united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

H4.SMR/1330-20

" Sixth Workshop on Non-Linear Dynamics and Earthquake Prediction"

15 - 27 October 2001

Gauss, Gödel, Heisenberg

H. Moritz

Graz Technical University
Institute of Physical Geodesy
Graz, Austria

GAUSS, GÖDEL, HEISENBERG

Helmut Moritz
Institute of Theoretical Geodesy
Graz University of Technology, Graz, Austria

Abstract. The three most important expressions of uncertainty or inaccuracy are (1) the classical theory of measuring errors due to C.F. Gauss (in astronomy, geodesy, physics, etc.); (2) the famous uncertainty relation of W. Heisenberg (in quantum mechanics), and (3) the incompleteness theorem of K. Gödel (in the foundations of mathematics). Scientific and philosophical applications and implications of these three theories are outlined: Laplace's demon and chaos, the role of probability in classical and quantum mechanics, the three-world theory of Popper and Eccles, and verification and falsification of physical theories.

1 Introduction

After earlier attempts by R. Bošković and A.M. Legendre, C.F. Gauss (1777–1855) created a theory of errors in a perfect and comprehensive form which is valid even today, in spite of the great progress of statistics since then. The principle is that *every* measurement or empirical determination of a physical quantity is affected by measuring errors of random character, which are unknown but subject to statistical laws.

Error theory has always been basic in geodesy and astronomy (Bošković and Gauss discovered error theory for their geodetic work!), but has been less popular in physics. Here it is frequently thought that, at least in principle, the experimental arrangements can always be made so accurate that measuring errors can be neglected. This is, usually implicitly, assumed in any book on theoretical physics. You will hardly find a chapter of error theory in a course of theoretical physics.

Unavoidable observational errors came to the attention of physicists first around 1925 when W. Heisenberg established his famous uncertainty relation:

$$\Delta p \Delta q \doteq \frac{h}{2\pi}$$

where h is Planck's constant basic in quantum theory. It states that a coordinate q and a momentum p (mass times velocity) cannot *both* be measured with arbitrary precision. If q is very accurate ($\Delta q \to 0$), then the error Δp in p will be very great:

$$\Delta p = \frac{h/2\pi}{\Delta q} \to \infty \quad ,$$

that is, an accurate measurement of position q makes the momentum p very uncertain.

Heisenberg's uncertainty relation is of fundamental conceptual importance and thus has become justly famous. In fact, Heisenberg's relation is much more popular with natural scientists and natural philosophers than Gauss' error theory, although the latter, as the geophysicist Jeffreys (1961, pp. 13–14) remarked, is certainly more important in everyday experimental practice than Heisenberg's uncertainty relation. Ordinary observational errors are usually much larger than Heisenberg's quantum uncertainties.

Gauss and Heisenberg have pointed out the fundamental importance of uncertainty in our experiments with nature and perhaps even in nature itself. Thus physics and other empirical sciences such as astronomy are basically uncertain, and least-squares adjustment procedures which go back to Gauss and Legendre are just an ingenious way to cope with this uncertainty.

On the other hand, mathematics has always been regarded as the prototype of an *exact* science. This belief received a deadly blow by K. Gödel's *incompleteness theorem* published in 1931. Gödel showed that mathematics can never be fully axiomatized: it is either incomplete or inconsistent. This implies that there may be true mathematical theorems which cannot be deduced from a finite set of mathematical axioms. Furthermore, mathematics, including set theory, as used in contemporary practice, cannot be proved to be consistent by an algorithmic procedure as used, for instance, in a computer. H. Weyl, one of the pioneers of modern mathematics and physics, was so pessimistic about the foundations of logic and mathematics that he wrote: "How much more convincing and closer to facts are the heuristic arguments and the subsequent systematic constructions in Einstein's general relativity theory, or the Heisenberg–Schrödinger quantum mechanics" (Weyl 1949, p. 235).

In the working practice of mathematicians, however, Gödel's incompleteness is largely ignored, in the same ways as in the working practice of physicists (except quantum physicists), Heisenberg's uncertainty plays a negligible role. Nevertheless, both facts are with us and make us aware of a theoretical "skeleton in the cupboard" which lurks at the back of all our scientific work, of a basic element of insecurity.

Both kinds of uncertainty, however, are very subtle and usually very small "*second-order effects*". Less well advertized, but usually much larger, is the effect of Gaussian obvservational errors. So to speak, the latter are a "*first-order effect*".

Many instances of uncertainties of Gödel and Heisenberg type are treated in (Moritz 1995).

2 Laplace's Demon and Chaos

Let our theoretical basis be "classical" Euclidean geometry, classical (Newtonian) mechanics and Gaussian error theory. The fundamental dogma of this way of thinking has been the belief that Gaussian errors can be made as small as we wish so that, at least theoretically, they can be completely disregarded. The events of nature proceed in a deterministic way, subject to causality according to classical mechanics. Euclidean geometry and Newtonian mechanics are not essentially affected by measuring uncertainties. Even if the initial conditions are not known with absolute precision, this does not essentially affect the result computed according to the laws of classical meachanics. The computed final results will not be essentially less accurate than the initial data.

This is the point of view of deterministic causality. It has found its classical expression in the form of *Laplace's demon*:

> An intelligent being which, for some given moment of time, knew all the forces by which nature is driven, and the relative position of the objects by which it is composed (provided the being's intelligence were so vast as to be able to analyze all the data), would be able to comprise, in a single formula, the movements of the largest bodies in the universe and those of the lightest atom: nothing would be uncertain to it, and both the future and the past would be present to its eyes. The human mind offers in the perfection which it has been able to give to astronomy, a feeble inkling of such an intelligence. (P. Laplace, 1749–1827).

The Newtonian theory has proved particular useful in astronomy, where the planets moving around the sun may be regarded as mass points, and where friction can be disregarded. On the basis of our present orbital determinations (the "initial conditions"), the movements of planets can be predicted with very high precision hundreds of years ahead. This seems to be an ideal case *of stability.*

This is in stark contrast with meteorological weather prediction which works only a few days ahead and is a typical case of *instability.* A small error in the initial conditions may cause an arbitrarily great error in the predicted results. This is E.N. Lorenz' "*butterfly effect*": a butterfly flapping ints wings in Austria may cause a tornado in the United States.

Lorenz' work in 1963 was one of the starting points of modern *chaos theory*, or deterministic chaos (Schuster 1988).

Curiously enough, chaos theory nevertheless goes back to astronomy since Henri Poincaré (1892) showed that the usual trigonometric series of celestial mechanics may frequently be divergent. This introduces uncertainties of chaos type even in astronomical predictions, but only for very long-range predictions (on the order of thousands of years, perhaps).

Already H. Bruns pointed out in 1884 that an astronomical series may be convergent or divergent, depending on whether a certain empirical parameter is a rational or irrational number. Now, to any irrational number, there can be found an arbitrarily close rational number, so that the question of whether a certain astronomical series is mathematically convergent or divergent, is physically meaningless!

This is closely related to the geodetic question whether the spherical-harmonic expansion of the earth's external gravitational potential is convergent or divergent at the earth's surface. As T. Krarup has shown in 1969, a grain of sand placed on the earth's surface may change convergence into divergence and vice versa. This geodetic form of the "butterfly effect" has sometimes been called the "sand-grain effect"; cf. (Moritz 1980, p. 64).

Now since we know that not everything in nature is stable, instabilities and chaos are seen everywhere in nature.

What is characteristic for chaos may be expressed as: "small causes $\longrightarrow$ large effects" (for example: butterfly $\longrightarrow$ tornado). Another phenomenon of this kind is the throw of dice. If, with one set of initial conditions (position and velocity of the hand throwing the die) we get a 5, with another set of initial conditions (even if it is practically identical, e.g., using a dice–throwing machine) we may throw a 3.

So the initial conditions become irrelevant, and symmetry takes over: all six faces of the die have equal probability. Thus probability arises from deterministic but chaotic motion. This also, as well as meteorological instability, was clearly recognized already by Poincaré.

Chaotic effects in nature thus are frequently responsible for probabilistic laws, and also random errors are of this kind. Reading an angle with a theodolite involves various movements (the hand turning a micrometer screw, rapid unvoluntary eye movements, etc.) which are (at least according to classical physics) completely determined, if not in practice, then at least in theory. Nevertheless we have *random* errors because a deterministic analysis simply is not practically feasible (even if it were theoretically possible which I doubt).

So modern chaos theory does throw a strong light on the relation between determinism and randomness, including Gaussian errors.

3 Approximate or Exact Science?

How exactly does a law of physics fit nature? If the data are inexact, are at least the laws exact? The well–known contemporary mathematician Penrose (1989, p. 183) gave a fine mathematical argument, based on Poincaré's ideas, that classical mechanics cannot be applicable to the real world. This proof is based on the *internal* structure of classical mechanics.

By *external* considerations it is also easy to see (and well known in physics), that classical mechanics is only an approximate limiting case of relativity theory for small velocities v ($v \ll c$, c being the light velocity) and a limiting case of quantum mechanics for $h \longrightarrow 0$; cf. (Moritz and Hofmann–Wellenhof 1993, pp. 233 and 311).

Unfortunately, general relativity and quantum mechanics are incompatible, so at least one of them must be inexact, too.

But how can a physical theory be exact if even the concepts which it uses cannot be defined exactly? Have you ever seen a point mass? Not even a geometric point can be defined exactly! So the approximate character of any physical theory is not really surprising.

4 The Role of Probability

The theory of probability plays an important role in chaotic systems and in the theory of random errors, as we have mentioned in sec. 2.

In quantum mechanics, probability is still more fundamental. There, a measurable quantity is given by a *linear operator* L, and the possible outcomes of the measurement are the *eigenvalues* λ_1, λ_2, ... of the operator L. If the state of the system is given by a *state function* ψ, then the probability that an eigenvalue will be measured is given by the *inner product* of ψ and the eigenfunction ϕ_k corresponding to the k-th eigenvalue λ_k:

$$p_k = < \psi, \phi_k > \quad .$$

Thus the eigenvalue λ_k ***cannot be predicted exactly***, but only with probability p_k!

The reader who is not familiar with linear operators L and state functions ψ, may understand the basic mathematical structure if he thinks of L as a $n \times n$ *square matrix* and of the state function ψ as a *unit vector* of n components. (In fact, the analogy becomes perfect if we let $n \to \infty$, because a linear operator is fully equivalent to an infinite square matrix, and a state function ψ corresponds to a unit vector in infinitely dimensional *Hilbert space*).

Also ϕ_k becomes a unit vector, so that $< \psi, \phi_k >$ is nothing else than the inner product of two unit vectors (for $n \to \infty$, to be sure).

Thus probability is absolutely basic to the conceptual structure of quantum mechanics. What is being discussed is whether probability in quantum mechanics is *subjective* or *objective*.

If the state function represents, so to speak, our *knowledge* of the quantum system under consideration, then the corresponding probabilities are subjective. This is the famous *Copenhagen interpretation* of Bohr and Heisenberg, which is essentially accepted by the majority of working quantum physicists.

The quantum probabilities, however, may also be considered *objective* facts of nature. This interpretation is favored by Karl Popper (quantum probabilities are objective *propensities* of the system) and by the Russian school, e.g. Blokhintsev (1968).

This is only a very brief outline; for more details cf. (Moritz 1995, sec. 3.5).

5 The Three Worlds of Popper and Eccles

The terminology has become popular by the famous (though not uncontroversial) work (Popper and Eccles 1977).

World 1 is the external world of nature in which we move, live, and die. It is the "real world" described by natural science (physics, chemistry, biology, geology, etc.).

World 2 is our internal world of thoughts, perceptions, emotions, headaches, joys, etc.

World 3 is the world of interpersonal human culture. It contains mathematics, languages, computer programs, poetry, music, etc. It is very similar to Plato's world of ideas.

Philosophers disagree on the extent in which these three worlds are "real". Some do not recognize World 3; they say that the World 3 object "mathematics" is only the collection of all books on mathematics ever written and published, that is, a collection of physical (World 1) objects. (But what about the mistakes contained in those books?)

Some deny the reality of internal experiences. Those persons are lucky because they never seem to have headaches or fear the dentist, and unlucky because they never enjoy a good meal. (I don't go so far as to say that they are not even thinking.)

Some philosophers even deny the reality of the external world.

At any rate, the three–world concept furnishes a very convenient terminology even for those who disagree with it.

6 Can We Draw a Circle?

I think that Gauss' error theory is able to contribute essentially to basic philosophical questions. Let us start with the Three–World theory outlined in the preceding section.

Consider mathematical reasoning. Logical and mathematical thinking are proverbially rigorous. How can our brain perform exact thinking?

To see the problem, take any mathematical theorem about a circle, e.g., its definition: the circle is the geometrical locus of all points whose distance from a given point is constant; in other terms, the circle is a curve of constant radius.

Now comes the paradox: *nobody*, not even the greatest mathematician, *has ever seen or drawn a mathematical circle.* Nobody (I really mean *nobody*) has ever seen or marked a point, and I dare say that probably nobody will ever by able to do so.

What is the reason? Logical, mathematical, and other axiomatic systems are *rigorous*, that is, absolutely accurate, at least in principle. For instance, $2 + 1 = 3$ and not 2.993. Logical and mathematical objects belong to World 3. The fact that a mathematician, whose mind belongs to World 2, is able to perform a rigorous logical deduction or find a rigorous mathematical proof which is recognized as such also by his fellow mathematicians, is very remarkable indeed. Mathematicians have discovered all properties of and theorems about a circle, without ever having been able to construct one on paper.

But what about the circles constantly used in illustrations in books on geometry etc.? *They are not exact circles*, as one easily sees by looking at them with a magnifying glass or under a microscope. At best, they are "fuzzy" realizations of exact, or "real", circles!

Some mathematicians write books full of geometric theorems and proofs, which do not contain a single figure. All theorems must be derivable from the axioms by logical deduction only. It is true that most such books do contain figures, but only as an aid to better visualize the geometric situation.

Thus logicians, mathematicians etc. appear to be capable of exact thinking, of dealing with World 3 objects directly. Thus there seems to be an intimate relation between World 3 and World 2. In a way, exact circles, being objects of World 3, *can be transferred exactly to World 2.*

Now comes the surprise. Circles *cannot be transferred exactly to World 1*! Realizations in World 1 of abstract World 3 objects such as points, straight lines, or circles are *always* approximate only!

Thus we have the following scheme of objects:

in World 3:	exact,
in World 2:	exact (at least in principle),
in World 1:	fuzzy.

This seems to be a clear indication that World 1 and World 2 are essentially different. This seems to be a rather significant philosophical result.

This may be illustrated by the following Figure 1. It is obviously impossible to trace an exact circle even with a very firm and sure hand.

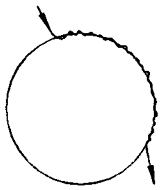

Figure 1: Tracing a circle with a shaky hand

By the way, the Gaussian error theory cannot be immediately applied to this case. The hand traces a random *function* whereas the classical error theory deals with *vectors* only. As we have mentioned, a function is essentially equivalent to a vector in Hilbert space: the coefficients of a Fourier expansion form an infinitely–dimensional vector. This fact was used in (Moritz 1961) to construct an error theory in Hilbert space.

7 Verification and Falsification

A physical theory must be verified (confirmed) by experiment. Such a verification can never be complete: the next experiment may not be in agreement with the theory.

Therefore, K. Popper (1977) tried to replace verification by falsification: a single experiment is sufficient to "falsify" a theory by showing that the experiment is incompatible with the theory.

Now the Gaussian error theory enters into the picture. A theory cannot only be exactly verified, it cannot be exactly falsified either: the experiment can appear to confirm a theory although it is incompatible with it! The measuring errors can give the false impression that a theory is correct.

Since the successful theories (classical mechanics, relativity, quantum theory) are so accurately confirmed by experiments, it is very difficult to overthrow them. Any crucial measurement must be so precise that it is in the "gray zone" between fact and fiction. Any small deviation from the theoretical result may as well be ascribed to random measuring errors, together with systematic effects. Think of the history of the measurement of the light velocity (now it is assumed as an absolutely correct value), or of the present attempts to empirically distinguish Einstein's general theory of relativity from competing theories of "Post–Newtonian Approximation".

So even falsification is not absolute. In practice, theoreticians need not worry about either verification or falsification of their theories anyway: their experimental friends will be eager to verify them, and their antagonists will be most happy to falsify them.

These two examples, distinction of World 1 and World 2 as discussed in the preceding section, and verification and falsification as outlined in the present section, show that the consideration of measuring errors can have important consequences in the philosophy of natural science.

References

Blokhintsev D I (1968) The Philosophy of Quantum Mechanics, Reidel, Dordrecht, Holland.

Jeffreys H (1961) Theory of Probability, 3rd ed., Oxford Univ. Press.

Moritz H (1961) Fehlertheorie der graphisch–mechanischen Integration; Grundzüge einer allgemeinen Fehlertheorie im Funktionenraum, Sonderheft 22, Österreichische Zeitschrift für Vermessungswesen, Wien.

Moritz H (1980) Advanced Physical Geodesy, Wichmann, Karlsruhe.

Moritz H (1995) Science, Mind and the Universe: an Introduction to Natural Philosophy, Wichmann, Heidelberg.

Moritz H and Hofmann–Wellenhof B (1993) Geometry, Relativity, Geodesy, Wichmann, Karlsruhe.

Penrose R (1989) The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics, Oxford Univ. Press.

Poincaré H (1892) Les Méthodes Nouvelles de la Mécanique Céleste, Gauthier–Villars, Paris.

Popper K R (1977) The Logic of Scientific Discovery, 9th ed., Hutchinson, London.

Popper K R and Eccles J C (1977) The Self and Its Brain, Springer, Berlin.

Schuster H G (1988) Deterministic Chaos, 2nd ed., VCH, Weinheim.

Weyl H (1949) Philosophy of Mathematics and Natural Science, Princeton Univ. Press.