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Applications of Normal Mode Relaxation Theory to Solid-Earth Geophysics

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Chapter 2

NORMAL MODE MODEL

1. DERIVATION OF THE EQUATION OF MOTION

In the following the mathematical model is described for the response of the viscoelastic linear Maxwell Earth model to a delta function type of force. After having derived the Green functions, the response of the Earth to arbitrary loads or forces in space and time is found by convolving these functions with the loads or forces.

We assume that the rheological laws (stress - strain and stress - strain rate relations) are linear and that the strains are infinitesimal. We do not deal with *non-linear* rheologies and *finite* strain theory, but that does not imply that these are not important for the Earth Sciences. However, for an introductory treatment of solid-earth relaxation processes we can neglect both.

For long time scale processes the inertial forces vanish, and conservation of linear momentum requires that the body forces $\vec{F}$ acting on the element of the body are balanced by the stresses that act on the surface of the element. At any instant of time we thus have stresses σ_{ij} on the infinitesimal block with density ρ

$$\nabla \cdot \sigma + \rho \vec{F} = \vec{0} \tag{2.1}$$

Assume now that the block is situated somewhere inside the Earth and that we displace the block by an infinitesimal amount $\vec{u}$. Furthermore, assume that the Earth is compressible, laterally homogeneous (but radially stratified!) and *hydrostatically pre-stressed.* We also assume that the Earth is not rotating (we will study rotation at a later stage). We will consider the *elastic* equations of motion, since any linear viscoelastic problems, which are of interest to us, is equivalent to an elastic problem in the Laplace domain, in agreement with the *Correspondence Principle,* as in will be shown in the following. The stress tensor $\sigma_{i,j}$ is the sum of the initial pressure, due to the hydrostatically prestressed conditions, plus a perturbation $\sigma^1_{i,j}$, so that $\sigma_{i,j}$ reads

$$\sigma_{ij} = \sigma^1_{ij} - p_0\delta_{ij} \tag{2.2}$$

σ^1_{ij} denotes a tensor which describes the acquired, non-hydrostatic, stress, that will be related to the strain by means of the appropriate *constitutive equations*. The pressure enter the equation above with the minus sign, since it denotes a compressive stress, which is negative according to the convention that stressea re positive when they act in the same direction of the outward normal to the surface. On the elementary surface enclosing the elementary volume in which the equation of equilibrium holds, the stress due to the load on the overlying material, namely the pressure, is negative according to this convention. The equation of conservation of linear momentum thus reads

$$\nabla \cdot \sigma^1 - \nabla p_0 + \vec{F} = 0 \tag{2.3}$$

If the body is subject to an elastic displacement $\vec{u}$ in t_0, then the pressure in $t_0 + \delta t$ at a fixed point in space, is given by

$$p_0(t_0 + \delta t) = p_0(t_0) - \vec{u} \cdot \nabla p_0 \tag{2.4}$$

The minus sign accounts for the fact that the pressure has to increase at a fixed point in space if the elastic displacement occurs in the opposite direction with respect to the pressure gradient.

The equation of conservation of linear momentum after the elastic displacement reads with $p_0(t_0 + \delta t)$ instead of $p_0(t_0)$

$$\nabla \cdot \sigma^1 - \nabla p_0(t_0) + \nabla(\vec{u} \cdot \nabla p_0) + \vec{F} = o \tag{2.5}$$

The gradient of the initial pressure is given by

$$\nabla p_0 = -\rho_0 g \hat{e}_r \tag{2.6}$$

where $\hat{e}_r$ denotes the unit vector, positive outward from the Earth center. With this explicit expression of the gradient of the initial pressure, the equation of equilibrium becomes

$$\nabla \cdot \sigma^1 - \nabla p_0(t_0) - \nabla(\rho_0 g \vec{u} \cdot \hat{e}_r) + \vec{F} = 0 \tag{2.7}$$

Combining (3.2) - (3.7) with (3.1) gives the following balance of forces in the fixed position in space

$$-\nabla p_0 + \nabla \cdot \sigma_1 - \nabla(\rho_0 g \vec{u} \cdot \hat{e}_r) + \rho\vec{F} = 0 \tag{2.8}$$

The force $\vec{F}$ can generally be split into gravity and all kind of other forcings and loads (e.g. tidal forces, centrifugal forces, loads due to ice-water redistribution, earthquake forcings, etc.). Let us, for the moment, assume that the force $\vec{F}$ is the gravity (so essentially the condition of a free, self-gravitating Earth with no other forcings or loads acting on its surface or interior) and that, as it is a conservative force, it can be expressed as the negative gradient of the potential field ϕ:

$$\vec{F} = -\nabla\phi \tag{2.9}$$

The potential field ϕ can be written as

$$\phi = \phi_0 + \phi_1 \tag{2.10}$$

with ϕ_0 the field in the initial state and ϕ_1 the infinitesimal perturbation.

Combining (3.4) with (3.9) - (3.10) and inserting this in (3.8) leads to the following linearized equation of momentum:

$$\nabla \cdot \sigma_1 - \nabla(\rho_0 g \vec{u} \cdot \hat{e}_r) - \rho_0 \nabla\phi_1 - \rho_1 g \hat{e}_r = 0 \tag{2.11}$$

whereby use is made of the fact that, according to (3.1), in the initial state

$$\nabla \cdot \sigma_0 + \rho_0 \vec{F}_0 = \vec{0} \tag{2.12}$$

with $\vec{F}_0 = \vec{g}_0$ the volume force in the initial state. Note that there is not a term with ρ_1 in the advective term of (3.8), as this would combine with $\vec{u}$ to a second-order term. For the same reason the term $\rho_1\phi_1$ does not occur in (3.11). Note also that the first term of (3.8) is canceled by the term $-\rho_0\nabla\phi_0$, as

$$\nabla p_0 = \rho_0 \nabla\phi_0 \tag{2.13}$$

according to (3.2) and (3.9).

The first term of (3.12) describes the contribution from the stress, the second term the advection of the (hydrostatic) pre-stress, the third term the changed gravity (*self-gravitation*) and the fourth term the changed density (compressibility). In cases where self-gravitation is neglected, the third term will be zero, while in the case of incompressibility the fourth term will be zero.

The perturbed gravitational potential ϕ_1 satisfies the *Poisson equation*

$$\nabla^2\phi_1 = 4\pi G\rho_1 \tag{2.14}$$

with G the universal gravitational constant. In the case of incompressibility the right-hand term will be zero and (3.14) reduces to the *Laplace equation*

$$\nabla^2\phi_1 = 0 \tag{2.15}$$

Equations (3.11) and (3.14) (or (3.15) for incompressible deformation) need to supplemented with a constitutional equation describing how stress and strain (or strain rate) are related to each other, and for this we can, for instance, use the Maxwell model.

For the 3-D Maxwell model, stress and strain rate are related by

$$\dot{\sigma}_{ij} + \frac{\mu}{\eta}(\sigma_{ij} - \frac{1}{3}\sum_{k=1}^{3}\sigma_{kk}\delta_{ij}) = 2\mu\dot{\epsilon}_{ij} + \lambda\sum_{k=1}^{3}\dot{\epsilon}_{kk}\delta_{ij} \qquad (2.16)$$

2. FUNDAMENTAL SOLUTIONS IN THE LAPLACE DOMAIN

In principle, deformation, stress field and gravity field for free Earth models can be solved by means of numerical integration techniques from the three equations (2.8), (2.10) or (2.11), and (2.12) with appropriate initial, boundary and continuity conditions. However, we will see that it is also possible to solve these equations virtually completely analytically by means of *normal model modeling* in the *Laplace transformed domain.* This analytical way of solving has a few great advantages: it leads us to a deeper insight in the mechanisms of the relaxation process with additional checking possibilities, and certainly for spherical (global) models they often prove easier to use than numerical integration techniques. Numerical integration techniques have also their advantages. For instance, they can generally easier deal with more elaborate models (e.g., those that use non-linear rheologies or lateral variations) and often prove simpler to use in half-space (regional) models. So the numerical and analytical models are more to be appreciated as being complimentary than redundant.

The Laplace transform $\tilde{F}(s)$ of a function $f(t)$ is defined by

$$\tilde{F}(s) = \int_0^\infty f(t)e^{-st}dt \qquad (2.17)$$

with t time and s the Laplace variable (which has the dimension of inverse time).

Laplace transformation of (2.12) gives:

$$(s + \frac{\mu}{\eta})\tilde{\sigma}_{ij}(s) - \frac{1}{3}\frac{\mu}{\eta}\sum_{k=1}^{3}\tilde{\sigma}_{kk}(s)\delta_{ij} = 2\mu s\tilde{\epsilon}_{ij}(s) + \lambda s\sum_{k=1}^{3}\tilde{\epsilon}_{kk}\delta_{ij} \qquad (2.18)$$

or

$$\tilde{\sigma}_{ij}(s) = \tilde{\lambda}(s)\sum_{k=1}^{3}\tilde{\epsilon}_{kk}(s)\delta_{ij} + 2\tilde{\mu}(s)\tilde{\epsilon}_{ij}(s) \qquad (2.19)$$

with the Laplace transformed Lamé parameters (also called *compliances*)

$$\tilde{\mu}(s) = \frac{\mu s}{s + \mu/\eta} \tag{2.20}$$

and

$$\tilde{\lambda}(s) = \frac{\lambda s + \mu k/\eta}{s + \mu/\eta} \tag{2.21}$$

with k the incompressibility (or bulk modulus).

Note that (2.15) has the form of a Hookean (linearly elastic) rheological equation in the Laplace-transformed domain. This is a very important aspect which greatly facilitates calculations. So we can derive equations for linear Maxwell viscoelastic bodies in the time domain with formulas for linear Hooke elastic bodies in the Laplace-transformed domain. It can be shown that this is generally valid for all linear viscoelastic bodies (so also for, e.g., the Kelvin and Burgers models). The so-called *Correspondence Principle* states that by calculating the associated elastic solutions in the Laplace-transformed domain the time dependent viscoelastic solutions can be found by Laplace inversion in a unique way. Now on, in the other section of this chapter, the tilde over the quantities will be neglected, in order to not overwhelm the text, although all the equations and quantities are defined in the Laplace transform domain.

3. EXPANSION IN SPHERICAL HARMONICS

In a normal mode expansion, using spherical coordinates with axial symmetry, the displacement field $\mathbf{u}$ can be expanded into a spheroidal and toroidal part

$$\mathbf{u} = \mathbf{u_S} + \mathbf{u_T} = \nabla \times \nabla \times [S(r)\mathbf{e_r}] + \nabla \times [T(r)\mathbf{e_r}] \tag{2.22}$$

where S and T stand for spheroidal and toroidal

$\mathbf{e_r}$ denotes the unit radial vector. The two scalar functions T S can be expanded into spherical harmonics

$$\left\{ \begin{array}{c} T(r) \\ S(r) \end{array} \right\} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \begin{array}{c} T_l^m(r) \\ S_l^m(r) \end{array} \right\} Y_l^m(\theta, \phi) \tag{2.23}$$

where $T_l^m(r)$ and $S_k^m(r)$ are the radial spectral coefficients, θ the colatitude and ϕ the longitude, with $Y_l^m(\theta, \phi)$ the spherical harmonics defined by

$$Y_l^m(\theta, \phi) = (-1)^m P_l^m(cos\theta)\exp(im\phi) \tag{2.24}$$

where

$$P_l^m(z) = \frac{(1 - z^2)^{m/n}}{2^l l!} \frac{d_{l+m}}{dz_{l+m}}(z^2 - 1)^l \tag{2.25}$$

are the associated Legendre polynomials of degree l and order m. Substituting the expansions of the spheroidal and toroidal components in the expression for the displacement results into

$$\left\{ \begin{array}{l} (u^S)_r \\ (u^S)_\theta \\ (u^S)_\phi \end{array} \right\} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \begin{array}{c} u_l^m(r) \\ v_l^m(r)\nabla_\theta \\ v_l^m(r)\nabla_\phi \end{array} \right\} Y_l^m(\theta, \phi) \tag{2.26}$$

and

$$\left\{ \begin{array}{l} (u^T)_r \\ (u^T)_\theta \\ (u^T)_\phi \end{array} \right\} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \begin{array}{l} 0 \\ t_l^m(r)\nabla_\phi \\ -t_l^m(r)\nabla_\theta \end{array} \right\} Y_l^m(\theta, \phi). \tag{2.27}$$

From now on, the superscript S and T for spheroidal and toroidal will be neglected, being self-explanatory the case in which the spheroidal and toroidal problem will be considered. The spheroidal and toroidal differential equations will be solved separately, sinse the medium does not carry lateral heterogeneities. Assuming axial symmetry, post-glacial rebound requires the solution of the spheroidal equations, while dislocation sources, that mimic the occurrence of faulting in the lithosphere, include the toroidal component of the solution.

4. SPHEROIDAL EQUATIONS

The equilibrium and Poisson equations can be written in spherical coordinates (Fung, 1965), with derivatives with respect to r and θ denoted by ∂_r and ∂_θ. Assuming that there is no longitudinal components in the fields as well as in their derivatives and taking account of the continuity equation written on the following way, based on the assumption that the umperturbed initial density ρ_0 is constant in time

$$\rho_1 = -\nabla \cdot (\rho_0 \vec{u}) = -\vec{u} \cdot \hat{e}_r \, \partial_r \rho_0 - \rho_0 \nabla \cdot \vec{u} \tag{2.28}$$

where $\Delta = \nabla \cdot \mathbf{u}$ and ρ_1 denotes the perturbed density, the two components of the momentum and Poisson equations become

$$\begin{aligned} 0 = & -\rho_0 \partial_r \phi_1 + \rho_0 g_0 \Delta - \rho_0 \partial_r (u g_0) + \partial_r \sigma_{rr} \\ & + r^{-1} \partial_\theta \sigma_{r\theta} + r^{-1} (2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\psi\psi} + \sigma_{r\theta} \cot \theta) \end{aligned} \tag{2.29}$$

$$\begin{aligned} 0 = & -\rho_0 r^{-1} \partial_\theta \phi_1 - \rho_0 g_0 r^{-1} \partial_\theta u \\ & + \partial_r \sigma_{r\theta} + r^{-1} \partial_\theta \sigma_{\theta\theta} + r^{-1} ((\sigma_{\theta\theta} - \sigma_{\psi\psi}) \cot \theta + 3\sigma_{r\theta}) \end{aligned} \tag{2.30}$$

$$r^{-2}\partial_r(r^2\partial_r\phi_1) + (r^2\sin\theta)^{-1}\partial_\theta(\sin\theta\partial_\theta\phi_1) = -4\pi G(\rho_0\Delta + u\partial_r\rho_0) \quad (2.31)$$

and the stress components in spherical coordinates are expressed as follows

$$\sigma_{rr} = \lambda\Delta + 2\mu\epsilon_{rr} \quad (2.32)$$

$$\sigma_{\theta\theta} = \lambda\Delta + 2\mu\epsilon_{\theta\theta} \quad (2.33)$$

$$\sigma_{\psi\psi} = \lambda\Delta + 2\mu\epsilon_{\psi\psi} \quad (2.34)$$

$$\sigma_{r\theta} = 2\mu\epsilon_{r\theta} \quad (2.35)$$

The strain tensor components are given in terms of the radial and horizontal (along meridian) components of the displacement

$$\epsilon_{rr} = \partial_r u \quad (2.36)$$

$$\epsilon_{\theta\theta} = r^{-1}(\partial_\theta v + u) \quad (2.37)$$

$$\epsilon_{\psi\psi} = (r^{-1}(v\ \cot\theta + u) \quad (2.38)$$

$$\epsilon_{r\theta} = \frac{1}{2}(\partial_r v - r^{-1}v + r^{-1}\partial_\theta u) \quad (2.39)$$

where u and v denote the radial and tangential (along meridian) components of the displacement vector. The fields defined above can be expanded in spherical harmonics

$$u = \sum_{l=0}^{\infty} U_l(r)P_l(\cos\theta) \quad (2.40)$$

$$v = \sum_{l=0}^{\infty} V_l(r)\partial_\theta P_l(\cos\theta) \quad (2.41)$$

$$\Delta = \sum_{l=0}^{\infty} \chi_l(r)P_l(\cos\theta) \quad (2.42)$$

$$\phi_1 = -\sum_{l=0}^{\infty} \phi_l(r)P_l(\cos\theta) \quad (2.43)$$

The r and θ components of the momentum equations become

$$
\begin{aligned}
0 &= -\rho_0 \partial_r \phi_1 + \rho_0 g_0 \Delta - \rho_0 \partial_r (u g_0) + \partial_r (\lambda \Delta + 2\mu \partial_r u) \\
&+ \tfrac{\mu}{r^2}[4r\partial_r u - 4u + r\partial_\theta \partial_r v + \partial_r v \cot\theta) \\
&+ \partial_\theta^2 u + (\partial_\theta u)\cot\theta - 3(\partial_\theta v + v\cot\theta)]
\end{aligned} \tag{2.44}
$$

The θ component of the equilibrium equation is given by

$$
\begin{aligned}
0 &= -(\tfrac{\rho_0}{r})\partial_\theta \phi_1 - (\tfrac{\rho_0 g_0}{r})\partial_\theta u + \mu \partial_r (\dot{v} - \tfrac{v}{r} + \tfrac{1}{r}\partial_\theta u) \\
&+ \tfrac{1}{r}\partial_\theta(\lambda\Delta) + \tfrac{2\mu}{r^2}(\partial_\theta^2 v + \partial_\theta v \cot\theta - v\cot^2\theta - v) \\
&+ \tfrac{3\mu}{r}\partial_r v + \tfrac{5\mu}{r^2}\partial_\theta u - \tfrac{\mu}{r^2} v
\end{aligned} \tag{2.45}
$$

Making use of the expansion of the displacement components u, v and Δ and ϕ_1 defined above and of the properties of the Legendre equation and of its derivative

$$\frac{d^2}{d\theta^2}P_l(\theta) + \cot\theta \frac{d}{d\theta}P_l(\theta) = -l(l+1)P_l(\theta) \tag{2.46}$$

$$\frac{d}{d\theta}[\frac{d^2}{d\theta^2}P_l(\theta) + \cot\theta\frac{d}{d\theta}P_l(\theta) + l(l+1)P_l(\theta)] = 0 \tag{2.47}$$

$$\frac{d^3}{d\theta^3}P_l + \frac{d^2}{d\theta^2}P_l\cot\theta - \frac{d}{d\theta}P_l[1 + \cot^2\theta + l(l+1)] = 0 \tag{2.48}$$

the momentum and Poisson equations become

$$
\begin{aligned}
0 &= \rho_0 \partial_r \phi_l + \rho_0 g_0 \chi_l - \rho_0 \partial_r (g_0 U_l) + \partial_r (\lambda \chi_l + 2\mu \partial_r U_l) \\
&+ r^{-2}\mu\{4\partial_r U_l r - 4U_l + l(l+1)(-U_l - r\partial_r V_l + 3V_l)\}
\end{aligned} \tag{2.49}
$$

$$
\begin{aligned}
0 &= \rho_0 \phi_l - \rho_0 g_0 U_l + \lambda \chi_l + r\partial_r\{\mu(\partial_r V_l - r^{-1}V_l + r^{-1}U_l)\} \\
&+ r^{-1}\mu\{5U_l + 3r\partial_r V_l - V_l - 2l(l+1)V_l\}
\end{aligned} \tag{2.50}
$$

$$\partial_r^2 \phi_l + \frac{2}{r}\partial_r \phi_l - \frac{l(l+1)}{r^2}\phi_l = 4\pi G(\rho_0 \chi_l + \partial_r \rho_0 U_l) \tag{2.51}$$

where all the terms in in the θ component of the momentum equation have been with respect to θ and multiplied by r. In particular the derivative of the Legendre equation has been used to deal with the term in brackets in (3.45) that contains ∂_θ^2.

From the expression of the divergence $\nabla\cdot$ in spherical coordinates it is possible to express Δ in terms of the harmonic components of the displacement vector. From $\Delta = \nabla \cdot \mathbf{u}$ and from the expression of the divergence in spherical coordinates, equatio A.123 in Ben-Menahem and Singh (1981), where the Legendre equation is used to deal with the v component of the displacement, it is obtainedo

$$\chi_l = \partial_r U_l + 2r^{-1} U_l - l(l+1) V_l r^{-1} \tag{2.52}$$

The solution vector is defined by

$$\begin{aligned}
y_1 &= U_l \\
y_2 &= V_l \\
y_3 &= \Pi_l + 2\mu\partial_r U_l \\
y_4 &= \mu(\partial_r V_l - \tfrac{V_l}{r} + \tfrac{U_l}{r}) \\
y_5 &= -\phi_l \\
y_6 &= -\partial_r \phi_l - \tfrac{l(l+1)}{r}\phi_l + 4\pi G\rho_0 U_l
\end{aligned} \tag{2.53}$$

where $\Pi_l = \lambda\chi_l$. The quantity y_6 is for obvious reasons sometimes nicknamed the *potential stress*. Why this parameter Q is chosen rather than $d\phi/dr$ will become clear when the boundary conditions are discussed in the next section.

Exercise. Prove that with the above definition of solution vector, the momentum and Laplace equations can be cast in the matrix form

$$\frac{d}{dr}\mathbf{y} = \mathbf{A} \cdot \mathbf{y} \tag{2.54}$$

where

$$\mathbf{A}_l(r,s) = \begin{pmatrix}
-\frac{2\lambda}{\beta r} & \frac{l(l+1)\lambda}{\beta r} & \frac{1}{\beta} & 0 & 0 & 0 \\
-\frac{1}{r} & \frac{1}{r} & 0 & \frac{1}{\mu} & 0 & 0 \\
\frac{4}{r}\left(\frac{\gamma}{r} - \rho g\right) & -\frac{l(l+1)}{r}\left(\frac{2\gamma}{r} - \rho g\right) & -\frac{4\mu}{\beta r} & \frac{l(l+1)}{r} & \frac{\rho(l+1)}{r} & -\rho \\
-\frac{1}{r}\left(\frac{2\gamma}{r} - \rho g\right) & -\frac{2\mu - l(l+1)(\gamma+\mu)}{r^2} & -\frac{\lambda}{\beta r} & -\frac{3}{r} & -\frac{\rho}{r} & 0 \\
-4\pi G\rho & 0 & 0 & 0 & -\frac{l+1}{r} & 1 \\
-\frac{4\pi G\rho(l+1)}{r} & \frac{4\pi G\rho l(l+1)}{r} & 0 & 0 & 0 & \frac{l-1}{r}
\end{pmatrix} \tag{2.55}$$

with

$$\beta(s) = \lambda(s) + 2\mu(s) \tag{2.56}$$

and

$$\gamma(s) = \mu(s)\frac{3\lambda(s) + 2\mu(s)}{\lambda(s) + 2\mu(s)} \tag{2.57}$$

In the incompressible case the Lamé parameter λ becomes infinitely large in such a way that $\lambda\nabla \cdot u(s)$ is finite and equal in magnitude to the isotropic pressure. For the incompressible case, (2.17) results in $\lambda(s) \to \infty$, implying that $\beta(s) \to \infty$ according to (2.37), and $\gamma(s) \to 3\mu(s)$ according to (2.38). With this, the matrix (2.36) becomes for the incompressible case

$$\mathbf{A}_l(r,s) = \begin{pmatrix} -\frac{2}{r} & \frac{l(l+1)}{r} & 0 & 0 & 0 & 0 \\ -\frac{1}{r} & \frac{1}{r} & 0 & \frac{1}{\mu} & 0 & 0 \\ \frac{4}{r}\left(\frac{3\mu}{r} - \rho g\right) & -\frac{l(l+1)}{r}\left(\frac{6\tilde{\mu}}{r} - \rho g\right) & 0 & \frac{l(l+1)}{r} & \frac{\rho(l+1)}{r} & -\rho \\ -\frac{1}{r}\left(\frac{6\mu}{r} - \rho g\right) & \frac{2(2l^2+2l-1)\mu}{r^2} & -\frac{1}{r} & -\frac{3}{r} & -\frac{\rho}{r} & 0 \\ -4\pi G\rho & 0 & 0 & 0 & -\frac{l+1}{r} & 1 \\ -\frac{4\pi G\rho(l+1)}{r} & \frac{4\pi G\rho l(l+1)}{r} & 0 & 0 & 0 & \frac{l-1}{r} \end{pmatrix} \tag{2.58}$$

5. ANALYTICAL SOLUTION FOR THE INCOMPRESSIBLE CASE

From the condition of incompressibility

$$\chi_l = 0 \tag{2.59}$$

and homogeneity of each layer

$$\partial_r \rho_0 = 0 \tag{2.60}$$

we obtain for the Laplace and momentum equations

$$\partial_r^2 \phi_l + \frac{2}{r}\partial_r \phi_l - \frac{l(l+1)}{r^2}\phi_l = 0 \tag{2.61}$$

$$\begin{aligned} 0 &= \rho_0 \partial_r \phi_l - \rho_0 \partial_r (g_0 U_l) + \partial_r (\Pi_l + 2\mu\partial_r U_l) \\ &+ \tfrac{\mu}{r^2}\{4r\partial_r U_{l_r} - 4U_l + l(l+1)(-U_l - r\partial_r V_l + 3V_l)\} \end{aligned} \tag{2.62}$$

$$0 = \rho_0\phi_l - \rho_0 g_0 U_l + \Pi_l + \mu r\partial_r(\partial_r V_l - \tfrac{V_l}{r} + \tfrac{U_l}{r}) \\ +\tfrac{\mu}{r}5U_l + 3r\partial_r V_l - V_l - 2l(l+1)V_l \quad (2.63)$$

where we have taken into account that the product $\lambda\chi_l$ remains finite for an incompressible body. From the equation (A.125) of Ben-Menahem and Singh (1981) and the Legendre equation

$$\nabla_r^2 = \partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2} \quad (2.64)$$

Deriving (3.64) with respect to r and summing the result of this derivation to (3.65) multiplied by $2/r$ and with (3.65) multiplied by $-l(l+1)/r^2$, we obtain, taking into account the equation (3.66) above

$$\nabla_r^2(\rho_0\phi_l - \rho_0 g_0 U_l + \Pi_l) = 0 \quad (2.65)$$

where it has been made use of the relationship between V_l and U_l derived from the condition of incompressibility $\chi_l = 0$ that gives

$$V_l = \frac{r\partial_r U_l + 2U_l}{l(l+1)} \quad (2.66)$$

This results derives from (A.123) by Ben-Menahem and Singh (1981)

$$\nabla \cdot \vec{u} = \partial_r u + \frac{2}{r}u + \frac{1}{r}\partial_\theta v + \frac{cot(\theta)}{r}v \quad (2.67)$$

From (3.64), collecting the derivative with respect to r

$$\partial_r(\rho_0\phi_l - \rho_0 g_0 U_l + \Pi_l) = -2\mu\partial_r^2 U_l - \frac{\mu}{r^2}\{4r\partial_r U_l - 4U_l + l(l+1)(-U_l - r\partial_r V_l + 3V_l)\} \quad (2.68)$$

The right hand side can be put in the form

$$-2\mu\partial_r^2 U_l - \frac{\mu}{r^2}[4\partial_r U_l r - 4U_l - l(l+1)U_l - rl(l+1)\partial_r V_l + 3l(l+1)V_l] \quad (2.69)$$

that becomes with (3.65) and (3.61)

$$-\mu\partial_r^2 U_l - 4\frac{\mu}{r}\partial_r U_l - 2\frac{\mu}{r^2}U_l + \frac{\mu}{r^2}l(l+1)U_l \quad (2.70)$$

Multiplying (3.70) by r^2 and equating to the left member of equation (3.68), taken with the opposite sign and multiplyed by r^2, we obtain

$$\mu r^2\partial_r^2 U_l + 4\mu r\partial_r U_l + 2\mu U_l - \mu l(l+1)U_l = r^2\partial_r(\rho_0 g_0 U_l - \Pi_l - \rho_0\phi_l) \quad (2.71)$$

We define

$$\rho_0 g_0 U_l - \rho_0 \phi_l - \Pi_l = \Gamma_l \tag{2.72}$$

Exercise. Show that the solution of the Laplace equation (3.63) takes the form

$$\phi = c_3 r^l + c_3^* r^{-(l+1)} \tag{2.73}$$

where r^l denotes the regular solution in $r = 0$ and $r^{-(l+1)}$ denotes the singular one.

The subscript 3 in (3.73) is used for ϕ_l for convenience. Γ satisfies the Laplace equation (3.67) and thus takes the following form, with the same dependence of ϕ with respect to r, where the constants c_1 and c_1^*, with the subscript 1, are multiplied by μ for convenience, as it will be apparent in the following

$$\Gamma = \mu c_1 r^l + \mu c_1^* r^{-(l+1)} \tag{2.74}$$

The homogeneous equation

$$r^2 \partial_r^2 U_l + 4r\partial_r U_l + 2U_l - l(l+1)U_l = 0 \tag{2.75}$$

obtained from (3.72) has two solutions, a regular one

$$c_2 r^{(l-1)} \tag{2.76}$$

and a singular one

$$c_2^* r^{-(l+2)} \tag{2.77}$$

A particular solution for the regular component can be obtained substituting the regular component of Γ, providing

$$r^2 \partial_r^2 U_l + 4r\partial_r U_l + 2U_l - l(l+1)U_l = r^2 c_1 l r^{(l-1)} \tag{2.78}$$

The regular solution is thus

$$\frac{c_1 l}{2(2l+3)} r^{(l+1)} \tag{2.79}$$

The singular component of the solution becomes, with the same procedure

$$c_1^* \frac{(l+1)}{2(2l-1)} r^{-l} \tag{2.80}$$

Summing up all the contributions we obtain

$$U_l = c_1 \frac{l r^{(l+1)}}{2(2l+3)} + c_2 r^{(l-1)} + c_1^* \frac{(l+1)}{2(2l-1)} r^{-l} + c_2^* r^{-(l+2)} \tag{2.81}$$

From (3.55) and (3.82) we obtain

$$V_l = c_1 \frac{l+3}{2(2l+3)(l+1)} r^{l+1} + \frac{c_2}{l} r^{l-1} + c_1^* \frac{2-l}{2l(2l-1)} r^{-l} - \frac{c_2^*}{l+1} r^{-(l+2)} \tag{2.82}$$

Exercise. Verify that with the definitions of the solution vector (3.55) and (3.81), (3.82), the components y_3, y_4 and y_6 take the form

$$\begin{aligned} y_3 &= c_1 \left[\frac{(l\rho g r + 2(l^2-l-3)\mu) r^l}{2(2l+3)}\right] + c_2[\rho_0 g r + 2(l-1)\mu) r^{l-2}]] \\ &+ c_3 \left[-\rho_0 r^l\right] + c_1^* \left[\frac{(l+1)\rho_0 g r - 2(l^2+3l-1)\mu}{2(2l-1) r^{l+1}}\right] \\ &+ c_2^* \left[\frac{\rho_0 g r - 2(l+2)\mu}{r^{l+3}}\right] + c_3^* \left[-\rho_0 r^{-(l+1)}\right] \end{aligned} \tag{2.83}$$

$$\begin{aligned} y_4 &= c_1 \frac{l(l+2)}{(2l+3)(l+1)} r^l + c_2 \frac{2(l-1)}{l} r^{(l-2)} \\ &+ c_1^* \frac{(l^2-1)}{l(2l-1)} r^{-(l+1)} + c_2^* \frac{2(l+2)}{l+1} r^{-(l+3)} \end{aligned} \tag{2.84}$$

$$\begin{aligned} y_6 &= +c_1 \frac{2\pi G \rho r^{l+1}}{2(2l+3)} + c_2 4\pi G \rho r^{l-1} - c_3(2l+1) r^{l-1} \\ &+ c_1^* \frac{2\pi G(l+1)}{2(2l-1)} r^{-l} + 4\pi G \rho c_2^* r^{-(l+2)} \end{aligned} \tag{2.85}$$

For each of the N layers of the Earth model (assuming that each layer has material parameters which are constant inside it, while also the gravity g is assumed to be constant inside such a layer), the solution can be written as

$$\mathbf{y}_l(r,s) = \mathbf{Y}_l(r,s) \cdot \mathbf{C}_l(r) \tag{2.86}$$

in which $\mathbf{Y}_l$ is the *fundamental matrix* and $\mathbf{C}_l$ a 6-vector integration constant.

The fundamental matrix $\mathbf{Y}_l(r,s)$ reads

$$\left(\begin{array}{cccccc} \frac{l r^{l+1}}{2(2l+3)} & r^{l-1} & 0 & \frac{(l+1)r^{-l}}{2(2l-1)} & r^{-l-2} & 0 \\ \frac{(l+3)r^{l+1}}{2(2l+3)(l+1)} & \frac{r^{l-1}}{l} & 0 & \frac{(2-l)r^{-l}}{2l(2l-1)} & -\frac{r^{-l-2}}{l+1)} & 0 \\ \frac{(l\rho g r+2(l^2-l-3)\mu)r^l}{2(2l+3)} & (\rho_0 g r+2(l-1)\mu)r^{l-2} & -\rho_0 r^l & \frac{(l+1)\rho_0 g r-2(l^2+3l-1)\mu}{2(2l-1)r^{l+1}} & \frac{\rho_0 g r-2(l+2)\mu}{r^{l+3}} & -\frac{\rho_0}{r^{l+1}} \\ \frac{l(l+2)\mu r^l}{(2l+3)(l+1)} & \frac{2(l-1)\mu r^{l-2}}{l} & 0 & \frac{(l^2-1)\mu}{l(2l-1)r^{l+1}} & \frac{2(l+2)\mu}{(l+1)r^{l+3}} & 0 \\ 0 & 0 & -r^l & 0 & 0 & -\frac{1}{r^l} \\ \frac{2\pi G\rho_0 l r^{l+1}}{2l+3} & 4\pi G\rho_0 r^{l-1} & -(2l+1)r^{l-1} & \frac{2\pi G\rho_0(l+1)}{(2l-1)r^l} & \frac{4\pi G\rho_0}{r^{l+2}} & 0 \end{array} \right. \tag{2.87}$$

Each column of this fundamental matrix represents an independent solution of the system (3.56) of ordinary differential equations. The analytical expression of the fundamental solution (3.87), which includes the regular and singular part in $r = 0$, has been first obtained in Sabadini, Yuen and Boschi (1982), while the regular part, which is appropriate for the solution of an homogeneous, viscoelastic sphere, has been first obtained by Wu and Peltier (1982). The inverse of the fundamental matrix $\mathbf{Y}$ has the form

$$\mathbf{Y}_l^{-1}(r,s) = \mathbf{D}_l(r)\bar{\mathbf{Y}}_l(r,s) \tag{2.88}$$

with $\mathbf{D}$ being a diagonal matrix with elements

$$\mathrm{diag}(\mathbf{D}_l(r)) = \frac{1}{2l+1}\left(\frac{l+1}{r^{l+1}},\ \frac{l(l+1)}{2(2l-1)r^{l-1}},\ -\frac{1}{r^{l-1}},\ l r^l,\ \frac{l(l+1)}{2(2l+3)}r^{l+2},\ -r^{l+1}\right) \tag{2.89}$$

and

$$\bar{\mathbf{Y}}_l(r,s) = \begin{pmatrix} \frac{\rho g r}{\mu} - 2(l+2) & 2l(l+2) & -\frac{r}{\mu} & \frac{l r}{\mu} & \frac{\rho r}{\mu} & 0 \\ -\frac{\rho g r}{\mu} + \frac{2(l^2+3l-1)}{l+1} & -2(l^2-1) & \frac{r}{\mu} & \frac{(2-l)r}{\mu} & -\frac{\rho r}{\mu} & 0 \\ 4\pi G\rho & 0 & 0 & 0 & 0 & -1 \\ \frac{\rho g r}{\mu} + 2(l-1) & 2(l^2-1) & -\frac{r}{\mu} & -\frac{(l+1)r}{\mu} & \frac{\rho r}{\mu} & 0 \\ -\frac{\rho g r}{\mu} - \frac{2(l^2-l-3)}{l} & -2l(l+2) & \frac{r}{\mu} & \frac{(l+3)r}{\mu} & -\frac{\rho r}{\mu} & 0 \\ 4\pi G\rho r & 0 & 0 & 0 & 2l+1 & -r \end{pmatrix} \tag{2.90}$$

Although it would be quite laborious to derive such an analytical compact form of a 6×6 inverse matrix ‘by hand’, this can be done nowadays by means of an algebraic software package like *Mathematica*. This was first done by Spada et al. (1992). Of course, it is not so difficult to show analytically that $\mathbf{Y} \times \mathbf{Y}^{-1} = \mathbf{I}$, with $\mathbf{I}$ the identity matrix, by hand!

6. TOROIDAL SOLUTION FOR THE INCOMPRESSIBLE CASE

The analogous of the A matrix for the toroidal case has been obtained by Alterman, Jarosh and Pekeris (1959) for the elastic case, that remains valid also for the viscoelastic case once the Correspondence Principle is considered. It reads, with the superscript T to distinguish the toroidal case

$$\mathbf{A}_l^T(r,s) = \begin{pmatrix} \frac{1}{r} & \frac{1}{\mu(s)} \\ \frac{\mu(s)(l(l+1)-2)}{r^2} & \frac{-3}{r} \end{pmatrix} \tag{2.91}$$

The vector solution $\mathbf{y}^T$ is given by

$$y_1 = t_l^m \tag{2.92}$$

$$y_2 = \mu(s)(\frac{dt_l^m}{dr} - \frac{t_l^m}{r}) \tag{2.93}$$

Exercise. Show that the l component of the fundamental solution is given by

$$\mathbf{Y}_l^T(r,s) = \begin{pmatrix} r^l & r^{-l-1} \\ \mu(s)(l-1)r^{l-1} & -\mu(s)(l+2)r^{-l-2} \end{pmatrix} \tag{2.94}$$

The inverse matrix of the fundamental solution reads

$$\mathbf{Y}^T(r,s) = \begin{pmatrix} \frac{2+l}{r^l(1+2l)} & \frac{r^{1-l}}{\mu(1+2l)} \\ \frac{r^{1+l}(l-1)}{1+2l} & \frac{r^{2+l}}{-\mu(1+2l)} \end{pmatrix} \tag{2.95}$$

7. SOLUTION FOR AN ARBITRARY FORCING SOURCE

After the solution for the homogeneous system of ordinary differential equations has been provided, it is now necessary to derive the solution of the non-homogeneous equations that account for the forcing term $\vec{F}$ entering equation (3.1), to deal with surface or internal loads, centrifugal forces and seismic dislocations.

The general solution of the non-homogeneous system of ordinary differential equations, where $\mathbf{f}$ is the vector characterizing the source

$$\frac{d}{dr}\mathbf{y} = \mathbf{A}\cdot\mathbf{y} + \mathbf{f} \tag{2.96}$$

is given by

$$\mathbf{y}(r) = \mathbf{Y}(r)[\int_{r_0}^{r} \mathbf{Y}^{-1}(r')\mathbf{f}(r')dr' + \mathbf{Y}^{-1}(r_0)\mathbf{y}(r_0)] \tag{2.97}$$

In the following derivation it is assumed that the source is embedded in the outermost layer of radius a, denoting the radius of the Earth, and internal radius b, denoting the interface between the bottom of the lithosphere and underlying layer. This procedure can generalized to a source embedded in an arbitrary internal layer. If the vector $\mathbf{f}$ has this form

$$\mathbf{f} = \mathbf{f}\delta(r - r_s) \tag{2.98}$$

with r_s denoting the radius of the source the non homogeneous system of ordinary differential equations takes the following form

$$\mathbf{y}(r) = \begin{cases} \mathbf{Y}(r)[\mathbf{Y}^{-1}(r_s)\mathbf{I}\mathbf{f} + \mathbf{Y}^{-1}(b)\mathbf{y}(b)], & r_s \le r \le a; \\ \mathbf{Y}(r)\mathbf{Y}^{-1}(b)\mathbf{y}(b), & b \le r \le r_s; \end{cases} \tag{2.99}$$

Exercise. Show that, if the forcing vector has the form

$$\mathbf{f} = \mathbf{f}\delta(r - r_s) + \mathbf{f}'\delta'(r - r_s) \tag{2.100}$$

the solution is given by

$$\mathbf{y}(r) = \begin{cases} \mathbf{Y}(r)[\mathbf{Y}^{-1}(r_s)(\mathbf{I}\mathbf{f} + \mathbf{A}(r_s)\mathbf{f}') + \mathbf{Y}^{-1}(b)\mathbf{y}(b)] & r_s \le r \le a; \\ \mathbf{Y}(r)\mathbf{Y}^{-1}(b)\mathbf{y}(b), & b \le r \le r_s; \end{cases} \tag{2.101}$$

8. PROPAGATOR MATRIX TECHNIQUE

For each layer of a spherical Earth model the solution vector (2.39) can be determined from the fundamental matrix. This solution vector expresses the most general solution for the displacements (radial and lateral), the stresses (radial and lateral), the gravity and the parameter y_6 from which the gravity gradient can be derived, for each layer of the spherical model and for each harmonic degree l in the Laplace domain. Each viscoelastic layer of the model is bounded by either another internal viscoelastic layer or an external layer (free outer surface, inviscid outer core layer at the core-mantle boundary). For each of these cases we need to determine the *boundary conditions.*

The *internal boundary conditions* are quite easy: for a boundary between two viscoelastic layers we require that U_l, V_l, σ_{rrl}, $\sigma_{r\theta l}$ and ϕ_l are continuous. This implies that during deformation there will be no 'cavitation' and no slip, while it is also assumed that no material crosses the boundary (otherwise we should have considered continuity of flow, ρU_l, rather than U_l). Internal boundaries where no material crosses are called *chemical boundaries.* Internal boundaries where material does cross, undergoing a phase change, are called *phase-change boundaries.* The boundary between the upper mantle and lower mantle at about 670 km depth is likely to be partly a chemical and partly a phase-change boundary, but we will assume here that in our Earth models there are only chemical boundaries.

As was already alluded to when the parameter $y_6$was defined in (2.34), we do not take the gravity gradient as sixth component of the solution vector but a combination of gravity, gravity gradient and radial displacement. The reason becomes clear when the boundary condition for the gravity gradient at the free outer surface of the model is considered. If ϕ^e denotes the gravity of the external layer and ϕ of the top layer of the Earth model, then at the free surface

$$\frac{\partial \phi_l^e}{\partial r} - \frac{\partial \phi_l}{\partial r} = -4\pi G \rho U_l \tag{2.102}$$

As the gravity gradient of the external layer satisfies (note that ϕ proportional to $1/r^{n+1}$ is a solution of (2.10), while the other solution, being proportial to r^n, becomes irregular at infinity)

$$\frac{\partial \phi_l^e}{\partial r} = -\frac{l+1}{r}\phi_l^e \tag{2.103}$$

and

$$\phi_l^e = \phi_l \tag{2.104}$$

we can express the external boundary condition as

$$y_6 = -\frac{\partial \phi}{\partial r} - \frac{l+1}{r}\phi + 4\pi G \rho U_l = 0 \tag{2.105}$$

With this it is clear that also y_6 is continuous for internal boundaries between viscoelastic layers.

At the interface $r = r_i$, the top layer i, in which

$$\mathbf{y}^{(i)}(r_{i+1}, s) = \mathbf{Y}^{(i)}(r_{i+1}, s) C^{(i)}(r_{i+1}) \tag{2.106}$$

can be linked to the layer $i+1$ below it, with

$$\vec{y}^{(i+1)}(r_{i+1}, s) = \mathbf{Y}^{(i+1)}(r_{i+1}, s) C^{(i+1)}(r_{i+1}) \tag{2.107}$$

by

$$y^{(i)} = y^{(i+1)} \tag{2.108}$$

as a consequence of the boundary conditions at the internal boundaries. With (2.50) it is possible to express the unknown constant vector $C^{(i)}$ into the unknown constant vector $C^{(i+1)}$. Doing this for every internal boundary of an N layer model (layer 1 is the top layer (crust or lithosphere), layers $2, 3, ..., N-1$ the layers below it, and layer N the core), the solution vector at the surface of the Earth at $r = a$ can be related to the conditions $C_l^{(N)}(r_c)$ at the core - mantle boundary (CMB) $r = r_c$ as

$$\mathbf{y}(a,s) = \left(\prod_{i=1}^{N-1} \mathbf{Y}_l^{(i)}(r_i, s) \mathbf{Y}^{(i)^{-1}}(r_{i+1}, s) \right) \mathbf{Y}^{(N)}(r_c, s) C^{(N)}(r_c) \tag{2.109}$$

The conditions at the CMB have been disputed among geophysicists since the 1960's. This controversy concentrates on the treatment of the continuity conditions for the vertical deformation at the CMB. Without going into details, if it is required that the vertical deformation at the CMB should be continuous, then this restricts the core to being either into a state of neutral equilibrium (homogeneous with neutral adiabatic temperature gradient) or that the radial stress at the CMB is zero. Both could be the case, but such restrictions are obviously not always the case in reality. Therefore the vertical deformation should in general not be continuous at the CMB. This might seem strange, as one would think that this could lead to 'cavitation' or to overlap of layers occurring. The way out of this conundrum is that the fluid core layers are rather to be interpreted as *equipotentials* rather than *material layers.*

The gravity should be continuous at the CMB, at least: if we assume that there are no additional masses *positioned at* the CMB. Inside the core, the gravity should be proportional to r^l, as the other solution of (2.10) is irregular at the center of the Earth. Note that this is in contrast to the surface gravity that we used to derive (2.45). So for the lowermost mantle layer at the CMB we get

$$y_5^{(N)}(r_c) = K_1 r_c^l \tag{2.110}$$

with $y_5^{(N)}$ the fifth component of the vector $\mathbf{y}^{(N)}$ and K_1 a constant.

Assuming that the core is inviscid (fluid), we can readily deduce that the tangential displacement of the mantle is not restricted, so for the lowermost mantle layer at the CMB we can set

$$y_2^{(N)}(r_c) = K_2 \tag{2.111}$$

with K_2 a constant and $y_2^{(N)}$ the second component of the vector $\mathbf{y}^{(N)}$.

This leads to the following condition for the lowermost mantle layer at the CMB (note the minus sign of ϕ in (2.33)):

$$y_1^{(N)}(r_c) = \frac{y_5^{(N)}}{g_c} + K_3 = -\frac{3r_c^{l-1}}{4\pi G\rho_c}K_1 + K_3 \tag{2.112}$$

with g_c the gravity at the CMB, K_3 a constant, and $y_1^{(N)}$ the first component of the vector $y^{(N)}$.

The radial stress (pressure) should be continuous over the CMB. With (2.54) this leads for the lowermost mantle layer at the CMB to the condition

$$y_3^{(N)}(r_c) = g_c\rho_c K_3 = \frac{4}{3}\pi G\rho_c^2 r_c K_3 \tag{2.113}$$

with $y_3^{(N)}$ the third component of the vector $y^{(N)}$.

The tangential stress in the fluid core is zero, and thus continuity of stress requires for the lowermost mantle layer at the CMB that

$$y_4^{(N)}(r_c) = 0 \tag{2.114}$$

with $y_4^{(N)}$ the fourth component of the vector $y^{(N)}$.

Finally, the parameter Q should also be continuous at the CMB, leading for the lowermost mantle layer at the CMB to the condition ((2.52) and (2.54) in (2.34)):

$$y_6^{(N)}(r_c) = 2(l-1)r_c^{l-1}K_1 + 4\pi G\rho_c K_3 \tag{2.115}$$

with $y_6^{(N)}$ the sixth component of the vector $\mathbf{y}^{(N)}$.

If we treat the core as the innermost boundary layer, then with (2.52) - (2.57) the conditions at the CMB can be expressed as a 6×3 interface matrix $\mathbf{I}_{c,l}(r_c)$ as

$$\mathbf{Y}^{(N)}(r_c, s)\mathbf{C}^{(N)}(r_c) = \mathbf{I}_{c,l}(r_c) \cdot \mathbf{C_c} \tag{2.116}$$

with

$$\mathbf{I}_{c,l}(r_c) = \begin{pmatrix} -r_c^{l-1}/A_c & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \rho_c A_c r_c \\ 0 & 0 & 0 \\ r_c^l & 0 & 0 \\ 2(l-1)r_c^{l-1} & 0 & 3A_c \end{pmatrix} \tag{2.117}$$

with ρ_c the (uniform) density of the core, $A_c = \frac{4}{3}\pi G\rho_c$, and $\vec{C}_c = (K_1, K_2, K_3)$ a 3-vector constant.

The solution vector $\mathbf{y}(a, s)$ (2.51) with (2.58) can either express the conditions for a free surface, or express the conditions for a (tidal) forcing or (surface) loading. The loading/forcing case will be treated in section 2.5.

The solution vector $\mathbf{y}(R, s)$ can be split into two parts: one part that contains the unconstrained parameters U_l, V_l and ϕ_l (which we are solving for), and the other containing the constrained $y_3 = \sigma_{rr}$, $y_4 = \sigma_{r\theta}$ and y_6.

For a free surface, the components of the latter, as we have seen already, are all zero at the surface. If P_1 denotes the *projection vector* of (2.51) with (2.58) on the third, fourth and sixth component of (2.51) with (2.58), then we get the following condition:

$$\mathbf{0} = \mathbf{P_1 y(a,s)} = \mathbf{P_1} \left(\prod_{\mathbf{i=1}}^{\mathbf{N-1}} \mathbf{Y^{(i)}(r_i, s) Y^{(i)^{-1}}(r_{i+1}, s)} \right) \mathbf{I_{c,l}(r_c)} \cdot \mathbf{C_c} \tag{2.118}$$

and this condition puts constraints on the s-values in the sense that only those s-values for which (with (2.58))

$$\det \left(P_1 \left(\prod_{i=1}^{N-1} \mathbf{Y}_l^{(i)}(r_i, s) \mathbf{Y}_l^{(i)^{-1}}(r_{i+1}, s) \right) \mathbf{I}_{c,l}(r_c) \right) = 0 \tag{2.119}$$

are non-zero solutions of (2.60). The expression (2.61) is called the *secular equation* and the determinant the *secular determinant.* Its solutions $s = s_j$ ($j = 1, 2, 3, ..., M$) are the inverse relaxation times of the M relaxation modes of the Earth model. These s_j are dependent on the harmonic degree l (and thus must be determined for each harmonic degree), but the index l is left away in order not to complicate the indexing. The total number of relaxation modes for each harmonic degree, M, is the same for each harmonic degree (with the exception of degree 1, but we will not digress on the differences between degree 1 on the one hand and degrees 2 and higher any further).

Experience and (extremely laborious) analytical proofs have led to the following results:

- The surface contributes one mode, labeled $M0$.
- If there is an elastic lithosphere on top of a viscoelastic mantle, then there is one mode triggered by the lithosphere - mantle boundary, labeled $L0$.
- At the boundary of two viscoelastic layers, one *buoyancy mode* is triggered if the density on both sides of the boundary is different. Buoyancy modes between two mantle layers are usually labeled Mi, with $i = 1, 2, 3, ...$, whereby $M1$ is usually the buoyancy mode associated with the 670 km

discontinuity (upper / lower mantle) and $M2$ with the 400 km discontinuity (shallow upper mantle / mantle transition zone).

- At the same boundary two additional *viscoelastic modes* are triggered if the Maxwell time on both sides of the boundary is different (so if the viscosity and rigidity are different, but the ratio of viscosity and rigidity not, then these viscoelastic modes are absent). These 'paired' modes are also called *transient modes* as they have relatively short relaxation times, and are therefore usually labeled Ti, with $i = 1, 2, 3, ...$

- The boundary between the lowermost mantle layer and the inviscid core contributes one mode, labeled $C0$.

It is thus possible, with the above rules, to determine the total amount of modes of (2.61). This is of importance, as solving (2.61) has to be done numerically. However, this root-solving is the only non-analytical part of the viscoelastic relaxation method as described in this chapter.

The root-solving procedure usually consists of two parts: *grid-spacing*, followed by a *bisection algorithm*. In the grid-spacing part, the s-domain is split into a number of discrete intervals. For each s-value at a boundary of an interval, the value of the determinant of (2.61) is calculated, after which this value is multiplied with the value of the determinant of the s-value of the boundary next to it. If this product is positive, then the determinant has not either not changed in sign (or has changed an even amount of times). If the product is negative, then we are sure that there is (at least) one root inside the interval bounded by the two s-values for which the determinant was calculated. In that case, the interval is split up in two parts, and the procedure of determining the product of the determinant of the bounding s-values is repeated. The interval where the determinant changes sign will result again in a negative product, and for this interval the procedure of cutting the interval in two, etc., is repeated. Thus the s-value where the determinant of (2.61) is equal to zero becomes progressively better estimated with each further step in this bisection algorithm. Of course, it can happen that the determinant of (2.61) changes sign over a small s-interval twice or even more times. It is thus necessary to choose the grids small in the s-domain (in practice, it appears that especially the two modes of each T-mode pair have inverse relaxation times (s-values) that are very close to each other). Only after the complete number (determined with the rules above) of roots/modes of (2.61) has been found, can one be sure that the complete signal will be retrieved after inverse-Laplace transformation. For this final step in the relaxation modeling procedure we use the so-called method of *complex contour integration*. Those readers who are not acquainted with this technique will find an overview in Appendix I.

9. PROPAGATION OF THE TOROIDAL SOLUTION

The same procedure discussed above can be used to propagate the toroidal solution. At the CMB the boundary condition is

$$y_2(c) = 0 \tag{2.120}$$

that states that at the core-mantle boundary the tangential stress are zero (Smylie and Manshina, 1971).

Exercise. In analogy with the spheroidal case, it is possible to build $\mathbf{I}_c(r_c)$, which is now a vector, that allows to make use of the same propagation procedure described for the spheroidal case. Making use of the boundary condition at the CMB for the tangential stress, show that $\mathbf{I}_c(r_c)$ takes the form

$$\mathbf{I}_c(r_c) = \begin{pmatrix} \frac{(l-1)r_c^{3l+1}}{l+2} + r_c^{-(l+1)} \\ 0 \end{pmatrix} \tag{2.121}$$

10. INVERSE RELAXATION TIMES FOR SIMPLE, INCOMPRESSIBLE EARTH MODELS

In order to gain insights into the physics of th erelaxatio processes, it is important to have a close look at the relaxation times corresponding to the modes excited by discontinuities in teh physicsal parameters of simple Earth models. We will considel only the spheroidal case. The relaxation times for a four and five layer model, depicted in Fig. (3.1), are shown In Fig. (3.2) and (3.3), as a function of the harmonic degree l. The relaxation times T_r are expressed in years, ranging from $l = 2$ to $l = 100$. Fig. () deals with a viscosity increase in the lower mantle, with the ratio between the lower and uppr mantle viscosity ranging from 1 to 200. OM stands for an old viscosity model, in which the upper mantle viscosity is fixed at 10^{21} Pa s, while NM stands for a new viscosity model, in which ν_{UM} is fixed at 0.5×10^{21} Pa s, in agreement with the recent analyzes by Lambeck et al. (1990), Vermeersen et al. (1999) and Devoti et al. (2000), based on postglacial rebound modeling from different perspectives, sea-level changes in the far field and long wavelength geopotential variations due to Pleistocene deglaciation. These models are chemically stratified at 420 and 670 km depth and the viscosity is uniform in the whole upper mantle; this stratification supports nine relaxation modes. The slowest modes have been named M1 and M2 by Wu and Peltier (1982) and are associated with the two internal chemical boundaries. At low degrees they are followed by the lithospheric (L0) mode and by the core (C0) and mantle (M0) modes, as portrayed in the panel NM by $B = 1$, with $B = \nu_{LM}/\nu_{UM}$ denoting the ratio between the lower to upper mantle viscosity. When B is increased

Schematic diagram that shows the rheological models which include a hard transition zone, model (a) and a two layer mantle, model (b), considered for the evaluation of the relaxation times, carried out in the following twi figures.

from 1 to 200, all the curves are moded upward toward slower relaxtion times. This upward migration occurs first fow longer wavelengths, say lower than $l = 10$, followed by the shorter ones which are less affected by lower mantle viscosity. For shorter wavelengths only the M1, M2 and core modes have slower relaxation times, while the lithospheric and mantle modes are rather unaffected, being the deformation at such high harmonic degrees concentrated in teh upper mantle and thus unaffected by lower mantle viscosity variations. The NM curves, in the left panel, can be obtained from their counterparts in the right panel by a uniform downward shift towards faster relaxation times, in agreement with the lowering of the global mantle viscosity of this model.

Fig. () carries out the effects of a viscosity increase in the transition zone for the new model NM, with $C = \nu_{TZ}/\nu_{UM}$ denotes the ratio between the viscosity in the transition zone with respect to the viscosity in the upper mantle. These models with a stiff transition zone at the upper lower mantle boundary are based on the laboratories studies by Karato (1989) and Meade and Jeanloz (1990), that suggest that the transition zone may form a layer of relatively high viscosity between the upper and lower mantle. The C parameter is varied between 1 and 200. The panel LB corresponds to an upper mantle viscosity of 0.5×10^{21} Pa s and $\nu_{LM} = 2 \times 10^{21}$ Pa s, while UB corresponds to the same upper mantle viscosity and to a higher lower mantle viscosity of $\nu_{LM} = 2 \times 10^{22}$ Pa s. Viscosity increase in the hard layer influences all the modes for all the models, in particular, the M1 and M2 modes, which is not surprising, being these modes excited by the discontinuities that bound the region where the viscosity is varied. With respect to the previous figure all the modes are now affected by the viscosity increase in the transition layer which lying close to the surface, is able to affect also the short wavelegth, high degree modes.

11. SURFACE LOADING

In the case of surface loading, (2.60) can be replaced by the condition

$$\mathbf{b} = \mathbf{P_1}\mathbf{y}_l(R,s) = P_1\left(\prod_{i=1}^{N-1}\mathbf{Y}_l^{(i)}(r_i,s)\mathbf{Y}_l^{(i)^{-1}}(r_{i+1},s)\right)\mathbf{I}_{c,l}(r_c)\cdot\mathbf{C}_c \tag{2.122}$$

where $\mathbf{b}$ constrains $\tilde{\sigma}_{rrl}$, $\sigma_{r\theta l}$ and y_6 at the surface. For a Heaviside function mass load, the vector $\mathbf{b}$ reads

Figure 2.1. Relaxation times in years as a function of the harmonic degree l and varying lower mantle viscosity. The parameter $B = \nu_{LM}/\nu_{UM}$ is varied from 1 to 200. OM corresponds to $\nu_{UM} = 10^{21}$ Pa s, while NM corresponds to $\nu_{UM} = 0.5 \times 10^{21}$ Pa s.

$$\mathbf{b} = (-\frac{1}{4\pi}g(R)(2l+1)/R^2\ ,\ 0\ ,\ -G(2l+1)/R^2)^T \tag{2.123}$$

The derivation of (2.89) will follow later.

Figure 2.2. Relaxation times in years as a function of the harmonic degree l and varying lower mantle viscosity. The parameter $C = \nu_{TZ}/\nu_{UM}$ is varied from 1 to 200. LB corresponds to $\nu_{UM} = \nu_{LM} = 10^{21}$ Pa s, while UB corresponds to $\nu_{UM} = 0.5\times10^{21}$ Pa s and $\nu_{LM} = 2\times10^{22}$ Pa s.

The unconstrained parameters U_l, V_l and ϕ_l at the surface can be expressed as

$$(U_l, V_l, \quad - \quad \phi_l)^T(R,s) = P_2\mathbf{y}(R,s) =$$
$$= \left(P_2 \prod_{i=1}^{N-1} \mathbf{Y}_l^{(i)}(r_i,s)\mathbf{Y}_l^{(i)^{-1}}(r_{i+1},s)\mathbf{I}_{c,l}(r_c)\right) \cdot \mathbf{C}_c \tag{2.124}$$

with P_2 the projection vector of (2.51) with (2.58) on the first, second and fifth component of (2.51) with (2.58).

Elimination of $\mathbf{C}_c$ from (2.88) and (2.90) results in

$$(U_l, V_l, \quad - \quad \phi_l)^T(R,s) = \left(\mathbf{P_2} \prod_{i=1}^{N-1} \mathbf{Y}^{(i)}(r_i,s)\mathbf{Y}^{(i)^{-1}}(r_{i+1},s)\mathbf{I}_{c,l}(r_c)\right) \cdot$$
$$\cdot \quad \left(P_1 \prod_{i=1}^{N-1} \mathbf{Y}^{(i)}(r_i,s)\mathbf{Y}^{(i)^{-1}}(r_{i+1},s)\mathbf{I}_{c,l}(r_c)\right)^{-1} \cdot \mathbf{b} \tag{2.125}$$

Each of the M solutions s_j of (2.119) represents a singularity for the right hand member of equation (2.125). The quantity in the second brackets, can be written as $\prod_{i=1}^{M}(s - s_j)$, where the s_j are the solutions of equation (2.119). Each s_j in (2.125) is thus responsible for the appearance of a singularity that corresponds to a first order pole (see in subsection 14.1 for further discussion concerning this point).

The inverse Laplace transform of (2.125) can be carried out by means of the *residue theorem*, as shown hereafter. The inverse Laplace transform $f(t)$ of a function $F(s)$ is formally defined by complex contour integration by (cf. (2.12))

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds \tag{2.126}$$

in which the real constant γ is chosen such that singularities of $F(s)e^{st}$ are either *all* on the left or *all* on the right side of the vertical line running from $\gamma - i\infty$ to $\gamma + i\infty$. Closing the contour with a half-circle (either on the left of the line or on the right, depending on where the singularities are situated) leads to a complex contour that is known as the *Bromwich path.*

The *residue theorem* states that if $F(s)$ in (2.131), in our case the right hand side of (2.125), is an analytical function with M singularities of first order, then

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds = \sum_{i=1}^{M}[Resf(t)]_{s=s_j} \tag{2.127}$$

where the residue in the pole of first order $s = s_j$ is given by

$$Resf(t) = lim_{s\to s_j}(s - s_j)f(t) \tag{2.128}$$

For the Heaviside surface loading the solution of the field U_l, V_l and $-\phi_l$ can thus be cast in the following form

$$(U_l, V_l, -\phi_l)^T(r,t) = \mathbf{K}^e(r)\delta(t) + \sum_{j=1}^{M} \mathbf{K}^j(r)e^{s_j t} \tag{2.129}$$

in which the $\mathbf{K}^j(r)$ are the vector residues of the solution kernel vector $\mathbf{y}(r,s)$ given by

$$\left(\frac{\mathbf{P_2BI}_c(r_c) \cdot (\mathbf{P_1BI}_c(r_c))^\dagger}{\frac{d}{ds}\det(\mathbf{P_1BI}_c(r_c))} \right)_{s=s_j} \cdot \mathbf{b} \tag{2.130}$$

with

$$\mathbf{B} = \prod_{i=1}^{N-1} \mathbf{Y}^{(i)}(r_i, s)\mathbf{Y}^{(i)^{-1}}(r_{i+1}, s) \tag{2.131}$$

and

$$(\mathbf{P_1BI}_c(r_c))^\dagger = (\mathbf{P_1BI}_c(r_c))^{-1} \cdot \det(\mathbf{P_1BI}_c(r_c)) \tag{2.132}$$

and $\mathbf{K}^e(r)$ the elastic limits

$$\mathbf{K}^e(r) = \lim_{s\to\infty} (\mathbf{P_2BI}_c(r_c) \cdot (\mathbf{P_1BI}_c(r_c))^{-1}) \cdot \mathbf{b} \tag{2.133}$$

This gives the radially dependent part of the *Green functions* for the variables for each degree l. Multiplying the Green functions with the Laplace transformed forcing functions (which is the same as a convolution in the space - time domain) and performing an inverse Laplace transformation gives the sought-for expressions.

Solution (2.96) shows that for each harmonic degree l, the horizontal displacement, vertical displacement and change in gravity consist of an immediate response to the (Heaviside) load (the elastic response), followed by M exponentially decaying (viscous) responses. At least, the viscous responses are decaying only if the inverse relaxation times s_j for each harmonic degree are *negative*. For incompressible models this turns out to be always the case if the Earth layers show no density inversions in the radial Earth profile. However, if there is a layer with a greater density than its neighboring layer below, then the buoyancy mode for the interface will have a positive inverse relaxation time for each harmonic degree l. Such a positive relaxation time leads, according to (2.96), to an exponentially increasing response in the displacements and gravity variations, and thus the interface becomes *Rayleigh-Taylor* unstable. If this occurs, convective motions will be triggered in the Earth model, and the linearization assumed in the normal-mode theory as developed in this chapter breaks down.

12. DISLOCATION SOURCE

With respect to the surface loading, the boundary conditions for dislocations are the vanishing of the stress components and y_6 at the Earth's surface

$$y_3(a) = y_4(a) = y_6(a) = 0 \tag{2.134}$$

These conditions can be cast in the following form

$$\mathbf{P_2Y(a)}[\mathbf{Y}^{-1}(r_s)(\mathbf{If} + \mathbf{A}(r_s)\mathbf{f'}) + \mathbf{Y}^{-1}(b)\mathbf{y}(b)] = 0 \tag{2.135}$$

where $\mathbf{P_2}$ denotes the projection operator on the third, fourth and sixth component of the solution vector.

If the three component vector $\mathbf{b_F}$ is defined in the following way

$$\mathbf{b_F} = -\mathbf{P_2Y(a)Y}^{-1}(r_s)(\mathbf{If} + \mathbf{A}(r_s)\mathbf{f'}) \tag{2.136}$$

the boundary conditions at the surface become

$$\mathbf{P_2Y(a)Y}^{-1}(b)\mathbf{y}(b)] = \mathbf{b_F} \tag{2.137}$$

With these definitions, the boundary conditions at the surface for dislocation sources become formally equivalent to those appropriate for surface loading.

13. APPROXIMATION METHOD FOR HIGH-DEGREE HARMONICS

When using spherical harmonics to describe Earth surface deformations we always have to face the problem of how many terms we should sum up in order to obtain an accurate solution. Since every harmonic represents a standing wave on the earth's surface, whose equator is about 40,000 km long, it is easy to determine the resolution given by each term of that series where the wavelength is given by the length of the equator divided by the harmonic degree. Concerning pointlike seismic sources we find in the modelling carried out in Chapter 13 that this wavelength is uniquely related to the source depth for the elastic response and to the thickness of the elastic layer for (viscoelastic) relaxation: the summation of several thousands of harmonics is thus required to get saturated convergence of the solution in case of shallow earthquakes.

The analytical propagator matrix technique, due to the stiffness of the fundamental matrices, doesn't not allow, in practice, a straigthforward calculation of more than a few thousands degrees. This is due to the $r^{\pm l}$ dependence of the fundamental matrix, that causes numerical problems of over- and underflow for high order harmonic degrees. However, it is possible to mathematically demonstrate that the irregular fundamental solutions in non-homogeneous Earth models are not necessary for calculating all the harmonic degrees, their weight getting smaller and smaller with increasing order. From a certain degree onwards, namely $l > 10^2 - 10^3$, depending on the Earth model, it is possible to obtain an approximated expression of the fundamental solutions by keeping only the regular part. This allows to remove the r^l growth of this part by rescaling procedures, as shown in detail in Riva and Vermeersen (2001).

14. MULTI-LAYER MODELS

14.1 INTRODUCTION

Multi-layer, spherically stratified, self-gravitating relaxation models with a large amount of layers (more than 100) can be dealt with analytically. Relaxation processes are studied for both Heaviside surface loads and tidal forcings. Simulations of the relaxation process of a realistic Earth model with an incompressible Maxwell rheology show that models containing about 30 to 40 layers have reached continuum limits on all timescales and for all harmonic degrees up to at least 150 whenever an elastic lithosphere is present, irrespective of the viscosity profile in the mantle. Especially fine-graded stratification of the shallow layers proofs to be important for high harmonic degrees in these models. The models produce correct long-time (fluid) limits. It is shown that differences in transient behavior of the various models are due to the applied volume-averaging procedure of the rheological parameters. Our earlier proposed hypothesis that purported shortcomings in the fundamental physics of (discrete) normal mode theory are artificial consequences of numerical inaccuracies, theoretical mis-interpretations and using incomplete sets of normal modes is reinforced by the results presented. We show explicitly that the models produce both continuous behavior resulting from continuous rheological stratifications and discrete behavior resulting from sharp density contrasts, as at the outer surface and the core mantle boundary. The differences between volume-averaged models and fixed-boundary contrast models are outlined. Reducing many-layer models with a volume-averaging procedure before employing a normal mode analysis is both economical and highly accurate on all timescales and for all spherical harmonic degrees. The procedure minimises chances of missing contributing modes, while using models with more layers will not result in any substantial increase of accuracy.

In Vermeersen *et al.* (1996a) it was explicitly shown that development and building of analytical models containing a large amount of layers are practically possible. We have shown results on 30-layer models in Vermeersen *et al.* (1996a), here we go up to models containing more than 100 layers. Formulas for these models have been expressed in Vermeersen *et al.* (1996a) in a concise form, where it was shown that simulations have high accuracies, and many aspects that remain elusive in numerical models become understable. A nice example to illustrate this may be found in the purely compressible relaxation