue of the Laplacian $-\Delta$ with the abovementioned boundary condition (we easily find $\lambda > 0$). Let λ_j be an eigenvalue and φ_j be the corresponding normalized eigenfunction ($\|\varphi_j\|^2 = 1$). With setting $u = a\varphi_j$, and demanding $\mathcal{H}(u) = 1$, we obtain $a = 1$ and $\mathcal{G}(u) = \lambda_j$. Hence, λ_j must be the smallest eigenvalue λ_1.

2. Ill-posed variational principle

Next, we consider a reversed problem and try to find a minimizer of $\mathcal{H}(u)$ with restricting $\mathcal{G}(u) = 1$. This is an ill-posed problem, because it assumes a constraint using the functional $\mathcal{G}$ that is not continuous in the topology of $L^2(\Omega)$ (the map to the boundary value $u|_\Gamma$ neither is continuous); [2] the constraints are not detectable in the minimization sequence. If one observe the minimization process in the topology of $H^1(\Omega)$, the target functional $\mathcal{H}(u)$, that is equivalent to the norm of $L^2(\Omega)$, is not coercive, and hence, its minimization sequence may not be bounded in the topology of $H^1(\Omega)$.

For pathological analysis, let us proceed with formal calculations. The Euler-Lagrange equation seems to be $-\Delta u = \mu^{-1}u$. Let $\mu^{-1} = \lambda_j$ (an eigenvalue of $-\Delta$), and $u = a\varphi_j$. The condition $\mathcal{G}(u) = 1$ yields $a = \lambda_j^{-1/2}$, and $\mathcal{H}(u) = 1/\lambda_j$. Hence, the minimum of $\mathcal{H}(u)$ is achieved by the largest eigenvalue that is unbounded, viz., $\inf \mathcal{H}(u) = 0$ and the minimizer is $\lim_{\lambda_j\to\infty} \lambda_j^{-1/2}\varphi_j = 0$ that is nothing but the minimizer of $\mathcal{H}(u)$ without any restriction. The constraint $\mathcal{G}(u) = 1$ does not work in this minimization problem.

These examples teach that constraints must be robuster than the target functional. This is not the case when the constraint includes higher-order derivatives relative to the target, because an infinitesimal perturbation with a small length scale can contribute any value to the constraint.

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[2] J.L. Lions and E. Magenes, *Non-Homegeneous Boundary Value Problems and Applications I*, Springer Verlag, Berlin, 1970.

C. Beltrami fields (two dimensional)

Let us study the relaxed state model of two dimensional vortex dynamics (Sec. II.F) using a variational principle. The equation was [see (69)]

$$\partial_t W + \{\Phi, W\} = 0, \tag{79}$$

with the circulation and boundary conditions

$$\oint_\Gamma \boldsymbol{n} \cdot \nabla\Phi \, d\gamma = K \text{ (given constant)}, \tag{80}$$

$$\Phi|_\Gamma = C \text{ (unknown constant)}. \tag{81}$$

The general stationary solution (equilibrium flow) is given by $\{\Phi, W\} = 0$ that implies $W = w(\Phi)$ with a certain smooth function w. Amongst them, the simplest nontrivial equilibrium is the "Beltrami flow" defined by

$$-\Delta\Phi (= W) = \mu\Phi \quad (\mu = \text{real constant}). \tag{82}$$

The Beltrami equation (82) with the circulation and boundary conditions (80)-(81) reads as an inhomogeneous equation (writing $\Phi = \varphi + C$, C is a certain constant)

$$\begin{aligned} &(-\Delta - \mu)\varphi = \mu C, \\ &\qquad \varphi|_\Gamma = 0, \quad \oint_\Gamma \boldsymbol{n} \cdot \nabla\varphi \, d\gamma = K. \end{aligned}$$

If μ is the eigenvalue of the Laplacian $-\Delta$ with the Dirichlet boundary condition, we demand $C = 0$ to obtain a solution. Otherwise, with $C \neq 0$, we have a solution $\varphi = -\mu(\Delta+\mu)^{-1}C$. The constant C can be matched to give the prescribed K. We, thus, have a nontrivial solution for every complex number μ. (This implies that the point spectrum of the Laplacian operator with the inhomogeneous circulation and boundary conditions (70)-(71) is the totality of complex numbers.) In what follows, we assume that μ is a real number (then φ is a real function).

The evolution equation (69) has two essential constants of motion (see Sec.II.E.2);

$$H_1 = \|W\|^2 \equiv \int_\Omega |W|^2 \, dx, \tag{83}$$

$$H_0 = (W, \mathcal{P}\Phi) \equiv \int_\Omega W \cdot (\mathcal{P}\Phi) \, dx, \tag{84}$$

where $\mathcal{P}\Phi = \Phi - C$ (C is chosen so that $\mathcal{P}\Phi|_\Gamma = 0$) is a projecton to homogenize the boundary condition (71). The H_1 and $H_0 = \|\nabla\Phi\|^2$ are, respectively, the enstrophy and the energy of the flow.

The Beltrami equation (82) is reproduced as the Euler-Lagrange equation of a variational principle

$$\delta(H_1 - \mu H_0) = 0 \tag{85}$$

with the circulation and boundary conditions (70)-(71). This variational principle can be regarded as a minimization of the enstrophy H_1 with restricting the energy H_0 (see the relaxation model of Sec. II.F). Similar to the example of Sec.III.B.1, this gives a well-posed variational principle.

D. Stability theory

1. Two-dimensional Beltrami flow

To study the stability of a Beltrami flow (denote the Hamiltonian by Φ_0), we linearize (79) with writing $\Phi = \Phi_0 + \varphi$ and $-\Delta\varphi = \omega$ (the circulation $\int_\Omega \omega dx$ must be zero);

$$\partial_t \omega + \{\Phi_0, \omega\} + \{\varphi, -\Delta\Phi_0\} = 0. \tag{86}$$

Using (82), we can write

$$\partial_t \omega + \{\Phi_0, \omega - \mu\varphi\} = 0. \tag{87}$$

We easily verify that

$$G(\varphi) = (\omega, \omega - \mu\mathcal{P}\varphi) = \|\omega\|^2 - \mu\|\nabla\varphi\|^2 \tag{88}$$

is a constant of motion ($dG(\varphi)/dt = 0$) associated with the linearized dynamics (87). In a bounded domain, we have an inequality

$$\| - \Delta\varphi\|^2 \geq \lambda\|\nabla\varphi\|^2$$

with λ being the smallest eigenvalue of the Laplacian $-\Delta$ with the Dirichlet boundary condition (one easily find $\lambda > 0$). We, thus, have

$$G(\varphi) \geq (\lambda - \mu)\|\nabla\varphi\|^2. \tag{89}$$

If $\mu < \lambda$, the a priori estimate (89) gives a bound for the energy $\|\nabla\varphi\|^2$, because $G(\varphi)$ is a constant determined by the initial condition of the perturbation φ. The bound of the Beltrami parameter $\mu < \lambda$ gives a sufficient condition for the stability of the Beltrami flow.

We can generalize this argument to a variety of second-order nonlinear systems. We first cast the method in an abstract theorem.

2. Abstract theory

Let $f(a, b)$ be a bilinear map. We define $\mathcal{F}(u) = f(u, u)$, and consider an abstract nonlinear evolution equation

$$\partial_t u = \mathcal{F}(u). \tag{90}$$

Suppose that there are symmetric bilinear forms $h_j(a, b)$ ($j = 1, \cdots, \nu$) such that

$$h_j(u, \mathcal{F}(u)) = 0 \quad (j = 1, \cdots, \nu,\ \forall u). \tag{91}$$

With writing $H_j(u) = h_j(u, u)$, we observe, for the solution of (90),

$$\begin{aligned}\frac{d}{dt} H_j(u) &= 2h_j(u, \partial_t u) \\ &= 2h_j(u, \mathcal{F}(u)) = 0.\end{aligned} \tag{92}$$

Hence, $H_j(u)$ $(j = 1, \cdots, \nu)$ are the constants of motion associated with the evolution equation (90).

Let u_0 be a stationary point (equilibrium) of (90), i.e., $\mathcal{F}(u_0) = 0$. We assume that u_0 also solves

$$\delta \left[\sum_{j=1}^{\nu} \mu_j H_j(u) \right] = 0 \tag{93}$$

with some fixed real numbers μ_j $(j = 1, \cdots, \nu)$; cf. (85). We call such u_0 as a "Beltrami field".

Theorem 2. *Suppose that $u = u_0 + \tilde{u}$ (u_0 is a Beltrami field) satisfies either (90) or its "linearized" equation*

$$\partial_t \tilde{u} = f(u_0, \tilde{u}) + f(\tilde{u}, u_0). \tag{94}$$

Then,

$$G(\tilde{u}) = \sum_{j=1}^{\nu} \mu_j H_j(\tilde{u}) \tag{95}$$

is a constant of motion.

(proof) Using (91), we observe

$$\begin{aligned}0 &= \sum \mu_j h_j(u, \mathcal{F}(u)) \\ &= \sum \mu_j h_j(u_0 + \tilde{u}, \mathcal{F}(u_0 + \tilde{u})) \\ &= \sum \mu_j h_j(u_0, \mathcal{F}(u_0 + \tilde{u})) \\ &\qquad + \sum \mu_j h_j(\tilde{u}, \mathcal{F}(u_0 + \tilde{u})).\end{aligned} \tag{96}$$

Since (93) implies $\sum \mu_j h_j(u_0, \delta) = 0$ $(\forall \delta)$, the first sum of (96) vanishes. Hence, if u solves (90), we obtain

$$\begin{aligned}\frac{d}{dt} G(\tilde{u}) &= 2 \sum \mu_j h_j(\tilde{u}, \partial_t \tilde{u}) \\ &= 2 \sum \mu_j h_j(\tilde{u}, \mathcal{F}(u_0 + \tilde{u})) = 0.\end{aligned} \tag{97}$$

We can rewrite (96) as

$$\begin{aligned}0 = &\sum \mu_j h_j(\tilde{u}, f(u_0, \tilde{u}) + f(\tilde{u}, u_0)) \\ &+ \sum \mu_j h_j(\tilde{u}, \mathcal{F}(\tilde{u})).\end{aligned} \tag{98}$$

By (91), the second term of (98) vanishes. If $\tilde{u}$ is a solution of (94), we obtain

$$\frac{d}{dt} G(\tilde{u}) = 2 \sum \mu_j h_j(\tilde{u}, f(u_0, \tilde{u}) + f(\tilde{u}, u_0)) = 0. \tag{99}$$

(QED)

3. Application for MHD

An interesting application of Theorem 2 is made in the stability analysis of a three dimensional plasma equilibrium with a flow. Let Ω be a bounded three-dimensional domain with a smooth boundary Γ. We assume that Ω is multiply connected with cuts Σ_ℓ [$\ell = 1, \cdots, m$ (the first Betti number)], i.e., $\Omega \setminus \cup(\Sigma_\ell)$ is simply connected.

When the domain Ω is multiply connected, we can assume that the Beltrami parameters μ_j are arbitrary real numbers [see (107)]. This fact is in analogy with the previous example (82). Here we remark an interesting characteristic of the eigenfunction of the curl operator. [3] In a multiply connected domain $\Omega(\subset \mathbf{R}^3)$, the curl operator has a point spectrum that covers the entire complex plane. This is because of the existence of a non-zero harmonic field ($\nabla \times \boldsymbol{h} = 0$, $\nabla \cdot \boldsymbol{h} = 0$ in Ω, and $\boldsymbol{n} \cdot \boldsymbol{h} = 0$ on Γ), which plays a role of inhomogeneous term in the eigenvalue problem

$$\nabla \times \boldsymbol{u} = \lambda \boldsymbol{u}.$$

We decompose the solenoidal field $\boldsymbol{u}$ into the harmonic component $\boldsymbol{h}$ and its orthogonal compliment $\boldsymbol{u}_\Sigma$. We can show that the latter component is a member of a Hilbert space

$$L^2_\Sigma(\Omega) = \{\nabla \times \boldsymbol{a} \in L^2(\Omega);\ \boldsymbol{n} \times \boldsymbol{a} = 0 \text{ on } \Gamma\}.$$

The eigenvalue problem now reads as

$$\nabla \times \boldsymbol{u}_\Sigma = \lambda(\boldsymbol{u}_\Sigma + \boldsymbol{h}).$$

If we take $\boldsymbol{h} = 0$, we find a nontrivial solution only for $\lambda_j \in \sigma_p$, where σ_p a countably infinite set of real numbers. The σ_p is the point spectrum of the self-adjoint curl operator that is defined in the Hilbert space $L^2_\Sigma(\Omega)$. For $\lambda' \notin \sigma_p$, we set $\boldsymbol{h} \neq 0$ and find a solution $\boldsymbol{u}_\Sigma = (\text{curl} - \lambda')^{-1}\lambda' \boldsymbol{h}$, where curl is the self-adjoint curl operator.

We consider an ideal plasma (magnetofluid) which obey

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nabla p = 0, \qquad (100)$$

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0, \qquad (101)$$

where $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{v}$ is the incompressible flow velocity, and p is the pressure. We have normalized $\boldsymbol{B}$ by its representative value B^*, $\boldsymbol{v}$ by the Alfvén speed $B^*/\sqrt{\mu_0 \rho}$ (ion mass density ρ is assumed to be a constant), p by B^{*2}/μ_0, and t by the ion gyration time. The length scale is arbitrary. We assume boundary conditions

$$\boldsymbol{n} \cdot \boldsymbol{v} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{B} = 0 \quad \text{on } \Gamma \qquad (102)$$

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[3] Z. Yoshida and Y. Giga, Math. Z. 204, 235 (1990).

and flux coditions

$$\int_{\Sigma_\ell} \boldsymbol{n} \cdot \boldsymbol{B} \, ds = K_\ell \quad (\ell = 1, \cdots, m), \tag{103}$$

where the fluxes through the cuts are given constants.

We have three important constants of motion;

$$H_0 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{B}\|^2 \quad \text{(energy)}, \tag{104}$$

$$H_1 = (\mathcal{P}\boldsymbol{A}, \boldsymbol{B}) \quad \text{(magnetic helicity)}, \tag{105}$$

$$H_2 = 2(\boldsymbol{v}, \boldsymbol{B}) \quad \text{(cross helicity)}, \tag{106}$$

where $\boldsymbol{A}$ is the vector potential of $\boldsymbol{B}$ and $\mathcal{P}$ is the orthogonal projection in $L^2(\Omega)$ onto $L^2_\Sigma(\Omega)$. Taking $\mathcal{P}\boldsymbol{A}$ as the vector potential makes the helicity H_1 gauge-invariant.

The variational principle

$$\delta(H_0 - \mu_1 H_1 - \mu_2 H_2) = 0 \tag{107}$$

gives Beltrami fields defined by

$$(1 - \mu_2^2)\nabla \times \boldsymbol{B} = \mu_1 \boldsymbol{B}, \tag{108}$$

$$\boldsymbol{v} = \mu_2 \boldsymbol{B}. \tag{109}$$

The $\boldsymbol{B}$ satisfying (108) is the "force-free field". We have a field aligned flow $\boldsymbol{v}$ whose magnitude is scaled by μ_2 in the local Alfvén speed unit.

Due to Theorem 2, the integral

$$G(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{v}}) = \|\tilde{\boldsymbol{v}}\|^2 + \|\tilde{\boldsymbol{B}}\|^2 - \mu_1(\mathcal{P}\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}) - 2\mu_2(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{B}}) \tag{110}$$

is a constant of motion for the perturbations $\tilde{\boldsymbol{B}}$ and $\tilde{\boldsymbol{v}}$ satisfying the nonlinear equation (100)-(101), or their linearlized equations.

We have an inequality

$$(\mathcal{P}\tilde{\boldsymbol{A}}, \nabla \times \tilde{\boldsymbol{A}}) \leq |\lambda|^{-1}\|\tilde{\boldsymbol{B}}\|^2,$$

where $|\lambda| = \min_j\{|\lambda_j|\}$ [λ_j ($j = 1, 2, \cdots$) are the eigenvalues of the self-adjoint curl operator]. To prove this relation, we invoke the spectral resolution theorem due to Yoshida-Giga to expand $\boldsymbol{u} = \sum(\boldsymbol{u}, \boldsymbol{\psi}_j)\boldsymbol{\psi}_j$ ($\forall \boldsymbol{u} \in L^2_\Sigma(\Omega)$), where $\boldsymbol{\psi}_j$ is the eigenfunction of the self-adjoint curl operator belonging to an eigenvaule λ_j. We, thus, can write

$$\mathcal{P}\boldsymbol{B} = \sum(\boldsymbol{B}, \boldsymbol{\psi}_j)\boldsymbol{\psi}_j,$$

and

$$\mathcal{P}\boldsymbol{A} = \sum(\boldsymbol{B}, \boldsymbol{\psi}_j)\boldsymbol{\psi}_j/\lambda_j.$$

Hence, we have

$$\begin{aligned}
(\mathcal{P}\boldsymbol{A}, \boldsymbol{B}) &= (\mathcal{P}\boldsymbol{A}, \mathcal{P}\boldsymbol{B}) \\
&\leq \|\mathcal{P}\boldsymbol{A}\| \cdot \|\mathcal{P}\boldsymbol{B}\| \\
&= (\sum(\boldsymbol{B}, \boldsymbol{\psi}_j)^2/\lambda_j^2)^{-1/2}(\sum(\boldsymbol{B}, \boldsymbol{\psi}_j)^2)^{-1/2} \\
&\leq |\lambda|^{-1}(\sum(\boldsymbol{B}, \boldsymbol{\psi}_j)^2) = |\lambda|^{-1}\|\mathcal{P}\boldsymbol{B}\|^2 \\
&\leq |\lambda|^{-1}\|\boldsymbol{B}\|^2.
\end{aligned}$$

Using

$$2(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{B}}) \leq \alpha\|\tilde{\boldsymbol{v}}\|^2 + \alpha^{-1}\|\tilde{\boldsymbol{B}}\|^2 \quad (\forall \alpha > 0),$$

we observe

$$G(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{v}}) \geq (1 - \alpha|\mu_2|)\, \|\tilde{\boldsymbol{v}}\|^2 + \left(1 - \frac{|\mu_2|}{\alpha} - \frac{|\mu_1|}{|\lambda|}\right) \|\tilde{\boldsymbol{B}}\|^2. \tag{111}$$

Taking $\alpha = 1/|\mu_2|$, (111) reads

$$G(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{v}}) \geq \left(1 - \mu_2^2 - \frac{|\mu_1|}{|\lambda|}\right) \|\tilde{\boldsymbol{B}}\|^2. \tag{112}$$

If $1 - \mu_2^2 - |\mu_1|/|\lambda| > 0$, then (112) gives a bound for the energy of $\tilde{\boldsymbol{B}}$. Under this condition, we write $1 - \mu_2^2 - |\mu_1|/|\lambda| = 1/\beta\ (> 0)$, and set $\alpha = |\mu_2|$. Then (111) yields

$$\left(1 + \frac{|\mu_1|}{|\lambda|}\beta\right) G(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{v}}) \geq (1 - \mu_2^2)\, \|\tilde{\boldsymbol{v}}\|^2. \tag{113}$$

If $1 - \mu_2^2 > 0$, (113) gives a bound for the energy of $\tilde{\boldsymbol{v}}$. This condition is weaker than the previous condition.

Now, we have sufficient conditions for the stability;

$$\sigma \equiv \frac{|\mu_1|}{1 - \mu_2^2} < |\lambda|, \tag{114}$$

$$\mu_2^2 < 1. \tag{115}$$

When both conditions are satisfied, the constant of motion $G(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{v}})$ bounds the energy of perturbations, implying the stability. In (114), σ stands for the eigenvalue of the Beltrami equation (108). The stability condition means that σ must not exceed the minimum of $|\lambda_j|$ (λ_j is the eigenvalue of the self-adjoint curl operator). The second condition implies that the flow velocity must not exceed the local Alfvén speed; see (109).

Math. Z. 204, 235–245 (1990)

Mathematische Zeitschrift

Remarks on Spectra of Operator Rot

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1. Introduction

This paper studies an eigenvalue problem

$$\operatorname{rot}\phi = \lambda\phi \qquad \text{in } \Omega \subset \mathbb{R}^3\,, \tag{1.1}$$

with a boundary condition

$$n \cdot \phi = 0 \qquad \text{on } \partial\Omega\,, \tag{1.2}$$

from a rigorous mathematical point of view. Here, $\phi = (\phi_1(x_1, x_2, x_3), \phi_2(x_1, x_2, x_3), \phi_3(x_1, x_2, x_3))$ is a 3-vector function, Ω is a 3-dimensional bounded domain with a smooth boundary $\partial\Omega$, and n is a unit normal vector on $\partial\Omega$. The rotation operator rot is defined, in Cartecian coordinates, by

$$[\operatorname{rot}\phi]_j = \frac{\partial\phi_{j+2}}{\partial x_{j+1}} - \frac{\partial\phi_{j+1}}{\partial x_{j+2}} \qquad (j\text{: integer mod } 3)\,.$$

The operator rot is one of the most important first-order differential operator, which frequently appears in many different physics models. For example in the Maxwell equations of electricity and magnetism, the operator

$$A = i\begin{bmatrix} 0 & -\operatorname{rot} \\ \operatorname{rot} & 0 \end{bmatrix}$$

acting on functions (ϕ, ψ) with boundary conditions $n \times \phi = 0$ is known to be a self-adjoint operator in the L^2 Hilbert space; see Duvaut–Lions [4]. The eigenfunctions of A are eigenmodes of electromagnetic waves. In fluid and plasma physics, rot appears to measure the volticity of various flows. This paper studies the spectra of rot, and give fundamental remarks on eigenfunctions of rot. This problem has important applications in plasma physics; see Appendix. The eigenfunctions of rot are called *free-decay fields*, which have been studied by Chandrasekhar–Kendall [3] for astrophysical plasmas. They give explicit calculations of the eigenfunctions for a periodic straight cylinder region when λ is real. In the theory of fusion plasmas, a free-decay field is called Taylor state that is considered to be the ultimate minimum-energy plasma equilibrium [9]. The free-decay fields are also useful to study turbulences in plasmas; for example see Ref. [7]. So far, mathematical backgrounds, however, was not studied.

In this paper, we study spectral properties of rot operator in various function spaces. We show that (1.1), (1.2) has a nontrivial solution for all complex λ, when Ω is multiply connected (Theorem 2). Eigenfunctions corresponding to different eigenvalues are not always orthogonal; otherwise, it contradicts to the separability of the Lebesgue space $L^2(\Omega)$. It turns out that there is a subset Λ of real eigenvalues such that the set $\{\phi_\lambda; \lambda \in \Lambda\}$ of the corresponding normalized eigenfunctions is a complete ortho-normal basis $L^2_\Sigma(\Omega)$ that is the orthogonal complement of the irrotational fields in $L^2(\Omega)$; cf. (2.3). To prove this fact, we introduce a self-adjoint operator S in $L^2_\Sigma(\Omega)$ associated with rot by choosing suitable additional boundary conditions. The spectral resolution is given by

$$Su = \sum_{\lambda \in \Lambda} (u, \phi_\lambda)\phi_\lambda ,$$

where (,) is the standard inner product of $L^2(\Omega)$; see Theorem 1.

In Section 2, we give a concise summary of basic function spaces. Section 3 is devoted to the study of a self-adjoint definition of rot. In Section 4, we extend the space and domain of rot, and study the original problem (1.1) and (1.2), with a help of the theory developed for the self-adjoint rot, and prove Theorem 2.

2. Basic Function Spaces

Throughout this paper, we consider linear spaces over the complex number field $\mathbb{C}$. We denote by $L^2(\Omega)$ the Lebesgue space of square integrable functions on Ω, which we equip with the usual inner product $(a, b) = \int_\Omega a \cdot \bar{b}\, dx$, where $\bar{b}$ is the complex conjugate of b, and the norm $\|a\| = (a, a)^{1/2}$. Here and hereafter we do not distinguish between function spaces of vector-valued and scaler-valued functions.

The Sobolev space of order s ($s \geqq 0$) is denoted by $H^s(\Omega)$. We denote by $H^s_0(\Omega)$ the closure in $H^s(\Omega)$ of the space $C^\infty_0(\Omega)$ of compactly supported smooth functions. The negative order Sobolev space $H^{-s}(\Omega)$ is the dual space of $H^s_0(\Omega)$. The H^s-norm $\|u\|_s$ of a vector function u is estimated as

$$C\|u\|_s \leqq \|\mathrm{rot}\, u\|_{s-1} + \|\mathrm{div}\, u\|_{s-1} + |n \cdot u|_{s-1/2} + \|u\|_{s-1} , \tag{2.1}$$

where C is a certain positive constant, $n \cdot u$ is the trace onto $\partial\Omega$ of the normal component of vector u, and $|n \cdot u|_{s-1/2}$ denotes the norm of $n \cdot u$ in $H^{s-1/2}(\partial\Omega)$; see Bourguignon–Brezis [2], and Foias–Temam [5].

The Weyl decomposition of the space $L^2(\Omega)$ of 3-vector functions is

$$L^2(\Omega) = L^2_\sigma(\Omega) \oplus \{\mathrm{grad}\ p;\ p \in H^1(\Omega)\} , \tag{2.2}$$

with

$$L^2_\sigma(\Omega) = \{u \in L^2(\Omega);\ \mathrm{div}\, u = 0,\ n \cdot u = 0\} .$$

We also derive another direct-sum decomposition of the space $L^2(\Omega)$ of 3-vector functions (see Foias–Temam [5]);

$$L^2(\Omega) = L^2_\Sigma(\Omega) \oplus \mathrm{Ker}(\mathrm{rot})\,, \tag{2.3}$$

where Ker(rot) is the kernel of rot in $L^2(\Omega)$, viz.,

$$\mathrm{Ker}(\mathrm{rot}) = \{u \in L^2(\Omega); \mathrm{rot}\, u = 0 \text{ in } \Omega\}\,,$$

and $L^2_\Sigma(\Omega)$ is the orthogonal complement of Ker(rot). Explicitly, we have an expression

$$L^2_\Sigma(\Omega) = \{u \in L^2(\Omega); \operatorname{div} u = 0, n \cdot u = 0\,,$$

$$\int_\Sigma v \cdot u\, ds = 0 \,(\text{for any } \Sigma)\}\,,$$

where $\int_\Sigma v \cdot u\, ds$ is the integral of the normal component $v \cdot u$ of u over an arbitrary smooth simple surface Σ in Ω whose boundary curve is on $\partial\Omega$. By the Hodge–Kodaira decomposition theory, we have

$$L^2_\Sigma(\Omega) = \{\mathrm{rot}\, w; w \in H^1(\Omega), \operatorname{div} w = 0, n \times w = 0\}\,, \tag{2.4}$$

where $n \times w$ is the trace onto $\partial\Omega$ of the tangential component of the vector function w.

The zero-flux condition $\int_\Sigma v \cdot u\, ds = 0$ is not trivial when the domain Ω is multiply connected. Therefore, in general, the space $L^2_\Sigma(\Omega)$ is smaller than $L^2_\sigma(\Omega)$. We write

$$L^2_\sigma(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega)\,, \tag{2.5}$$

where $L^2_H(\Omega)$ is the orthogonal complement of $L^2_\Sigma(\Omega)$ in $L^2_\sigma(\Omega)$ (see Morrey [8], Chap. 7, Theor. 7.7.7). By the definition, we see

$$L^2_H(\Omega) = \{h \in L^2(\Omega); \mathrm{rot}\, h = 0, \operatorname{div} h = 0, n \cdot h = 0\}\,.$$

The dimension of the space $L^2_H(\Omega)$ is finite and equals the genus of $\partial\Omega$ (see eg. [1]). If Ω is simply connected, the dimension $\dim(L^2_H(\Omega)) = 0$, so $L^2_\sigma(\Omega) = L^2_\Sigma(\Omega)$.

For $u \in L^2_\Sigma(\Omega) \cap H^1(\Omega)$, the estimate (2.1), together with a Poincare-type inequality

$$c\,\|u\| \leqq \|\mathrm{rot}\, u\|\,,$$

leads

$$c'\,\|u\|_1 \leqq \|\mathrm{rot}\, u\|\,, \tag{2.1'}$$

where c and c' are certain positive constants (Foias–Temam [5], Lemma 1.6).

3. Formulation of Self-adjoint Operator and Its Spectral Resolution

For a pair of smooth vector functions u_1, u_2, integrating by parts yields

$$(\mathrm{rot}\, u_1, u_2) = (u_1, \mathrm{rot}\, u_2) + \int_{\partial\Omega} v_1 \times \bar{v}_2 \cdot n\, ds\,,$$

where $v_i = n \times u_i (i = 1, 2)$, and $\bar{v}_2$ is the complex conjugate of v_2. It is obvious that rot, with the boundary condition (1.2) alone, is not symmetric. When we assume in addition $v_1 = 0$ on $\partial\Omega$, the boundary integral vanishes and rot is symmetric, however, generally the operator is not even closed (see Prop. 2). In this section, we find an appropriate additional boundary condition that makes rot a self-adjoint operator. The spectral resolution of the self-adjoint rot operator gives a complete set of eigenfunctions that spans the space $L^2_\Sigma(\Omega)$.

Definition. We define a space of 3-vector functions:

$$H^1_{\Sigma\Sigma}(\Omega) = \{u \in L^2_\Sigma(\Omega);\ \mathrm{rot}\, u \in L^2_\Sigma(\Omega)\}\ .$$

We define an operator S in the Hilbert space $L^2_\Sigma(\Omega)$ by

$$Su = \mathrm{rot}\, u, \qquad \text{for } u \in D(S) = H^1_{\Sigma\Sigma}(\Omega)\ .$$

Theorem 1. *The operator S is self-adjoint in the space $L^2_\Sigma(\Omega)$. The spectrum $\sigma(S)$ of S consists of only point spectrum $\sigma_p(S) \subset \mathbb{R}$. Therefore, the set of eigenfunctions of S gives an orthogonal complete basis of the space $L^2_\Sigma(\Omega)$.*

Before proving this theorem, we prepare the following lemmas concerning the domain $D(S) = H^1_{\Sigma\Sigma}(\Omega)$.

Lemma 1. *Concerning the space $H^1_{\Sigma\Sigma}(\Omega)$ we have*

(1) *The space $H^1_{\Sigma\Sigma}(\Omega)$ is a subspace of $H^1(\Omega)$, and is dense in $L^2_\Sigma(\Omega)$.*
(2) *The range $R(S)$ of the operator S is just equal to $L^2_\Sigma(\Omega)$. The operator S has a compact inverse from $L^2_\Sigma(\Omega)$ to $H^1_{\Sigma\Sigma}(\Omega)$.*
(3) *An alternative expression of $H^1_{\Sigma\Sigma}(\Omega)$ is*

$$H^1_{\Sigma\Sigma}(\Omega) = \{u = P_\Sigma w;\ w \in H^1(\Omega), \mathrm{div}\, w = 0, n \times w = 0\}\ ,$$

where P_Σ is the orthogonal projector from $L^2(\Omega)$, onto $L^2_\Sigma(\Omega)$.

Proof. By the estimate (2.1'), we see that $H^1_{\Sigma\Sigma}(\Omega) \subset H^1(\Omega)$. The space $C_0^\infty(\Omega) \cap L^2_\Sigma(\Omega)$ is dense in $L^2_\Sigma(\Omega)$, and is contained in $H^1_{\Sigma\Sigma}(\Omega)$, therefore $H^1_{\Sigma\Sigma}(\Omega)$ is dense in $L^2_\Sigma(\Omega)$; this proves the first part of the lemma.

By the definition of $D(S)$, it is obvious that $R(S) \subset L^2_\Sigma(\Omega)$. Let us write

$$V(\Omega) = \{u = P_\Sigma w;\ w \in H^1(\Omega), \mathrm{div}\, w = 0, n \times w = 0\}\ .$$

By the definition (2.3), $w - P_\Sigma w \in \mathrm{Ker(rot)}$. We see that $V(\Omega) \subset D(S) = H^1_{\Sigma\Sigma}(\Omega)$, because, for $u = P_\Sigma w \in V$, we have $u \in L^2_\Sigma(\Omega)$ and

$$\mathrm{rot}\, u = \mathrm{rot}\, w \in L^2_\Sigma(\Omega)\ .$$

In view of (2.4), we observe that

$$\begin{aligned} S(V(\Omega)) &= \{Sv;\ v \in V(\Omega)\} \\ &= (\mathrm{rot}\, P_\Sigma w = \mathrm{rot}\, w;\ w \in H^1(\Omega), \mathrm{div}\, w = 0, n \times w = 0\} \\ &= L^2_\Sigma(\Omega)\ , \end{aligned}$$

which proves that $L^2_\Sigma(\Omega) \subset R(S)$, therefore $L^2_\Sigma(\Omega) = R(S)$. Since $L^2_\Sigma(\Omega)$ is orthogonal to Ker(rot), S has a unique inverse S^{-1} defined on $L^2_\Sigma(\Omega)$. This obviously shows that the third part of the lemma:

$$H^1_{\Sigma\Sigma}(\Omega) = V(\Omega) .$$

Finally, we show that S^{-1} is compact. Let $y_i (i = 1, 2, . .)$ be a bounded sequence in $L^2_\Sigma(\Omega)$. By (2.1'), the sequence $x_i = S^{-1} y_i$ is bounded in $H^1(\Omega)$. Therefore, in view of Rellich theorem, we see that a subsequence of x_i is strongly convergent in the topology of $L^2_\Sigma(\Omega)$, viz., S^{-1} is compact. □

By Lemma 1(3), we notice that, for $u \in H^1_{\Sigma\Sigma}(\Omega)$, the tangential trace $n \times u$ is equal to $n \times \gamma$ with $\gamma \in \mathrm{Ker}\,(\mathrm{rot})$. We next summarize results for the tangential trace of $u \in H^1_{\Sigma\Sigma}(\Omega)$. We identify $n \times u$ with an $H^{1/2}$-class 1-form v on the boundary $\partial\Omega$. This form v is shown to be closed and to represent some cohomology class on $\partial\Omega$. The cohomology class represented by v comes from tangential traces of functions in $L^2_H(\Omega)$. We write

$$g_i = n \times h_i \qquad \text{for } h_i \in L^2_H(\Omega) \quad (i = 1, 2, \ldots, N) ,$$

where h_i $(i = 1, 2, \ldots, N)$ is the orthogonal basis of $L^2_H(\Omega)$; g_i is regarded as a 1-form on $\partial\Omega$.

Lemma 2. *Concerning the tangential traces of functions in $L^2_\Sigma(\Omega)$, we have*

(1) *Let $u \in H^1_{\Sigma\Sigma}(\Omega)$, and $v = n \times u$. Then, v is a closed differential 1-form on $\partial\Omega$. Moreover, we have an expression*

$$v = d\omega + \sum_{i=1}^{N} \alpha_i g_i , \tag{3.1}$$

where $d\omega$ is the exact part of v expressed by a coboundary of an $H^{3/2}$-class 0-form ω on $\partial\Omega$, g_i is the tangential trace of $h_i \in L^2_H(\Omega)$, and $\alpha_i \in \mathbb{C}$.

(2) *The tangential trace $n \times u$ for $u \in H^1_{\Sigma\Sigma}(\Omega)$ is a surjection to the space of closed differential forms such that (3.1) holds, viz., for every closed differential form v of $H^{1/2}(\partial\Omega)$-class such that (3.1) holds, we find an extension $\tilde{v} \in H^1_{\Sigma\Sigma}(\Omega)$ whose tangential trace is identified with v.*

Proof. Let $u \in H^1_{\Sigma\Sigma}(\Omega)$ and $v = n \times u \in H^{1/2}(\partial\Omega)$. Since rot $u \in L^2_\Sigma(\Omega)$, we have

$$n \cdot \mathrm{rot}\, u = dv = 0 (\in H^{-1/2}(\partial\Omega)) ,$$

which proves that v is a closed differential form. Since rot $u \in L^2_\Sigma(\Omega)$, rot u is orthogonal to the space $L^2_H(\Omega)$. Integration by parts yields

$$\begin{aligned} 0 &= \int_\Omega \mathrm{rot}\, u \cdot \bar{h}_i \, dx \\ &= \int_\Omega u \cdot \mathrm{rot}\, \bar{h}_i \, dx + \int_{\partial\Omega} (u \times \bar{h}_i) \cdot n \, dS \\ &= \int_\Omega u \cdot \mathrm{rot}\, \bar{h}_i \, dx + \int_{\partial\Omega} [(u \times n) \times (\bar{h}_i \times n)] \cdot n \, dS \\ &= \int_{\partial\Omega} v \wedge \bar{g}_i . \end{aligned} \tag{3.2}$$

Here, $\wedge$ denotes exterior product of forms on $\partial\Omega$. The final integration is an integral of 2-form on $\partial\Omega$. Since v is closed, (3.2) implies

$$v = d\omega + \sum_{i=1}^{N} \alpha_i g_i .$$

It remains to show the second part of the lemma. By a standard extension argument (see Lions–Magenes [6], Chap. I., 3.2 and 8.2), we find $w \in H_0^1(\Omega) \cap H^2(\Omega)$ such that $n \times \operatorname{rot} w = (n, \operatorname{grad})(n \times (n \times w)) = v$ for every 1-form of $H^{1/2}(\partial\Omega)$ class. Since $n \times w = 0$ on $\partial\Omega$, we see that rot $w \in L^2_\sigma(\Omega)$. Tracing (3.2) from bottom to top, we see the condition (3.1) implies that rot w is orthogonal to $L^2_H(\Omega)$. We thus conclude that rot $w \in H^1_{\Sigma\Sigma}(\Omega)$ which is the desired extension of v into Ω. □

We now give the proof of Theorem 1.

Proof of Theorem 1. By Lemma 1 (1), the domain $D(S) = H^1_{\Sigma\Sigma}(\Omega)$ is dense in $L^2_\Sigma(\Omega)$. We first prove that S is closed. Let $x_i (i = 1, 2, \ldots)$ be a sequence in $D(S)$ such that $x_i \to x$ with $Sx_i \to y$ in $L^2_\Sigma(\Omega)$. Directly from the definition of $D(S)$, we see that $x \in D(S)$, so that $Sx = y$, which proves that S is closed.

The adjoint operator S^* of S is defined by

$$(Su_1, u_2) = (u_1, S^* u_2) \qquad \text{for any } u_1 \in D(S) ,$$

for $u_2 \in L^2_\Sigma(\Omega)$ as far as a function $S^* u_2 \in L^2_\Sigma(\Omega)$ exists. For $u_2 \in D(S^*)$ the linear form $u_1 \to (Su_1, u_2)$ is continuous on $D(S)$ for the topology of $L^2_\Sigma(\Omega)$. In particular for $u_1 \in C_0^\infty(\Omega) \cap D(S)$, we have

$$(Su_1, u_2) = (u_1, \operatorname{rot} u_2) .$$

We thus see that rot $u_2 \in L^2(\Omega)$ for $u_2 \in D(S^*)$, which, together with (2.1'), implies that

$$D(S^*) \subset H^1(\Omega) .$$

Let $u_1 \in D(S)$ and $u_2 \in D(S^*)$. We have tangential traces

$$n \times u_1 = v_1 \in H^{1/2}(\partial\Omega), n \times u_2 = v_2 \in H^{1/2}(\partial\Omega) .$$

We have

$$(Su_1, u_2) = (u_1, \operatorname{rot} u_2) + \int_{\partial\Omega} (v_1 \times \bar{v}_2) \cdot n \, ds .$$

We may consider both v_1 and v_2 are 1-forms of $H^{1/2}$-class on $\partial\Omega$, so that we may write

$$\int_{\partial\Omega} (v_1 \times \bar{v}_2) \cdot n \, ds = \int_{\partial\Omega} v_1 \wedge \overline{v_2} .$$

By Lemma 2 (1) the differential form v_1 is closed, and permits an expression

$$v_1 = d\omega_1 + g ,$$

where $d\omega_1$ is the coboundary of an $H^{3/2}$-class 0-form ω_1, and g is the cohomology part. We now have

$$(Su_1, u_2) = (u_1, \operatorname{rot} u_2) + \int_{\partial\Omega} (d\omega_1 + g) \wedge \bar{v}_2$$
$$= (u_1, \operatorname{rot} u_2) - \int_{\partial\Omega} \omega_1 \wedge d\bar{v}_2 + \int_{\partial\Omega} g \wedge \bar{v}_2 , \tag{3.3}$$

where we used an integration by parts

$$\int_{\partial\Omega} d(\omega_1 \wedge \bar{v}_2) = \int_{\partial\partial\Omega} \omega_1 \wedge \bar{v}_2 = 0 ,$$

since $\partial\partial\Omega = \emptyset$. Because $\operatorname{rot} u_2 \in L^2(\Omega)$, $(u_1, \operatorname{rot} u_2)$ is continuous for u_1 on $D(S)$ in the topology of $L^2_\Sigma(\Omega)$; which should also be the case for the surface integral term of (3.3). By Lemma 2 (2), we see that the surface integral is continuous for u_1 on $D(S)$ in the topology of $L^2_\Sigma(\Omega)$ if and only if the differential form v_2 satisfies

$$d\bar{v}_2 = 0, \qquad \text{and} \qquad \int_{\partial\Omega} g_i \wedge \bar{v}_2 = 0, \quad \text{for every } g_i .$$

By Lemma 2 (1), this implies that $\operatorname{rot} u_2 \in L^2_\sigma(\Omega)$. Therefore we get

$$S^* = S, \qquad \text{with } D(S^*) = D(S) ,$$

which completes the proof of the self-adjointness of S.

Since S^{-1} is a compact operator (Lemma 1 (2)), the spectrum of S^{-1} consists of only point spectrum which does not accumulate besides 0. The spectral resolution of S is given by the inverse of the spectral resolution of S^{-1}. Therefore, the second assertion of the theorem is proved. □

4. Extension to Non-symmetric Operators and Continuum of Spectra

We consider extensions of the self adjoint rotation operator S, and study the spectra of them. Our goal in this paper is to study the eigenvalue problem (1.1) and (1.2). Therefore we need to extend the function space $L^2_\Sigma(\Omega)$ to $L^2_\sigma(\Omega)$, of course with giving up the self-adjointness of the operator. We first note the following point:

Remark 1. Every eigenfunction, in the space $L^2_\Sigma(\Omega)$, of rot is a member of the set of eigenfunctions of S. The proof is straightforward by the definition of the domain of S.

Here we consider the eigenvalue problem in the space $L^2_\sigma(\Omega)$.

Definition. We define

$$H^1_{\Sigma\sigma}(\Omega) = \{u \in L^2_\Sigma(\Omega);\ \operatorname{rot} u \in L^2_\sigma(\Omega)\} .$$

We consider an operator T in the Hilbert space $L^2_\sigma(\Omega)$ defined by

$$Tu = \operatorname{rot} u, \qquad \text{for } u \in D(T) = H^1_{\Sigma\sigma}(\Omega) .$$

Remark 2. When $\dim(L^2_H(\Omega)) = 0$, $T = S$. Otherwise, T is an extension of S. Obviously by the definition, T is a closed operator. When $\dim(L^2_H(\Omega))$ is not zero, the domain $D(T)$ is not dense in $L^2_\sigma(\Omega)$, and T is not a symmetric operator; for the symmetry of rot, it is essential that rot u is orthogonal to $L^2_H(\Omega)$; see the proof of Lemma 2(1) and Theorem 1.

Proposition 1. *The resolvent set* $\rho(T)$ *of the operator* T *is equal to* $\rho(S)$. *For* $\lambda \in \rho(T)$, *the resolvent operator* $(T-\lambda)^{-1}$ *is compact.*

Proof. Let us first show the existence of the inverse T^{-1}. We consider a problem

$$Tu = f \in L^2_\sigma(\Omega)\,. \tag{4.1}$$

We extend the region Ω to $\mathbb{R}^3$. Let $\tilde{f}$ be the zero extension of f, viz.,

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega\,. \end{cases}$$

Since $f \in L^2_\sigma(\Omega)$, $\tilde{f}$ is approximated in L^2 sense by smooth divergence-free functions supported in Ω. We thus see div $\tilde{f} = 0$ in $\mathbb{R}^3$ in distribution sense. We denote by $(-\Delta)^{-1}$ the vector Newtonian potential. Let w_0 be the restriction on Ω of function $\operatorname{rot}(-\Delta)^{-1}\tilde{f}$, and $u_0 = P_\Sigma w_0$, where P_Σ is the projector onto $L^2_\Sigma(\Omega)$. Since div commutes with $(-\Delta)^{-1}$, we have $\operatorname{div}(-\Delta)^{-1}\tilde{f} = 0$, which deduces rot $w_0 = f$ in Ω. We see that u_0 is a solution of (4.1), which is unique since $D(T)$ is orthogonal to Ker(rot).

We next consider the resolvent of T. Let $\lambda \in \rho(S)$, and consider an equation

$$(T-\lambda)u = f \in L^2_\sigma(\Omega)\,. \tag{4.2}$$

We write $f = g + h$, where $g = P_\Sigma f$ so that $h \in L^2_H(\Omega)$. Let $u_0 = T^{-1}h$, and $w = u - u_0$. We now consider

$$(T-\lambda)w = g + \lambda u_0 \in L^2_\Sigma(\Omega)\,,$$

which is solved by

$$w = (S-\lambda)^{-1}(g + \lambda u_0) \in D(S) = H^1_{\Sigma\Sigma}(\Omega)\,.$$

We now have a unique solution to (4.2);

$$u = u_0 + (S-\lambda)^{-1}(g + \lambda u_0)\,.$$

An H^1-regularity argument, with the Rellich theorem, shows that the resolvent $(T-\lambda)^{-1}$ for $\lambda \in \rho(T) = \rho(S)$ is compact. □

Definition. We consider a further extension of T. We define

$$H^1_{\sigma\sigma}(\Omega) = \{u \in L^2_\sigma(\Omega);\ \operatorname{rot} u \in L^2_\sigma(\Omega)\}\,.$$

We consider an operator $\tilde{T}$ in the Hilbert space $L^2_\sigma(\Omega)$ defined by

$$\tilde{T}u = \operatorname{rot} u, \qquad \text{for } u \in D(\tilde{T}) = H^1_{\sigma\sigma}(\Omega)\,.$$

Remark 3. Obviously by the definition, $\tilde{T}$ is a closed operator. Since a subspace $\{u + h;\ u \in H^1_{\Sigma\Sigma}(\Omega),\ h \in L^2_\Sigma(\Omega)\}$ of $H^1_{\sigma\sigma}(\Omega)$ is dense in $L^2_\sigma(\Omega)$, we see that the domain $H^1_{\sigma\sigma}(\Omega)$ of $\tilde{T}$ is dense in $L^2_\sigma(\Omega)$.

Theorem 2. *The spectrum of $\tilde{T}$ consists of only point spectrum $\sigma_p(\tilde{T})$. When* $\dim(L^2_H(\Omega)) = 0$, $\tilde{T} = S$, *so that* $\sigma_p(\tilde{T}) = \sigma_p(S)$. *When* $\dim(L^2_H(\Omega))$ *is not zero,* $\sigma_p(\tilde{T}) = \mathbb{C}$, *viz., for every* $\lambda \in \mathbb{C}$, *we have eigenfunction* u_λ:

$$(\tilde{T} - \lambda)u_\lambda = 0 . \tag{4.3}$$

Proof. When $\dim(L^2_H(\Omega)) = 0$, $\tilde{T} = S$ by the definition. Otherwise $\tilde{T}$ is an extension of S. First, for $\lambda \in \sigma_p(S)$, the eigenfunction of the operator S is the solution of (4.3). Next, by Proposition 1, for $\lambda \in \rho(T) = \rho(S)$, we have a nontrivial solution $u \in D(T) = H^1_{\Sigma\sigma}(\Omega)$ of

$$(T - \lambda)u = \lambda h \in L^2_H(\Omega) .$$

Let $u_\lambda = u + h \in L^2_\sigma(\Omega) \cap H^1(\Omega)$. Then u_λ is a nontrivial solution of (4.3). □

5. Additional Remarks and Summary

We may also consider a restriction of S;

Definition. We define a rotation operator Q in $L^2_\Sigma(\Omega)$ by

$$Qu = \operatorname{rot} u \qquad \text{for } u \in D(Q) = H^1_0(\Omega) \cap L^2_\Sigma(\Omega) .$$

Proposition 2. *The operator Q is symmetric, however, is not closed. The resolvent set $\rho(Q)$ of Q is equal to $\rho(S)$. The spectra of Q consists of point spectrum $\sigma_p(Q)$ and residual spectrum $\sigma_r(Q)$. We have relations*

$$\sigma_p(Q) \subset \sigma_p(S), \qquad \sigma_r(Q) = \sigma_p(S) - \sigma_p(Q) .$$

Proof. Obviously Q is symmetric, however is not even closed; to be proved later. By Remark 1, we see $\sigma_p(Q) \subset \sigma_p(S)$. Let us show that $\sigma_p(S) - \sigma_p(Q) \subset \sigma_r(Q)$. Set $\lambda \in \sigma_p(S) - \sigma_p(Q)$, and denote by ϕ_λ the eigenfunction of S belonging to the eigenvalue λ. We have, for every $u \in D(Q)$,

$$((Q - \lambda)u, \phi_\lambda) = ((\operatorname{rot} - \lambda)u, \phi_\lambda) = (u, (\operatorname{rot} - \lambda)\phi_\lambda) = 0 ,$$

which shows that the domain of the resolvent $(Q - \lambda)^{-1}$ is orthogonal to ϕ_λ, so that λ is a residual spectrum of Q.

It remains to prove that $\rho(Q) = \rho(S)$. Let $\lambda \in \rho(S)$. For every $f \in L^2_\Sigma(\Omega)$, we have $u = (S - \lambda)^{-1} f \in H^1_{\Sigma\Sigma}(\Omega)$. Note that u may not be in $H^1_0(\Omega)$. Since $D(Q)$ is dense in $L^2_\Sigma(\Omega)$, there is a sequence $\{u_i \in D(Q)\}$ converging to u in the topology of $L^2_\Sigma(\Omega)$. Since $D(Q) \subset D(S)$, we have

$$\begin{aligned} \|(Q - \lambda)u_i - f\| &= \|(S - \lambda)u_i - f\| \\ &= \|(S - \lambda)(u_i - u)\| \to 0 , \end{aligned}$$

which shows that the domain of the resolvent $(Q - \lambda)^{-1}$ is dense in $L^2_\Sigma(\Omega)$. With taking $\lambda = 0$, this also shows that Q is not closed, since we may take $u \in D(S)$ such that u does not belong to $H^1_0(\Omega)$ (see Lemma 1(3)). For $f \in D((Q - \lambda)^{-1})$, we have an estimate

$$\|(Q-\lambda)^{-1}f\| = \|(S-\lambda)^{-1}f\| \leqq \alpha^{-1}\|f\|,$$

$$\alpha = \min_{\lambda_i \in \sigma_p(S)} |\lambda - \lambda_i|,$$

which shows that $(Q-\lambda)^{-1}$ is continuous on $D((Q-\lambda)^{-1})$, therefore $\lambda \in \rho(Q)$. Obviously $\rho(Q) \subset \rho(S)$, so the proof is now complete. □

Remark 4. Since the operator Q is not closed, the equation

$$\operatorname{rot} u = f \in L^2_\Sigma(\Omega) \tag{5.1}$$

may not have a solution $u \in D(Q) = H^1_0(\Omega) \cap L^2_\Sigma(\Omega)$, although $0 \in \rho(Q) = \rho(S)$. When we remove the condition that the solution be in $L^2_\Sigma(\Omega)$, we get a solution in $H^1_0(\Omega)$. We find the solution by changing the gauge of $u_0 = S^{-1}f \in H^1_{\Sigma\Sigma}(\Omega)$. By Lemma 1(3), we see $n \times u_0 = n \times \gamma$ with some $\gamma \in \operatorname{Ker}(\operatorname{rot})$. We may choose γ such that $n \cdot \gamma = 0$ (div γ may not be 0). Then $u_0 - \gamma$ is the desired solution. Borchers and Sohr [1] prove the solvability of (5.1) in $H^{1,r}_0(\Omega)$ that is the L^r-Sobolev space.

We summarize in Tables 1 and 2 the results obtained for various definitions of the space and the domain of the operator rot.

Table 1. Spaces, domains, and spectra of various rot operators. Here, we assume that $\dim(L^2_H(\Omega))$ is not zero, viz., genus of $\partial\Omega$ is greater than 0. The operator S is self-adjoint, so that $\sigma_p(S) \subset \mathbb{R}$

Operator	Space	Domain	Point spectrum	Continuous spectrum	Residual spectrum
Q	L^2_Σ	$H^1_0 \cap L^2_\Sigma$	$\sigma_p(Q)$	$\varnothing$	$\sigma_p(S) - \sigma_p(Q)$
S	L^2_Σ	$H^1_{\Sigma\Sigma}$	$\sigma_p(S)$	$\varnothing$	$\varnothing$
T	L^2_σ	$H^1_{\Sigma\sigma}$	$\sigma_p(S)$	$\varnothing$	$\varnothing$
$\tilde{T}$	L^2_σ	$H^1_{\sigma\sigma}$	$\mathbb{C}$	$\varnothing$	$\varnothing$

Table 2. Summary of various rot operators

Operator	Densely defined?	Closed?	Symmetric?	Self-adjoint?
Q	Yes	No	Yes	No
S	Yes	Yes	Yes	Yes
T	No	Yes	No	No
$\tilde{T}$	Yes	Yes	No	No

Appendix (Physical Background)

Here we give a short review of the background of the problem (1.1) and (1.2) in the plasma physics. Ampére's law relates a magnetic field H with a current density j;

$$\operatorname{rot} H = j,$$

where we neglected the so-called displacement current term that is small when the considered system is slowly evolving. The magnetic force acting on a plasma is given by

$$j \times H = (\operatorname{rot} H) \times H .$$

Here the permeability is assumed one for simplicity. When H satisfies

$$\operatorname{rot} H = \mu(x) H ,$$

with a scaler function $\mu(x)$, we see that $j \times H = 0$. Such magnetic fields are called *force-free magnetic fields.* It is obvious that the scalar function $\mu(x)$ should satisfy $(H, \operatorname{grad})\mu = 0$, because $\operatorname{div} H = 0$. The simplest force-free fields are characterized by $\mu(x) = \lambda$: real constants. These force-free fields are called *free-decay fields,* because they have the following significant character, which has been originally studied by Chandrasekhar–Kendall [3] for astrophysical plasmas. The magnetohydrodynamic equations of plasmas are

$$\partial H/\partial t = -\operatorname{rot} \eta \operatorname{rot} H + \operatorname{rot}(v \times H),$$

$$\rho\,[\partial v/\partial t + (v, \operatorname{grad})v] = \nu \Delta v + (\operatorname{rot} H) \times H - \operatorname{grad} p ,$$

and, for incompressible case,

$$\operatorname{div} v = 0 .$$

Here η, ν, ρ, p, v are respectively resistivity, viscosity, mass density, pressure, and the fluid velocity of the plasma. When η is a constant, a free-decay field H_f gives a solution;

$$H(x, t) = H_f(x) \cdot e^{-\eta\lambda^2 t}, \qquad v = 0, \quad p = 0 ,$$

which is a simply decaying magnetic field without causing any magnetic force and fluid motion.

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Received February 23, 1989; in final form June 5, 1989