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SPRING SCHOOL ON SUPERSTRINGS AND RELATED MATTERS

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LECTURES ON MIRROR SYMMETRY

Lecture 1

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Please note: These are preliminary notes intended for internal distribution only.

1 Aspects of $\mathcal{N}=2$ Supersymmetry

1.1 $(2,2)$ Supersymmetry in $1+1$ Dimensions

Throughtout the course of the lecture, the worldsheet theories we will consider have $(2,2)$ supersymmetry which is generated by four supercharges, $Q_+, \overline{Q}_+, Q_-, \overline{Q}_-$. They are complex but related to each other by hermitian conjugation

$$Q_\pm^\dagger = \overline{Q}_\pm. \tag{1.1}$$

Together with Hamiltonian H, momemtum P and Lorentz generator M, they obey the following (anti-)commutation relations:

$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0, \tag{1.2}$$
$$\{Q_\pm, \overline{Q}_\pm\} = H \pm P, \tag{1.3}$$
$$\{\overline{Q}_+, \overline{Q}_-\} = Z, \quad \{Q_+, Q_-\} = Z^*, \tag{1.4}$$
$$\{Q_-, \overline{Q}_+\} = \widetilde{Z}, \quad \{Q_+, \overline{Q}_-\} = \widetilde{Z}^*, \tag{1.5}$$
$$[iM, Q_\pm] = \mp Q_\pm, \quad [iM, \overline{Q}_\pm] = \mp \overline{Q}_\pm, \tag{1.6}$$

Z and $\widetilde{Z}$ are central charges. By (1.1) and (1.3), the spectrum is positive semi-definite, $H \geq 0$, and zero energy states are those annihilated by all the supercharges $Q_\pm, \overline{Q}_\pm$, i.e. supersymmetric ground states. There are also two R-charges, F_V (vector) and F_A (axial):

$$[iF_V, Q_\pm] = -iQ_\pm, \quad [iF_V, \overline{Q}_\pm] = i\overline{Q}_\pm, \tag{1.7}$$
$$[iF_A, Q_\pm] = \mp iQ_\pm, \quad [iF_A, \overline{Q}_\pm] = \pm i\overline{Q}_\pm. \tag{1.8}$$

Whether F_V and/or F_A are conserved depends on the system, although the fermion numbers $e^{i\pi F_V}$ and $e^{i\pi F_A}$ should always be conserved. This algebra may be considered as the dimensional reduction of $d=4$ $\mathcal{N}=1$ supersymmetry: F_A and $\widetilde{Z}$ are the rotation and momenta in the reduced directions while F_V descends from the $\mathcal{N}=1$ R-charge.

The $(2,2)$ algebra has an interesting automorphism called the *mirror automorphism*:

$$Q_- \longleftrightarrow \overline{Q}_-,$$
$$F_V \longleftrightarrow F_A, \tag{1.9}$$
$$Z \longleftrightarrow \widetilde{Z}. \tag{1.10}$$

Mirror symmetry is an equivalence of two theories where the $(2,2)$ generators are exchanged in this way.

There are also parity automorphisms, in fact two of them — A-parity and B-parity. A parity flips the space coordinate and therefore does $P \to -P$ and $M \to -M$. A-parity maps other generators as

$$\begin{aligned} &Q_- \leftrightarrow \overline{Q}_+, \quad \overline{Q}_- \leftrightarrow Q_+, \\ &F_V \to -F_V, \quad Z \to Z^*, \end{aligned} \tag{1.11}$$

while B-parity does

$$\begin{aligned} &Q_- \leftrightarrow Q_+, \quad \overline{Q}_- \leftrightarrow \overline{Q}_+, \\ &F_A \to -F_A, \quad \widetilde{Z} \to \widetilde{Z}^*. \end{aligned} \tag{1.12}$$

Like R-symmetries, whether there is a parity symmetry depends on the system. The two parities are exchanged under the mirror automorphism. Generators that are invariant under an A-parity are H, F_A, $\widetilde{Z}$,

$$Q_A = \overline{Q}_+ + Q_-, \tag{1.13}$$

and $Q_A^\dagger = Q_+ + \overline{Q}_-$. They form a closed algebra that includes $\{Q_A, Q_A^\dagger\} = 2H$ and $Q_A^2 = \widetilde{Z}$. Generators invariant under a B-parity are H, F_V, Z,

$$Q_B = \overline{Q}_+ + \overline{Q}_-, \tag{1.14}$$

and $Q_B^\dagger = Q_+ + Q_-$. They also form a closed algebra.

Supsrsymmetric Ground States

Let us quantize the system on a circle with a periodic boundary condition which sets $Z = \widetilde{Z} = 0$. Now, the operators $(Q, F) = (Q_A, F_A)$ or (Q_B, F_V) obey the following commutation relations

$$\{Q, Q^\dagger\} = 2H, \tag{1.15}$$

$$Q^2 = 0, \tag{1.16}$$

$$[F, Q] = Q. \tag{1.17}$$

By the second and the third equation, the Hilbert space of states $\mathcal{H}$ can be regarded as the Q-complex;

$$\cdots \xrightarrow{Q} \mathcal{H}^{q-1} \xrightarrow{Q} \mathcal{H}^q \xrightarrow{Q} \mathcal{H}^{q+1} \xrightarrow{Q} \cdots, \tag{1.18}$$

where $\mathcal{H}^q$ is the subspace of R-charge $F = q$. As noted above, F is not necessarily a conserved charge and the grading q may not be a $\mathbf{Z}$-grading. However, the fermion number $e^{i\pi F}$ is always conserved and thus there is at least a $\mathbf{Z}_2$ grading. By the equation (1.15), Q-cohomology classes are in one to one correspondence with the supersymmetric ground states;

$$\mathcal{H}^q_{\mathrm{SUSY}} \cong H^q(Q) := \frac{\mathrm{Ker}Q : \mathcal{H}^q \to \mathcal{H}^{q+1}}{\mathrm{Im}Q : \mathcal{H}^{q-1} \to \mathcal{H}^q}. \tag{1.19}$$

Witten index is the same as the Q-index

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F \,\mathrm{e}^{-\beta H} = \sum_q (-1)^q \dim H^q(Q), \tag{1.20}$$

and is independent of any supersymmetric deformation of the system.

Chiral Ring

An operator $\mathcal{O}$ is called a *chiral operator* if $[Q_B, \mathcal{O}\} = 0$ and *twisted chiral operator* if $[Q_A, \mathcal{O}\} = 0$. It represents a $Q = Q_B/Q_A$ cohomology class of operators. One can show from the supersymmetry algebra (with $Z = \widetilde{Z} = 0$) that if $\mathcal{O}$ is a chiral operator, $[Q_B, \mathcal{O}\} = 0$, then

$$[(H \pm P), \mathcal{O}] = \{Q_B, [Q_\pm, \mathcal{O}\}]. \tag{1.21}$$

Thus, the worlsheet translations do not change the Q_B-cohomology classes. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two chiral operators, the product $\mathcal{O}_1\mathcal{O}_2$ is also a chiral operator. Same can be said on twisted chiral operators. Thus, Q-cohomology classes of operators form a ring, called the *chiral ring* for $Q = Q_B$ and *twisted chiral ring* for $Q = Q_A$.

Twisting

The Euclidean theory obtained by Wick rotation still has a supersymmetry, with the same algebra and same hermiticity, where the rotation generator is given by $M_E = iM$. If a vector R-charge F_V is conserved and integral, one can twist the theory by declaring that $M_E + F_V$ to be the new rotation generator. This is called the *A-twist*. The same procedure for axial R-symmetry is called the *B-twist*. After A(B)-twist, Q_A (Q_B) becomes scalar and thus there is a supersymmetry even when the worldsheet is curved. Correlation functions with insertions of only twisted chiral (chiral) operators are independent of the choice of worldsheet metric. For this reason the twisted model is sometimes called topological A(B)-model. Sphere 3-point functions determine the structure constants of the twisted chiral ring (chiral ring).

Field/State Correspondence

When a twisting is possible, there is a one to one correspondence with (twisted) chiral ring elements and supersymmetric ground states. Suppose the system is B-twistable. Consider a worlsheet of semi-infinite cigar geometry as in Fig. 1, and perform the B-twisted path-integral in the interior of the cigar, with a chiral ring element ϕ_i inserted at the tip. This leads to a wavefunction at the circle boundary. The flat cylinder region is not affected by twisting, and thus the wavefunction can be regarded as a state of the untwisted theory. Because of the twisting in the curved region, the fermions are periodic along the circle — namely the state belongs to RR sector. In the limit of infinite length, all the excited states are projected out and we are left with the zero energy state $|i\rangle$. This

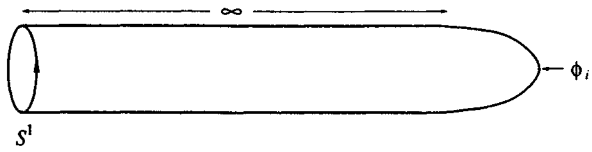

Figure 1: The semi-infinite cigar

is the supersymmetric ground state corresponding to ϕ_i.

1.1.1 $(2,2)$ Superspace and Superfields

In constructing supersymmetric field theory, it is useful to use the superfield formalism. Let us extend Minkowski space with one time and one space coordinates x^0, x^1 by including four fermionic coordinates $\theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-$ (related by complex conjugations $\overline{\theta}^\pm = (\theta^\pm)^\dagger$). Superfields are functions on this superspace. We introduce differential operators

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm \partial_\pm \tag{1.22}$$

$$\overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm \partial_\pm. \tag{1.23}$$

where $\partial_\pm$ are differentiations by $x^\pm := x^0 \pm x^1$: $\partial_\pm = \frac{\partial}{\partial x^\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$. These differential operators satisfy the following anti-commutation relations,

$$\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} = -2i\partial_\pm, \tag{1.24}$$

with all other anti-commutators vanishing. We define another set of differential operators

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm \partial_\pm \tag{1.25}$$

$$\overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm \partial_\pm, \tag{1.26}$$

which anti-commute with $\mathcal{Q}_\pm$ and $\overline{\mathcal{Q}}_\pm$, i.e. $\{D_\pm, \mathcal{Q}_\pm\} = 0$, etc. These obey the similar anti-commutation relations $\{D_\pm, \overline{D}_\pm\} = 2i\partial_\pm$.

A *chiral superfield* Φ is a superfield which satisfies the following equations,

$$\overline{D}_\pm \Phi = 0\,. \tag{1.27}$$

It has the following form

$$\Phi(x^\mu, \theta^\pm, \overline{\theta}^\pm) = \phi(y^\mu) + \theta^\alpha \psi_\alpha(y^\mu) + \theta^+\theta^- F(y^\mu), \tag{1.28}$$

where $y^\pm = x^\pm - i\theta^\pm\overline{\theta}^\pm$. A *twisted chiral superfield* $\widetilde{\Phi}$ is a superfield which satisfy

$$\overline{D}_+\widetilde{\Phi} = D_-\widetilde{\Phi} = 0. \tag{1.29}$$

It has a similar expansion as (1.28) with θ^- and $\overline{\theta}^-$ exchanged. Holomorphic combinations of (twsited) chiral superfields are also (twsietd) chiral superfields.

We now construct action functionals of supefields which are invariant under the transformation $\delta = \epsilon_+\mathcal{Q}_- - \epsilon_-\mathcal{Q}_+ - \overline{\epsilon}_+\overline{\mathcal{Q}}_- + \overline{\epsilon}_-\overline{\mathcal{Q}}_+$. There are three kinds of them: D-term, F-term and twisted F-terms. They are expressed as follows respectively

$$\int \mathrm{d}^2x\,\mathrm{d}^4\theta\, K(\mathcal{F}_i) = \int \mathrm{d}^2x\,\mathrm{d}\theta^+\mathrm{d}\theta^-\mathrm{d}\overline{\theta}^-\mathrm{d}\overline{\theta}^+\, K(\mathcal{F}_i), \tag{1.30}$$

$$\int \mathrm{d}^2x\mathrm{d}^2\theta\, W(\Phi_i) = \int \mathrm{d}^2x\,\mathrm{d}\theta^-\mathrm{d}\theta^+\, W(\Phi_i)\Big|_{\overline{\theta}^\pm=0}, \tag{1.31}$$

$$\int \mathrm{d}^2x\,\mathrm{d}^2\widetilde{\theta}\,\widetilde{W}(\widetilde{\Phi}_i) = \int \mathrm{d}^2x\,\mathrm{d}\overline{\theta}^-\mathrm{d}\theta^+\,\widetilde{W}(\widetilde{\Phi}_i)\Big|_{\overline{\theta}^+=\theta^-=0},. \tag{1.32}$$

Here $K(-)$ is an arbitrary differentiable function of arbitrary superfields $\mathcal{F}_i$, $W(\Phi_i)$ is a holomorphic function of chiral superfields Φ_i, and $\widetilde{W}(\widetilde{\Phi}_i)$ is a holomorphic function of twisted chiral superfields $\widetilde{\Phi}_i$. W ($\widetilde{W}$) is called a superpotential (twisted superpotential). Invariance under δ can be shown essentially by using the Stokes theorem $\int \mathrm{d}^2x\partial_\pm F = \int \mathrm{d}\theta\frac{\partial}{\partial\theta}G = 0$, provided we use the (twisted) chirality $\overline{D}_\pm W = 0$, $\overline{D}_+\widetilde{W} = D_-\widetilde{W} = 0$.

Given such an invariant Lagrangian one can find corresponding Noether charges, which become operators $Q_\pm, \overline{Q}_\pm$ in the quantum theory such that $\delta\mathcal{O} = i[\hat{\delta}, \mathcal{O}]$ where $\hat{\delta} = \epsilon_+Q_- - \epsilon_-Q_+ - \overline{\epsilon}_+\overline{Q}_- + \overline{\epsilon}_-\overline{Q}_+$. The algebra of $\mathcal{Q}_\pm$, $\overline{\mathcal{Q}}_\pm$ implies that the charges $Q_\pm$, $\overline{Q}_\pm$ obey the $(2,2)$ supersymmetry algebra. The lowest component ϕ of a chiral superfield Φ obeys $[\overline{Q}_\pm, \phi] = 0$ and in particular is a chiral operator. The lowest component of a twisted chiral superfield is likewise a twisted chiral operator.

R-symmetries are realized on the superspace as the phase rotations of the fermionic coordinates. Suppose the action $S(\mathcal{G}_I)$ is invariant under the following transformations

$$\mathcal{G}_I(x^\mu, \theta^\pm, \overline{\theta}^\pm) \mapsto \mathrm{e}^{i\alpha q_{V,I}}\mathcal{G}_I(x^\mu,\, \mathrm{e}^{-i\alpha}\theta^\pm,\, \mathrm{e}^{i\alpha}\overline{\theta}^\pm) \tag{1.33}$$

$$\mathcal{G}_I(x^\mu, \theta^\pm, \overline{\theta}^\pm) \mapsto \mathrm{e}^{i\beta q_{A,I}}\mathcal{G}_I(x^\mu,\, \mathrm{e}^{\mp i\beta}\theta^\pm,\, \mathrm{e}^{\pm i\beta}\overline{\theta}^\pm). \tag{1.34}$$

Then, the cooresponding Noether charges F_V and F_A are the vector and axial R-charges.

1.1.2 Deceoupling of Parameters

An extremely strong property is the decoupling of the chiral and twisted chiral parameters. Suppose we have a $(2,2)$ supersymmetric theory and integrate out heavy fields or high frequency modes to obtain a low energy effective theory. Then, parameters of superpotential at high energy theory cannot enter into twisted chiral superpotential of low energy theory. This can be shown by using the idea of promoting parameters to fields. As we have seen, for $(2,2)$ supersymmetry (twisted) superpotential has to be a holomorphic function of (twisted) chiral superfields. Thus a parameter of superpotential must be promoted to a chiral superfields, but that cannot enter into the twisted superpotential at lower energy. Likewise, parameters of twisted superpotential cannot enter into the superpotential of low energy theory. Using this fact, one can derive various kinds of supersymmetric non-renormalization theorems.

Alternative way to see this "decoupling" is to look at the correlation functions of topological models: *A(B)-model correlation functions depend only on twisted chiral (chiral) parameters and depend on them holomorphically.* This is because all the supersymmetric deformations except by twisted chiral (chiral) parameters are by $Q_A(Q_B)$-exact operators. For example, consider deformation by a chiral parameter. It is given by an F-term $\int d\theta^- d\theta^+ \, \delta W$. Using the lowest component δw of δW, this can be written as

$$\int d^2x\{Q_-,[Q_+,\delta w]\} = \int d^2x(\{Q_A,[Q_+,\delta w]\} - \{\overline{Q}_+,[Q_+,\delta w]\}) = \int d^2x\{Q_A,[Q_+,\delta w]\},$$

where we have used $[\overline{Q}_+,\delta w]=0$, $\{Q_+,\overline{Q}_+\}=H+P$ and stokes theorem. Since it is Q_A-exact, it annihilates the correlation function of Q_A-closed operators.

1.2 Non-Linear Sigma Models and Landau-Ginzburg Models

We introduce supersymmetric non-linear sigma models on Kahler manifolds and Landau-Ginzburg models. We write down the classical action and supercharges, and compute the Witten index. We also study anomaly of some fermion number symmetry, the space of supersymmetric ground states, and renormalization group flow.

1.2.1 The Models

Non-linear Sigma Model

Consider a function $K(\phi^i,\overline{\phi}^{\bar{\imath}})$ of n complex variables $\phi^1,\ldots,\phi^n$ such that the matrix $g_{i\bar{\jmath}} := \partial_i\partial_{\bar{\jmath}}K(\phi^i,\overline{\phi}^{\bar{\imath}})$ is positive definite. This matrix can be regarded as a Kähler metric $ds^2 = g_{i\bar{\jmath}}dz^i d\overline{z}^{\bar{\jmath}}$ on $\mathbf{C}^n = \{(z^1,\ldots,z^n)\}$, which defines the Levi-Civita connection $\Gamma^i_{jk} = g^{i\bar{\jmath}}\partial_j g_{k\bar{\jmath}}$ on the tangent bundle $T\mathbf{C}^n$. Under this assumption, we consider the Lagrangian density

$$\mathcal{L}_{kin} = \int d^4\theta\, K(\Phi^i, \overline{\Phi}^{\bar{\imath}}), \tag{1.35}$$

for n chiral superfields $\Phi^1, \ldots, \Phi^n$. In terms of component fields $\phi^i, \psi^i_\pm, F^i$ of Φ^i, $\mathcal{L}_{kin}$ can be expressed as

$$\begin{aligned}\mathcal{L}_{kin} &= -g_{i\bar{\jmath}}\partial^\mu\phi^i\partial_\mu\overline{\phi}^{\bar{\jmath}} + ig_{i\bar{\jmath}}\overline{\psi}^{\bar{\jmath}}_-(D_0 + D_1)\psi^i_- + ig_{i\bar{\jmath}}\overline{\psi}^{\bar{\jmath}}_+(D_0 - D_1)\psi^i_+ + R_{i\bar{\jmath}k\bar{l}}\psi^i_+\psi^k_-\overline{\psi}^{\bar{\jmath}}_-\overline{\psi}^{\bar{l}}_+ \\ &\quad + g_{i\bar{\jmath}}(F^i - \Gamma^i_{jk}\psi^j_+\psi^k_-)(\overline{F}^{\bar{\jmath}} - \Gamma^{\bar{\jmath}}_{\bar{k}\bar{l}}\overline{\psi}^{\bar{k}}_-\overline{\psi}^{\bar{\jmath}}_+),\end{aligned} \tag{1.36}$$

up to total derivatives in x^μ. In the above expression, $R_{i\bar{\jmath}k\bar{l}}$ is the Riemannian curvature of the metric $g_{i\bar{\jmath}}$ and D_μ is defined by

$$D_\mu\psi^i_\pm := \partial_\mu\psi^i_\pm + \partial_\mu\phi^j\Gamma^i_{jk}\psi^k_\pm. \tag{1.37}$$

We note here that the expression (1.36) is covariant except the last term which can be eliminated by the equation of motion. Also, the action is invariant under the "Kähler transformation" $K(\Phi^i, \overline{\Phi}^{\bar{\imath}}) \to K(\Phi^i, \overline{\Phi}^{\bar{\imath}}) + f(\Phi^i) + \overline{f}(\overline{\Phi}^{\bar{\imath}})$. This is manifest in the component expression (1.36) but can also be understood by the fact that $\int d^4\theta f(\Phi)$ is a total derivative if $f(\Phi^i)$ is holomorphic. Thus, we can apply this construction for each coordinate patch of a Kähler manifold M (possibly with more complicated topology than $\mathbf{C}^n$), and the glue the patches together by the invariance of the action under coordinate change and Kähler transformation. This will lead us to define an action for a map of the worldsheet to any Kähler manifold:

$$\phi : \Sigma \to M. \tag{1.38}$$

The fermions are the spinors with values in the pulled-back tangent bundle ϕ^*TM

$$\psi_\pm \in \Gamma(\Sigma, \phi^*TM^{(1,0)} \otimes S_\pm), \tag{1.39}$$

$$\overline{\psi}_\pm \in \Gamma(\Sigma, \phi^*TM^{(0,1)} \otimes S_\pm). \tag{1.40}$$

The derivative (1.37) is the covariant derivative with respect to the Levi-Civita connection pulled back to the worldsheet Σ by the map ϕ. This system is called the *supersymmetric non-linear sigma model on a Kähler manifold* (M, g).

If there is a non-trivial cohomology class $B \in H^2(M, \mathbf{R})$, one can modify the theory by putting the phase factor

$$\exp\left(i\int\phi^*B\right) \tag{1.41}$$

in the path-integral. This is invariant under a continuous deformation of the map ϕ. In particular, it is invariant under the supersymmetry variation and this modification does not break the supersymmetry. Also, the form of the supercurrent and the supercharges remains the same as above.

Landau-Ginzburg Model

Let $W(\phi^1, \dots, \phi^n)$ be a holomorphic function. Then, one can consider the F-term

$$\begin{aligned}\mathcal{L}_W &= \int \mathrm{d}^2\theta\, W(\Phi^i) \;+\; c.c. \\ &= F^i\partial_i W - \partial_i\partial_j\psi^i_+\psi^j_- + \overline{F}^{\bar{\imath}}\partial_{\bar{\imath}}\overline{W} - \partial_{\bar{\imath}}\partial_{\bar{\jmath}}\overline{W}\overline{\psi}^{\bar{\imath}}_-\overline{\psi}^{\bar{\jmath}}_+. \end{aligned} \tag{1.42}$$

As the total Lagrangian we take the sum

$$\mathcal{L} = \int \mathrm{d}^4\theta\, K(\Phi^i, \overline{\Phi}^{\bar{\imath}}) + \left(\int \mathrm{d}^2\theta\, W(\Phi^i) \;+\; c.c.\right). \tag{1.43}$$

After eliminating the auxiliary fields $F, \overline{F}$, we obtain the following expression for the total Lagrangian

$$\begin{aligned}\mathcal{L} &= -g_{i\bar{\jmath}}\partial^\mu\phi^i\partial_\mu\overline{\phi}^{\bar{\jmath}} + ig_{i\bar{\jmath}}\overline{\psi}^{\bar{\jmath}}_-(D_0 + D_1)\psi^i_- + ig_{i\bar{\jmath}}\overline{\psi}^{\bar{\jmath}}_+(D_0 - D_1)\psi^i_+ + R_{i\bar{\jmath}k\bar{l}}\psi^i_+\psi^k_-\overline{\psi}^{\bar{\jmath}}_-\overline{\psi}^{\bar{l}}_+ \\ &\quad -g^{\bar{\imath}j}\partial_{\bar{\imath}}W\partial_j W - D_i\partial_j W\psi^i_+\psi^j_- - D_{\bar{\imath}}\partial_{\bar{\jmath}}\overline{W}\overline{\psi}^{\bar{\imath}}_-\overline{\psi}^{\bar{\jmath}}_+ \end{aligned} \tag{1.44}$$

We see that the addition of the term $\mathcal{L}_W$ has led to the potential term

$$U(\phi) = g^{\bar{\imath}j}\partial_{\bar{\imath}}W\partial_j W, \tag{1.45}$$

plus the fermion mass (or Yaukawa coupling) term $-W''(\phi)\psi_+\psi_-$. The model is called the $\mathcal{N} = 2$ *supersymmetric Landau-Ginzburg model on* (M, g) *with superpotential* W.

Supercharges

Let us write down the expression for the supercharges. Since non-linear sigma model (NLSM) is the $W = 0$ case of Landau-Ginzbut (LG) model, it is enough to present the result for the latter. The supersymmetry variation of the component fields are

$$\begin{aligned} &\delta\phi^i = \epsilon_+\psi^i_- - \epsilon_-\psi^i_+, && \delta\overline{\phi}^{\bar{\imath}} = -\overline{\epsilon}_+\overline{\psi}^{\bar{\imath}}_- + \overline{\epsilon}_-\overline{\psi}^{\bar{\imath}}_+, \\ &\delta\psi^i_+ = 2i\overline{\epsilon}_-\partial_+\phi^i + \epsilon_+F^i, && \delta\overline{\psi}^{\bar{\imath}}_+ = -2i\epsilon_-\partial_+\overline{\phi}^{\bar{\imath}} + \overline{\epsilon}_+\overline{F}^{\bar{\imath}}, \\ &\delta\psi^i_- = -2i\overline{\epsilon}_+\partial_-\phi^i + \epsilon_-F^i, && \delta\overline{\psi}^{\bar{\imath}}_- = 2i\epsilon_+\partial_-\overline{\phi}^{\bar{\imath}} + \overline{\epsilon}_-\overline{F}^{\bar{\imath}}, \end{aligned} \tag{1.46}$$

where F^i can be replaced by the solution $F^i = \Gamma^i_{jk}\psi^j_+\psi^k_- - g^{i\bar{l}}\partial_{\bar{l}}\overline{W}$ ($\overline{F}^{\bar{\imath}}$ is its complex conjugate). Following the Noether procedure, we find the consereved currents $G^\mu_\pm$ and $\overline{G}^\mu_\pm$ - *supercurrents*

$$\begin{aligned} G^0_\pm &= g_{i\bar{\jmath}}(\partial_0 \pm \partial_1)\overline{\phi}^{\bar{\jmath}}\,\psi^i_\pm \mp \overline{\psi}^{\bar{\imath}}_\mp\partial_{\bar{\imath}}\overline{W}, \\ G^1_\pm &= \mp g_{i\bar{\jmath}}(\partial_0 \pm \partial_1)\overline{\phi}^{\bar{\jmath}}\psi^i_\pm - \overline{\psi}^{\bar{\imath}}_\mp\partial_{\bar{\imath}}\overline{W}, \\ \overline{G}^0_\pm &= g_{i\bar{\jmath}}\overline{\psi}^{\bar{\imath}}_\pm(\partial_0 \pm \partial_1)\phi^j \pm \psi^i_\mp\partial_i W, \\ \overline{G}^1_\pm &= \mp g_{\bar{\imath}j}\overline{\psi}^{\bar{\imath}}_\pm(\partial_0 \pm \partial_1)\phi^j \pm \psi^i_\mp\partial_i W \end{aligned} \tag{1.47}$$

The supercharges are given by $Q_\pm = \int dx^1 G^0_\pm$ and $\overline{Q}_\pm = \int dx^1\overline{G}^0_\pm$.

1.2.2 Witten Index

Let us compute the Witten index $I = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}(-1)^F \mathrm{e}^{-\beta H}$, regarded as the torus path-integral with periodic boundary condition in both directions. Using the deformation invariance we can take the zero size limit of the torus. Then, it reduces to the integral of only constant modes.

Let us first consider NLSM on a compact Kahler manifold (M, g). The space of bosonic constant modes is nothing but M itself. Fermionic constant modes span the sum of two copies of tangent bundles (one from $(\psi_-, \overline{\psi}_+)$ another from $(\psi_+, \overline{\psi}_-)$). For a constant mode the Lagrangian (1.36) reduces to

$$\mathcal{L} = R_{i\bar{j}k\bar{l}}\psi_+^i\psi_-^k\overline{\psi}_-^{\bar{j}}\overline{\psi}_+^{\bar{l}}. \tag{1.48}$$

By the substitution $\psi_-^i \to \mathrm{d}z^i$, $\overline{\psi}_+^{\bar{i}} \to \mathrm{d}\bar{z}^{\bar{i}}$, $\psi_+^i \to \psi^i$, and $\overline{\psi}_-^{\bar{i}} \to g^{\bar{i}j}\tilde{\psi}_j$, it can be regarded as $R^j_{ik\bar{l}}\psi^i\tilde{\psi}_j\mathrm{d}z^k \wedge \mathrm{d}\bar{z}^{\bar{l}} = R^j{}_i\psi^i\tilde{\psi}_j$ where $R^i{}_j$ is the curvature 2-form of the bundle $T^{(1,0)}M$. Then the index is given by

$$I = \int_M \prod_{i=1}^{n} \mathrm{d}\psi^i \mathrm{d}\tilde{\psi}^{\bar{j}}\, \mathrm{e}^{\frac{i}{2\pi}R^j{}_i\psi^i\tilde{\psi}_j} = \int_M \det\left(\frac{i}{2\pi}R\right) = \chi(M) \tag{1.49}$$

The index is the Euler number of M.

Let us next consider the LG model. We assume that the potential $U = |\partial W|^2$ grows at infinity in the non-compact space M so that the index is well-defined. We also assume that all the critical points are non-degenerate — they are isolated and that the Hessian $\partial_i\partial_j W$ is of maximal rank. The classical action reduces on the constant modes to

$$\mathcal{L} = R_{i\bar{j}k\bar{l}}\psi_+^i\psi_-^k\overline{\psi}_-^{\bar{j}}\overline{\psi}_+^{\bar{l}} - g^{\bar{i}j}\partial_{\bar{i}}W\partial_j W - D_i\partial_j W\psi_+^i\psi_-^j - D_{\bar{i}}\partial_{\bar{j}}\overline{W}\overline{\psi}_-^{\bar{i}}\overline{\psi}_+^{\bar{j}}. \tag{1.50}$$

It may appear difficult to perform this integration. However, we note that this 0-dimensional system also has a supersymmetry: simply eliminate $\partial_\mu\phi^I$ in (1.46). Then, the integral localizes on the fixed point of the supersymmetry variation and the exact answer can be obtained by the quadratic approximation around each fixed point. The fixed points are at the critical point of W. Thus, the index is given by

$$I = \sum_{p_c:\text{crit pt}} \frac{\det\partial_i\partial_j W \det\partial_{\bar{i}}\partial_{\bar{j}}\overline{W}}{|\det\partial_i\partial_j W|^2} = \#(\text{critical points of } W). \tag{1.51}$$

In the firts expression, the nemerator and denominator are from the fermionic and bosonic integrals respectively, both in the quadratic approximation.

1.2.3 R-Symmetry

Recall that $(2,2)$ algebra has vector and axial R-charges. In NLSM and LG model, we examine the condition for conservation of R-charges.

Classical Level

We start with examining the invariance of the classical Lagrangian under a vector (1.33) and an axial (1.34) R-rotations. The D-term is always invariant (for $q_V = q_A = 0$) since $\mathrm{d}^4\theta$ is invariant under phase rotation of $\theta^{\pm}$. Thus, the NLSM Lagrangian, which consists only of $\int \mathrm{d}^4\theta\, K$, is invariant under both vector and axial R-rotations. LG model has F-term $\int \mathrm{d}^2\theta W(\Phi)$. The measure $\mathrm{d}^2\theta$ is neutral under axial R-rotation, and thus there is an axial R-symmetry at the classical level. On the other hand, under the vector R-rotation the measure has charge -2. Thus, Lagrangian is invariant only if it is possible to transform the fields Φ^i in such a way that $W(\Phi^i)$ is rotated by charge 2. This transformation should be holomorphic and preserve the Kahler metric. Thus, the LG model has classical vector R-symmetry if there is a one-parameter family of holomorphic automorphisms $f_\beta : X \to X$ such that $f_\beta^* g = g$ and

$$f_\beta^* W = \mathrm{e}^{2i\beta} W. \tag{1.52}$$

Such a superpotential is called *quasi-homogeneous.*

Anomaly

Exact symmetry of the quantum theory must preserve the path-integral measure $\mathcal{D}X\, \mathrm{e}^{-S[X]}$, not just the classical action $S[X]$. Since R-symmetry acts on the fermions $\psi_\pm$ as phase rotation, we should examine the fermion measure $\mathcal{D}\Psi$. We choose as the worldsheet the compact Euclidean torus $\Sigma = T^2$, and fix a bosonic background, namely fix a map $\phi : \Sigma \to M$. The fermion kinetic term is

$$-2ig_{i\bar{j}}\overline{\psi}^{\bar{j}}_- D_{\bar{z}}\psi^i_- + 2ig_{i\bar{j}}\overline{\psi}^{\bar{j}}_+ D_z\psi^i_+, \tag{1.53}$$

where z is a complex coordinate of the worldsheet, and D_μ is the covariant derivative on $\phi^* TM^{(1,0)}$ (1.37). Atiyah-Singer index theorem says

$$\mathrm{Index} D_{\bar{z}} = -\mathrm{Index} D_z = \int_\Sigma c_1(\phi^* TM^{(1,0)}) = \int_\Sigma \phi^* c_1(M), \tag{1.54}$$

where $c_1(M)$ is the first Chern class of M (represented by the Ricci form $c_1(M) = \frac{i}{2\pi} R_{i\bar{j}} \mathrm{d}z^i \mathrm{d}\bar{z}^{\bar{j}}$). It shows the following miss-match in the number of fermion zero modes

$$\#(\psi_-\text{-zero}) - \#(\overline{\psi}_-\text{-zero}) = \#(\overline{\psi}_+\text{-zero}) - \#(\psi_+\text{-zero}) = \textstyle\int_\Sigma \phi^* c_1(M). \tag{1.55}$$

If $k := \int \phi^* c_1(M) > 0$, there are generically k zero modes for ψ_- and $\overline{\psi}_+$ but none for $\overline{\psi}_-$ and ψ_+. The path-integral measure is invariant under the vector R-rotation which rotates

ψ_- and $\overline{\psi}_+$ oppositely. However, under the axial rotation $\psi_\pm \to e^{\mp i\beta}\psi_\pm$, the measure is rotated as

$$\mathcal{D}\Psi \to e^{-2ik\beta}\mathcal{D}\Psi. \tag{1.56}$$

Equivalently, the axial R-rotation shifts the B-field class as

$$[B] \to [B] - 2\beta c_1(M). \tag{1.57}$$

Thus, the axial R-symmetry is anomalous if $c_1(M) \neq 0$. It is anomaly free only if $c_1(M) = 0$, i.e. only if M is Calabi-Yau. However, even for non-Calabi-Yau, not all the axial rotations are anomalous. For example, if $\int \phi^* c_1(M)$ is always an integer multiple of $p \in \mathbf{Z}$, then $\mathbf{Z}_{2p}$ subgroup of $U(1)_A$ is anomaly free. In particular, the $\mathbf{Z}_2$ subgroup is always unbroken, as required.

Summary

In NLSM on M, there is always $U(1)_V$ R-symmetry but $U(1)_A$ is broekn to its discrete subgroup $\mathbf{Z}_{2p}$ with $p := \gcd \int \phi^* c_1(M)$. For LG model, $U(1)_V$ is unbroken iff W is quasi-homogeneous, while $U(1)_A$ is not broken by superpotential. Typical cases are presented in the table.

	$U(1)_V$	$U(1)_A$
CY sigma model	○	○
sigma model on M with $c_1(M) \neq 0$	○	×
LG model on CY with generic W	×	○
LG model on CY with quasi-homogeneous W	○	○

Notice the symmetry of the table: it is invariant under left-right/top-bottom exchange. This is actually not a coincidence — some NLSMs and some LG models are exchanged under mirror symmetry that exchanges $U(1)_V$ and $U(1)_A$, mapping quantum effect of one to the classical property of the other, or vice versa. In particular, we note that the mirror of NLSM on M with $c_1(M) \neq 0$ cannot have $U(1)_V$ symmetry and therefore cannot be a NLSM again; if it exists it should have some $U(1)_V$ breaking F-term.

1.2.4 Supersymmetric Ground States

The space of supersymmetric ground states of NLSM on M is isomorphic to the cohomology group of M which is inturn the same as the space of harmonic forms on M;

$$\mathcal{H}_{\text{SUSY}} \cong \bigoplus_{p,q=1}^{n} H^{p,q}(M). \tag{1.58}$$

Here $H^{p,q}(M)$ is the space of harmonic (p,q) forms, or (p,q)-th Dolbeault cohomology group. If M is Calabi-Yau, the vector and axial R-charges of the ground states are

$$\begin{aligned} q_V &= -p+q, \\ q_A &= p+q-n \end{aligned} \quad \text{on } H^{p,q}(M). \tag{1.59}$$

If M is not Calabi-Yau, the axial R-symmetry is anomalous, and only the expression for q_V makes sense. In any case, Witten index is given by $I = \sum_{p,q}(-1)^{-p+q}\dim H^{p,q}(M) = \sum_{i=1}^{2n}(-1)^i H^i(M) = \chi(M)$, reproducing the formula (1.49). If two Calabi-Yau manifolds M and $\widetilde{M}$ are mirror to each other, the ground states in $H^{p,q}(M)$ are mapped to the ground states in $H^{n-p,q}(\widetilde{M})$ so that the vector and axial R-charges are exchanged. In particular, there is a relation between M and $\widetilde{M}$ in the Hodge numbers $h^{p,q} = \dim H^{p,q}$:

$$h^{p,q}(M) = h^{n-p,q}(\widetilde{M}). \tag{1.60}$$

The supersymmetric ground states of LG model are in one to one correspondence with the critical points of the superpotential W, if all the critical points are non-degenerate. If M is a non-compact Calabi-Yau, the axial R-charges are conserved and they are all zero

$$q_A = 0 \quad \text{on the ground states.} \tag{1.61}$$

The reason is that the ground state wavefunctions in the dimensionally reduced model (supersymmetric quantum mechanics) is given by middle-dimensional forms. This also reproduces the index formula $I = \#(\text{crit. pts. of } W)$.

If a NLSM on M is mirror to a LG model, then the vector R-charge of the NLSM ground states has to be zero. Namely, $H^{p,q}(M) = 0$ if $p \neq q$ (the Hodge diamond of M is diagonal).

1.2.5 Renormalization Group Flow

The non-linear sigma model is scale invariant at the classical level. However, the metric is renormalized with the beta function given by

$$\beta_{IJ} = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_{IJ} = \frac{1}{2\pi} R_{IJ} + \cdots \tag{1.62}$$

where Ricci tensor term is the one-loop effect and $+\cdots$ are from higher loops and are convention dependent. If Ricci tensor is positive definite, at higher energies the metric is larger and the sigma model coupling is weaker. Thus NLSM is asymptotically free for Ricci positive manifolds. For a Calabi-Yau manifold ($R_{IJ} = 0$), the sigma model is scale invariant at the one-loop level.

This applies also to supersymmetric NLSM on a Kahler manifold M. The $+\cdots$ terms are modified but still convention dependent. However, there is a nicer story: *The complex structure of M is not renormalized, and the Kahler class is renormalized only at the one-loop level.* The latter means that the *cohomology class* of the Kahler form $\omega = \frac{i}{2} g_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}$ flows exactly as

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu}[\omega] = c_1(M). \tag{1.63}$$

If M is Calabi-Yau, the Kahler class is invariant under RG flow. In fact, it is believed that a CY sigma model flows in the infra-red limit to a non-trivial superconformal field theory which is determined uniquely by the complex structure, Kahler class and the class of B-field.

The Landau-Ginzburg superpotential is invariant under renormalization group flow, except for the overall scaling. This is the non-renormalization theorem of superpotentials, which is one of the strongest properties of (2, 2) theories or any dimensional reduction of $4d$ $\mathcal{N} = 1$ theoories. This was first shown by Grisaru-Seigel-Rocek using superfield perturbation theory and a simpler and sometimes stronger argument using holomorphy was found by Seiberg.

1.2.6 Complexified Kahler Class and Complex Structure

Complex structure of M is parametrized by chiral parameters. This is obvious since the complex coordinates themselves are represented by chiral superfields, and the information of complex structure resides in the transition function at the overlap of patches. In the LG model, the parameters of the superpotential are of course chiral.

To see what are the twisted chiral parameters of the system, let us look at the A-model correlation functions which depend only on twisted chiral parameters, holomorphically. They receive contributions of holomorphic maps ϕ for which e^{-S} is given by

$$\exp\left(-\int_\Sigma \phi^*\omega + i\int_\Sigma \phi^* B\right).$$

Thus, the correlators depend only on the complex combination

$$[\omega] - i[B] \tag{1.64}$$

of the cohomology classes, and the depencence is holomorphic. This shows that the complexified Kahler class yields twisted chiral parameters.

Thus, complex structure and the parameters in the LG superpotential are chiral, while the complexified Kahler class (1.64) is twisted chiral. We will see that the linear sigma model makes this result more transparent for a certain class of target spaces.

Since Mirror Symmetry exchanges chiral and twisted chiral, the complexified Kahler class of the one manifold is mapped to the complex structure of the mirror, and vice versa. This is consistent with the relation $h^{1,1}(M) = h^{n-1,1}(\widetilde{M})$.

1.3 D-Branes and Orientifolds

One can study D-branes and orientifolds from the worldsheet point of view. They are respectively boundary conditions on the worldsheet boundary and quotient by parity symmetries of the worldsheet. We will consider those preserving a half of the $(2,2)$ supercharges. The relevant halves are $Q_A = \overline{Q}_+ + Q_-$ and $Q_A^\dagger$ or $Q_B = \overline{Q}_+ + \overline{Q}_-$ and $Q_B^\dagger$. A D-brane/orientifold preserving them will be called A-brane/A-orientifold or B-brane/B-orientifold. In the case of Calabi-Yau sigma models, we are also interested in D-branes/orientifolds that preserve some *space-time supercharges.*

1.3.1 A-branes and B-branes

Let us consider a D-brane wrapped on a submanifold γ of M. The open string boundary condition associated with this D-brane is

$$\begin{array}{l} \partial_0\phi^I \text{ and } (\psi_- + \psi_+)^I \text{ are tangent to } \gamma, \\ \partial_1\phi^I \text{ and } (\psi_- - \psi_+)^I \text{ are normal to } \gamma. \end{array} \tag{1.65}$$

Here we are using the real coordinates of M. The condition on the fermions is required from $\mathcal{N} = 1$ supersymmetry. We examine the condition for extended supersymmetry.

Q_A and $Q_A^\dagger$ generate the variation δ with $\epsilon_+ = \overline{\epsilon}_-$, which does $\delta_A\phi^i = \epsilon_+\psi_-^i - \overline{\epsilon}_+\psi_+^i = \epsilon_1(\psi_-^i - \psi_+^i) + i\epsilon_2(\psi_-^i + \psi_+^i)$, where ϵ_1 and ϵ_2 are the real and imaginary parts of ϵ_+. In the real coordinates this reads as

$$\delta_A\phi^I = -i\epsilon_1 J^I{}_K(\psi_- - \psi_+)^K + i\epsilon_2(\psi_- + \psi_+)^I, \tag{1.66}$$

where J is the complex structure of the Kahler manifold. This should be tangent to γ. Using the $\mathcal{N} = 1$ condition (1.65), we find that J applied to normal vectors to γ are tangent to γ. If we assume that γ is middle dimensional, this also means that J maps tangent vectors to normal vectors as well. Then, for two tangent vectors v_1 and v_2 to γ, we find $\omega(v_1, v_2) = g(Jv_1, v_2) = 0$. Namely, γ is a Lagrangian submanifold of the symplectic manifold (M, ω). A more careful analysis shows that the assumption was not necessary as long as B is zero and the gauge field on the brane is flat, and Q_A is a symmetry if and only if γ is Lagrangian. If there is a superpotential W (the case of LG model) there is an additional condition that $\mathrm{Im}(W)$ has to be locally constant.

Q_B and $Q_B^\dagger$ generate the variation δ with $\epsilon_- = -\epsilon_+$. The bosonic fields ϕ^i transforms as $\delta_B\phi^i = \epsilon_+(\psi_-^i + \psi_+^i)$, or in the real coordinates

$$\delta_B\phi^I = -i\epsilon_1 J^I{}_K(\psi_- + \psi_+)^K + i\epsilon_2(\psi_- + \psi_+)^I. \tag{1.67}$$

Combining with the condition from $\mathcal{N} = 1$ supersymmetry we find that J applied to tangent vectors to γ are tangent to γ. Namely, γ must be a complex submanifold of (M, J). In fact this is sufficient for B-type sypersymmetry. One can also have gauge field A on the brane: Q_B invariance requires the curvature F_A to be a $(1,1)$-form. A superpotential W has to be locally constant on B-branes.

If the model has space-time interpretation (e.g. Calabi-Yau sigma model), it is more interesting to consider D-branes preserving a part of space-time supersymmetry. This imposes an additional constraint. For A-branes, space-time supersymmetry requires the Lagrangian submanifold γ to be *special Lagrangian*, $\Omega|_\gamma = \mathrm{e}^{i\theta}\mathrm{vol}(\gamma)$ for a constant phase $\mathrm{e}^{i\theta}$ wher Ω is the top holomorphic form of M. (There is a further condition of criticality of the space-time superpotential which is generated by dics instantons.) For B-branes, it requires the gauge field to obey Hermitian-Yang-Mills equation, which depends on the choice of Kahler structure.

1.3.2 A-orientifolds and B-orientifolds

Orientifolds are associed with parity symmetries of the worldsheet theory. As we have seen there are two kinds of parity in $(2,2)$ theories. A-parity exchanges $\overline{Q}_+$ and Q_- whereas B-parity exchanges $\overline{Q}_+$ and $\overline{Q}_-$. The supercurrent are given in (1.47) and the supercharges takes the form

$$\begin{aligned} Q_\pm &= \int \mathrm{d}x^1 \left[(\partial_0 \pm \partial_1)\overline{\phi}\psi_\pm \mp i\overline{\psi}_\mp \overline{W}'\right], \\ \overline{Q}_\pm &= \int \mathrm{d}x^1 \left[\overline{\psi}_\pm(\partial_0 \pm \partial_1)\phi \pm i\psi_\mp W'\right] \end{aligned} \tag{1.68}$$

We consider the parity symmetry of the form $\tau\Omega$ where Ω is the map $x^1 \to -x^1$, $\psi_+ \leftrightarrow \psi_-$, and τ is some isometry of M. Exchange of $\overline{Q}_+$ and Q_- requires that τ maps holomorphic coordinates to anti-holomorphic coordinates. Since τ is an isometry this is equivalent to the condition that τ flips the sign of the Kahler form, $\tau^*\omega = -\omega$. If there is a superpotential W, $\tau : W \to \overline{W}$ is also required. Exchange of $\overline{Q}_+$ and $\overline{Q}_-$ occurs if τ preserves the holomorphic structure of M. Also it should flip the sign of the superpotemtial, $\tau : W \to -W$.

1.4 Summary

We have studied some aspects of $(2,2)$ supersymmetric field theories. One thing we have noticed is that a $(2,2)$ theory contains two sectors — let us call them A-sector and B-sector — which are decoupled from each other in a certain sense. Each of them has its own "holomorphy" and this enables us to controle many things. The two "holomorphies" never mix. See the table below. Introduction of A-brane/A-orientifold or B-brane/B-orientifold keeps one holomorphy and looses the other.

Mirror Symmetry is an equivalence of two theories under which the two sectors are exchanged. It is usually the case that B-sector is easiler to handle compared to A-sector. This is because the "size of a manifold" resides in the A-sector, and B-sector quantities can be computed by going to the large size limit where the sigma model is weakly coupled. Thus, if we know a mirror pair, by combining the knowledge on B-sector of the two we can learn about both sectors of both theories.

This practical aspect is of vital importance in studying Calabi-Yau compactification of superstring theories, with and without D-branes/orientifolds. Note that Type II string theory on Calabi-Yau 3-folds yields $\mathcal{N} = 2$ theories in $3 + 1$ dimensions, while addition of branes and taking orientifold projection break $\mathcal{N} = 2$ supersymmetry, in some cases keeping $\mathcal{N} = 1$ supersymmetry.

Note that A-sector is of interest from the point of view of symplectic geometry while B-sector is of interest from complex analysis or algebraic geometry. Thus one may say that Mirror Symmetry is a "symmetry" between symplectic geometry and algebraic geometry. Regarding the long and different histories of these two fields in mathematics, this is quite an interesting and fascinating suggestion. If we pick up one aspect which is well-known in one side, it hits the other side with a complete surprise!

A	B
$Q_A = \overline{Q}_+ + Q_-$	$Q_B = \overline{Q}_+ + \overline{Q}_-$
twisted chiral $[Q_A, \mathcal{O}] = 0$	chiral $[Q_B, \mathcal{O}] = 0$
twisted superpotential term $\int d^2\widetilde{\theta}\, \widetilde{W}$	superpotential term $\int d^2\theta\, W$
A-twist $M_E \to M_E + F_V$	B-twist $M_E \to M_E + F_A$
complexified Kahler class $[\omega] - i[B]$	complex structure J
A-model: GW-invariants (counting holomorphic curves)	B-model: Variations of Hodge Structure (period integrals)
A-brane : Lagrangian submanifold of (M, ω) with flat connection $\mathrm{Im}(W) = \mathrm{const}$	B-brane : complex submanifold of (M, J) with holomorphic connection $W = \mathrm{const}$
A-orientifold: involution $\omega \to -\omega$ $W \to \overline{W}$	B-orientifold: holomorphic involution $W \to -W$

Exercise: Continue the table.