united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

SMR.1402 - 9

SPRING SCHOOL ON SUPERSTRINGS AND RELATED MATTERS

18 - 26 March 2002

LECTURES ON MIRROR SYMMETRY

Lecture 2

K. HORI
Institute for Advanced Study
Princeton, NJ 08540
USA

Please note: These are preliminary notes intended for internal distribution only.

2 Linear Sigma Models

2.1 Classical Theory

Let us consider the Lagrangian $L = \int d^4\theta \, \overline{\Phi}\Phi$ for a chiral superfield Φ. For a chiral superfield A, the transformation

$$\Phi \to e^{iA}\Phi \tag{2.1}$$

sends a chiral superfield to a chiral superfield, but the Lagrangian is not invariant under this. Now, we introduce a *real* superfield V that transforms as

$$V \to V + i(\overline{A} - A). \tag{2.2}$$

Then, the modified Lagrangian

$$L = \int d^4\theta \; \overline{\Phi} e^V \Phi \tag{2.3}$$

is invariant under the transformation (2.1) and (2.2). A real scalar superfield V that transforms as (2.2) is called a *vector superfield.* Gauge transformations can eliminate the low components of V and make it into the form

$$\begin{aligned} V &= \theta^-\overline{\theta}^-(v_0 - v_1) + \theta^+\overline{\theta}^+(v_0 + v_1) - \theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} \\ &\quad + i\theta^-\theta^+(\overline{\theta}^-\overline{\lambda}_- + \overline{\theta}^+\overline{\lambda}_+) + i\overline{\theta}^+\overline{\theta}^-(\theta^-\lambda_- + \theta^+\lambda_+) + \theta^-\theta^+\overline{\theta}^+\overline{\theta}^- D. \end{aligned} \tag{2.4}$$

v_μ is a one-form field, σ is a complex scalar, $\lambda_\pm$ and $\overline{\lambda}_\pm$ form a Dirac fermion, and D is a real scalar. This fix the gauge symmetry (2.1)-(2.2) up to the ordinary one $A = \alpha(x^\mu)$, and is called the *Wess-Zumino gauge.* WZ gauge is not invariant under the variation $\delta = \epsilon_+ Q_- - \epsilon_- Q_+ - \overline{\epsilon}_+\overline{Q}_- + \overline{\epsilon}_-\overline{Q}_+$. but one can bring $V + \delta V$ back into the WZ gauge by some gauge transformation. In this way we find the supersymmetry transformation of the component fields in the WZ gauge:

$$\begin{aligned} &\delta\, v_\pm = i\overline{\epsilon}_\pm\lambda_\pm + i\epsilon_\pm\overline{\lambda}_\pm, \\ &\delta\sigma = -i\overline{\epsilon}_+\lambda_- - i\epsilon_-\overline{\lambda}_+, \\ &\delta D = -\overline{\epsilon}_+\partial_-\lambda_+ - \overline{\epsilon}_-\partial_+\lambda_- + \epsilon_+\partial_-\overline{\lambda}_+ + \epsilon_-\partial_+\overline{\lambda}_-, \\ &\delta\lambda_+ = i\epsilon_+(D + iv_{01}) + 2\epsilon_-\partial_+\overline{\sigma}, \\ &\delta\lambda_- = i\epsilon_-(D - iv_{01}) + 2\epsilon_+\partial_-\sigma, \\ &\delta\phi = \epsilon_+\psi_- - \epsilon_-\psi_+, \\ &\delta\psi_+ = i\overline{\epsilon}_-(D_0 + D_1)\phi + \epsilon_+ F - \overline{\epsilon}_+\overline{\sigma}\phi, \end{aligned}$$

$$\delta\psi_- = -i\bar{\epsilon}_+(D_0 - D_1)\phi + \epsilon_- F + \bar{\epsilon}_-\sigma\phi,$$
$$\delta F = -i\bar{\epsilon}_+(D_0 - D_1)\psi_+ - i\bar{\epsilon}_-(D_0 + D_1)\psi_- + \bar{\epsilon}_+\bar{\sigma}\psi_- + \bar{\epsilon}_-\sigma\psi_+ + i(\bar{\epsilon}_-\bar{\lambda}_+ - \bar{\epsilon}_+\bar{\lambda}_-)\phi,$$

where $D_\mu\phi$ and $D_\mu\psi_\pm$ are the covariant derivative $D_\mu := \partial_\mu + iv_\mu$. The superfield

$$\Sigma := \overline{D}_+ D_- V \tag{2.5}$$

is invariant under the gauge transformation $V \to V + i(\overline{A} - A)$, and is twisted chiral $\overline{D}_+\Sigma = D_-\Sigma = 0$. It is expressed in WZW gauge as

$$\Sigma = \sigma + i\theta^+\overline{\lambda}_+ - i\overline{\theta}^-\lambda_- + \theta^+\overline{\theta}^-[D - iv_{01}] + \cdots. \tag{2.6}$$

Here $v_{01} := \partial_0 v_1 - \partial_1 v_0$ is the fieldstrength or the curvature of v_μ. The superfield Σ is called the *super-field-strength* of V.

We will consider supersymmetric gauge theories with Lagrangian of the following type;

$$L = \int \mathrm{d}^4\theta \left(\overline{\Phi}\, \mathrm{e}^V \Phi - \frac{1}{2e^2}\overline{\Sigma}\Sigma\right) + \mathrm{Re}\left(-t\int \mathrm{d}^2\widetilde{\theta}\,\Sigma\right), \tag{2.7}$$

where e is a real parameter of mass dimension 1, and t is a dimensionless complex parameter. This is manifestly gauge invariant and supersymmetric. In component fields, the three terms can be written as

$$\begin{aligned}\int \mathrm{d}^4\theta\, \overline{\Phi}\,\mathrm{e}^V\Phi \;=\;& -D^\mu\overline{\phi}D_\mu\phi + i\overline{\psi}_-(D_0 + D_1)\psi_- + i\overline{\psi}_+(D_0 - D_1)\psi_+ + D|\phi|^2 + |F|^2\\ & -|\sigma|^2|\phi|^2 - \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_- - i\overline{\phi}\lambda_-\psi_+ + i\overline{\phi}\lambda_+\psi_- + i\overline{\psi}_+\overline{\lambda}_-\phi - i\overline{\psi}_-\overline{\lambda}_+\phi,\end{aligned} \tag{2.8}$$

$$-\frac{1}{2e^2}\int \mathrm{d}^4\theta\, \overline{\Sigma}\Sigma \;=\; \frac{1}{2e^2}\left(-\partial^\mu\overline{\sigma}\partial_\mu\sigma + i\overline{\lambda}_-(\partial_0 + \partial_1)\lambda_- + i\overline{\lambda}_+(\partial_0 - \partial_1)\lambda_+ + v_{01}^2 + D^2\right), \tag{2.9}$$

$$\mathrm{Re}\left(-t\int d^2\widetilde{\theta}\,\Sigma\right) \;=\; -rD + \theta v_{01}. \tag{2.10}$$

In the last expression we have used

$$t = r - i\theta. \tag{2.11}$$

The parameter r is called the Fayet-Illiopoulos parameter and θ is the Theta angle in $1+1$ dimensions. After elimination of the auxiliary fields D and F, we find the following expreesion for the total Lagrangian

$$\begin{aligned}L \;=\;& -D^\mu\overline{\phi}D_\mu\phi + i\overline{\psi}_-(D_0 + D_1)\psi_- + i\overline{\psi}_+(D_0 - D_1)\psi_+ - \frac{e^2}{2}\left(|\phi|^2 - r\right)^2\\ & -|\sigma|^2|\phi|^2 - \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_- - i\overline{\phi}\lambda_-\psi_+ + i\overline{\phi}\lambda_+\psi_- + i\overline{\psi}_+\overline{\lambda}_-\phi - i\overline{\psi}_-\overline{\lambda}_+\phi\\ & + \frac{1}{2e^2}\left(-\partial^\mu\overline{\sigma}\partial_\mu\sigma + i\overline{\lambda}_-(\partial_0 + \partial_1)\lambda_- + i\overline{\lambda}_+(\partial_0 - \partial_1)\lambda_+ + v_{01}^2\right) + \theta v_{01}\end{aligned} \tag{2.12}$$

There is a potential for the scalar fields $U = |\sigma|^2|\phi|^2 + \frac{e^2}{2}(|\phi|^2 - r)^2$.

It is straightforward to generalize this to the case of many charged chiral multiplets $\Phi_1, \ldots, \Phi_N$ with various charges $Q_1, \ldots, Q_N$. The Lagrangian

$$L = \int \mathrm{d}^4\theta \left(\sum_{i=1}^{N} \overline{\Phi}_i \, \mathrm{e}^{Q_i V} \Phi_i - \frac{1}{2e^2} \overline{\Sigma}\Sigma \right) + \mathrm{Re} \left(-t \int \mathrm{d}^2\widetilde{\theta}\, \Sigma \right), \tag{2.13}$$

is manifestly supersymmetric and invariant under gauge transformations $\Phi_i \to \mathrm{e}^{iQ_i A}\Phi_i$, $V \to V - i(A - \overline{A})$. One can also add an F-term

$$L_W = \int \mathrm{d}^2\theta \, W(\Phi_i) + c.c., \tag{2.14}$$

provided there is a gauge invariant superpotential $W(\lambda^{Q_i}\Phi_i) = W(\Phi_i)$. The component expression for the total Lagrangian $L + L_W$ is similar to (2.12) where the scalar potential is now

$$U(\phi_i, \sigma) = \sum_{i=1}^{N} |Q_i \sigma|^2 |\phi_i|^2 + \frac{e^2}{2} \Big(\sum_{i=1}^{N} Q_i |\phi|^2 - r \Big)^2 + \sum_{i=1}^{N} \left| \frac{\partial W}{\partial \phi_i} \right|^2 . \tag{2.15}$$

One can also consider further generalization to many $U(1)$ gauge groups or non-abelian gauge groups. We will not consider such cases in this lecture.

Note that the system has two kinds of parameters; One is t which is a twisted chiral parameter, and the others are the parameters of the superpotential W that are chiral parameters. As stressed before, t and the superpotential parameters do not mix.

The Lagrangian is invariant under axial R-rotations if we assign the axial R-charge 2 to Σ. It is also invariant under vector R-rotations as long as $W(\Phi_i)$ is quasi-homogeneous. Thus, in such a case the classical system has both $U(1)_A$ and $U(1)_V$ R-symmetries.

2.2 Renormalization and Axial Anomaly

The system is super-renormalizable with respect to the gauge coupling that has mass dimension 1. However, FI-parameter will be renormalized in many cases as we will see. A related phenomenon is the axial anomaly.

Let us first consider our basic model — $U(1)$ gauge theory with a single charge 1 chiral superfield Φ. We look at the effective Lagrangian at a high but finite energy scale μ, which is obtained by integrating out the modes of the fields with the frequencies in the range $\mu \leq |k| \leq \Lambda_{\mathrm{UV}}$ where Λ_{UV} is the ultraviolet cut-off. Let us look at the terms in the Lagrangian involving the D field. At the cut-off scale it is

$$\frac{1}{2e^2} D^2 + D(|\phi|^2 - r(\Lambda_{UV})). \tag{2.16}$$

Integrating out the modes of ϕ, the term $D|\phi|^2$ is corrected by $D\langle|\phi|^2\rangle$ where

$$\langle|\phi|^2\rangle = \int_{\mu\leq|k|\leq\Lambda_{\rm UV}} \frac{d^2k}{(2\pi)^2}\frac{2\pi}{k^2} = \log\left(\frac{\Lambda_{\rm UV}}{\mu}\right). \tag{2.17}$$

Thus, the D dependent terms in the effective action at the scale μ is given by

$$\frac{1}{2e^2}D^2 + D\left(|\phi|^2 + \log\left(\frac{\Lambda_{UV}}{\mu}\right) - r(\Lambda_{UV})\right). \tag{2.18}$$

We have seen that the FI parameter runs as

$$r(\mu) = r(\Lambda_{UV}) - \log\left(\frac{\Lambda_{UV}}{\mu}\right). \tag{2.19}$$

In other words, FI parameter is a function of the scale of the form $r(\mu) = \log(\mu/\Lambda)$. The dimensionless parameter r of the classical theory is replaced by the scale parameter Λ in the quantum theory. Dimensional transmutation is at work.

A related quantum effect is the anomaly of the axial R-symmetry. Recall that the classical Lagrangian is invariant under the axial R-rotation with the axial R-charge of Σ being 2 (but the charge of Φ being arbitrary). This symmetry is broken by anomaly since there is a charged fermion. Counting the number of fermionic zero modes, we find that the axial R-rotation changes the measure as

$$\mathcal{D}\psi\mathcal{D}\overline{\psi} \longrightarrow {\rm e}^{-2ki\alpha}\mathcal{D}\psi\mathcal{D}\overline{\psi}, \tag{2.20}$$

in the background with quantized curvature $(-1/2\pi)\int v_{12}{\rm d}x^1{\rm d}x^2 = k$. Since the Theta term in the Euclidean action is $i(\theta/2\pi)\int v_{12}{\rm d}x^1{\rm d}x^2 = -ik\theta$ (the path-integral weight is ${\rm e}^{ik\theta}$), the rotation (2.20) amounts to the shift in Theta angle

$$\theta \longrightarrow \theta - 2\alpha. \tag{2.21}$$

Thus, the $U(1)_A$ R-symmetry of the classical system is broken to $\mathbf{Z}_2$ in the quantum theory. Physics does not depend on the Theta angle θ since a shift of θ can be absorbed by the axial rotation, or a field redefinition.

Thus, the dimensionless parameters r and θ of the classical theory are not any more a parameter of the quantum theory. They are replaced by the single scale parameter Λ.

One can repeat this argument in the case where there are N chiral superfields Φ_i of charge Q_i ($i = 1, \ldots, N$). The term $D|\phi|^2$ in (2.16) is now replaced by $D\sum_{i=1}^N Q_i|\phi_i|^2$, and thus the renormalization group flow of the FI parameter is given by

$$r(\mu) = \sum_{i=1}^N Q_i \log\left(\frac{\mu}{\Lambda}\right). \tag{2.22}$$

The axial rotaion shifts the Theta angle as

$$\theta \longrightarrow \theta - 2\sum_{i=1}^{N} Q_i \alpha. \tag{2.23}$$

Thus, if $b_1 := \sum_{i=1}^{N} Q_i \neq 0$, dimensional transmutation is at work and the $U(1)_A$ symmetry is anomalously broken to $\mathbf{Z}_{2b_1}$. The FI and Theta parameters are replaced by the single scalar parameter Λ. If $b_1 = 0$, the FI parameter does not run as a function of the scale and the full $U(1)_A$ symmetry is unbroken. The FI and Theta parameters r and θ remain as the parameters of the quantum theory.

The above argument applies independently of whether or not the superpotential term $\int d^2\theta\, W(\Phi_i)$ is present. The interaction induced from this does not yield divergences that renormalize the FI parameters, which is the content of the decoupling theorem presented before. Furthermore, the superpotential $W(\Phi_i)$ itself is not renormalized as long as we keep all the fields.

2.3 Non-Linear Sigma Models from Gauge Theories

We show that the linear sigma models realize non-linear sigma models on a certain class of target spaces. In the first part, we discuss the cases without superpotential for the chiral fields, in which case the target space is a toric manifold. In the second part, we turn on superpotentials. This will give us the sigma model on a submanifold of a toric manifold.

2.3.1 $\mathbb{CP}^{N-1}$

Let us consider the $U(1)$ gauge theory with N chiral superfields $\Phi_1, \dots, \Phi_N$ of charge 1, with no superpotential. We first look at classical supersymmetric vacua given by configurations where the potential energy

$$U = \sum_{i=1}^{N} |\sigma|^2 |\phi_i|^2 + \frac{e^2}{2}\left(\sum_{i=1}^{N} |\phi_i|^2 - r\right)^2 \tag{2.24}$$

vanishes. If r is positive $U = 0$ is attained by a configuration which obeys $\sigma = 0$ and

$$\sum_{i=1}^{N} |\phi_i|^2 = r. \tag{2.25}$$

The set of all supersymmertric vacua modulo $U(1)$ gauge group action makes the vacuum manifold. It is nothing but the complex projective space of dimension $N-1$;

$$\mathbb{CP}^{N-1} = \left\{ (\phi_1, \ldots, \phi_N) \,\middle|\, \sum_{i=1}^{N} |\phi_i|^2 = r \right\} \Big/ U(1) \,. \tag{2.26}$$

The modes of ϕ_i's tangent to this vacuum manifold are massless. The field σ and the mode of ϕ_i's transverse to $\sum_{i=1}^N |\phi_i|^2 = r$ have mass $e\sqrt{2r}$ as can be seen from the potential in (2.24). The gauge field v_μ acquires mass $e\sqrt{2r}$ by eating the Goldstone mode (Higgs mechanism). For fermions, the modes of $\psi_{i\pm}$ and $\overline{\psi}_{i\pm}$ obeying

$$\sum_{i=1}^{N} \overline{\phi}_i \psi_{i\pm} = 0, \quad \sum_{i=1}^{N} \overline{\psi}_{i\pm} \phi_i = 0, \tag{2.27}$$

are massless. Other modes including the fermions in the vector multiplet have mass $e\sqrt{2r}$. The equations (2.27) mean that the vectors $(\psi_{j\pm}, \overline{\psi}_{j\pm})$ are tangent to $\sum_{j=1}^N |\phi_j|^2 = r$ and are orthogonal to the gauge orbit. Namely, they are tangent vectors of the vacuum manifold $\mathbb{CP}^{N-1}$ at ϕ_i. These together with the tangent modes of ϕ_i's constitute massless supermultiplets. The massive bosonic and fermionic modes constitute a supermultiplet of mass $e\sqrt{2r}$. This is the supersymmetric version of the Higgs mechanism.

In the limit

$$e \to \infty, \tag{2.28}$$

the massive modes decouples and the classical theory reduces to that of the massless modes only. In this limit, the gauge kinetic term vanish and the vector multiplet becomes non-dynamical; the equations of motion simply yield algebraic constraints. The equations of motion for D and $\lambda_\pm$ yield the constraints (2.25) and (2.27). The equations for v_μ and σ give constraints on themselves;

$$v_\mu = \frac{i}{2} \frac{\sum_{i=1}^N \left(\overline{\phi}_i \partial_\mu \phi_i - \partial_\mu \overline{\phi}_i \phi_i \right)}{\sum_{j=1}^N |\phi_j|^2}, \tag{2.29}$$

$$\sigma = -\frac{\sum_{i=1}^N \overline{\psi}_{i+} \psi_{i-}}{\sum_{j=1}^N |\phi_j|^2}. \tag{2.30}$$

The action on the massless modes is that of the supersymmetric non-linear sigma model on $\mathbb{CP}^{N-1}$. The metric of $\mathbb{CP}^{N-1}$ is read from the scalar kinetic term and is given by $ds^2 = \sum_{i=1}^N |D\phi_i|^2$, where $D\phi_i$ is the covariant derivative of ϕ_i. Eqn (2.29) means that $D_\mu \phi_j$ is orthogonal to the gauge orbit. Thus, the metric $\sum_{i=1}^N |D\phi_i|^2$ measures the length of a tangent vector of $\mathbb{CP}^{N-1}$ by lifting it to a tangent vector of $\{\sum_{i=1}^N |\phi_i|^2 = r\}$ orthogonal to the gauge orbit. This is equal to r times the normalized Fubini-Study metric $g^{\rm FS}$

$$ds^2 = r\, g^{\rm FS}. \tag{2.31}$$

Using (2.29), one can also show that the Theta term $(\theta/2\pi)\int dv$ yields the B-field

$$B = \theta\, \omega^{\rm FS}, \tag{2.32}$$

where $\omega^{\rm FS}$ is the Kahler form for the Fubini-Study metric. Finally, the background value (2.30) for σ yields the four-fermi term of the non-linear sigma model. Thus, the classical theory reduces in the limit $e \to \infty$ to the supersymmetric non-linear sigma model on $\mathbb{CP}^{N-1}$ with the metric (2.31) and the B-field (2.32).

Let us examine the story at the quantum level. The main quantum effect is the one-loop renormalization of the FI-parameter: (2.22) or

$$r(\mu) = r(\mu') + N \log\left(\frac{\mu}{\mu'}\right) = N\log(\mu/\Lambda). \tag{2.33}$$

First thing to notice is that r is positive and large at high eneough energies. By this, we can consider the theory at high energy $\mu \gg \Lambda$ as the $\mathbb{CP}^{N-1}$ sigma model plus transverse modes, and the effect of the latter is suppressed by $\mu/e\sqrt{\log(\mu/\Lambda)}$. Let us check this with the RG flow of the NLSM. Since the metric of $\mathbb{CP}^{N-1}$ is given by (2.31), the flow of FI parameter (2.33) is the flow of the metric $g_{i\bar{\jmath}}(\mu) = g_{i\bar{\jmath}}(\mu') + N\log(\frac{\mu}{\mu'})g^{\rm FS}_{i\bar{\jmath}}$. Since the Fubini-Study metric obey the Einstein equation $R_{i\bar{\jmath}} = N g^{\rm FS}_{i\bar{\jmath}}$, the flow is written as

$$g_{i\bar{\jmath}}(\mu) = g_{i\bar{\jmath}}(\mu') + \log\left(\frac{\mu}{\mu'}\right) R_{i\bar{\jmath}}. \tag{2.34}$$

This is nothing but the flow of the metric in NLSM (1.62).

Finally, let us make an important observation. The Kahler form ω for the metric (2.31) is proportional to the Fubini-Study form; $\omega = r\omega^{\rm FS}$. Noting also (2.32), we find that the complexified Kahler class is given by

$$[\omega] - i[B] = t[\omega^{\rm FS}]. \tag{2.35}$$

In other words, t is nothing but the complexified Kahler class parameter. Thus, it is now manifest that the complexified Kahler class is parametrized by a twisted chiral parameter. We do not have to compute the topological correlation function to see this. This is a great advantage of the linear sigma model.

2.3.2 $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{CP}^1$ (Resolved Conifold)

We next consider the $U(1)$ gauge theory with four chiral superfields $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ with charge $1, 1, -1, -1$. Since the sum of charges vanish, $1+1-1-1=0$, r is not renormalized and axial R-symmetry is anomaly free. Thus, $t = r - i\theta$ is a genuin twisted chiral parameter of the system. The scalar potential of the model is

$$U = \sum_{i=1}^{4} |\sigma|^2 |\phi_i|^2 + \frac{e^2}{2} \left(|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 - r \right)^2 \tag{2.36}$$

The interpretation of the system is different depending on whether $r \gg 0$, $r = 0$ or $r \ll 0$.
$\underline{r \gg 0}$ The D-term constraint with $r > 0$ requires ϕ_1 or ϕ_2 to be non-zero. The gauge symmetry is Higgsed and $\sigma = 0$ is forced. The vacuum manifold is the total space of the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{CP}^1$; (ϕ_1, ϕ_2) span the base while ϕ_3 and ϕ_4 give the fibre coordinates. The model is interpreted as the sigma model on this vacuum manifold. This manifold is a non-compact CY, which is consistent with the fact that r does not run. t is again interpreted as the complexified Kahler parameter.
$\underline{r = 0}$ A configuration with $\phi_i \equiv 0$ is allowed. There the gauged symmetry is unbroken and there is a new flat direction spanned by σ, the "Coulomb branch". The standard branch is the zero base size limit of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and has a singularity, the vanished $\mathbb{CP}^1$. It is attatched to the "Coulomb brach" at that singularity.
$\underline{r \ll 0}$ The system is similar to the case $r \gg 0$ but the role of (ϕ_1, ϕ_2) and (ϕ_3, ϕ_4) is switched.
In all these cases, $x = \phi_1\phi_3$, $y = \phi_2\phi_4$, $z = \phi_1\phi_4$, $w = \phi_2\phi_3$ are gauge invariant variables and they are related by

$$xy = zw.$$

This is the equation difiening the conifold singularity. In the cases $r \gg 0$ and $r \ll 0$, the singularity is actually resolved. The resolution at $r \gg 0$ is different from that at $r \ll 0$: In the former, (ϕ_1, ϕ_2) becomes the base $\mathbb{CP}^1$ but in the latter, (ϕ_3, ϕ_4) becomes the base $\mathbb{CP}^1$. The two are related by "flop" — blown-down and blow-up.

We find that the parameter space is separated into two regions by the singularity at $r = 0$. At this level of analysis, we find that the regions related by "flop" are separated. We will see that a more careful analysis of gauge dynamic modifies this picture.

2.3.3 $\mathcal{O}(-d)$ over $\mathbb{CP}^{N-1}$ or $\mathbf{C}^N/\mathbf{Z}_d$ Orbifold

We next consider $U(1)$ gauge theory with $N+1$ fields $\Phi_1, ..., \Phi_N, P$ of charge $1, ..., 1, -d$, without superpotential term. The scalar potential of the model is

$$U = |\sigma|^2 \sum_{i=1}^{N} |\phi_i|^2 + |\sigma|^2 d^2 |p|^2 + \frac{e^2}{2} \left(\sum_{i=1}^{N} |\phi_i|^2 - d|p|^2 - r \right)^2 . \tag{2.37}$$

The structure of the vacuum manifold depends on whether $r > 0$, $r = 0$ or $r < 0$:
$\underline{r > 0}$ The D-term constaint requires some $\phi_i \neq 0$, which breaks the $U(1)$ gauge symmetry and giving mass to the gauge multiplet. In particular $\sigma = 0$. The vacuum manifold is the total space of the line bundle $\mathcal{O}(-d)$ over $\mathbb{CP}^{N-1}$ where $(\phi_1, ..., \phi_N)$ span the base $\mathbb{CP}^{N-1}$ and p is the fibre coordinate. The zise of $\mathbb{CP}^{N-1}$ is r.
$\underline{r = 0}$ $\phi_i = p = 0$ is alloed. If $\phi_i = p = 0$, the gauge symmetry is not broken and there is a flat direction parameterized by σ; let us call it a "Coulomb branch". There is also the standard "Higgs branch" where the gauge symmetrty is broken by ϕ_i or p. It is the zero base size limit of $\mathcal{O}(-d)$. The total vacuum manifold is a union of the two branches attatched at one-point.
$\underline{r < 0}$ The D-term constraint requires $p \neq 0$, breaking $U(1)$ gauge symmetry into $\mathbf{Z}_d$. The gauge multiplet is again massive and $\sigma = 0$ is enforced. The vacuum manifold has the $\mathbf{Z}_d$ orbifold singularity at $\phi_i = 0$. The metric is exactly the flat orbifold $\mathbf{C}^N/\mathbf{Z}_d$ in the limit $r \to -\infty$.

We recall that the FI parameter is renormalized as

$$r(\mu) = r(\mu') + (N - d)\log(\mu/\mu'). \tag{2.38}$$

Thus, the quantum theory depends crucially on whether $d < N$, $d = N$ or $d > N$. We discuss these cases separately.

• $d < N$.
$r > 0$ at high energies and thus we can interpret the model as the NLSM on the total space of $\mathcal{O}(-d)$ over $\mathbb{CP}^{N-1}$. The FI-Theta parameter t is identified there as the complexfied Kahler class.

• $d = N$.
In this case r does not run and the axial R-symmetry is anomaly free. $t = r - i\theta$ is thus a genuin parameter of the theory. The theory is interpreted as the sigma model on the vacuum manifold for the respective value of r. The theory is singular if $r = 0$ because a new branch, the "Coulomb" branch, develops. We find that the parameter space is separated into two regions by the singularity at $r = 0$. We will see that this will be

modified when further quantum effect is taken into account.

• $d > N$.
$r < 0$ at high energies and the model is interpreted as the sigma model on a space with $\mathbf{Z}_d$ orbifold singularity. In fact since $r \to -\infty$ in the continuum limit, the model is understood as some relevant perturbation of the $\mathbf{C}^N/\mathbf{Z}_d$ orbifold CFT.

In all these cases, a careful analysis is required to understand what happens in the infra-red limit. This will be done shortly.

2.3.4 Hypersurfaces and Complete Intersections

So far, we have been considering gauge theories without F-terms. We can actually obtain non-linear sigma models on a certain class of submanifolds of toric manifolds by turning on a certain type of superpotential. We focus on the basic example of hypersurfaces of $\mathbb{CP}^{N-1}$, which captures the essential point.

Hypersurfaces in $\mathbb{CP}^{N-1}$

Let us consider a degree d polynomial of $\phi_1, \ldots, \phi_N$;

$$G(\phi_1, \ldots, \phi_N) = \sum_{i_1,\ldots,i_d} a_{i_1 \ldots i_d} \phi_{i_1} \cdots \phi_{i_d}. \tag{2.39}$$

We assume that $G(\phi_i)$ is generic in the sense that

$$G = \frac{\partial G}{\partial \phi_1} = \cdots = \frac{\partial G}{\partial \phi_N} = 0 \quad \text{implies} \quad \phi_1 = \cdots = \phi_N = 0. \tag{2.40}$$

Then, the complex hypersurface M of $\mathbb{CP}^{N-1}$ defined by

$$G(\phi_1, \ldots, \phi_N) = 0 \tag{2.41}$$

is a smooth complex manifold of complex dimension $N - 2$. The Kahler form of $\mathbb{CP}^{N-1}$ restricts to a Kahler form on M. It is known that the second cohomology group is one-dimensional and is generated by the restriction of the class $[H] := c_1(\mathcal{O}(1))$ which is represented by a positive-definite Kahler form (up to normalization). The first Chern class of M is equal to

$$c_1(M) = (N - d)[H]|_M. \tag{2.42}$$

So, M is Ricci positive for $d < N$, Calabi-Yau for $d = N$, and Ricci negative for $d > N$. The non-linear sigma model on M is asymptotically free, scale invariant, and infra-red free, respectively.

Linear Sigma Model for the Hypersurface

Now, let us again consider a $U(1)$ gauge theory with $N+1$ chiral multiplets $\Phi_1, \cdots, \Phi_N, P$ of charge $1, \cdots, 1, -d$. This time, we include the gauge invariant superpotential

$$W = P\,G(\Phi_1, \ldots, \Phi_N). \tag{2.43}$$

The scalar potential of the model is

$$\begin{aligned} U &= |\sigma|^2 \sum_{i=1}^N |\phi_i|^2 + |\sigma|^2 d^2 |p|^2 + \frac{e^2}{2}\left(\sum_{i=1}^N |\phi_i|^2 - d|p|^2 - r\right)^2 \\ &\quad + \Big|G(\phi_1, \ldots, \phi_N)\Big|^2 + \sum_{i=1}^N |p|^2 |\partial_i G|^2 . \end{aligned} \tag{2.44}$$

Let us analyze the spectrum of the classical theory. The structure of the classical supersymmetric vacuum manifold $U = 0$ is different for $r > 0$ and $r < 0$, and we will treat these two cases (along with the case $r = 0$) separately.

$\underline{r > 0}$ $U = 0$ requires some $\phi_i \neq 0$ and therefore $\sigma = 0$. If $p \neq 0$, $U = 0$ further requires $G = \partial_1 G = \cdots \partial_N G = 0$ which implies by the condition (2.40) that all $\phi_i = 0$. However, this contradicts with $\phi_i \neq 0$ for some i. Thus p must be zero. We thus find that $U = 0$ is attained by $\sigma = p = 0$ and

$$\sum_{i=1}^N |\phi_i|^2 = r, \quad G(\phi_1, \ldots, \phi_N) = 0. \tag{2.45}$$

The vacuum manifold is the set of (ϕ_i) obeying these equations, divided by the $U(1)$ gauge group action. This is nothing but the hypersurface M. The modes of ϕ_i tangent to the manifold M are massless. Other modes are massive. Some have mass of order $e\sqrt{r}$ as in the case without superpotential, but some others have mass determined by G or its coefficients $a_{i_1 \ldots i_d}$ in (2.39). If we send e and $a_{i_1 \ldots i_d}$ to infinity by an overall scaling, all the massive modes decouple and the classical theory reduces to the non-linear sigma model on the hypersurface M, with the complexified Kahler class given by $[\omega] - i[B] = t[\omega^{\rm FS}]|_M$.

$\underline{r < 0}$ $U = 0$ requires $p \neq 0$ and thus $\sigma = 0$. Under the condition (2.40), $U = 0$ then requires all $\phi_i = 0$. p is thus constrained in $|p|^2 = |r|/d$. Up to the gauge transformation the vacuum manifold is a point. A choice of vacuum value of p, say $\langle p \rangle = \sqrt{|r|/d}$, breaks the $U(1)$ gauge symmetry into $\mathbf{Z}_d$. The vector multiplet fields together with the P-multiplet fields have a mass $e\sqrt{|r|/d}$ by the super-Higgs mechanism. The fields Φ_i are all massless as long as the degree d of the polynomial $G(\Phi_i)$ is larger than two, $d > 2$. If we take the limit $e \to \infty$, the classical theory reduces to the theory of Φ_i's only. It is the

Landau-Ginzburg theory with the superpotential

$$W = \langle p \rangle G(\Phi_1, \ldots, \Phi_N), \tag{2.46}$$

where $\langle p \rangle$ is the vacuum value of p (say $\langle p \rangle = \sqrt{|r|/d}$). We should keep in mind that there is a residual $\mathbf{Z}_d$ gauge symmetry, and it acts non-trivially on Φ_i (as charge 1 fields). Thus, the low energy theory is not the ordinary Landau-Ginzburg model but its $\mathbf{Z}_d$-orbifold, or "Landau-Ginzburg orbifold".

$\underline{r = 0}$ $U = 0$ requires $\sum_{i=1}^N |\phi_i|^2 = d|p|$. If $p \neq 0$, some $\phi_i \neq 0$. However, $U = 0$ with $p \neq 0$ requires $G = \partial_1 G = \cdots = \partial_N G = 0$ which means by the condition (2.40) $\phi_1 = \cdots = \phi_N = 0$, a contradiction. Thus p must be zero and $\phi_i = 0$. Then σ is free. The vacuum manifold is the complex σ-plane. Σ multiplet fields are always massless. At $\sigma \neq 0$ other modes are massive, but they become massless at $\sigma = 0$.

In the quantum theory, we must take into account the renormalization of the FI parameter r. It depends on whether $b_1 = N - d$ is positive, zero, or negative. We separate the discussion into these three cases.

• $d < N$.
In this case, the theory is parameterized by the dynamically generated scale Λ which determines the RG flow of the FI parameter

$$r(\mu) = (N - d) \log(\mu/\Lambda). \tag{2.47}$$

At the scale much larger than Λ, the FI parameter is positive and very large: $r \gg 1$. Thus, the first case of the above argument applies. In particular, by taking the limit where $e/\Lambda \to \infty$ and $a_{i_1 \ldots i_d}/\Lambda \to \infty$, the theory reduces to the non-linear sigma model on the hypersurface M. Since

$$c_1(M) = (N - d)[H]|_M \tag{2.48}$$

is positive, the sigma model is asymptotically free. The logarithmic running of the Kahler parameter of the non-linear sigma model is proportional to (2.48) and matches precisely with the logarithmic running (2.47) of the FI parameter.

• $d = N$.
In this case, the FI parameter does not run and the theory is parametrized by $t = r - i\theta$. In particular, we can choose the value of r as we wish. We separate the discussion into three cases.
$\underline{\text{For } r \gg 0}$, the theory reduces in the limit $e\sqrt{r} \to \infty$ and $a_{i_1 \ldots i_d} \to \infty$ to the non-linear sigma model on the hypersurface M. Since M is a Calabi-Yau manifold $c_1(M) = 0$, the

Kahler class of the sigma model does not run, which agrees with the fact that r does not run, either. The complexified Kahler class is identified as t at large r.
For $r \ll 0$, the theory reduces in the limit $e\sqrt{|r|} \to \infty$ to the LG orbifold.
For $r = 0$, the σ branch develops. It is a non-compact flat direction and the theory must exhibit some kind of singularity when approached from $r \gg 0$ or $r \ll 0$. The behavior of the theory near $r = 0$ is modified by several quantum effects and the Theta angle θ plays an important role. This will be discussed later in this section.

• $d > N$.
In this case, the FI parameter at the cut-off scale is large and negative. Thus, the theory at high energies does not describe the non-linear sigma model on the hypersurface M but looks closer to the LG orbifold. The LG orbifold itself is a superconformal field theory and must preserve the axial R-symmetry. On the other hand, the gauge theory preserves only the discrete subgroup $\mathbf{Z}_{2(d-N)}$ and contains a running coupling (the FI parameter). Thus, it would be appropriate to identify the model as the LG orbifold perturbed by a relevant operator that breaks the $U(1)$ axial R-symmetry to $\mathbf{Z}_{2(d-N)}$.

2.4 Low Energy Dynamics

In the previous discussion, we have identified the gauge theories as NLSMs (or orbifold/LG models) by looking at energies which are smaller than the coupling $e\sqrt{r}$ but are considered as high energies from the point of view of the NLSMs. We now attempt to describe the physics of the LSMs at much lower energies in order to learn about the low energy dynamics of the NLSMs models. In the case where the theory undergoes the dimensional transmutation we will look at energies μ smaller than the dynamical scale Λ.

2.4.1 The Behaviour at Large Σ

It turns out that it is useful in many ways to look at the behavior of the theory where the lowest component σ of the super-field-strength Σ is taken to be large and slowly varying. The σ dependent terms in the kinetic term of the charged matter field Φ are

$$-|\sigma|^2|\phi|^2 - \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_-. \tag{2.49}$$

We see that σ plays the role of the mass for the field Φ. Taking σ large means making Φ heavy. We are thus considering gauge theory with heavy charged matter fields.

To be specific, let us consider a $U(1)$ gauge theory with several charged chiral superfields Φ_i. At large σ the charged matter fields are heavy and the massless degrees of freedom are only the Σ multiplet itself. The theory is that of a $U(1)$ gauge theory in $1+1$ dimensions with heavy charged fields.

Let us compute the vacuum energy of the system. Since Φ_i's are heavy, they are frozen at the zero expectation value and one can set $\Phi_i = 0$ classically. Then, the potential energy is given by

$$U_r = \frac{e^2}{2} r^2. \tag{2.50}$$

The contributions to the vacuum energy from σ and $\lambda_\pm$ cancell against each other because of the supersymmetry. There is actually a contribution to the energy density from the gauge field v_μ. The terms in the action that depend on the gauge field are

$$S = \frac{1}{2\pi} \int d^2x \left(\frac{1}{2e^2} v_{01}^2 + \theta v_{01} \right). \tag{2.51}$$

Let us quantize the system by compactifying the spacial direction on S^1 so that x^1 is a periodic coordinate of period 2π, $x^1 \equiv x^1 + 2\pi$. By using gauge transformations $v_\mu \to v_\mu - \partial_\mu \gamma$, one can set

$$v_0 = 0, \quad v_1 = a(t), \tag{2.52}$$

where $a(t)$ depends only on $t = x^0$. The gauge transformation $\gamma = mx^1$ preserves this form. This is an allowed gauge transformation provided m is an integer since $\mathrm{e}^{i\gamma} = \mathrm{e}^{imx^1}$ is single valued if $m \in \mathbf{Z}$. Thus there is a gauge equivalence relation

$$a(t) \equiv a(t) + m, \quad m \in \mathbf{Z}. \tag{2.53}$$

In terms of this variable, the action is given by

$$S = \int dt \left(\frac{1}{2e^2} \dot{a}^2 + \theta \dot{a} \right). \tag{2.54}$$

The transition amplitude from a state Ψ_i at time t_i to a state Ψ_f at time t_f is given by the path-integral

$$\begin{aligned} \langle \Psi_f, \mathrm{e}^{-i(t_f - t_i)H} \Psi_i \rangle &= \int da_f da_i \Psi_f^*(a_f) \int_{\substack{a(t_f)=a_f \\ a(t_i)=a_i}} \mathcal{D}a \; \mathrm{e}^{iS} \, \Psi_i(a_i) \\ &= \int da_f da_i \Psi_f^*(a_f) \, \mathrm{e}^{i\theta a_f} \int_{\substack{a(t_f)=a_f \\ a(t_i)=a_i}} \mathcal{D}a \; \mathrm{e}^{i \int_{t_i}^{t_f} \frac{1}{2e^2} \dot{a}^2 dt} \; \mathrm{e}^{-i\theta a_i} \Psi_i(a_i) \end{aligned} \tag{2.55}$$

This shows that the Hamiltonian acts on the phase-rotated wavefunctions $\widetilde{\Psi}(a) = e^{-i\theta a}\Psi(a)$ as $\frac{e^2}{2}\left(-i\frac{d}{da}\right)^2$. Namely, it acts on the ordinary wavefunctions as

$$H\Psi(a) = \frac{e^2}{2}\left(-i\frac{d}{da} - \theta\right)^2 \Psi(a). \tag{2.56}$$

We recall that a is a periodic variable (2.53). Thus, single-valued wavefunctions $\Psi(a)$ are expressed as linear combinations of the Fourier modes $e^{2\pi nia}$ with $n \in \mathbf{Z}$. Those Fourier modes are actually the energy eigenfunctions. Thus, the spectrum is

$$E_n = \frac{e^2}{2}(2\pi n - \theta)^2. \tag{2.57}$$

The ground state energy is therefore given by

$$E_{\text{vac}} = \frac{e^2}{2}\hat{\theta}^2 \tag{2.58}$$

where $\hat{\theta}^2$ is defined by

$$\hat{\theta}^2 := \min_{n\in\mathbf{Z}}\{(\theta - 2\pi n)^2\}. \tag{2.59}$$

This total energy E_{vac} can be considered also as the vacuum energy density since $\frac{1}{2\pi}\int dx^1 = 1$ in the present set-up. What is the value of the field strength at the ground state? To see this, we note that the conjugate momentum for a is given by $p_a = \frac{\partial L}{\partial \dot{a}} = \frac{\dot{a}}{e^2} + \theta$. [1] From this we see that

$$v_{01} = -e^2\theta + e^2 p_a. \tag{2.60}$$

Namely, the field strength $v_{01} = \dot{a}$ is equal to $-e^2\theta$ up to integer multiples of $2\pi e^2$. In particular, the magnitude of the vacuum value of v_{01} is

$$|v_{01}|_{\text{vac}} = e^2|\hat{\theta}|. \tag{2.61}$$

The vacuum value of v_{01} is thus discontinuous as a function of θ. There is an intuitive understanding of this discontinuity, due to Coleman, which applies when the theory is formulated on $\mathbf{R}^2$. We assume that the mass M of the charged particle is much larger than the gauge coupling, $M \gg e$, so that the charged particles can be treated semi-classically. If we put a charged particle of charge Q at $x^1 = 0$, it generates a field strength v_{01} which obeys

$$\partial_1 v_{01} = 2\pi Q e^2 \delta(x^1). \tag{2.62}$$

Namely, it generates a gap of v_{01} by $2\pi Q e^2$. Now suppose θ is positive but smaller than π. Then there is a unique ground state with the field strength $v_{01} = -e^2\theta$ and the energy

[1] In fact a naive canonical quantization also leads to the result (2.56); $H = p_a\dot{a} - L = \frac{e^2}{2}(p_a - \theta)^2$.

density $U = \frac{e^2}{2}\theta^2$. One cannot have a single charged particle since that would make $v_{01}(+\infty)$ to be different from $v_{01}(-\infty)$ but v_{01} is required to take the (unique) vacuum value at both spacial infinity. However, one can have particles of total charge zero. For instance, let us consider the situation where we have one with charge 1 at $x^1 = -L/2$ and one with charge -1 at $x^1 = L/2$. Outside the interval $-L/2 \leq x^1 \leq L/2$ the field strength takes the vacuum value $-e^2\theta$ while it takes the value $-e^2\theta + 2\pi e^2$ inside that interval. The energy of that configuration compared to the one for the vacuum state with $v_{01} \equiv -e^2\theta$ is

$$\Delta E = \left(\frac{e^2}{2}(2\pi - \theta)^2 - \frac{e^2}{2}\theta^2\right) L. \tag{2.63}$$

As long as $\theta < \pi$, this is positive and is proportional to the separation L. To descrese the energy, the separation L is reduced to zero. Namely, there is an attractive force between the particles of opposite charge. Charged particles cannot exist in isolation; they are *confined*. Now let us increase θ so that $\theta > \pi$. Then ΔE is negative. It is now energetically favorable for the separation L to be larger. There is a replusive force now. Eventually, the two particles are infinitely separated and disappear to the negative and positive infinities in x^1. What is left is the field strength with the value $v_{01} = -e^2\theta + 2\pi e^2$. The absolute value is nothing but $e^2|\hat{\theta}|$ for θ in the range $\pi < \theta < 3\pi$. Even if we started without the particles of opposite charges, they can be created and go away to infinity. Creating a pair costs an energy $2M$, but the negative enregy ΔE for large L is enough to cancel it. Effectively, the field strength is reduced by $2\pi e^2$. This is the intuitive explanation of the discontinuity. A similar thing happens when θ goes beyond $3\pi, 5\pi, \ldots$ or when θ decreases in the negative direction and goes below $-\pi$, $-3\pi, \ldots$. [2]

The total energy density is thus the sum of (2.50) and (2.58)

$$U = \frac{e^2}{2}\left(r^2 + \hat{\theta}^2\right) = \frac{e^2}{2}\left|\hat{t}\right|^2. \tag{2.64}$$

We notice that this expression is almost the same as the potential energy of the Landau-Ginzburg model obtained by setting Φ_i to zero and considering Σ as the ordinary twisted chiral superfield having the twisted superpotential

$$W(\Sigma) = -t\Sigma. \tag{2.65}$$

[2]Here we are assuming that there is a matter field of charge 1, or the greatest common divisor of the charges Q_i is 1. If the g.c.d. of Q_i's is $q > 1$, the critical value of θ is $q\pi$ (times an odd integer) and the definition of $\hat{\theta}^2$ is replaced by

$$\hat{\theta}^2 := \min_{n\in\mathbf{Z}}\{(\theta + 2\pi qn)^2\}.$$

Thus, in such a case the physics is periodic in θ with period $2\pi q$.

That Σ is not really an ordinary twisted chiral superfield but the super-field-strength (the imaginary part of the auxiliary field is the curvature v_{01}) has only a minor effect; the shift in θ by 2π times an integer.

This story, however, can be further modified by quantum effects. In the above discussion we have considered Φ_i to be totally frozen. But of course we must take into account the quantum fluctuation of Φ_i's. What it does is to modify the FI-Theta parameter as a function of σ. Let us now analyze this.

Effective Action for Σ

Let us first consider the basic example of the $U(1)$ gauge theory with a single chiral superfield Φ of charge 1, without F-term (which is not allowed in this case). Let us take σ to be slowly varying and large compared to the energy scale μ where we look at the effective theory. The Φ multiplet has a mass of order $\sigma \gg \mu$ and therefore it is appropriate to describe the effective theory in terms of the low frequency modes of Σ only. Thus, the effective action at energy μ is obtained by integrating out the entire modes of Φ and the modes of Σ with the frequencies in the range $\mu \leq |k| \leq \Lambda_{\rm UV}$. By supersymmetry, the terms with at most two derivatives and not more than four fermions are constrained to be of the form

$$S_{\it eff}(\Sigma) = \int d^4\theta(-K_{\it eff}(\Sigma,\overline{\Sigma})) + \frac{1}{2}\left(\int d^2\widetilde{\theta}\, \widetilde{W}_{\it eff}(\Sigma) \;+\; c.c.\right). \tag{2.66}$$

We try to compute these terms in two steps: integrate out Φ first, then the high frequency modes of Σ. Since the action $S(\Sigma,\Phi)$ is quadratic in Φ, the first step can be carried out exactly by the one-loop computation

$$e^{iS^{(1)}_{\it eff}(\Sigma)} = \int \mathcal{D}\Phi\; e^{iS(\Sigma,\Phi)}. \tag{2.67}$$

As we will see, the effective superpotential $\widetilde{W}^{(1)}_{\it eff}(\Sigma)$ will not be further corrected by the second step (a non-renormalization theorem). Thus, the focus will be on obtaining $\widetilde{W}_{\it eff}$ by the first step.

Since $\Sigma = \sigma + \theta^+\overline{\theta}^-(D - iv_{01}) + \cdots$, the dependence of the effective action on D and v_{01} is as follows. From the D-term we obtain

$$\int d^4\theta(-K_{\it eff}(\Sigma,\overline{\Sigma})) = \partial_\sigma\partial_{\overline{\sigma}}K_{\it eff}(\sigma,\overline{\sigma})|D - iv_{01}|^2 + \cdots. \tag{2.68}$$

From the twisted F-terms we have

$$\frac{1}{2}\left(\int d^2\widetilde{\theta}\,\widetilde{W}_{\it eff}(\Sigma) \;+\; c.c.\right) \;=\; \frac{1}{2}\Big(\partial_\sigma\widetilde{W}_{\it eff}(\sigma)(D - iv_{01}) \;+\; c.c.\Big)$$

$$= D\,\mathrm{Re}\left[\partial_\sigma \widetilde{W}_{\!eff}(\sigma)\right] + v_{01}\,\mathrm{Im}\left[\partial_\sigma \widetilde{W}_{\!eff}(\sigma)\right] + \cdots. \tag{2.69}$$

Thus, in order to determine $\widetilde{W}_{eff}$ it is enough to look at the D and v_{01} linear terms in the effective action. The Kahler potential can be determined by the D quadratic term. To simplify the computation one can set $\lambda_\pm = \overline{\lambda}_\pm = 0$ without losing any information. In this case the Φ-dependent part of the (Euclidean) classical action is

$$\begin{aligned} L^E_{kin} &= |D_\mu\phi|^2 + |\sigma|^2|\phi|^2 - D|\phi|^2 \\ &\quad -2i\overline{\psi}_- D_{\bar z}\psi_- + 2i\overline{\psi}_+ D_z\psi_+ + \overline{\psi}_-\sigma\psi_+ + \overline{\psi}_+\overline{\sigma}\psi_-. \end{aligned} \tag{2.70}$$

We are going to evaluate

$$\mathrm{e}^{-\frac{1}{2\pi}\int \Delta L^{(1)}_E d^2x} := \int \mathcal{D}^2\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}\ \mathrm{e}^{-\frac{1}{2\pi}\int L^E_{kin} d^2x}. \tag{2.71}$$

The dependence of $\Delta L^{(1)}_E$ on the phase of $\sigma = |\sigma|\,\mathrm{e}^{i\gamma}$ is easy to obtain. At the classical level, this phase can be absorbed by the phase rotation of the fermions

$$\psi_\pm \to \mathrm{e}^{\mp i\gamma/2}\psi_\pm,\quad \overline{\psi}_\pm \to \mathrm{e}^{\pm i\gamma/2}\overline{\psi}_\pm. \tag{2.72}$$

However, this is the axial rotation which is anomalous. The effect is thus the shift in the Theta angle noted before. In other words the effective action for σ is related to that for $|\sigma|$ by

$$\Delta L^{(1)}_E(\sigma) = \Delta L^{(1)}_E(|\sigma|) - i\gamma\, v_{12} = \Delta L^{(1)}_E(|\sigma|) - i\arg(\sigma) v_{12}. \tag{2.73}$$

Now, $\Delta L^{(1)}_E(|\sigma|)$ is given by

$$e^{-\frac{1}{2\pi}\int \Delta L^{(1)}_E(|\sigma|)d^2x} = \frac{\det\begin{pmatrix} |\sigma| & 2iD_{\bar z} \\ 2iD_z & -|\sigma| \end{pmatrix}}{\det\Big(-D_\mu D_\mu + |\sigma|^2 - D\Big)}. \tag{2.74}$$

The square of the Dirac operator is

$$\begin{aligned} \begin{pmatrix} |\sigma| & 2iD_{\bar z} \\ 2iD_z & -|\sigma| \end{pmatrix}^2 &= \begin{pmatrix} -4D_{\bar z}D_z + |\sigma|^2 & 0 \\ 0 & -4D_zD_{\bar z} + |\sigma|^2 \end{pmatrix} \\ &= \begin{pmatrix} -D_\mu D_\mu + |\sigma|^2 - v_{12} & 0 \\ 0 & -D_\mu D_\mu + |\sigma|^2 + v_{12} \end{pmatrix}, \end{aligned} \tag{2.75}$$

where we have used the relation $D_{\bar z}D_z = \frac{1}{2}(D_{\bar z}D_z + D_zD_{\bar z}) + \frac{1}{2}[D_{\bar z}, D_z] = \frac{1}{4}D_\mu D_\mu + \frac{1}{2}iv_{z\bar z}$. Thus, we obtain

$$\begin{aligned} \frac{1}{2\pi}\int \Delta L^{(1)}_E(|\sigma|)d^2x &= \log\det(-D_\mu D_\mu + |\sigma|^2 - D) \\ &\quad -\frac{1}{2}\log\det(-D_\mu D_\mu + |\sigma|^2 - v_{12}) - \frac{1}{2}\log\det(-D_\mu D_\mu + |\sigma|^2 + v_{12}). \end{aligned} \tag{2.76}$$

There is no v_{12} linear term in this but there is a D-linear term. It is given by

$$\frac{1}{2\pi}\int \Delta L_E^{(1)}(|\sigma|)d^2x\bigg|_{D-\text{linear}} = \mathrm{Tr}\left(\frac{-D}{-\partial_\mu\partial_\mu + |\sigma|^2}\right). \tag{2.77}$$

Namely, we have

$$\Delta L_E^{(1)}(|\sigma|)\bigg|_{D-\text{linear}} = -D\int\limits_{|k|\leq\Lambda_{\rm UV}} \frac{d^2k}{(2\pi)^2}\frac{2\pi}{k^2+|\sigma|^2} = -\frac{1}{2}D\log\left(\frac{\Lambda_{\rm UV}^2+|\sigma|^2}{|\sigma|^2}\right). \tag{2.78}$$

Similarly we can read the D quadratic term from (2.76) as

$$\Delta L_E^{(1)}(|\sigma|)\bigg|_{D-\text{quadratic}} = -\frac{1}{2}D^2\int\limits_{|k|\leq\Lambda_{\rm UV}} \frac{d^2k}{(2\pi)^2}\frac{2\pi}{(k^2+|\sigma|^2)^2} = -\frac{1}{4|\sigma|^2}D^2\frac{1}{1+\frac{|\sigma|^2}{\Lambda_{\rm UV}^2}}. \tag{2.79}$$

To summarize, we have

$$\Delta L_E^{(1)}(\sigma) = -\log\left(\frac{\Lambda_{\rm UV}}{|\sigma|}\right)D - i\arg(\sigma)v_{12} - \frac{1}{4|\sigma|^2}(D^2-v_{12}^2)+\cdots, \tag{2.80}$$

where $+\cdots$ are the terms which are not linear nor quadratic in $(D,\, v_{12})$, and we have neglected the powers of $|\sigma|/\Lambda_{\rm UV}$ which vanish in the continuum limit. Noting the relation of the Euclidean and Minkowski Lagrangians $L_E = -L|_{x^0=-ix^2}$ we see that

$$\partial_\sigma\Delta\widetilde{W}^{(1)} = \log\left(\frac{\Lambda_{\rm UV}}{|\sigma|}\right) - i\arg(\sigma) = \log\left(\frac{\Lambda_{\rm UV}}{\sigma}\right), \tag{2.81}$$

$$\partial_\sigma\partial_{\overline{\sigma}}\Delta K^{(1)} = \frac{1}{4|\sigma|^2}. \tag{2.82}$$

Thus we find

$$\partial_\sigma\widetilde{W}_{\!\mathit{eff}}^{(1)} = \log\left(\frac{\Lambda_{UV}}{\sigma}\right) - t_0 = \log\left(\frac{\mu}{\sigma}\right) - t(\mu), \tag{2.83}$$

$$\partial_\sigma\partial_{\overline{\sigma}}K_{\mathit{eff}}^{(1)} = \frac{1}{2e^2}+\frac{1}{4|\sigma|^2}. \tag{2.84}$$

In (2.83), the dependence on the ultra-violet cut-off $\Lambda_{\rm UV}$ has cancelled against the one from the bare coupling t_0. Similarly, it is independent of the choice of the scale μ; the $\log(\mu)$ dependence is cancelled by the $\log(\mu)$ dependence of $t(\mu)$ induced by the RG flow. In terms of $\mathit{\Lambda} := \mu\, \mathrm{e}^{-t(\mu)} = \Lambda\, \mathrm{e}^{i\theta}$, the complexified RG invariant scale parameter, (2.83) can be written as

$$\partial_\sigma\widetilde{W}_{\!\mathit{eff}}^{(1)}(\sigma) = \log(\mathit{\Lambda}/\sigma). \tag{2.85}$$

This effective superpotential captures the axial anomaly of the system; The axial rotation $\Sigma\to\mathrm{e}^{2i\beta}\Sigma$ shifts the Theta angle as $\theta\to\theta-2\beta$ (or $\widetilde{W}_{\!\mathit{eff}}^{(1)}(\Sigma)$ has the correct axial charge 2 if we let the axial R-rotation shift the Theta angle as $\theta\to\theta+2\beta$).

We have yet to integrate out the high frequency modes of the Σ multiplet fields. This will definitely affect the Kahler potential. However, it cannot affect the twisted superpotential. The correction should involve the gauge coupling constant e but that parameter cannot enter into $\widetilde{W}_{\mathit{eff}}$. To elaborate this point, we first note that the standard requirements (symmetry, holomorphy, asymptotic condition) constrain the form of the superpotential. Here we use the axial R-symmetry with $\sigma \to e^{2i\beta}\sigma$, $\Lambda \to e^{2i\beta}\Lambda$ and the condition that $\widetilde{W}_{\mathit{eff}}(\Sigma)$ approaches (2.85) at $\sigma/\Lambda \to \infty$. The constrained form is such that

$$\partial_\sigma \widetilde{W}_{\mathit{eff}}(\sigma) = \log(\Lambda/\sigma) + \sum_{n=1}^{\infty} a_n (\Lambda/\sigma)^n. \tag{2.86}$$

The correction terms take the form of non-perturbative corrections. However, in the present computation, we are simply integrating out the high frequency modes of Σ in a theory without a charged field, and there is no room for non-perturbative effects. Thus, we conclude that all $a_n = 0$. This establishes that (2.85) remains the same at lower energies.

We thus see that the effective superpotential is given by

$$\widetilde{W}_{\mathit{eff}}(\Sigma) = -\Sigma\left(\log\left(\frac{\Sigma}{\Lambda}\right) - 1\right). \tag{2.87}$$

We consider its first derivative as the effective FI-Theta parameter that varies as a function of σ;

$$t_{\mathit{eff}}(\sigma) := -\partial_\sigma \widetilde{W}_{\mathit{eff}}(\sigma) = \log(\sigma/\Lambda). \tag{2.88}$$

Using (2.64) we find that the energy density is given by

$$U = \frac{e_{\mathit{eff}}^2}{2}\,|\,\hat{t}_{\mathit{eff}}(\sigma)\,|^2, \tag{2.89}$$

where $(1/2e_{\mathit{eff}}^2) = \partial_\sigma\partial_{\bar\sigma}K_{\mathit{eff}}$, and the hat in $\hat{t}_{\mathit{eff}}$ stands for the shift by $2\pi n$ that is explained above. This shift resolves the apparent problem of the superpotential (2.87) not being single valued.

It is straightforward to generalize the above result to more general cases. If there are N chiral superfields of charge 1, the effective action is simply obtained by multiplying $\Delta L(\sigma)$ by N. Thus, the effective superpotential is

$$\widetilde{W}_{\mathit{eff}}(\Sigma) = -\Sigma\left[N\left(\log\left(\frac{\Sigma}{\mu}\right) - 1\right) + t(\mu)\right] = -N\Sigma\left(\log\left(\frac{\Sigma}{\Lambda}\right) - 1\right), \tag{2.90}$$

where $\Lambda := \mu\, e^{-t(\mu)/N}$ is the complexified RG invariant dynamical scale. For the most general case where the gauge group is $U(1)^k = \prod_{a=1}^k U(1)_a$ with the chiral matter fields

Φ_i of charge Q_{ia}. The effective superpotential is

$$\widetilde{W}_{eff}(\Sigma_1, \dots, \Sigma_k) = -\sum_{a=1}^{k} \Sigma_a \left[\sum_{i=1}^{N} Q_{ia} \left(\log\left(\frac{\sum_{b=1}^{k} Q_{ib}\Sigma_b}{\mu} \right) - 1 \right) + t_a(\mu) \right]. \qquad (2.91)$$

This is derived exactly using one-loop computation in the case where there is no superpotential term for Φ_i's. However, even if there is such an F-term, by the decoupling theorem of F-terms and twisted F-terms, the result (2.91) will not be affected.

2.4.2 The $\mathbb{CP}^{N-1}$ Model

Now let us study the low energy behavior of the $\mathbb{CP}^{N-1}$ model. As we have seen, this is realized by the $U(1)$ gauge theory with N chiral superfields of charge 1. The axial R-symmetry $U(1)_A$ is anomalously broken to $\mathbf{Z}_{2N}$ and the theory dynamically generates the scale parameter Λ. We look at the effective theory at energy $\mu \ll \Lambda$. The region in the field space where σ is slowly varying compared to $1/\mu$ and much larger than μ is described by the theory of the Σ multiplet determined above. Namely, the effective twisted superpotential is given by (2.90) with the effective FI-Theta parameter

$$t_{eff}(\sigma) := -\partial_\sigma \widetilde{W}_{eff}(\sigma) = N \log(\sigma/\Lambda). \qquad (2.92)$$

The supersymmetric ground states are found by looking for the value of σ which satisfy $U = (e_{eff}^2/2)|\hat{t}_{eff}(\sigma)|^2 = 0$. Namely, we look for solutions to $t_{eff}(\sigma) \in 2\pi i\mathbf{Z}$ or equivalently

$$e^{t_{eff}(\sigma)} = 1. \qquad (2.93)$$

This is solved by

$$\sigma = \Lambda \cdot e^{2\pi i n/N}, \quad n = 0, \dots, N-1. \qquad (2.94)$$

Since the scale μ is taken to be much smaller than Λ, these vacua are in the region where the analysis is valid. Thus, we find N supersymmetric vacua in this region. The $\mathbf{Z}_{2N}$ axial R-symmetry cyclically permutes these N vacua. Namely, a choice of a vacuum spontaneously breaks the axial R-symmetry to $\mathbf{Z}_2$;

$$\mathbf{Z}_{2N} \rightsquigarrow \mathbf{Z}_2. \qquad (2.95)$$

From this analysis alone, however, we cannot exclude the possibility of other vacua in the region with small σ. To describe the physics in such a region, we need to use a completely different set of variables. If we use the full variables Φ_i's and Σ, we need to find a minimum where the potential U in (2.24) vanishes. However, if $\mu \ll \Lambda$, $r(\mu)$ is large and negative and $U = 0$ can not be attained by any configuration. This is one indication that there is other vacuum state. Also, the above number, N, saturates the number of supersymmetric vacua

$$\dim H^*(\mathbb{CP}^{N-1}) = N \tag{2.96}$$

found from the direct analysis of non-linear sigma model. This also indicates that there is no other vacuum. However, to find out the decisive answer we need more information. That will be provided when we will prove the mirror symmetry of the $\mathbb{CP}^{N-1}$ model and the LG model of affine Toda superpotential. The determination of the supersymmetric vacua of the latter model is straightforward and it tells us that the above N vacua are indeed the whole set.

The Dynamics at Large N

We have seen that σ has a non-zero expectation values at these N vacua. This shows that the matter fields Φ_i, which include massless modes (the Goldstone modes for $SU(N)/\mathbf{Z}_N \rightsquigarrow U(N-1)/\mathbf{Z}_N$) classically, acquire a mass

$$m_\Phi \simeq \Lambda, \tag{2.97}$$

at the quantum level. Since there are no Goldstone bosons, the Global symmetry $SU(N)/\mathbf{Z}_N$ cannot be broken.

Let us try to analyze the gauge dynamics of these massive charged fields. For this we need to know also the gauge kinetic terms, not only the superpotential terms. From (2.84) we see that the effective gauge coupling constant at the one-loop level is given by

$$\frac{1}{e_{\mathit{eff}}^{(1)2}} = \frac{1}{e^2} + \frac{N}{2|\sigma|^2}. \tag{2.98}$$

As we noted above, this is further corrected by Σ-integrals and we do not know the actual form of the effective gauge coupling constant. However, there is a limit in which one can actually use (2.98) to analyze the dynamics. It is the large N limit. Since there are N matter fields of the same charge, the matter integral simply yields N times $\Delta L^{(1)}(\Sigma)$. Thus, any correction to (2.98) is suppressed by powers of $1/N$. Also, the gauge coupling near the vacua is of order $\Lambda/\sqrt{N}$ and can be made as small as one wishes, no matter how large is the bare gauge coupling e (this is particularly useful for our purpose; the $e \to \infty$ limit). In particular, in this limit, the mass of the charged matter fields is very large compared to the gauge coupling constant,

$$m_\Phi/e_{\mathit{eff}} \sim \sqrt{N} \gg 1. \tag{2.99}$$

Thus, we can treat the charged matter fields semi-classically.

Suppose the Φ_i or $\overline{\Phi}_i$ particles are located at $x^1 = x^1, \ldots, x^1_s$. Then, the equation of motion for the gauge field is given by

$$\frac{\partial}{\partial x^1}\left(\frac{v_{01}}{e_{\mathit{eff}}^2} + \theta_{\mathit{eff}}\right) = 2\pi \sum_{i=1}^{s} \epsilon_i \delta(x^1 - x^1_i), \tag{2.100}$$

where θ_{eff} is the effective Theta angle

$$\theta_{\mathit{eff}} = \mathrm{Im}\,(t_{\mathit{eff}}(\sigma)) = N \arg(\sigma/\Lambda), \tag{2.101}$$

and $\epsilon_i = \pm 1$ is the charge of the particle at $x^1 = x^1_i$. Thus, $v_{01}/e^2_{\mathit{eff}} + \theta_{\mathit{eff}}$ has a gap of $\pm 2\pi$ at the location of the particles. At any of the N vacua we have $v_{01} = e^2_{\mathit{eff}}|\hat{\theta}_{\mathit{eff}}|^2 = 0$, which means $\theta_{\mathit{eff}} = 2\pi n$ for some $n \in \mathbf{Z}$. Thus, in order to have a finite energy configuration, we need

$$\left.\begin{array}{l} v_{01} \to 0 \\ \theta_{\mathit{eff}} \to 2\pi n_{\pm} \end{array}\right\} \text{ at } x^1 \to \pm\infty, \tag{2.102}$$

where $n_{\pm}$ are some integers. For an arbitrary distribution of particles, we can find a solution to (2.100) obeying this condition. In particluar, a Φ_i particle (or a $\overline{\Phi}_i$ particle) can exist by itself. In the presence of a Φ_i particle, the vacuum at left infinity $x^1 \to -\infty$ is not the same as the vacuum at the right infinity $x^1 \to +\infty$. This is because

$$\theta_{\mathit{eff}}\Big|_{x^1=+\infty} - \theta_{\mathit{eff}}\Big|_{x^1=-\infty} = \int_{-\infty}^{+\infty} \frac{\partial}{\partial x^1}\left(\frac{v_{01}}{e^2_{\mathit{eff}}} + \theta_{\mathit{eff}}\right) dx^1 = \int 2\pi\delta(x^1 - x^1_0) = 2\pi, \tag{2.103}$$

where we have used $v_{01} \to 0$ at $x^1 \to \pm\infty$. If the left infinity is at $\sigma = \Lambda$, then the right infinity is at $\sigma = \Lambda\, \mathrm{e}^{2\pi i/N}$. A configuration connecting different vacua is called a *soliton.* We have shown that Φ_i is a soliton. We will see later that this soliton preserves a part of the supersymmetry and its mass can be computed exactly.

If one Φ_i particle and one $\overline{\Phi}_i$ particle are located at $x^1 = -L/2$ and $x^1 = L/2$ respectively, eqn.(2.100) can be solved by a configuration as shown in Fig.2. The configuration is at the vacuum in the region $-L/2 < x^1 < L/2$ and the total energy does not grow linearly as a function of the separation L. Thus, there is no long range force between them. Namely, *charged particles are not confined in this theory.* This is essentially the effect of the coupling

$$N \arg(\sigma/\Lambda) v_{01}. \tag{2.104}$$

This coupling screens the long range interaction between the charged particles.

Thus, the Φ_i particle exists as a particle state in the quantum Hilbert space. From the classical story, we expect that these states consitute the fundamental representaion of the

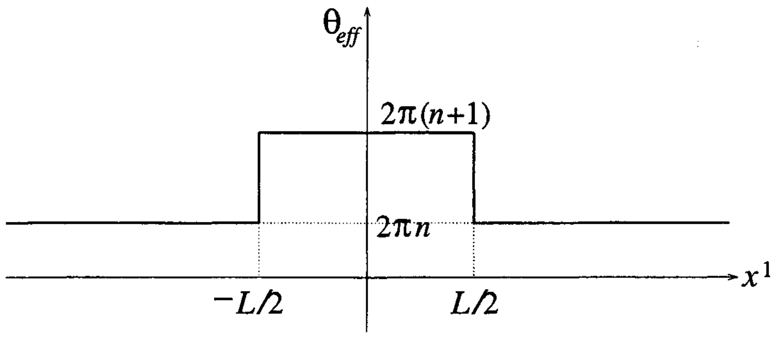

Figure 2: The configuration of $\theta_{eff} = N \arg(\sigma/\Lambda)$ for a pair of particles, charge 1 at $x^1 = -L/2$ and charge -1 at $x^1 = L/2$.

group $SU(N)$. Note that $SU(N)$ is not quite the same as the classical global symmetry group $SU(N)/\mathbf{Z}_N$. The symmetry group of the quantum theory is not $SU(N)/\mathbf{Z}_N$ but its universal covering group. Such a phenomenon is common in quantum field theories (known as *charge fractionalization*). In the present case this happens because there appeared a state transforming nontrivially under the "overlap" $\mathbf{Z}_N$ of $SU(N)$ and the gauge group, $U(1)$. Whether such a thing happens or not depends on the gauge dynamics. If the Φ_i particles were confined (as in the case without $\arg(\sigma)$-v_μ coupling), there would not be a state charged under $U(1)$ gauge group, and therefore all the states would be neutral under $\mathbf{Z}_N = SU(N) \cap U(1)$. In that case, the global symmetry group would remain as $SU(N)/\mathbf{Z}_N$.

2.5 The "Phases"

Let us consider a $U(1)$ gauge theory with several chiral superfields $\Phi_1, \ldots, \Phi_M$ with charges $Q_1, \ldots, Q_M$ that sum to zero:

$$\sum_{i=1}^{M} Q_i = 0. \tag{2.105}$$

In this case, the axial R-symmetry $U(1)_A$ is an exact symmetry of the quantum theory, and the FI parameter does not run along the RG flow. We have in mind two classes of theories: one is the linear sigma model for compact Calabi-Yau hypersurfaces in $\mathbb{CP}^{N-1}$ or weighted projective spaces; the other is the theory without F-terms, which yields the non-linear sigma model on non-compact Calabi-Yau manifolds.

Since the FI parameter does not run, one can choose r to be whatever value one wants. As we have seen in the previous discussion, the theory at $r \gg 0$ and the theory at $r \ll 0$

have completely different interpretations, and also at $r = 0$ the theory becomes singular due to a development of a new branch of vacuum manifold where σ is unconstrained. Thus, it appears that the parameter space is completely separated by a singular point $r = 0$ into two regions with different physics.

This picture is considerably modified when the Theta angle θ is taken into account. The actual parameter of the theory (in addition to the real and chiral parameterts that enter into D-terms and F-terms) is $t = r - i\theta$ and the parameter space is a complex torus or a cylinder. It may appear that the parameter space is still separated into two regions by the circle at $r = 0$. However, it turns out not to be the case when we think about the origin of the singularity at $r = 0$. The singularity is expected when there is a new branch of vacua where new massless degrees of freedom appears. In the classical analysis at $r = 0$, that is identified as the Σ multiplet since there is a non-compact flat direction where σ is free. However, at large σ, as we have analysed the actual energy density receives also a contribution from the electric field or Theta angle as in (2.64). Taking into account the more refined quantum correction, the energy density at large σ is

$$U = \frac{e_{\mathit{eff}}^2}{2}\left(r_{\mathit{eff}}^2 + \hat{\theta}_{\mathit{eff}}^2\right) = \frac{e_{\mathit{eff}}^2}{2}\left|\hat{t}_{\mathit{eff}}\right|^2, \tag{2.106}$$

where

$$t_{\mathit{eff}} = -\partial_\sigma \widetilde{W}_{\mathit{eff}}(\sigma) = t + \sum_{i=1}^{M} Q_i \log Q_i. \tag{2.107}$$

Here we have used the formula (2.91) for $\widetilde{W}_{\mathit{eff}}$, where the Σ/μ dependence disappears because of (2.105). Thus, the energy at large σ vanishes at $r = -\sum_{i=1}^{M} Q_i \log Q_i$ and *at a single value of* θ which is 0 or π (mod 2π) depending on Q_i's. Thus, except at a single point in the cylinder, there is no flat direction of σ. This means that the singularity is expected only at the single point. This yields a significant change to our picture; The two regions, $r \gg 0$ and $r \ll 0$, are not any more separated by a singularity, but are smoothly connected along a path avoiding the singular point. These two regions can be considered as a sort of analytic continuation of each other.

This change of picture has several applications, including correspondence between Calabi-Yau sigma models and Landau-Ginzburg orbifolds as well as analytic continuation to different topology. We now describe them here.

Topology Change

Let us revisit the $U(1)$ gauge theory with chiral superfields $\Phi_1, \ldots, \Phi_4$ of charge $1, 1, -1, -1$ without superpotential. We recall that the theory at $r \gg 0$ and theory at $r \ll 0$ both yields sigma model on resolved conifold but the two are related by "Flop". In the semi-classical analysis, we found that the two regions are separated by a singularity at $r = 0$. Now we know that the genuin singularity is at $t = 0$ of complex codimension 1, and the two regions $r \gg 0$ and $r \ll 0$ are no longer separated. The present case is a special case where the two resolutions are *isomorphic.* However, if this is embedded as a part of some larger geometry, "flop" usualy changes the global topology. In such a case, what we have seen shows that the sigma model on topologically distinct manifolds can be smoothly connected.

Calabi-Yau/Orbifold Correspondence

Let us reconsider the $U(1)$ gauge theory with chiral superfields $\Phi_1, \ldots, \Phi_N, P$ of charge $1, \ldots, 1, -N$, without superpotential. We have learned that the theory at $r \gg 0$ describes the sigma model on the total space of $\mathcal{O}(-N)$ over $\mathbb{CP}^{N-1}$, which is a non-compact Calabi-Yau manifold. On the other hand, the theory at $r \to -\infty$ is the free $\mathbf{C}^N/\mathbf{Z}_N$ orbifold theory. Thus, the sigma-model on the total space of $\mathcal{O}(-N)$ over $\mathbb{CP}^{N-1}$ and the one on the orbifold $\mathbf{C}^N/\mathbf{Z}_N$ are in the same moduli space of theories.

Calabi-Yau/Landau-Ginzburg Correspondence

Let us turn on the superpotential $W = PG(\Phi_i)$ to the model considered right above. As we have seen, the theory at $r \gg 0$ is identified as the non-linear sigma model on the Calabi-Yau hypersurface $G = 0$ of $\mathbb{CP}^{N-1}$, whereas the theory at $r \to -\infty$ is identified as the LG orbifold with group $\mathbf{Z}_N$ and the superpotential $W = \langle p \rangle G(\Phi_1, \ldots, \Phi_N)$. Thus, the Calabi-Yau sigma model and the LG orbifold are smoothly connected to each other. In other words, the LG orbifold and the Calabi-Yau sigma model are in the same moduli space of theories. The two are interpretations of different regions of the moduli space.

2.5.1 Landau-Ginzburg Orbifold as an IR fixed Point

As another example, let us consider a hypersurface of $\mathbb{CP}^{N-1}$ of degree d less than N, so that the sigma model is asymptotically free. As we have seen, the linear sigma model for this is the $U(1)$ gauge theory with chiral superfields $\Phi_1, \ldots, \Phi_N, P$ of charge $1, \ldots, 1, -d$ and the superpotential $W = PG(\Phi_i)$ where $G(\Phi_i)$ is the degree d polynomial defining the hypersurface. The axial R-symmetry $U(1)_A$ is anomalously broken to $\mathbf{Z}_{2(N-d)}$ and the theory dynamically generates the scale parameter Λ.

The effective theory for large and slowly varying Σ is the theory of a $U(1)$ gauge multiplet with the effective FI-Theta parameter given by

$$t_{\mathit{eff}}(\sigma) = (N-d)\log(\sigma/\Lambda) - d\log(-d). \tag{2.108}$$

The supersymmetric vacua are found by solving $e^{it_{\mathit{eff}}(\sigma)} = 1$ or

$$\sigma^{N-d} = (-d)^d \Lambda^{N-d}, \tag{2.109}$$

and we find $(N-d)$ of them in the admissible region. These are massive, and a choice of vacuum spontaneously breaks the axial R-symmetry as $\mathbf{Z}_{2(N-d)} \rightsquigarrow \mathbf{Z}_2$.

Now let us ask whether these $(N-d)$ are the whole set of vacua. There is an obvious reason to doubt it; the direct analysis of non-linear sigma model shows that the number of vacua is equal to the dimension of the cohomology group $H^*(M)$, which is larger than $(N-d)$. How can we find the rest? They must be in the region where the large σ analysis does not apply. Let us examine the potential (2.44) in terms of the full set of variables once again, now at low energies. At $\mu \ll \Lambda$ the FI parameter is negative, and the analysis of supersymmetric vacua $U=0$ is completely different from that at high energies. It is more like in the $d=N$ case with $r<0$ and we find a single supersymmetric vacuum at $\sigma=0$, $\phi_i=0$ and $|p| = \sqrt{|r|/d}$ where the axial R-symmetry group $\mathbf{Z}_{2(N-d)}$ is not spontaneously broken. Thus, we find at least one extra supersymmetric vacuum besides those found at $\sigma \sim \Lambda$. The theory around this vacuum is described by the LG orbifold of the fields $\Phi_1, \ldots, \Phi_N$ with the group $\mathbf{Z}_d$ and the superpotential $W \sim G(\Phi_1, \ldots, \Phi_N)$. For $d>2$ this LG orbifold is expected to flow to a non-trivial superconformal field theory where the axial $\mathbf{Z}_{2(N-d)}$ discrete R-symmetry enhances to the full $U(1)$ symmetry (or actually further to affine symmetry). One can actually analyze the spectrum of the supersymmetric vacua of this LG orbifold, and that in fact shows that the number of vacua is $\dim H^*(M) - (N-d)$, and the total number saturates the one derived from the direct analysis. Thus, we expect that this extra (degenerate) vacuum really exists in the quantum theory and is the only one that was missed by the large σ analysis. Of course, to be decisive we need more information. Again, we will see that mirror symmetry (which we will give an argument for) shows that this is in fact correct.

2.5.2 A Flow from Landau-Ginzburg Orbifold

As a final example, let us consider the case $d > N$ of the $U(1)$ gauge theory considered right above. As we have seen, the FI parameter at the cut-off scale is negative and the theory at high energy describes the LG orbifold perturbed by an operator that breaks the $U(1)$ axial R-symmetry to $\mathbf{Z}_{2(d-N)}$.

The large σ analysis shows that there are $(d-N)$ vacua determined by (2.109), each of which breaks $\mathbf{Z}_{2(d-N)}$ to $\mathbf{Z}_2$. We may also find supersymmetric vacua near $\sigma = 0$. In fact, the FI parameter becomes positive at low energies and we find the degree d hypersurface M in $\mathbb{CP}^{N-1}$ as the vacuum manifold at $\sigma = 0$. The non-linear sigma model on M is IR free and we expect this to be one of the IR fixed point of the theory.