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Relativistic Preliminaries

S.M. Mahajan

University of Texas at Austin,
Fusion Research Centre
U.S.A.

These are preliminary lecture notes, intended only for distribution to participants.

strada costiera, 11 - 34014 trieste italy - tel. +39 0402240111 fax +39 040224163 - sci_info@ictp.trieste.it - www.ictp.trieste.it

Notation

$V^\mu \rightarrow$ Contravariant Index (transforms as x^μ

$V_\mu \rightarrow$ Covariant Index " " p_μ

The Metric $\eta^{\mu\nu}$ = diag $\{-1, 1, 1, 1\}$ — Minkowsky
raises or lowers indices

$$V^\mu = \eta^\mu{}_\alpha V^\alpha$$

Summation Convention : A repeated index is to be summed over (0-3)

$$x^\mu = (ct, \overset{x^1}{x_1}, \overset{x^2}{x_2}, \overset{x^3}{x_3}) \equiv (x^0, \overset{x^1}{x_1}, \overset{x^2}{x_2}, \overset{x^3}{x_3})$$

For the spatial indices, the upper and lower are equivalent — For time they change sign.

Greece indices $\rightarrow$ 0-3 $\alpha\ \beta \cdots$

Latin indices $\rightarrow$ 1-3 $i\ j\ k \cdots$

A tensor of Rank n has n independent indices

$R^{\alpha\beta\gamma\cdots\delta}$ $\rightarrow$ all Contravariant

$R_{\alpha\beta\cdots\delta}$ $\rightarrow$ all Covariant

$R^{\alpha\beta\cdots}{}_{\gamma\delta\cdots}$ $\rightarrow$ Mixed

Contraction : In a mixed tensor if two indices are the same

$R^{\alpha}{}_{\alpha}{}^{\beta}{}_{\delta}$ is a tensor of rank two

$R^\alpha{}_\alpha$ is a scalar.

Order is important

Four Vectors

$$V^\mu = (V^0, \underset{\sim}{V})$$

→ three components.

The Contraction (four dot product)

$$V^\mu V_\mu = \eta^\mu{}_\alpha V^\alpha V_\mu = V^{0\,2} - V^2 .$$

Lorentz Transformations :

	S lab. frame	$S' \longrightarrow \underset{\sim}{v}$ with r.t S
Coordinates	x	x'

$$x'^\mu = \underset{\text{boosts}}{\Lambda^\mu{}_\nu} x^\nu + \underset{\text{translations}}{a^\mu}$$

$$\boxed{\begin{aligned} \Lambda^0{}_0 &= \gamma \\ \Lambda^i{}_j &= \delta^i{}_j + \frac{v_i v_j}{v^2}(\gamma - 1) \\ \Lambda^0{}_j &= \gamma v_j \end{aligned}}$$

symmetric

$c = 1$

For ~~on~~ a pure Lorentz boost only $\Lambda^\mu{}_\nu$ is needed.

$$\Lambda^\alpha{}_\beta \Lambda^\gamma{}_\delta \eta_{\alpha\gamma} = \eta_{\beta\delta} \qquad \text{orthogonality}$$

What is a Lorentz Scalar?

	S	S'
phase-space co-ordinate-momentum	x, p	x', p'
Some Object	F	F'

Scalar, then, is that object

$$F'(x', p') = F(x, p) = F'(\Lambda x, \Lambda p)$$

A vector V will then have the characteristic

$$V'(x', p') = \Lambda V(x, p)$$

$$Q'(x', p') = \Lambda \Lambda Q(x, p) \quad \text{etc.} \dots$$

In fact these are simply the transformation characterstics of physical (various) quantities.

Invariance: An Extremely important notion.

$G(x, p)$ is invariant if

$$G'(x', p') = G(x', p') \quad \Big\} \text{ functional form preserv}$$

For a Lorentz Scalar

$$G'(x', p') = G(x, p)$$

$$\Rightarrow \boxed{G(x', p') = G(x, p) = G(\Lambda x, \Lambda p)}$$

Lorentz Invariants are the staff of life.

Examples: $\quad p \cdot x \equiv p^\mu x_\mu \qquad d^4x = d^4x' \qquad |\Lambda| = 1$

Four Vectors — Four - Force

We know Newtonian force $\rightarrow$ a vector in 3-d co-ordinate space.

How then do we construct a four force (I will denote the force by $\underset{\sim}{Q}$)

$$\frac{d\underset{\sim}{p}}{dt} = \underset{\sim}{Q} \qquad \text{Newtons law}$$

What is the relativistic form $[d\tau = \gamma^{-1} dt]$?

$$\boxed{\frac{dp^\mu}{d\tau} = q^\mu} \quad \leftarrow \quad \text{four-force}$$

Unless we have a prescription to ~~know~~ derive q^μ from $\underset{\sim}{Q}$, we are stuck.

(i) In the system which is instantaneously at rest, let us also define a four-force

$$Q^0 = 0 \; , \; Q^i = \underset{\sim}{Q}$$

(ii) Let us boost it to a system moving with speed $\underset{\sim}{v}$ — Use Lorentz-transformation law

$$q^0 = \Lambda^0{}_\nu Q^\nu = \Lambda^0{}_i Q^i = \gamma \underset{\sim}{v} \cdot \underset{\sim}{Q}$$

$$q^i = \Lambda^i{}_\nu Q^\nu = \Lambda^i{}_j Q^j = Q^i + \frac{\gamma - 1}{v^2} v^i \underset{\sim}{v} \cdot \underset{\sim}{Q}$$

$$\underset{\sim}{q} = \underset{\sim}{Q} + \frac{\gamma-1}{v^2} (\underset{\sim}{v} \cdot \underset{\sim}{Q}) \underset{\sim}{v}$$

So we have calculate $q^\mu = [q^0, \underset{\sim}{q}]$ fully in terms of $\underset{\sim}{Q}$ and the velocity $\underset{\sim}{v}$.

What is four velocity ?

$$V^\mu = [\gamma, \gamma \underset{\sim}{v}]$$

$$V^\mu V_\mu = -\gamma^2 [1 - v^2] = -1 \qquad \text{Invariant}$$

$$V_R{}^\mu = [1, 0]$$

$$p_R{}^\mu = [m, 0] \quad \Rightarrow \quad p^\mu = [\gamma m, \gamma m \underset{\sim}{v}]$$

$$\therefore \quad p^0 = \gamma m \Rightarrow p^{0\,2} = \gamma^2 m^2 = m^2(\gamma^2 - 1) + m^2$$
$$= m^2 + \underset{\sim}{p}^2$$

$\Rightarrow$ Mass-shell condition (holds for physical particles)

$$p^\mu = [p^0, \underset{\sim}{p}] \qquad p^0 = \sqrt{p^2 + m^2}$$

will be often used

Flux: $\quad \Gamma^\mu = [\Gamma^0, \underset{\sim}{\Gamma}]$

$$\Gamma^0 = [n, n \underset{\sim}{v}] \equiv [\gamma n_R, \gamma n_R \underset{\sim}{v}]$$

Therefore <u>density</u> is not a scalar $\Longrightarrow$ it is the zeroth component of the Flux four-vector.

The Electromagnetic Field

$$\underset{\sim}{E},\ \underset{\sim}{B} \Leftrightarrow (\underset{\sim}{A},\ \phi)$$

We will find, however, that although, we do have a four vector

$$A^\mu = [A^0; \underset{\sim}{A}]$$

$\underset{\sim}{E}$ and $\underset{\sim}{B}$ cannot be changed to some equivalent four-vectors.

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad \partial_\mu = \left(\frac{\partial}{\partial t}, \underset{\sim}{\nabla}\right)$$

$$\partial^\mu = \left(-\frac{\partial}{\partial t}, \underset{\sim}{\nabla}\right)$$

a fully-antisymmetric 2nd rank tensor.
Its components are

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

$$\boxed{F^{0i} = E^i \ , \qquad F^{ij} = \epsilon^{ijk} B_k \ , \quad F^{\alpha\alpha} = 0}$$

You should show that

$$\dot{p} = q\,[\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}] \equiv \underset{\sim}{F}_L$$

$$\Longrightarrow$$

$$\frac{dp^\mu}{d\tau} = f_L^\mu \qquad \Gamma^\mu = \eta_R V^\mu$$

$$\boxed{f_L^\mu = F^\mu{}_\nu V^\nu}$$

Exercise: Show $\boxed{F^{\mu\nu} G_\nu = [\underset{\sim}{E} \cdot \underset{\sim}{G}\ ,\ \underset{\sim}{E} G_0 + B \times \underset{\sim}{G}]}$

Lorentz Invariance - Again

This is an essential constraint on physical theories
⇒ They have to be Lorentz Invariant ⇒
Restrictive and a great help in fixing what theories are possible and what are not

For our purpose here we shall have the simple workable definition:

[All Equations of the Theory Must be equalities between tensors of equal rank]

0 is a tensor of arbitrary Rank

$$R^{\alpha} = 0$$

$$R^{\mu\nu} = 0$$

$$R^{\mu\nu} + p^{\mu} = 0 \quad \text{is not fine}$$

Sounds like trivial ? Any thing but

that ⇒

(Tensors and Spinors → Representations of the Lorentz Group Poincare)

How About Equations of nonrelativistic plasma physics ?

Invariants of the Electromagnetic field

$$F^{\mu\nu} F_{\mu\nu} \propto B^2 - E^2 \qquad B^2 - E^2 = W$$

$$\epsilon^{\alpha\beta\lambda\nu} F_{\alpha\beta} F_{\lambda\nu} \propto \underset{\sim}{E} \cdot \underset{\sim}{B} \qquad \underset{\sim}{E} \cdot \underset{\sim}{B} = \lambda W$$

The quantity

$$\epsilon^{\alpha\beta\lambda\nu} F_{\alpha\beta} = \mathcal{F}^{\lambda\nu} \rightarrow \text{dual of } F.$$

$$\mathcal{F} = F\left(\underset{\sim}{E} \rightarrow \underset{\sim}{B},\ \underset{\sim}{B} \rightarrow -\underset{\sim}{E}\right)$$

No matter what frame you go to W and λ are always the same

These are the specific properties of a given electromagnetic field configuration and hence are very important labels.

For light waves in vaccum ($|B| = |E|$, $\underset{\sim}{B} \perp E$,)

$$B^2 - E^2 = 0$$

$$\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$$

⇒ exercise :- just think and figure out because of this, there is no frame in which the light wave (or an e-m wave in vacuum) can have a pure electric or pure magnetic field ⟶ In all other cases it is possible.

Quasi Projection Operators

- From ~~F~~ $F^{\mu\nu}$ and its dual $\mathcal{F}^{\mu\nu}$, we can construct ~~...~~ symmetric 2nd Rank tensors – Only two of these are obvious and it turns out they are also not independent:

$$\mathcal{E}^{\mu\nu} = F^{\mu}{}_{\alpha} F^{\alpha\nu}$$

$$e^{\mu\nu} = -\frac{1}{w} F^{\mu}{}_{\alpha} F^{\alpha\nu}$$

$$b^{\mu\nu} = \frac{1}{w} \mathcal{F}^{\mu}{}_{\alpha} \mathcal{F}^{\alpha\nu}$$

$$\Rightarrow \quad \boxed{e^{\mu\nu} + b^{\mu\nu} = \eta^{\mu\nu}} \qquad (1)$$

Exercise : show that if we have a magnetic field along, say, the z (3) direction and the electric field is zero

$$e^{\mu\nu} = \text{diag}\{0, 1, 1, 0\} \qquad \text{Perpendicular Projector}$$

$$b^{\mu\nu} = \text{diag}\{-1, 0, 0, 1\} \qquad \text{parallel ~~projector~~}$$

Exercise : show at your leisure that (1) holds for arbitrary electric and magnetic fields.

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ercise :
arder)

$$F^{\mu x} e_x{}^{\nu} = F^{\mu\nu} - \lambda \mathcal{F}^{\mu\nu} \xrightarrow{\lambda \to 0} F^{\mu\nu}$$

$$\mathcal{E}^{\mu x} \mathcal{E}_x{}^{\nu} = e^{\mu\nu} + \lambda^2 \eta^{\mu\nu} \xrightarrow{\lambda \to 0} e^{\mu\nu}$$

We will be dealing with only those systems for which $\lambda \to 0$.

Thus $\mathcal{E}$ are exact projection operators in the limit of $E_{\parallel} \to 0$: This will be one of the defining ~~constraints~~ of a magnetized plasma why $\epsilon^{\mu\nu}$ $(b^{\mu\nu})$?

(i) We have to eventually construct the $T^{\mu\nu}$ energy-momentum tensor for a plasma in the presence of electromagnetic fields.

(ii) $T^{\mu\nu}$ is a symmetric tensor

(iii) It will, therefore, be constructed from symmetric tensors — q.e.d

Energy Momentum Tensor $T^{\mu\nu}$

For an ideal Fluid : See Landau & Lifschitz, Weinberg

in its rest frame

$$\bar{T}^{00} = u \qquad \bar{T}^{ii} = p \quad , \quad T^{\mu\nu}_{\mu\neq\nu} = 0$$

u ⇓ proper energy density; p ⇓ pressure

$$\bar{T} = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Lab-Frame : S is moving with r.t S_R' at $-\underset{\sim}{v}$.

So to find $T^{\mu\nu}$ we must Lorentz boost $\bar{T}^{\mu\nu}$ with $-\underset{\sim}{v}$.

$$T^{\mu\nu} = \Lambda^{\mu}_{\alpha}(-\underset{\sim}{v})\, \Lambda^{\nu}_{\beta}(-\underset{\sim}{v})\, \bar{T}_{\alpha\beta}$$

$$\boxed{T^{\mu\nu} = \eta^{\alpha\beta} p + \underbrace{(u+p)}_{h} U^{\alpha} U^{\beta}} \quad ***$$

$h \to$ enthalpy density

This is the energy-momentum used, say, for the early universe : An isotropic fluid (ideal)