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One-Dimensional Spectral Studies in Single Fluid MHD for Stability of Fusion Plasmas

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These are preliminary lecture notes, intended only for distribution to participants.

One-dimensional Spectral Studies in Single Fluid MHD for Stability of Fusion Plasmas

Autumn College on Plasma Physics @ICTP

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0. Preliminaries

Consider

$$\mathrm{i}\partial_t \boldsymbol{\psi} = \mathcal{A}\boldsymbol{\psi}, \tag{1}$$

where $\mathcal{A}$: $N \times N$-matrix, $\boldsymbol{\psi}(t)$: N-dimensional vector.

Scalar product

$$(\boldsymbol{\phi} \,|\, \boldsymbol{\psi}) = \boldsymbol{\phi} \cdot \bar{\boldsymbol{\psi}} \tag{2}$$

Eigenvalues and eigenvectors

$$\mathcal{A}\boldsymbol{\varphi}_j = \lambda_j \boldsymbol{\varphi}_j \tag{3}$$

If $\mathcal{A}$ is Hermitian (selfadjoint), we can span whole vector space by orthogonal eigenvectors (N eigenvectors)

$$(\boldsymbol{\varphi}_i \,|\, \boldsymbol{\varphi}_j) = 0 \quad \text{for } i \neq j \ (i, j = 1, \ldots, N) \tag{4}$$

$$(\boldsymbol{\varphi}_j \,|\, \boldsymbol{\varphi}_j) = 1 \quad (j = 1, \ldots, N) \tag{5}$$

Projection

$$\mathcal{P}_j = (\cdot \,|\, \boldsymbol{\varphi}_j)\boldsymbol{\varphi}_j \tag{6}$$

$$\boldsymbol{\psi} = \sum_{j=1}^{N} (\boldsymbol{\psi} \,|\, \boldsymbol{\varphi}_j)\boldsymbol{\varphi}_j \quad \text{for any } \boldsymbol{\psi} \tag{7}$$

Spectral resolution

$$\mathcal{A} = \sum_{j=1}^{N} \lambda_j \boldsymbol{\varphi}_j (\cdot \,|\, \boldsymbol{\varphi}_j) \tag{8}$$

Evolution equation becomes

$$
\begin{aligned}
\mathrm{i}\partial_t \boldsymbol{\psi}(t) &= \sum_{j=1}^{N} \lambda_j \boldsymbol{\varphi}_j (\boldsymbol{\psi}(t) \,|\, \boldsymbol{\varphi}_j) \\
&= \sum_{j=1}^{N} \lambda_j a_j(t) \boldsymbol{\varphi}_j
\end{aligned} \tag{9}
$$

Taking a scalar product with $\boldsymbol{\varphi}_i$,

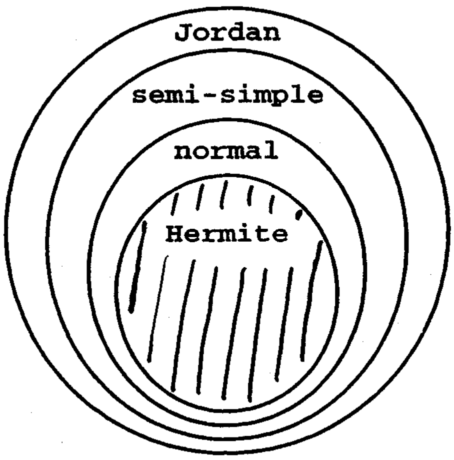
$$
\mathrm{i}\partial_t a_i(t) = \lambda_i a_i(t) \tag{10}
$$

$$\Downarrow$$

$$
a_i(t) = a_i(0) \exp(-\mathrm{i}\lambda_i t) \tag{11}
$$

Solution

$$
\boldsymbol{\psi}(t) = \sum_{j=1}^{N} a_j(0) \exp(-\mathrm{i}\lambda_j t) \boldsymbol{\varphi}_j \tag{12}
$$



1. MHD Equations

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{13}$$

$$\rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) = \boldsymbol{j} \times \boldsymbol{B} - \nabla p \ (+ \rho \boldsymbol{g}), \tag{14}$$

$$\partial_t p + \boldsymbol{v} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{v} = 0, \tag{15}$$

$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \tag{16}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}, \tag{17}$$

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \boldsymbol{0}, \tag{18}$$

Comment: We can remove $\boldsymbol{j}$, p, and $\boldsymbol{E}$ from the system.

$\Rightarrow$ seven waves

2. Linearized MHD Equation

Linearization

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 + \boldsymbol{\psi}_1, \tag{19}$$

where $|\boldsymbol{\psi}_1| \ll |\boldsymbol{\psi}_0|$.

Equilibrium is written by $\partial_t = 0$ in all equations.

Displacement vector $\boldsymbol{\xi}$

$$\partial_t \boldsymbol{\xi}(\boldsymbol{x}, t) = \boldsymbol{v}_1(\boldsymbol{x}, t), \quad \boldsymbol{\xi}(\boldsymbol{x}, 0) = \mathbf{0}. \tag{20}$$

Linearized MHD equation

$$\begin{aligned}\partial_t^2 \boldsymbol{\xi} &= \mathcal{F}\boldsymbol{\xi} \\ &= \frac{1}{\rho_0}\Big[\nabla(\gamma p_0 \nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla p_0) \\ &\quad + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}_0) \times [\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0)] \\ &\quad + \frac{1}{\mu_0}[\nabla \times (\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0))] \times \boldsymbol{B}_0\Big].\end{aligned} \tag{21}$$

Hermiticity of force operator $\mathcal{F}$

$$\langle \boldsymbol{\eta} \,|\, \mathcal{F}\boldsymbol{\xi} \rangle = \langle \mathcal{F}\boldsymbol{\eta} \,|\, \boldsymbol{\xi} \rangle, \tag{22}$$

$$\left(\langle \boldsymbol{\eta} \,|\, \boldsymbol{\xi} \rangle \equiv \frac{1}{2} \int_\Omega \rho_0 \boldsymbol{\eta} \cdot \bar{\boldsymbol{\xi}} \, \mathrm{d}V. \right)$$

Stability theory

1. Energy principle
2. Spectral analysis

2. Linearized MHD Equation

All eigenvalues of force operator $\mathcal{F}$ are real.

$$\mathcal{F}\boldsymbol{\xi} = \lambda\boldsymbol{\xi} \tag{23}$$

$$\left.\begin{aligned} \langle\boldsymbol{\xi}\,|\,\mathcal{F}\boldsymbol{\xi}\rangle &= \bar{\lambda}\langle\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle \\ \langle\mathcal{F}\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle &= \lambda\langle\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle \end{aligned}\right. \quad \Rightarrow \quad \lambda = \bar{\lambda} \ \text{: real}$$

Now the eigenvalue is $\lambda = -\omega^2$ with $\exp(-\mathrm{i}\omega t)$.

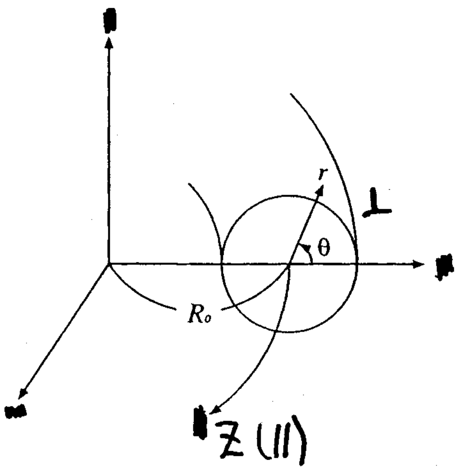
If we have <u>any</u> positive eigenvalue,

$$-\omega^2 = \lambda > 0 \quad \Rightarrow \omega = \pm\mathrm{i}\sqrt{\lambda}$$
$$\Rightarrow \exp(\mp\sqrt{\lambda}t) : \ \boxed{\text{unstable}}$$

If we have <u>all</u> eigenvalues negative,

$$-\omega^2 = \lambda < 0 \quad \Rightarrow \omega = \pm\sqrt{-\lambda}$$
$$\Rightarrow \exp(\pm\mathrm{i}\sqrt{-\lambda}t) : \ \boxed{\text{stable}}$$

3. Reduced MHD Equations[1]



Low-β tokamak ordering (ϵ = minor r./major r.)

$$B_z \sim 1 + \epsilon^2, \quad \nabla_\perp \sim 1, \quad B_\perp \sim \partial_z \sim \epsilon,$$
$$v_\perp \sim \epsilon, \quad p \sim \epsilon^2, \quad \partial_t \sim \boldsymbol{v} \cdot \nabla_\perp \sim \epsilon, \quad v_z \sim 0. \tag{24}$$

Static fluid $\boldsymbol{v}_0 = \boldsymbol{0}$. incompressible ($\nabla \cdot \boldsymbol{v} = 0$)

Stream and flux functions

$$\boldsymbol{v}_1 = \nabla\phi \times \boldsymbol{e}_z, \quad \boldsymbol{B}_1 = \nabla\psi \times \boldsymbol{e}_z. \tag{25}$$

Two-fields RMHD equations (after linearization)

$$\partial_t \Delta\phi = \boldsymbol{B}_0 \cdot \nabla\Delta\psi + (\nabla j_0 \times \boldsymbol{e}_z) \cdot \nabla\psi, \tag{26}$$
$$\partial_t \psi = \boldsymbol{B}_0 \cdot \nabla\phi. \tag{27}$$

Boundary conditions: $\phi = \psi = 0$ at the edge.

[1]H. R. Strauss, Phys. Fluids **19**, 134 (1976).

About norm

RMHD equation for homogeneous plasma

$$\partial_t u = \mathcal{A} u, \tag{28}$$

$$\mathcal{A} = \begin{pmatrix} 0 & \boldsymbol{B}\cdot\nabla\Delta \\ \boldsymbol{B}\cdot\nabla\Delta^{-1} & 0 \end{pmatrix}. \tag{29}$$

State vector $u = {}^T(\Delta\phi, \psi)$.

Norm should be taken with the metric

$$\boxed{\mathcal{M} = \begin{pmatrix} -\Delta^{-1} & 0 \\ 0 & -\Delta \end{pmatrix},} \tag{30}$$

as

$$\langle u \,|\, u^\dagger \rangle \equiv (\Delta\phi \,|\, -\Delta^{-1} \,|\, \Delta\phi^\dagger) + (\psi \,|\, -\Delta \,|\, \psi^\dagger). \tag{31}$$

where $(\phi \,|\, \phi^\dagger) = \int \phi \bar{\phi}^\dagger \, \mathrm{d}V$ denotes simple norm.

Physically, this norm corresponds to energy bilinear form

$$\begin{aligned} \langle u \,|\, u \rangle &= \int \Delta\bar{\phi}(-\Delta^{-1})\Delta\phi + \bar{\psi}(-\Delta)\psi \, \mathrm{d}V & (32) \\ &= \int |\nabla\phi|^2 + |\nabla\psi|^2 \, \mathrm{d}V, & (33) \end{aligned}$$

where $\boldsymbol{v} = \nabla\phi \times \boldsymbol{e}_z$, $\boldsymbol{B} = \nabla\psi \times \boldsymbol{e}_z$.

About norm

Difficult for state vector u in inhomogeneous case.

Combining two equaitons

$$\begin{aligned}\partial_t^2 \Delta\phi &= \mathcal{A}_{\mathrm{u}} \Delta\phi \\ &= \boldsymbol{B}\cdot\nabla\Delta\boldsymbol{B}\cdot\nabla\Delta^{-1}(\Delta\phi) \\ &\quad + (\nabla j \times \boldsymbol{e}_z)\cdot\nabla\boldsymbol{B}\cdot\nabla\Delta^{-1}(\Delta\phi) \qquad (34)\end{aligned}$$

Define a scalar product as

$$\langle \Delta\phi \,|\, \Delta\phi^\dagger \rangle = (\Delta\phi \,|\, -\Delta^{-1} \,|\, \Delta\phi^\dagger), \qquad (35)$$

kinetic energy norm

then $\mathcal{A}_{\mathrm{u}}$ becoms Hermitian.

4. Spectrum for Static Plasmas ($\boldsymbol{v}_0 = \mathbf{0}$)

(a) Continuous spectra in slab geometry

Equilibrium magnetic field

$$\boldsymbol{B} = (0, B_y(x), B_z) \tag{36}$$

Alfvén equation for eigenvalue ω^2

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(\omega^2 - \omega_{\mathrm{A}}^2)\frac{\mathrm{d}\phi}{\mathrm{d}x}\right] - k_y^2(\omega^2 - \omega_{\mathrm{A}}^2)\phi = 0, \tag{37}$$

where $\omega_{\mathrm{A}}(x) = \boldsymbol{k} \cdot \boldsymbol{B}(x)/\sqrt{\mu_0 \rho}$.

Regular singularity appears at $x = x_{\mathrm{s}}$ when $\omega = \omega_{\mathrm{A}}(x_{\mathrm{s}})$. Solution is (due to Frobenius)

$$\phi(x) = a_1 g_1(x) + a_2[g_1(x) \log|x - x_{\mathrm{s}}| + g_2(x)]. \tag{38}$$

where $g_1(x)$ and $g_2(x)$ are analytic functions around x_{s}. Note: this solution is non-square-integrable under previous norm

$$\int (\log|x - x_{\mathrm{s}}|)\Delta(\log|x - x_{\mathrm{s}}|)\,\mathrm{d}x \sim -\int \frac{1}{(x - x_{\mathrm{s}})^2}\,\mathrm{d}x \tag{39}$$

There is no other solution in slab Alfvén equation.

$$\underline{\omega}_{\mathrm{A}}^2 = \inf_x \omega_{\mathrm{A}}^2(x) : \quad \text{lower bound,} \tag{40}$$

Dividing the singular factor

$$\omega^2 - \omega_{\mathrm{A}}^2 = (\omega^2 - \underline{\omega}_{\mathrm{A}}^2) + (\underline{\omega}_{\mathrm{A}}^2 - \omega_{\mathrm{A}}^2), \tag{41}$$

multiplying $\bar{\phi}$, and integrating with respect to x

$$(\omega^2 - \underline{\omega}_{\mathrm{A}}^2) \int_\Omega \left(\left| \frac{\mathrm{d}\phi}{\mathrm{d}x} \right|^2 + k_y^2 |\phi|^2 \right) \mathrm{d}x$$

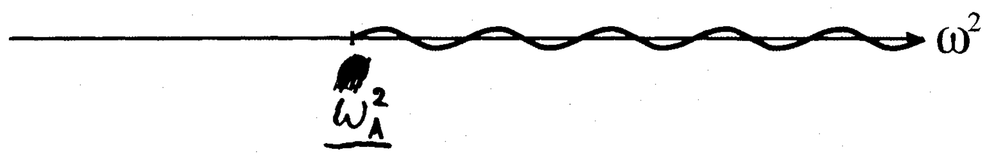
$$= - \int_\Omega (\underline{\omega}_{\mathrm{A}}^2 - \omega_{\mathrm{A}}^2) \left(\left| \frac{\mathrm{d}\phi}{\mathrm{d}x} \right|^2 + k_y^2 |\phi|^2 \right) \mathrm{d}x. \tag{42}$$

$$\Downarrow$$

$$\omega^2 \geq \underline{\omega}_{\mathrm{A}}^2 \tag{43}$$

Spectrum is

$$\sigma = \sigma_{\mathrm{c}} = \{\omega^2 \mid \min_{x \in \Omega} \omega_{\mathrm{A}}^2 \leq \omega^2 \leq \max_{x \in \Omega} \omega_{\mathrm{A}}^2\}. \tag{44}$$

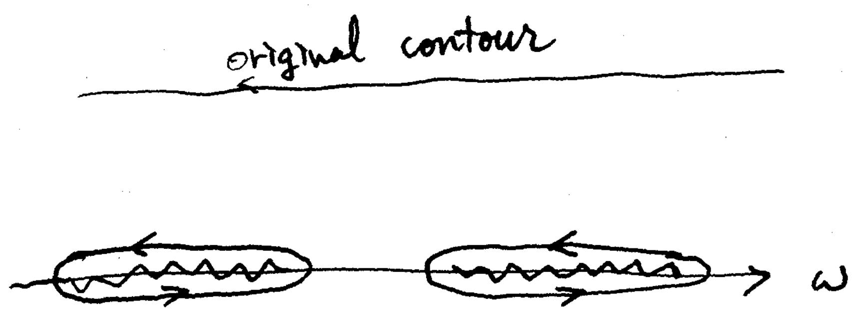


Solving by Laplace transform

$$\tilde{\phi}(\omega) = \int_0^\infty \phi(t)\, e^{\mathrm{i}\omega t}\, \mathrm{d}t, \tag{45}$$

$$\phi(t) = \frac{1}{2\pi \mathrm{i}} \int_C \tilde{\phi}(\omega)\, e^{-\mathrm{i}\omega t}\, \mathrm{d}\omega, \tag{46}$$

Branch cuts appear on $\omega = \omega_{\mathrm{A}}$



⇓

Continuum damping

$$\phi \propto \frac{1}{t} \exp[\mathrm{i}\omega_{\mathrm{A}}(x)t] + \frac{1}{t} \exp[-\mathrm{i}\omega_{\mathrm{A}}(x)t] \tag{47}$$

no stationary oscillation

(b) Instability in cylindrical geometry

Equilibrium magnetic field

$$\boldsymbol{B} = (0, B_\theta(r), B_z) \tag{48}$$

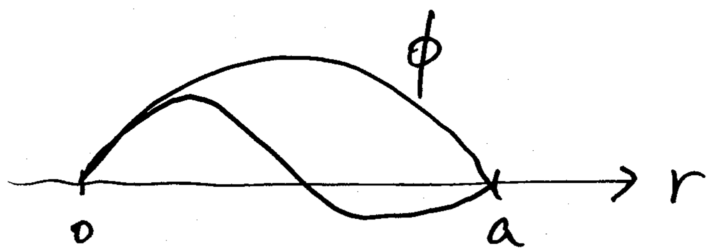
Alfvén equation for eigenvalue ω^2

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r(\omega^2-\omega_\mathrm{A}^2)\frac{\mathrm{d}\phi}{\mathrm{d}r}\right] - \frac{m^2}{r^2}(\omega^2-\omega_\mathrm{A}^2)\phi + \boxed{\underbrace{\frac{2}{r}\frac{\mathrm{d}F}{\mathrm{d}r}F\phi}_{\text{extra}}} = 0, \tag{49}$$

Point spectra (kink modes) appear due to extra term.

Boundary conditions for $m \geq 1$ are

$$\phi = 0 \quad \text{at } r = 0, a \tag{50}$$



Simple ODEs

$$u'' - (-1)u = 0, \quad v'' - (-4)v = 0 \tag{51}$$

$$(-1 > -4)$$

have solutions

$$u \sim \sin x, \quad v \sim \sin 2x, \tag{52}$$

slow rapid

$\Downarrow$ generalize

Oscillation theorem (Sturm)

For the ODE in the real domain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[K_1(r,\omega)\frac{\mathrm{d}u}{\mathrm{d}r}\right] - G_1(r,\omega)u = 0 \tag{53}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[K_2(r,\omega)\frac{\mathrm{d}v}{\mathrm{d}r}\right] - G_2(r,\omega)v = 0 \tag{54}$$

If $K_1 > K_2 > 0$ and $G_1 > G_2$ for any r, then v oscillates more rapidly than u.

some more considerations...

Forget about the boundary condition at $r = a$!

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r(\omega_{\mathrm{A}}^2(x) - \omega^2)\frac{\mathrm{d}\phi}{\mathrm{d}r}\right] - \frac{m^2}{r}(\omega_{\mathrm{A}}^2(x) - \omega^2)\phi - \frac{2}{r}\frac{\mathrm{d}F}{\mathrm{d}r}F\phi = 0, \tag{55}$$

$K > 0$ $\quad G\phi$

Suppose two neighboring solutions

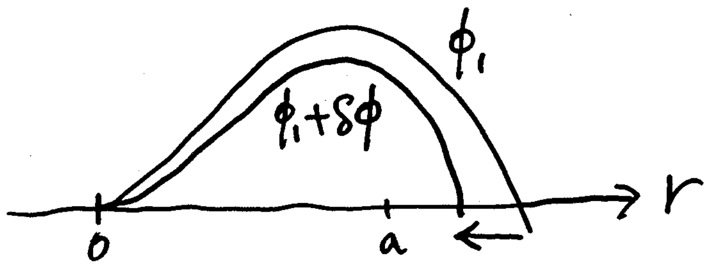
ϕ_1 :	solution for ω_1^2	with	$\phi_1(0) = 0$
$\phi_1 + \delta\phi$:	solution for $\omega_1^2 + \delta\omega^2$	with	$\phi_1 + \delta\phi(0) = 0$

If $\delta\omega^2 > 0$, then

$$K(\omega_1^2) > K(\omega_1^2 + \delta\omega^2), \quad G(\omega_1^2) > G(\omega_1^2 + \delta\omega^2) \tag{56}$$

$$\Downarrow$$

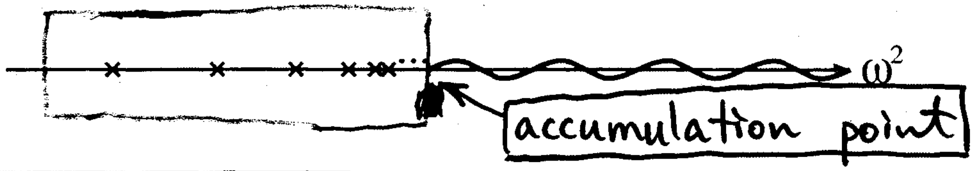
$\phi_1 + \delta\phi$ oscillates more rapidly than ϕ_1.



Spectra are

infinite numbers

$$\sigma = \sigma_{\mathrm{c}} \oplus \sigma_{\mathrm{p}} \tag{57}$$



[1]Goedbloed and Sakanaka, Phys. Fluids **17**, 908 (1974).

(c) Interchange Instability in Stellarators[2]

Stellarator ordering

$$B \sim B_z + \epsilon^{1/2} \underline{\nabla \eta} + \epsilon$$

$$p \sim \epsilon \qquad \text{helical field}$$

RMHD equations for stellarators

$$\partial_t \psi = \boldsymbol{B} \cdot \nabla \phi, \tag{58}$$

$$\rho \frac{\mathrm{d}\Delta\phi}{\mathrm{d}t} = -\boldsymbol{B} \cdot \nabla j_z + \boxed{\nabla \kappa \times \nabla p \cdot \boldsymbol{e}_z,} \tag{59}$$

drive

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 0, \tag{60}$$

where

$$\boldsymbol{B} \cdot \nabla = B_0 \partial_z + \nabla \psi \times \boldsymbol{e}_z \cdot \nabla, \tag{61}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \nabla \phi \times \boldsymbol{e}_z \cdot \nabla, \tag{62}$$

$$\kappa = \frac{2r\cos\theta}{R_0} + \boxed{\frac{\overline{(\nabla\eta)^2}}{B_0^2},} \tag{63}$$

helical curvature

$$j_z = -\Delta A_z, \tag{64}$$

$$A_z = \psi + \frac{1}{2B_0} \overline{\nabla \langle \eta \rangle \times \nabla \eta} \cdot \boldsymbol{e}_z. \tag{65}$$

[2]Tatsuno, *et al.*, Nucl. Fusion **39**, 1391 (1999).
Carreras *et al.*, Phys. Plasmas **8**, 990 (2001).

Eigenvalue equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \left[\frac{1}{r} - \frac{2m\iota'(n-m\iota)}{\gamma^2+(n-m\iota)^2}\right]\frac{\mathrm{d}\phi}{\mathrm{d}r} - \left\{\frac{m^2}{r^2} + \frac{1}{\gamma^2+(n-m\iota)^2} \times \left[\left(\frac{m\iota'}{r}+m\iota''\right)(n-m\iota) - \frac{D_\mathrm{s}m^2}{r^2}\right]\right\}\phi = 0$$

$m(n)$: poloidal(toroidal) mode numbers,

ι: Rotational transform,

$D_\mathrm{s} = -\frac{1}{2}\beta_0 N p'(4r\iota + r^2\iota')$: Instability drive,

β_0: central toroidal beta,

N: toroidal period number of the helical coils.

Numerical solution for the $(m, n) = (2, 1)$ mode.

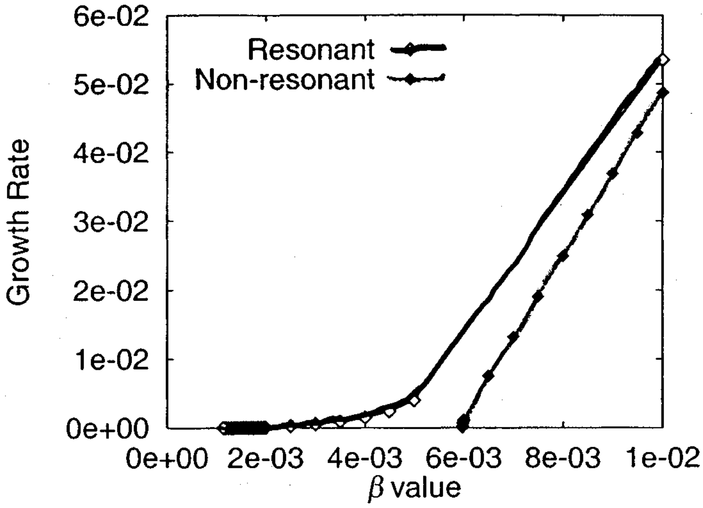
Equilibrium profiles are

$\iota = 0.499 + 0.2r^2$ (resonant), ← rational surface r~0.07

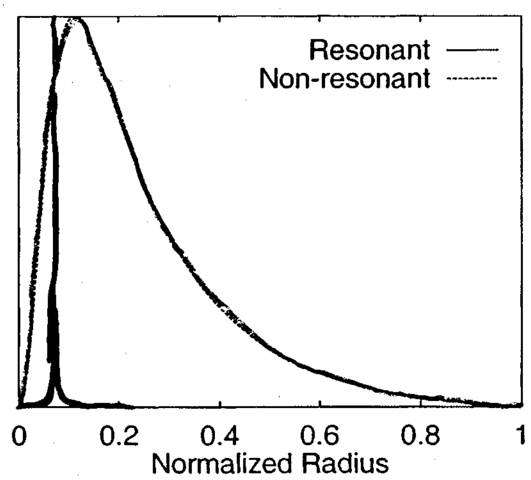
$\iota = 0.501 + 0.2r^2$ (non-resonant), ← no rational surface

$p = p_0(1 - r^4)$ (for both).

Beta dependence of growth rate



Eigenfunction near the beta limit



5. Spectral Studies for Shear Flow Plasmas

(a) Linear Shear Flow Profile and Kelvin's Method[3]

Consider linear shear flow profile in a slab plasma

$$\boldsymbol{v}_0 = (0, \sigma x, 0) \qquad \sigma = \text{const.} \tag{66}$$

Non-Hermiticity only enters from $\boldsymbol{v}_0 \cdot \nabla$ operator

$$\partial_t u + \boldsymbol{v}_0 \cdot \nabla u = \mathcal{A} u \tag{67}$$

coordinate transform ↙ ↘ spectral resolution

$\mathcal{A}$: Hermitian (selfadjoint) operator

Suppose we have a set of 'shearing modes' satisfying two conditions

- Characteristic equation

$$\partial_t \tilde{\varphi}(t; k, x) + \boldsymbol{v}_0 \cdot \nabla \tilde{\varphi}(t; k, x) = 0. \tag{68}$$

- Eigenequation

$$\mathcal{A} \tilde{\varphi}(t; k, x) = \lambda_k(t)\, \tilde{\varphi}(t; k, x). \tag{69}$$

[3]Volponi, Yoshida, & Tatsuno, Phys. Plasmas **7**, 2314 (2000). Tatsuno, Volponi, & Yoshida, Phys. Plasmas **8**, 399 (2001).

We can decompose any function as

$$u = \int \hat{u}_k(t)\, \tilde{\varphi}(t; k, x)\, \mathrm{d}k. \tag{70}$$

Plugging this expression into eq. (67),

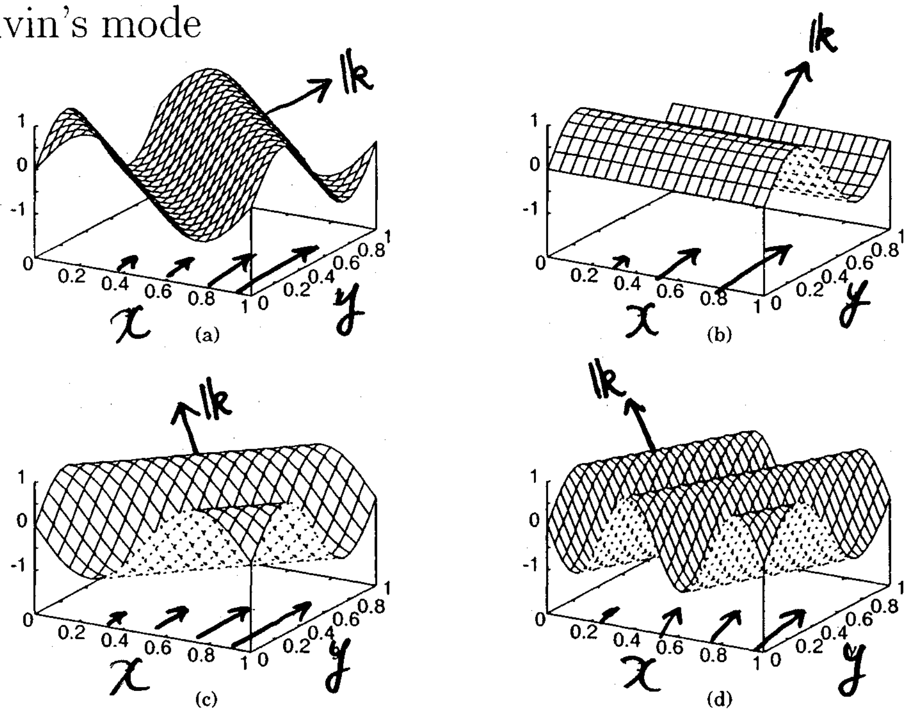
$$\int [\partial_t \hat{u}_k(t)]\, \tilde{\varphi}(t; k, x)\, \mathrm{d}k = \int \hat{u}_k(t) \lambda_k(t)\, \tilde{\varphi}(t; k, x)\, \mathrm{d}k, \tag{71}$$

we can obtain the following ODE on time for each mode due to the orthogonality of the eigenvectors;

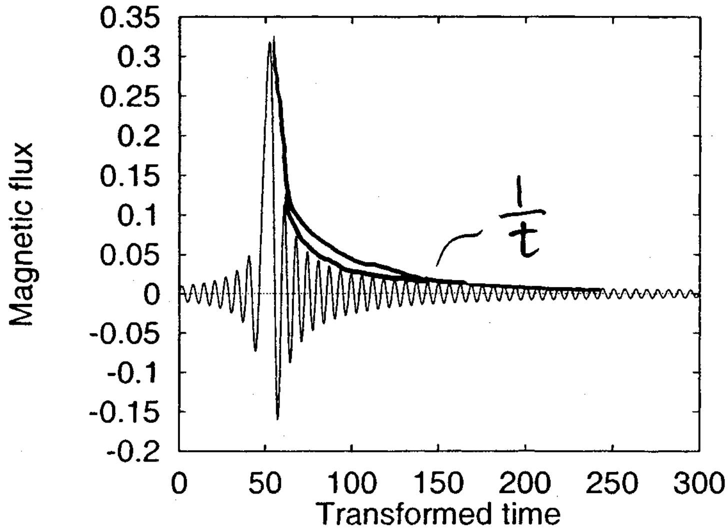
$$\mathrm{d}_t \hat{u}_k(t) = \lambda_k(t)\, \hat{u}_k(t). \tag{72}$$

This is no longer a simple exponential evolution.

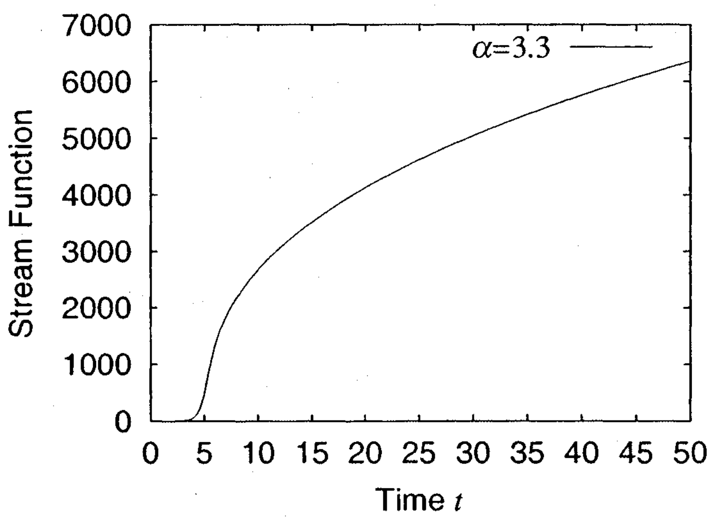
Kelvin's mode



Transients and Secularities of Kelvin's modes



$$\psi \sim e^{i\omega t}/t$$

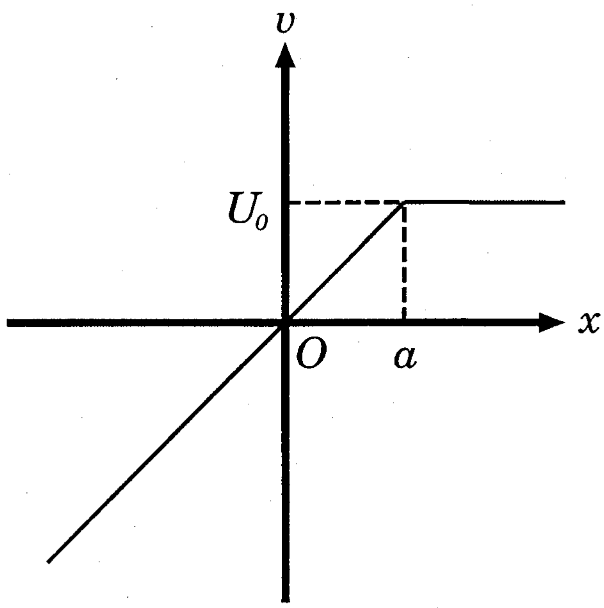


$$\phi \sim t^{0.35}$$

(b) Kelvin-Helmholtz Instability with Surface Wave Model

Consider 2-D Euler fluid

$$\boldsymbol{v}_0 = (0, v(x)) \tag{73}$$



Generalized Rayleigh equation

$$\mathrm{i}\partial_t \Psi = kv(x)\Psi + kw(x)\mathcal{K}\Psi. \tag{74}$$

$$(\mathcal{K}f)(x) = -\Delta^{-1} = \int_{-\infty}^{+\infty} \frac{e^{-k|x-\xi|}}{2k} f(\xi)\,\mathrm{d}\xi. \tag{75}$$

with $\Psi = -\Delta\phi$: vorticity.

$w(x) = v''(x)$ gives ordinary Rayleigh equation.

Norm again

In enstrophy norm

$$\langle\!\langle \varPsi \mid kv\varPsi^{\dagger} \rangle\!\rangle = \int kv\bar{\varPsi}\varPsi^{\dagger}\,\mathrm{d}x$$
$$= \langle\!\langle kv\varPsi \mid \varPsi^{\dagger} \rangle\!\rangle, \tag{76}$$

$$\langle\!\langle \varPsi \mid kw\Delta^{-1}\varPsi^{\dagger} \rangle\!\rangle = \langle\!\langle \Delta^{-1}kw\varPsi \mid \varPsi^{\dagger} \rangle\!\rangle$$
$$\neq \langle\!\langle kw\Delta^{-1}\varPsi \mid \varPsi^{\dagger} \rangle\!\rangle. \tag{77}$$

opeartor $kv(x)$	Hermitian
operator $kw(x)\mathcal{K}$	non-Hermitian

In energy norm

$$\langle \varPsi \mid kv\varPsi^{\dagger} \rangle = -\int \bar{\varPsi}\,\Delta^{-1}(kv\varPsi^{\dagger})\,\mathrm{d}V$$
$$= -\int (kv\Delta^{-1}\bar{\varPsi})\,\Delta\Delta^{-1}\varPsi^{\dagger}\,\mathrm{d}x$$
$$= -\int (\Delta kv\Delta^{-1}\bar{\varPsi})\,\Delta^{-1}\varPsi^{\dagger}\,\mathrm{d}x$$
$$= \langle \Delta kv\Delta^{-1}\varPsi \mid \varPsi^{\dagger} \rangle. \tag{78}$$

$$\langle \varPsi \mid kw\Delta^{-1}\varPsi^{\dagger} \rangle = -\int \bar{\varPsi}\,\Delta^{-1}(kw\Delta^{-1}\varPsi^{\dagger})\,\mathrm{d}x$$
$$= -\int (kw\Delta^{-1}\bar{\varPsi})\,\Delta^{-1}\varPsi^{\dagger}\,\mathrm{d}x$$
$$= \langle kw\Delta^{-1}\varPsi \mid \varPsi^{\dagger} \rangle, \tag{79}$$

operator $kv(x)$	non-Hermitian
operator $kw(x)\mathcal{K}$	Hermitian

Dividing vorticity field as

$$\boxed{\Psi = \alpha(t)\delta(x-a) \ + \ \tilde{\Psi}(x,t)} \tag{80}$$

Evolution equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) = \frac{U}{2a}(2ka-1)\alpha(t) - \frac{U}{2a}\int_{-\infty}^{\infty} e^{-k|a-\xi|}\tilde{\Psi}(\xi,t)\,\mathrm{d}\xi, \tag{81}$$

$$\mathrm{i}\partial_t\tilde{\Psi}(x,t) = kv(x)\tilde{\Psi}(x,t). \tag{82}$$

Eigenvalue problem

$$\begin{aligned}\lambda\varphi(x) &= \mathcal{A}\varphi(x)\\ &= kv(x)\varphi(x)\\ &\quad - \frac{U}{2a}\delta(x-a)\int_{-\infty}^{\infty} e^{-k|x-\xi|}\varphi(\xi)\,\mathrm{d}\xi.\end{aligned} \tag{83}$$

Two kinds of eigenmodes exist

	eigenvalue	eigenfunction
point	$\lambda_1 = kU - U/2a$	$\varphi_1 = \delta(x-a)$
continuum	$\lambda_\mu = kU\mu/a$	$\varphi_\mu = \delta(x-\mu) + \dfrac{e^{-k(a-\mu)}}{2k(a-\mu)-1}\delta(x-a)$

where $\mu < a \wedge \mu \neq a - 1/2k.$

When $\mu = a - 1/2k$, frequencies overlap $(kU\mu/a = \lambda_1)$.
Eigenfunction in a wider sense: $\varphi_2 = \delta(x - \mu_0)$

$$(\lambda_1 - \mathcal{A})\varphi_2 = \frac{U}{2a}e^{-k(a-\mu_0)}\varphi_1, \qquad (84)$$

$$(\lambda_1 - \mathcal{A})^2\varphi_2 = 0. \qquad (85)$$

Taking basis vector by $\delta(x-a)$ and $\delta(x-\mu)$,

$$\mathcal{A} = \begin{pmatrix} kU - \frac{U}{2a} & -\frac{U}{2a}e^{-k(a-\mu)} \\ 0 & kU\frac{\mu}{a} \end{pmatrix}. \qquad (86)$$

Including $\mu = a - 1/2k$.

frequencies overlap when $\mu = a - 1/2k$

⇓

For an initial condition $\Psi(0) = \varphi_2$,

$$\Psi(t) = \mathrm{i}\frac{U}{2a\sqrt{e}}t\varphi_2 . e^{ikU\mu_0 t/a} \qquad (87)$$

secularity

Linear Stability Theory

(from the viewpoint of spectral analysis)

to show instability — rather simple

you can show by finding one growing solution

to show stability — rather difficult

you must know all spectral properties

including exponential growth

secular

⋮

↓

non-Hermiticity