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SPRING SCHOOL ON SUPERSTRINGS AND RELATED MATTERS

18 - 26 March 2002

LECTURES ON MIRROR SYMMETRY

Lectures 3 and 4

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Please note: These are preliminary notes intended for internal distribution only.

MIRROR SYMMETRY

K. HORI

C. Vafa & KH 2000

A. Kapustin & KH 2001
2002

Mirror Symmetry

T-duality (d=2)

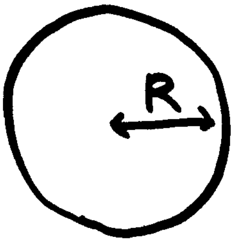
$$\left.\begin{array}{l}\varphi \equiv \varphi + 2\pi \quad \text{scalar},\\ \mathcal{B} : \text{aux. 1-form}\end{array}\right\} \text{on } \Sigma^2$$

$$S' = \frac{1}{2R^2}\int_\Sigma |\mathcal{B}|^2 + \int_\Sigma \mathcal{B} \wedge d\varphi$$

$\int \mathcal{D}\mathcal{B}$ ↙ ↘ $\int \mathcal{D}\varphi$

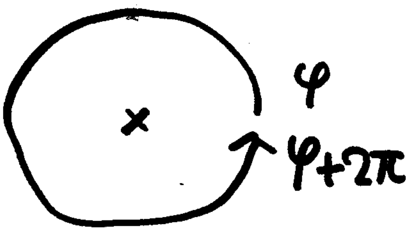
$$\mathcal{B} = R^2 * d\varphi \qquad\qquad \mathcal{B} = d\tilde{\varphi} \quad \tilde{\varphi} \equiv \tilde{\varphi} + 2\pi$$

$$S = \frac{R^2}{2}\int_\Sigma |d\varphi|^2 \qquad\qquad \tilde{S} = \frac{1}{2R^2}\int_\Sigma |d\tilde{\varphi}|^2$$



$$\boxed{R \leftrightarrow \frac{1}{R}}$$

<u>winding #</u> $\leftrightarrow$ <u>momentum</u>



$\leftrightarrow$ $\times\, \underline{\underline{e^{i\tilde{\varphi}}}}$

(2,2) SUSY

Roček - Verlinde

$$L' = \int d^4\theta \left(\frac{1}{4R^2} B^2 - \frac{1}{2} B (\Phi + \bar{\Phi}) \right)$$

B : real superfield

Φ : chiral

$\Phi \equiv \Phi + 2\pi i$

$\int \mathcal{D}B$

$$B = R^2 (\Phi + \bar{\Phi})$$

$$L = \frac{R^2}{2} \int d^4\theta \, |\Phi|^2$$

$$\boxed{\begin{array}{l} \sigma\text{-model on} \\ \mathbb{C}^\times = \mathbb{R} \times S^1_R \end{array}}$$

$\int \mathcal{D}\Phi \mathcal{D}\bar{\Phi}$

$$\bar{D}_+ \bar{D}_- B = D_+ D_- B = 0$$

$$B = \Theta + \bar{\Theta}$$

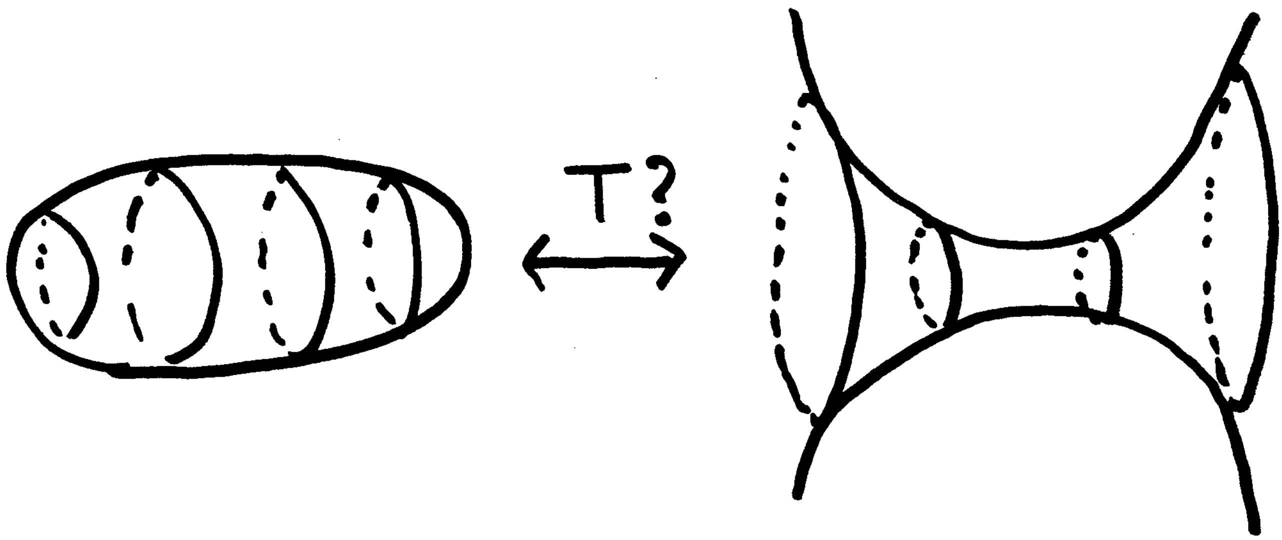
Θ : twisted chiral

$\Theta \equiv \Theta + 2\pi i$

$$\tilde{L} = \frac{1}{2R^2} \int d^4\theta \left(-|\Theta|^2 \right)$$

$$\boxed{\sigma\text{-model on } \tilde{\mathbb{C}}^\times = \mathbb{R} \times \tilde{S}^1_{1/R}}$$

$\therefore$ T-duality is Mirror Symmetry



momentum $\longleftrightarrow$ winding number

O.K.

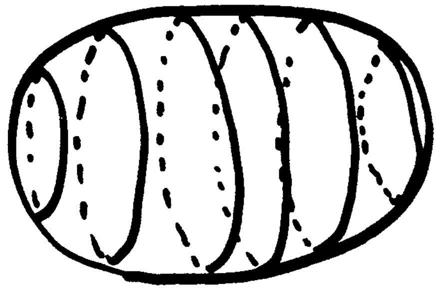
No winding $\longleftrightarrow$ No momentum ?

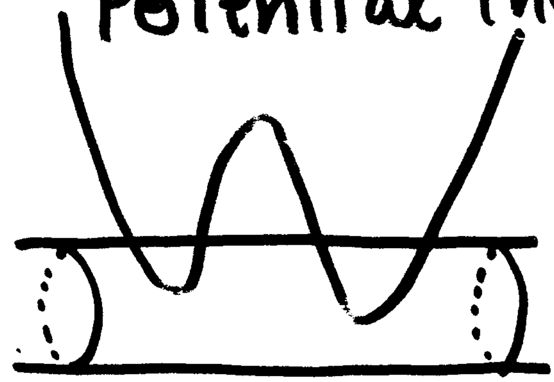
?

For supersymmetric non-linear sigma models on toric manifolds, we find ...

(2,2) SUSY

$\mathbb{CP}^1$ $\longleftrightarrow$ Potential Theory





size r

B-field θ

$$Y \equiv Y + 2\pi i$$

$$W = e^{-Y} + e^{-r+i\theta} \cdot e^{Y}$$

"Sine-Gordon"

$\mathbb{CP}^{N-1}$ $\longleftrightarrow$ <u>affine-Toda</u>

$$= \frac{|\phi_1|^2 + \cdots + |\phi_N|^2 = r}{U(1)}$$

B-field θ

$$Y_1 + \cdots + Y_N = r - i\theta$$

$$W = e^{-Y_1} + \cdots + e^{-Y_N}$$

General Toric $\longleftrightarrow$ SG-type {Liouville} <u>Landau-Ginzburg model</u>

$$\sum_{i=1}^{N} Q_i^a |\phi_i|^2 = r^a \Big/ U(1)^k$$

$a = 1, \cdots, k$

B-field θ^a

$$\sum_{i=1}^{N} Q_i^a Y_i = r^a - i\theta^a$$

$$W = \sum_{i=1}^{N} e^{-Y_i}$$

Derivation

Set-up: Linear Sigma Model

U(1) gauge theory

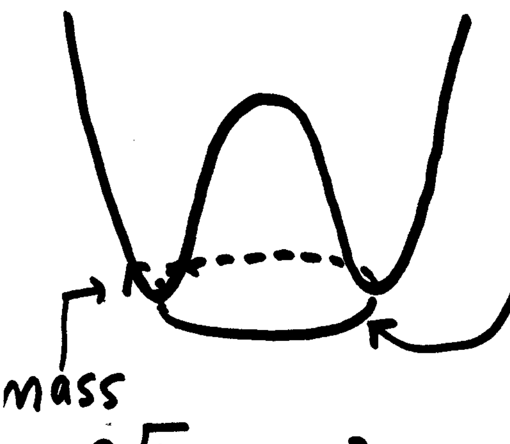
$\Phi_1, \dots, \Phi_N$ charge 1 scalars (+ fermions) chiral

V gauge potential (+ partners)

$\Sigma = \overline{D}_+ D_- V$ field strength (+ partners) twisted chiral

$$L = \int d^4\theta \left[\sum_{i=1}^{N} \overline{\Phi}_i e^V \Phi_i - \frac{1}{2e^2} |\Sigma|^2 \right] + \mathrm{Re} \int d^2\tilde{\theta} \left(-t \Sigma \right)$$

$$= \sum_{i=1}^{N} |D_\mu \phi_i|^2 \underline{- \frac{e^2}{2} \left(\sum_{i=1}^{N} |\phi_i|^2 - r \right)^2} + \underline{\underline{\theta F_{01}}}$$



$$M_{vac} = \frac{\sum_{i=1}^{N} |\phi_i|^2 = r}{U(1)}$$

$$= \mathbb{C}P^{N-1} \circlearrowleft (S^1)^{N-1}$$

$e^2 \to \infty$: Supersymmetric NLSM on $\mathbb{C}P^{N-1}$

θ : B-field

Dualise $\arg(\Phi_i)$ $\qquad \phi_i = \rho_i e^{i\varphi_i}$

$\boxed{\text{bosonic}}$ $\quad |D_\mu \phi_i|^2 = (\partial_\mu \rho_i)^2 + \rho_i^2 (\partial_\mu \varphi_i + A_\mu)^2$

$$S' = \frac{1}{2\rho_i^2}\int |B_i|^2 + \int B_i \wedge (d\varphi_i + A)$$

$\int \mathcal{D}B_i$ $\qquad\qquad$ $\int \mathcal{D}\varphi_i$

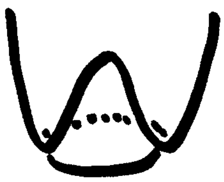
$$S = \frac{1}{2}\int \rho_i^2 |d\varphi_i + A|^2$$

$$B_i = d\tilde{\varphi}_i$$

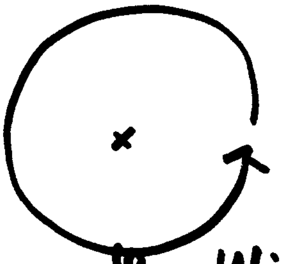
$$\tilde{S} = \frac{1}{2}\int \frac{1}{\rho_i^2}|d\tilde{\varphi}_i|^2 + \int d\tilde{\varphi}_i \wedge A$$

$$// \quad \boxed{-\int \tilde{\varphi}_i F_A}$$

① Dynamical Theta Angle



$\exists\ \phi_i$-vortex



φ_i-Winding $\quad \longleftrightarrow \quad$ momentum operator

$\times\ e^{i\tilde{\varphi}_i}$

② $e^{i\tilde{\varphi}_i}$-insertion

$\boxed{\text{SUSY}}$

$$\Phi_i = \phi_i + \cdots \quad \leftrightarrow \quad Y_i = \rho_i^2 - i\tilde{\varphi}_i + \cdots \quad (Y_i \equiv Y_i + 2\pi i)$$

<u>chiral</u> $\qquad$ <u>twisted chiral</u>

$$\left.\begin{array}{lll} \Phi_i \text{ charge } 1 & \leftrightarrow \text{ ① } & Y_i \Sigma \\ \text{SUSY } \Phi_i\text{-vortex} & \leftrightarrow \text{ ② } & e^{-Y_i} \end{array}\right\} \text{in superpotential}$$

$$\therefore \underline{W = -t\Sigma + \sum_{i=1}^{N} Y_i \Sigma + \sum_{i=1}^{N} e^{-Y_i}} \quad \text{exact}$$

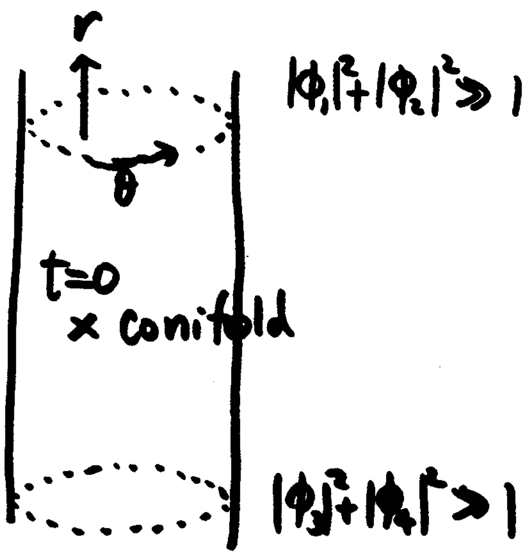
NLSM Limit $e^2 \to \infty$: Σ heavy $\rightsquigarrow$ integrate out

$$\rightsquigarrow \quad \underline{Y_1 + Y_2 + \cdots + Y_N = t}$$

$$\underline{W = e^{-Y_1} + e^{-Y_2} + \cdots + e^{-Y_N}}$$

This is the Mirror LG Model.
(for CP^{N-1})

non-compact CY



$U(1)$	Φ_1	Φ_2	Φ_3	Φ_4
	1	1	-1	-1

M_{vac} : $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$ resolved conifold

MIRROR $\longleftrightarrow$
$$W = e^{-Y_1}+e^{-Y_2}+e^{-Y_3}+e^{-Y_4} \Big/ Y_1+Y_2-Y_3-Y_4=t$$
$$= e^{-Y_0}\left(e^{-t}+e^{-\Theta_1-\Theta_2}+e^{-\Theta_1}+e^{-\Theta_2}\right)$$

IR equivalent $\dashrightarrow$
$$W' = e^{-Y_0}\left(e^{-t}+e^{-\Theta_1-\Theta_2}+e^{-\Theta_1}+e^{-\Theta_2}\right)+UV$$

<u>U, V : $\mathbb{C}$-valued</u>

<u>mass of BPS D-branes</u>

$$\Pi = \int dY_0\, d\Theta_1\, d\Theta_2\, dU\, dV\, e^{-iW'}$$

change of variables $\underline{U = e^{-Y_0}\tilde{U}\ ,\ V = \tilde{V}}$:

$$\Pi = \int e^{-Y_0} dY_0\, d\Theta_1 d\Theta_2 d\tilde{U} d\tilde{V}$$
$$\exp\left[-i e^{-Y_0}\left(e^{-t} + e^{-\Theta_1-\Theta_2} + e^{-\Theta_1} + e^{-\Theta_2} + \tilde{U}\tilde{V}\right)\right]$$

$$\int de^{-Y_0}$$
$$= \int d\Theta_1 d\Theta_2 d\tilde{U} d\tilde{V}\ \delta\left[e^{-t} + e^{-\Theta_1-\Theta_2} + e^{-\Theta_1} + e^{-\Theta_2} + \tilde{U}\tilde{V}\right]$$

.... Period Integral of

$$e^{-t} + e^{-\Theta_1-\Theta_2} + e^{-\Theta_1} + e^{-\Theta_2} + \tilde{U}\tilde{V} = 0$$

— deformed conifold

$$\left[\begin{array}{l} t=0: \\ (1+e^{-\Theta_1})(1+e^{-\Theta_2}) + \tilde{U}\tilde{V} = 0 \\ \text{conifold singularity at} \\ \qquad \Theta_1 = \Theta_2 = \pi i,\ \tilde{U} = \tilde{V} = 0 \end{array}\right]$$

Degree d hypersurface in $\mathbb{C}P^{N-1}$

$G(\Phi_1, \dots, \Phi_N)$ Polynomial of degree d

like $a_1 \Phi_1^d + \cdots + a_N \Phi_N^d + b_{i_1 \cdots i_d} \Phi_{i_1} \cdots \Phi_{i_d} + \cdots$

$U(1)$ gauge theory $\Phi_1, \dots, \Phi_N, P$ <u>chirals</u>

charge $1, \dots, 1, -d$

- Superpotential $W = P\,G(\Phi_1, \dots, \Phi_N)$

$$\Rightarrow \quad M_{vac} \overset{r>0}{=\!=} \left\{ \begin{array}{l} P = 0 \\ G(\phi_1, \cdots \phi_N) \end{array} \; \sum_i |\phi_i|^2 = r \right\} \Big/ U(1) \subset \mathbb{C}P^{N-1}$$

$$\underline{\text{degree } d \text{ hypersurface } X_d}$$

$$\left[\begin{array}{lll} c_1(X_d) \propto (N-d) & > 0 & d = 1, \cdots, N-1 \quad (X_1 = \mathbb{C}P^{N-2}) \\ & = 0 & d = N \end{array} \right]$$

If $W = 0$, $\quad M_{vac} \overset{r>0}{=} \left\{ \sum_{i=1}^N |\phi_i|^2 - d|P|^2 = r \right\} \Big/ U(1) =: V^N_{toric}$

The mirror is LG on $(\mathbb{C}^\times)^N$

- $Y_1 + \cdots + Y_N - d\,Y_P = t$
- $\widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_N} + e^{-Y_P}$

Having $W = P\,G(\Phi_1, \dots, \Phi_N)$

$\Rightarrow$ Change of variables (later)

$$e^{-Y_1} = X_1^d, \dots, e^{-Y_N} = X_N^d, \quad e^{-Y_P} = e^{t/d} X_1 \cdots X_N$$

$$\{(Y_i)\} \xleftrightarrow{1 : \mathbb{Z}_d^{N-1}} \{(X_i)\}$$

- $\mathbb{Z}_d^{N-1}$ orbifold of LG on $\mathbb{C}^N = \{(X_1, \dots, X_N)\}$

$$\widetilde{W} = X_1^d + \cdots + X_N^d + e^{t/d} X_1 \cdots X_N$$

- vacua $\Leftrightarrow d\widetilde{W} = 0$
  - $(N-d)$ massive vacua at $X_i^d = S$, $S^{N-d} = (-d)^d e^{-t}$
  - One vacuum at $X_1 = \cdots = X_N = 0$ $\quad (d \geq 2)$

$\xrightarrow{IR}$ $\underline{\text{non-trivial SCFT}}$ $\quad 2 < d \leq N$

$$c/3 = N(d-2)/d$$

(× $d = N$: $c/3 = N-2 = \dim_{\mathbb{C}} X_d$, $e^{t/d} X_1 \cdots X_N$ marginal

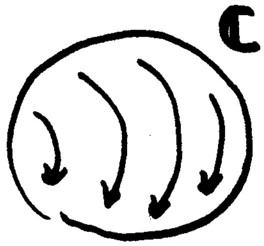
× $2 < d < N$: $e^{t/d} X_1 \cdots X_N$ irrelevant

$W_{IR} = X_1^d + \cdots + X_N^d \,/\, \mathbb{Z}_d^{N-1}$)

- $\mathrm{Tr}(-1)^F = (N-d) + \mathrm{Tr}_{SCFT}(-1)^F = \dfrac{(1-d)^N + Nd - 1}{d} = \chi(X_d)$

- $U(1)_A \xrightarrow[\text{anomaly}]{} \mathbb{Z}_{2(N-d)} \xrightarrow[IR]{} U(1)_A$

Kähler Potential

CP^1 | Sine-Gordon


$\longleftrightarrow$ $W = e^{-Y} + e^{-t+Y}$

U(1) isometry $\longleftrightarrow$ Winding # in $Y \to Y + 2\pi i$

$\cap$

SU(2) $\longleftrightarrow$?

Kobayashi-Uematsu

$\left.\begin{array}{l} W = e^{-Y} + e^{-t+Y} \\ K = |Y|^2/2k \end{array}\right\}$ $SU(2)_q$ quantum group symmetry

$\downarrow$

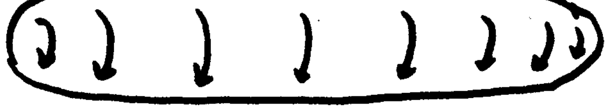
SU(2) $q \to 1$ as $k \to \infty$

× S-matrix ✓ Fendley-Intriligator

× We have seen $K \sim |Y|^2/2r_0$

<u>What is the mirror of S.G. with $k < \infty$?</u>

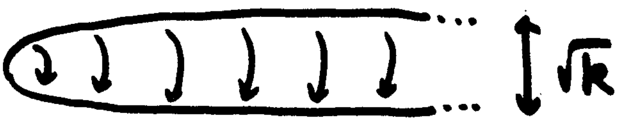
F.I.



$\updownarrow \sqrt{k}$

<u>sausage</u> Can we show this?

A simpler model : Cut in half



cigar

$\leftrightarrow$

$$\begin{cases} K = |Y|^2/2k \\ W = e^{-Y} \end{cases}$$

Liouville

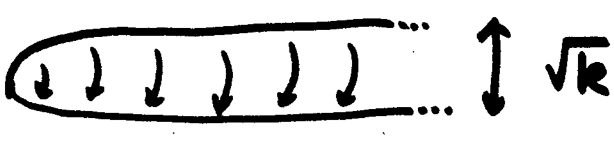
Linear σ-Model

U(1) gauge theory , Φ, $P \equiv P + 2\pi i$ chiral

gauge transf: $\begin{cases} \Phi \to e^{i\Lambda}\Phi & \text{(charge 1)} \\ P \to P + i\Lambda \end{cases}$

$$L = \int d^4\theta \left[\bar{\Phi} e^V \Phi + \frac{k}{4}(P + \bar{P} + V)^2 - \frac{1}{2e^2}|\Sigma|^2 \right]$$

- Space of classical vacua is a cigar



It flows under renormalization group to 2d Black Hole

<u>2d BH</u>: $SL(2,\mathbb{R})_{k+2}$ mod $U(1)$ supercoset

$$c = 3\left(1+\frac{2}{k}\right)$$

The flow can be shown in 3 steps

① One-loop β-function (valid at $k \gg 1$)

② Computation of $c = 3(1+2/k)$

identify $\mathcal{N}=2$ superconformal currents in $\bar{Q}_+$ cohomology ring

Witten
Silverstein-Witten

$$\left(\bar{Q}_+^2 = 0,\ \{\bar{Q}_+, Q_+\} = H+P\right)$$

$$J = \text{classical} + \underbrace{\frac{1}{2}[\bar{D}_-, D_-](P+\bar{P}+V)}_{\text{induced by Chiral/Konishi anomaly}}$$

linear dilaton

③ Rigidity of $SL(2,\mathbb{R})$ mod $U(1)$:

No SUSY, Parity even marginal deformation dualization + vortex-instanton effect

$$\Rightarrow \begin{cases} \widetilde{W} = \Sigma(Y+Y_P) + \mu e^{-Y} \\ K = -\frac{1}{2e^2}|\Sigma|^2 - \frac{1}{2k}|Y_P|^2 + \cdots \end{cases}$$

$\xrightarrow{e\to\infty}$ int. out Σ : $Y+Y_P=0$

$$\begin{cases} \widetilde{W} = \mu e^{-Y} \\ K = -\frac{1}{2k}|Y|^2 + \cdots \to 0 \end{cases}$$

Thus,

$$L = \int d^4\theta \left[\bar{\Phi} e^V \Phi + \frac{k}{4}(P+\bar{P}+V)^2 - \frac{1}{2e^2}|\Sigma|^2 \right]$$

$SL(2,\mathbb{R})_{k+2}$ mod $U(1)$ super coset (2d BH)	$\mathcal{N}=2$ Liouville theory $K = \|Y\|^2/2k$ $W = \mu e^{-Y}$

Mirror Symmetry

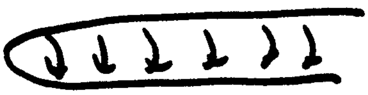
There is an $\mathcal{N}=0$ (No SUSY) version

Fateev
Zamolodchikov
Zamolodchikov

$SL(2,\mathbb{R})_k$ mod $U(1)$
bosonic coset

$\longleftrightarrow$

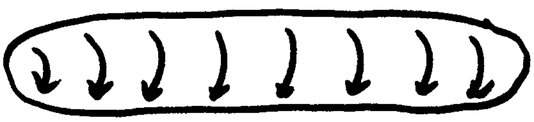
Sine-Liouville theory
$U = \mu^2 e^{-\phi} \cos\vartheta$



Generalization

LSM with charged fields (like Φ)
& inhomogeneous fields (like P)

NL σ-M on "squashed" toric manifold



$\updownarrow \sqrt{k}$

LG model of (affine) Toda type with finite Kähler potential $\sim \frac{1}{k}$

Application : NS5-branes

$$\frac{SL_2(\mathbb{R})_{m+2}}{U(1)} \Big/ \underline{\mathbb{Z}_m} \xleftrightarrow{\text{MIRROR}} W = \mu e^{-\underline{m}Y}$$

$$\frac{SU(2)_{m-2}}{U(1)} \Big/ \mathbb{Z}_m \xleftrightarrow{\text{MIRROR}} W = X^m$$

$$\left[\frac{SL_2(\mathbb{R})_{m+2}}{U(1)} \times \frac{SU(2)_{m-2}}{U(1)}\right] \Big/ \mathbb{Z}_m \sim R_\phi \times \left[U(1)_m \times \frac{SU(2)_{m-2}}{U(1)}\right] \Big/ \mathbb{Z}_m$$

$$\xleftrightarrow{T} R_\phi \times SU(2)_{m-2}$$

$$\underline{m \text{ NS5-branes}}$$

$$\|$$

$$\left[\frac{SL_2(\mathbb{R})_{m+2}}{U(1)} \Big/ \mathbb{Z}_m \times \frac{SU(2)_{m-2}}{U(1)} \Big/ \mathbb{Z}_m\right] \Big/ \mathbb{Z}_m \xleftrightarrow{\text{Mirror}} \left[W = \mu e^{-mY} + X^m\right] \Big/ \mathbb{Z}_m$$

$$\overset{\text{IR}}{\longleftrightarrow} \left[W = e^{-mY}(\mu + \tilde{X}^m + \tilde{U}\tilde{V})\right] \Big/ \mathbb{Z}_m$$

$$\longleftrightarrow \underline{\left\{ \mu + \tilde{X}^m + \tilde{U}\tilde{V} = 0 \right\} \ \sigma\text{-model}}$$

deformed $ALE(A_{m-1})$

- $\underline{\text{move NS5's}} \longleftrightarrow \mu + \tilde{X}^m + \underline{\sum_{\ell=0}^{m-2} u_\ell \tilde{X}^\ell} + \tilde{U}\tilde{V} = 0$

- $NS5(D_n, E_{6,7,8}) \longleftrightarrow ALE(D_n, E_{6,7,8})$

$\mathrm{LSM}_k \quad U(1) \quad \Phi_1 \ \Phi_2 \ P$
$\qquad\qquad\qquad\qquad 1 \quad 1 \quad \text{shift}$

$$\mathrm{LSM}_k / \mathbb{Z}_{2m} \xleftrightarrow{\text{MIRROR}} W = e^{-mZ}(e^{-Y}+e^{Y})$$

$$\left[\frac{\mathrm{LSM}_{4m}}{\mathbb{Z}_2} \times \frac{SU(2)_{m-2}}{U(1)}\right]/\mathbb{Z}_m \sim \mathbb{R}_\phi \times \mathbb{CP}^1 \times \left[U(1)_m \times \frac{SU(2)_{m-2}}{U(1)}\right]/\mathbb{Z}_m$$

$$\xleftrightarrow{T} \mathbb{R}_\phi \times \mathbb{CP}^1 \times SU(2)_{m-2}$$

<u>m NS5 wrapped on $\mathbb{CP}^1$</u>

↳ 4d $\mathcal{N}=2$ $SU(m)$ Super-YM

$\|$

$$\left[\frac{\mathrm{LSM}_{4m}}{\mathbb{Z}_{2m}} \times \frac{SU(2)_{m-2}}{U(1)}/\mathbb{Z}_m\right]/\mathbb{Z}_m \xleftrightarrow{\text{MIR}} \left[W = e^{-mZ}(e^{-Y}+e^{Y})+X^m\right]/\mathbb{Z}_m$$

$$\dashleftarrow\!\dashrightarrow \underline{\left\{ e^{-Y}+e^{Y}+\tilde{X}^m+\tilde{U}\tilde{V}=0 \right\}} \ \text{CY 3fold}$$

<u>move NS5's</u> $\leftrightarrow \ e^{-Y}+e^{Y}+\tilde{X}^m+\underline{\sum_{l=0}^{m-2} u_l \tilde{X}^l}+\tilde{U}\tilde{V}=0$

$\rightsquigarrow$ Seiberg-Witten solution of $\mathcal{N}=2$ super-YM

$\times$ $\left.\begin{array}{l}\text{zero size}\\ \text{zero B-field}\end{array}\right\}\ \mathbb{CP}^1 \leftrightarrow e^{-Y}+e^{Y}+\tilde{X}^m \pm 2+\tilde{U}\tilde{V}=0$

ADE

Argyres-Douglas SCFT ! ADE

D-BRANES
AND
MIRROR SYMMETRY

KH.

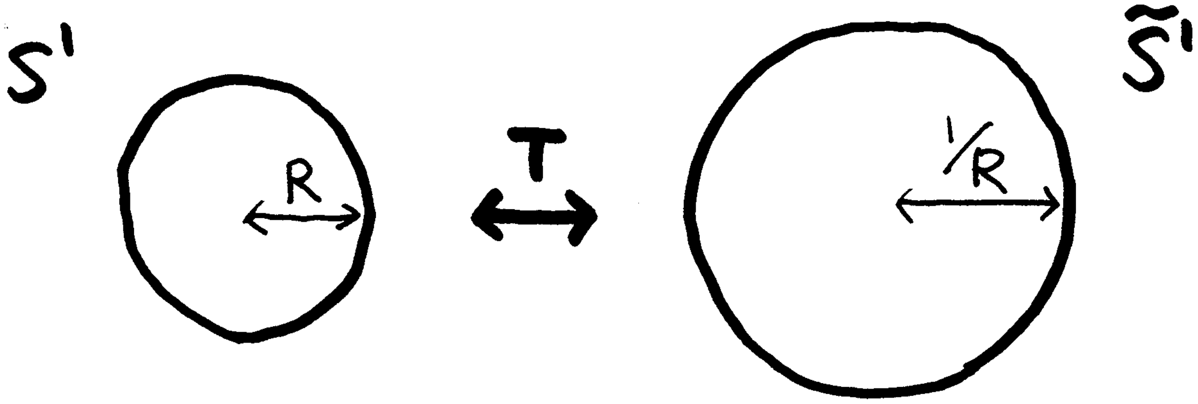
— A. Iqbal, C. Vafa & K.H May 2000

— KH Dec 2000

— "Ch. 40" of a book to appear 2002 (?)
by S. Katz, A. Klemm, R. Pandharipande, R. Thomas
R. Vakil, C. Vafa, E. Zaslow & KH

⋮

D-branes & T-duality

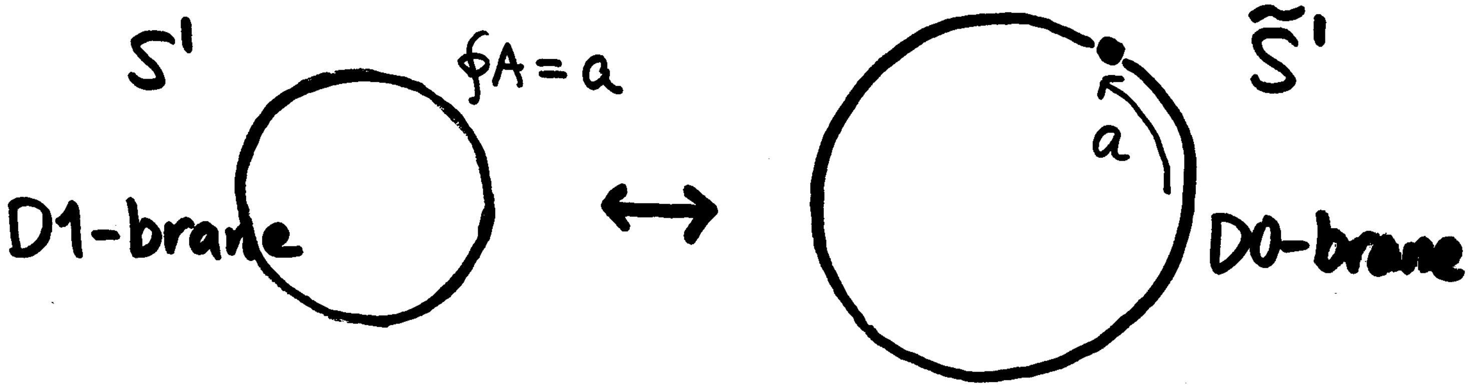


$$*dX = d\tilde{X}$$

Σ : World sheet with boundary

Neumann B.C.   Dirichlet B.C.

$$*dX|_{\partial\Sigma} = 0 \longleftrightarrow d\tilde{X}|_{\partial\Sigma} = 0$$



Wilson line $\longleftrightarrow$ Position

Dai-Leigh-Polchinski
Horava

Q: How are D-branes transfored under Mirror Symmetry?

We restrict our attention to D-branes that preserve a $\frac{1}{2}$ of the supersymmetry of the bulk of the world sheet

↳ (2,2) supersymmetry

$Q_\pm, \bar{Q}_\pm$ $\qquad \{Q_\pm, \bar{Q}_\pm\} = H \pm P$

(Mirror Symmery : $Q_- \leftrightarrow \bar{Q}_-$)

two kinds of "$\frac{1}{2}$" Ooguri Oz Yin

$$\left.\begin{aligned} Q_A &= \bar{Q}_+ + Q_- \\ Q_A^\dagger &= Q_+ + \bar{Q}_- \end{aligned}\right\} \text{A-branes}$$

mirror

$$\left.\begin{aligned} Q_B &= \bar{Q}_+ + \bar{Q}_- \\ Q_B^\dagger &= Q_+ + Q_- \end{aligned}\right\} \text{B-branes}$$

Derivation

$\mathcal{B}$ 1-form on Σ

$\tilde{\varphi} \equiv \tilde{\varphi} + 2\pi$ on Σ

$u \equiv u + 2\pi$ on $\partial\Sigma$

$$S' = \int_\Sigma \frac{R^2}{2}|\mathcal{B}|^2 + \int_\Sigma d\tilde{\varphi}\wedge\mathcal{B} + \int_{\partial\Sigma}(a-\tilde{\varphi})\,du$$

$\int \mathcal{D}\tilde{\varphi}$ $\swarrow$

$\mathcal{B} = d\varphi$ on Σ

$\mathcal{B}|_{\partial\Sigma} = du$

$$S = \int_\Sigma \frac{R^2}{2}|d\varphi|^2 + \underbrace{\int_{\partial\Sigma} a\,d\varphi}_{\textbf{Wilson line}}$$

$\searrow$ $\int \mathcal{D}\mathcal{B}\,\mathcal{D}u$

$\mathcal{B} = \frac{1}{R^2} * d\tilde{\varphi}$ on Σ

$\tilde{\varphi}|_{\partial\Sigma} = a$] **Position**

$$\tilde{S} = \int_\Sigma \frac{1}{2R^2}|d\tilde{\varphi}|^2$$

X Kähler manifold $\begin{cases} \omega & \text{symplectic structure} \\ J & \text{complex structure} \end{cases}$

[LG model
$W : X \to \mathbb{C}$ superpotential]

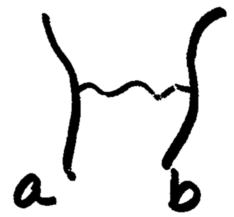
A D-brane wrapped on $\gamma \subset X$
supporting a $U(1)$ gauge potential A

is

<u>an A-brane</u> if $\gamma \subset (X, \omega)$ Lagrangian
A : flat $(F_A = 0)$
$[\ \mathrm{Im}\, W = \text{constant on } \gamma\ ]$

<u>a B-brane</u> if $\gamma \subset (X, J)$ complex submanifold
A : holomorphic $(F_A^{0,2} = 0)$
$[\ W = \text{constant on } \gamma\ ]$

$\mathbf{a} = (\gamma_a, A_a)$, $\mathbf{b} = (\gamma_b, A_b)$ both A-branes $\left(\begin{smallmatrix}\text{or}\\ \text{both B-branes}\end{smallmatrix}\right)$

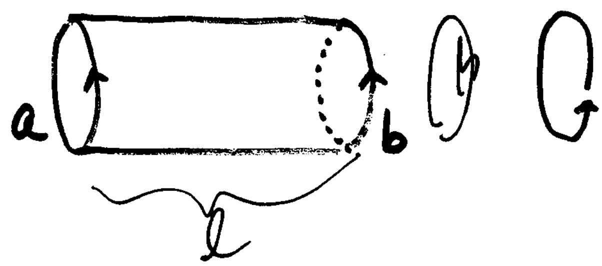


open string

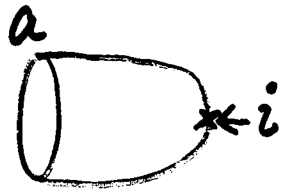
Hilbert space $\mathcal{H}_{a,b}$

$\{Q, Q^\dagger\} = 2H$ $\qquad (Q = Q_A\ (^{\text{or}}\ Q_B))$

Witten index $\mathbf{I(a,b)} = \mathrm{Tr}_{\mathcal{H}_{a,b(\ell)}} (-1)^F e^{-\beta H(\ell)}$

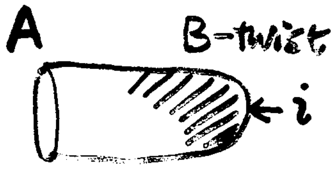
$=$ 

$=$ 

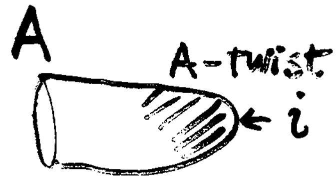
$\Pi^a_i =$ 

.... <u>RR charge</u> of the brane

($\propto$ mass if BPS in space-time)



- Independent of Kähler class
- $D_i \Pi^a_j = \underbrace{C_{ij}{}^k}_{\text{chiral ring}} \Pi^a_k$ for cplx deformation



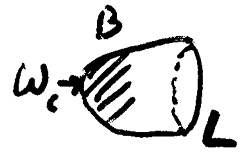
.... topological disc amplitudes

★ $L \subset X$ Lagrangian

$$I(L_1, L_2) = \#(L_1 \cap L_2)$$

$X = CY$: $\Pi_i^L =$ 

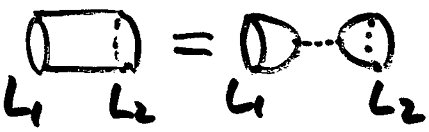
$= \pm i^* \int_L \omega_i$ Period

$\tilde{\Pi}_i^L =$ 

$= \pm i^* \int_L \omega_i$

$\eta_{ij} = \int_X \omega_i \wedge \omega_j$

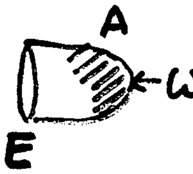
inverse



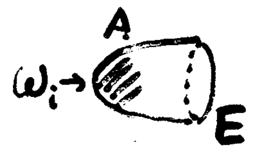
: $\#(L_1 \cap L_2) = \int_{L_1} \omega_i \, \eta^{ij} \int_{L_2} \omega_j$

★ $E \subset X$ holomorphic bundle

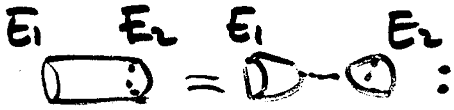
$$I(E_1, E_2) = \chi(E_1, E_2) = \int_X ch(E_1^\vee) ch(E_2) \, Td(X)$$

$\Pi_i^E =$ 

$= \int_X e^{B + i\omega} \omega_i \, ch(E^\vee) \sqrt{Td(X)} + \cdots$

$\tilde{\Pi}_i^E =$ 

$= \int_X e^{-B - i\omega} \omega_i \, ch(E) \sqrt{Td(X)} + \cdots$



:

$$\chi(E_1, E_2) = \int_X e^{B + i\omega} \omega_i \, ch(E_1^\vee) \sqrt{Td(X)} \; \eta^{ij} \int_X e^{-B - i\omega} \omega_j \, ch(E_2) \sqrt{Td(X)}$$

MIRROR SYMMETRY

non-linear σ-model on X^n_{toric} $\longleftrightarrow$ Landau-Ginzburg model $W : (\mathbb{C}^\times)^n \to \mathbb{C}$

e.g. $X = \mathbb{C}P^{N-1}$ $\longleftrightarrow$ $W = e^{-Y_1} + \cdots + e^{-Y_{N-1}} + e^{-t+Y_1+\cdots+Y_N}$

We will study the maps

B-branes in X $\longleftrightarrow$ A-branes in LG

holomorphic bundles — Lagrangians with $\mathrm{Im}\, W = \mathrm{const}$

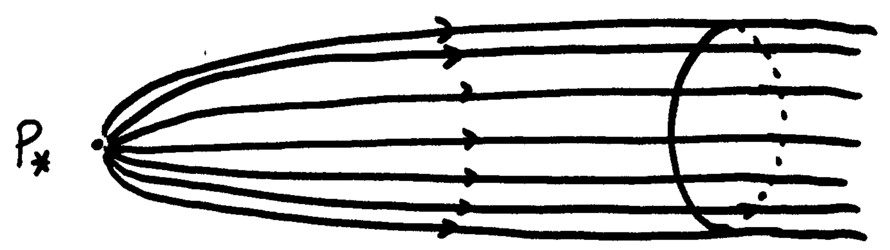
A-branes in X $\longleftrightarrow$ B-branes in LG

Lagrangian submfds — cplx submfds with $W = \mathrm{const}$

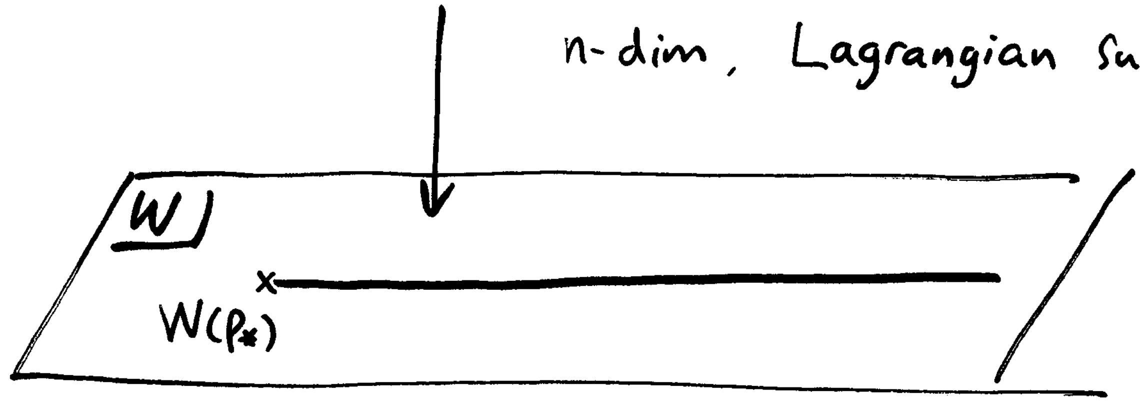
A-branes in LG $\quad W : Y^n \to \mathbb{C}$

p_* a non-degenerate critical point of W.

Gradient flow lines of $\underline{\mathrm{Re}(W)}$ starting from p_*



n-dim, Lagrangian submfd γ_{p_*}



$\therefore$ A-brane

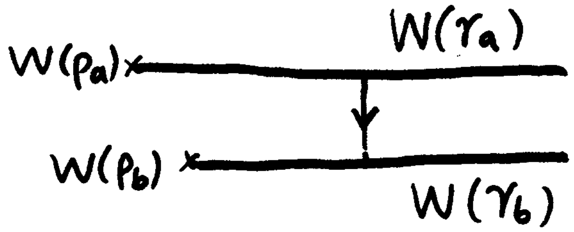
two such branes

Open string quantum mechanics

$$Q = Q_R = \int \bar{\psi}_+ (\partial_t \phi + \partial_\sigma \phi + i\bar{W}') + \psi_- (\partial_t \bar{\phi} - \partial_\sigma \bar{\phi} + iW')$$

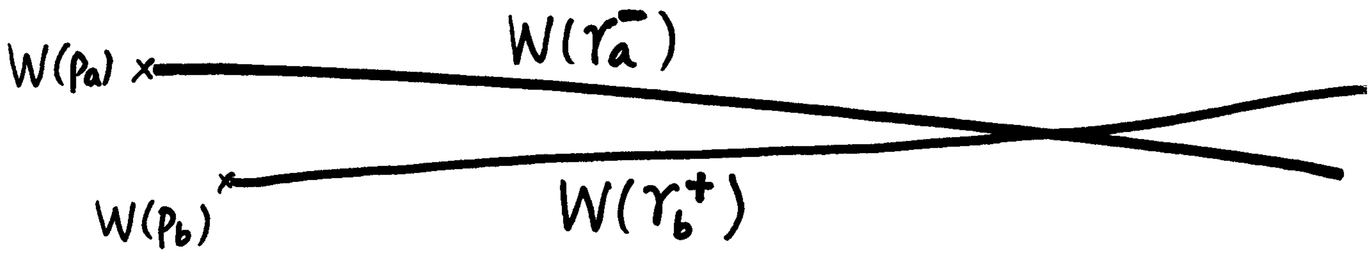
$$Q = Q^\dagger = 0 \quad \Rightarrow \quad \begin{aligned} &\partial_t \phi = 0 \\ &\partial_\sigma \phi = -\,\mathrm{grad}\, \underline{\mathrm{Im}(W)} \end{aligned}$$

SUSY ground states : Grad flows of $-\mathrm{Im}(W)$



$\left(\nexists \text{ if } \mathrm{Im}W(p_a) < \mathrm{Im}W(p_b) \right)$

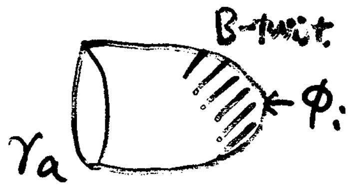
How many ?



- $\boxed{I(a,b) = \#(\gamma_a^- \cap \gamma_b^+)} = \#(\text{BPS solitons in a-b sector})$

- $\mathcal{H}_{SUSY} = HF^{\bullet}_W(\gamma_a, \gamma_b)$ dimension $|I(a,b)|$

Overlap with RR ground states

$\Pi_i^a =$  $= \int_{\gamma_a^-} e^{-iW} \phi_i \Omega$

$\widetilde{\Pi}_{\bar{i}}^a =$  $= \int_{\gamma_a^+} e^{-i\overline{W}} * \phi_{\bar{i}} \overline{\Omega}$

$I(a,b) = \Pi_i^a g^{i\bar{j}} \widetilde{\Pi}_{\bar{j}}^b$: Riemann's bilinear identity

B-branes in X ↔ A-branes in LG

$$X = \mathbb{C}P^{N-1} \quad \longleftrightarrow \quad W = e^{-Y_1} + \cdots + e^{-Y_{N-1}} + e^{-t+Y_1+\cdots+Y_N}$$

$\dim H^{\cdot}(X) = N$ $\qquad$ N : crit pts $e^{-Y_1} = \cdots = e^{-Y_{N-1}} = e^{-t/N} e^{2\pi i l/N}$

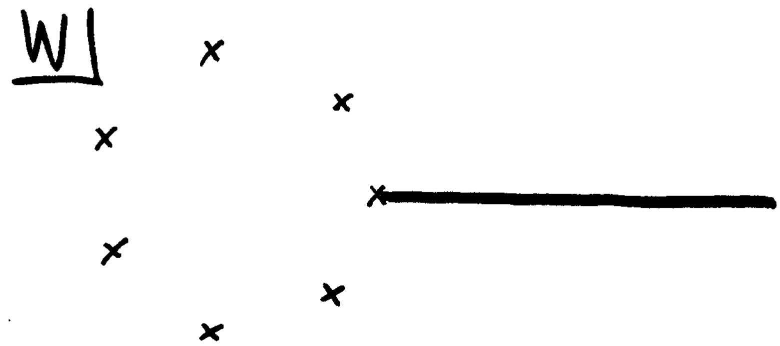
$l = 0, 1, \cdots, N-1$

set $B = 0$ $\qquad$ $\theta = 0 \quad (t \in \mathbb{R}_+)$

$D(2(N-1))$ brane wrapped on X

trivial gauge field

… pure Neumann B.C. ⟷ $Y_1, \cdots, Y_{N-1}$ all real



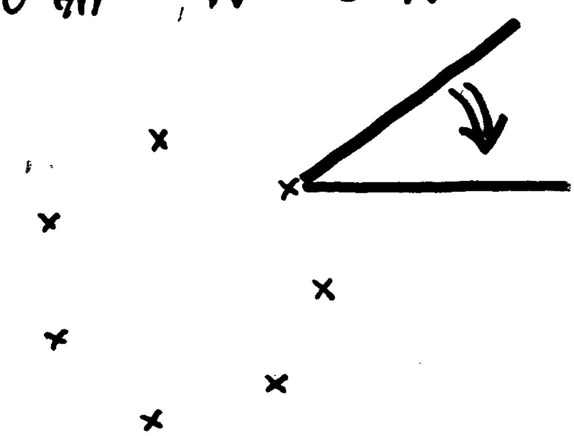
axial R-rotation

$$B \to B + Nr \ \longleftrightarrow\ \theta \to \theta + Nr, \ W \to e^{ir} W$$

$B = 2\pi$, D-brane on X
trivial gauge field

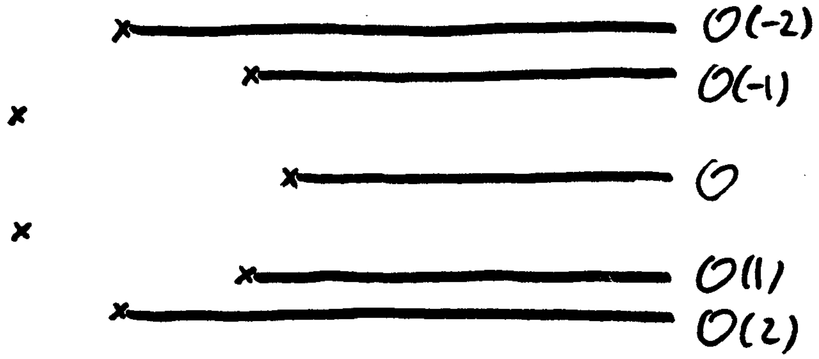
⟷

$\equiv$ $B = 0$, D-brane on X
supporting $\mathcal{O}(-1)$



preserves $\overline{Q}_+ + e^{\frac{2\pi i}{N}} \overline{Q}_-$

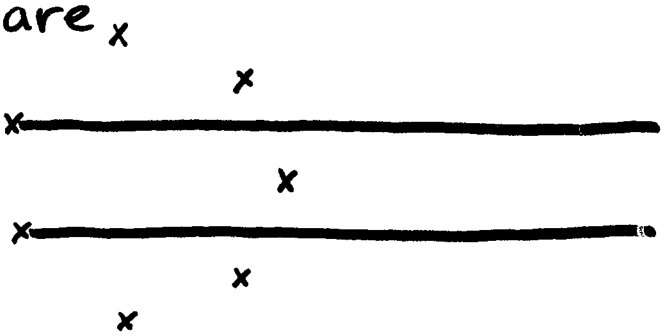
↳ 1



indeed

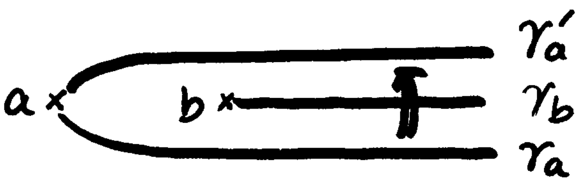
$$\chi(\mathcal{O}(i),\mathcal{O}(j)) = \delta_{i<j}\binom{N+j-i-1}{j-i} = I(\gamma_i,\gamma_j)$$

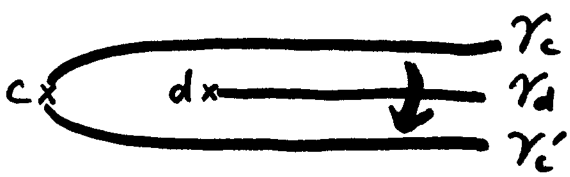
- What about $\mathcal{O}(\pm 3)$, $\mathcal{O}(\pm 4)$, ... ?
- What are



mirror to ?

Picard-Lefschetz :





$$\gamma_a \longrightarrow \gamma_a' + \gamma_b\, I(b,a) \qquad \gamma_c \longrightarrow \gamma_c' + \gamma_d\, I(c,d)$$

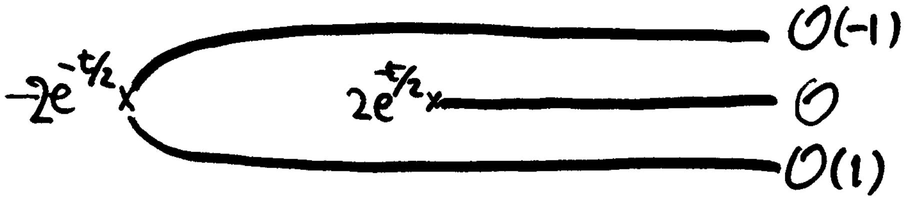
$$E_a \longrightarrow E_a' + E_b\,\chi(E_b,E_a) \qquad E_c \longrightarrow E_c' + E_d\,\chi(E_c,E_d)$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

$\pm L_{E_b}E_a$ <u>Mutation</u> $\pm R_{E_d}E_c$

e.g. $X = \mathbb{CP}^1 \longleftrightarrow W = e^{-Y} + e^{-t+Y}$

crit. pts: $e^{-Y} = \pm e^{-t/2}$



$$0 \to \mathcal{O}(-1) \to \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}(1)) \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$$

$$\mathcal{O}(-1) = L_{\mathcal{O}}\mathcal{O}(1)$$

$$(f, g) \longmapsto fX_0 + gX_1$$

$$\sigma \longmapsto (X_1\sigma, -X_0\sigma)$$

$$0 \to \mathcal{O}(-1) \to \mathrm{Ext}^0(\mathcal{O}(-1), \mathcal{O})^* \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$$

$$\mathcal{O}(1) = R_{\mathcal{O}}\mathcal{O}(-1)$$

$$\begin{tikzcd}
\cdots & \mathcal{O}(-2) & \mathcal{O}(-1) \arrow[rr, bend right, "R"'] & \mathcal{O} & \mathcal{O}(1) \arrow[ll, bend right, "L"'] & \mathcal{O}(2) & \cdots
\end{tikzcd}$$

<u>Helix of Period 2</u>

$X = \mathbb{CP}^{N-1}$

$$\cdots\ \mathcal{O}\ \ \mathcal{O}(1) \cdots \mathcal{O}(N) \cdots$$

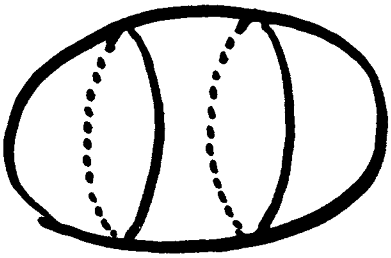
Helix of Period N

A-branes in X $\longleftrightarrow$ B-branes in LG, $W: Y \to \mathbb{C}$

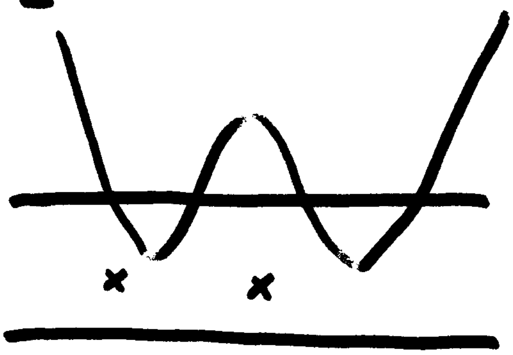
$L \subset X$ Lag $\qquad\qquad$ $Z \subset Y$ cplx

$\qquad\qquad\qquad\qquad$ $W|_Z = \text{const}$

1



$\longleftrightarrow$



torus fibers $\qquad\qquad$ points

$\sigma: X \to X$ $\longleftrightarrow$ $\hat{\sigma}: Y \to Y$ holo. inv.

anti-holo. involution $\qquad$ s.t. $W(\hat{\sigma} y) = -W(y)$

[$\exists$ Parity anomaly if $w_2(X) \neq 0$] I. Brunner & KH $\qquad$ [$\nexists$ such map]

2 $\{(x, \sigma x)\} \in X \times X$ $\longleftrightarrow$ $\{(y, \hat{\sigma} y)\} \in Y \times Y$

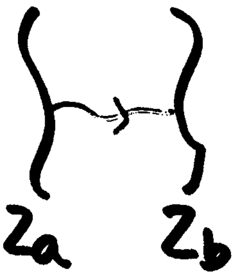
extra B-field if $w_2(X) \neq 0$

$W = W(y) + W(\hat{\sigma} y) = 0$

3 $X^{\sigma} \subset X$ $\longleftrightarrow$ $Y^{\hat{\sigma}} \subset Y$

B-branes in LG $W : Y \to \mathbb{C}$

$$Q = Q_B = \frac{1}{2\pi}\int_0^{\pi} dx^1 \left\{ (\bar\psi_- + \bar\psi_+)\partial_0\phi - (\bar\psi_- - \bar\psi_+)\partial_1\phi + \frac{i}{2}(\psi_- - \psi_+)W'_j \right.$$



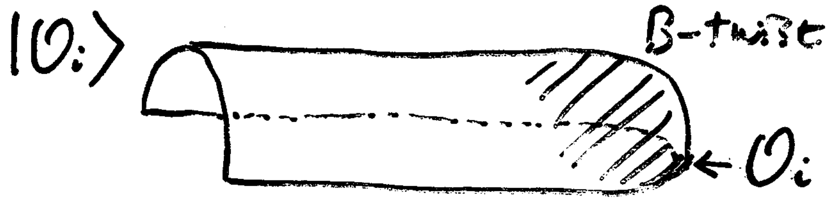
$$\underline{Q^2 = \frac{1}{2\pi i}\left(W|_{Z_b} - W|_{Z_a}\right)}$$

$$Q^2 = 0 \quad \text{iff} \quad W|_{Z_b} = W|_{Z_a}$$

This is the mirror statement of "$m_1^2 \neq 0$"

$Z_a = Z_b =: Z \Rightarrow Q^2 = 0$

What is the $\mathcal{H}_{SUSY}$?



B-twist $\eta^{\bar i} = -(\bar\psi_-^{\bar i} + \bar\psi_+^{\bar i})$, $\theta_i = g_{i\bar j}(\bar\psi_-^{\bar j} - \bar\psi_+^{\bar j})$, $\rho_z^i = \psi_-^i$ $\rho_{\bar z}^i = \psi_+^i$

$\delta = \bar\epsilon Q$

$$\boxed{\begin{array}{ll} \delta\phi^i = 0 & \delta\bar\phi^{\bar i} = \bar\epsilon\eta^{\bar i} \\ \delta\theta^i = \bar\epsilon\,\partial_i W & \delta\eta^{\bar i} = 0 \\ \delta\rho^i_\mu = -2\bar\epsilon J^\nu_\mu \partial_\nu\phi^i & \end{array}}$$

fixed pt : Constant map to $\underline{\mathrm{Crit}(W)}$

Boundary Condition

$$\left.\begin{array}{ll} \phi \in Z & \\ \eta^{\bar{i}} : \text{tan to } Z & \theta_i : \text{normal to } Z \\ \rho_n^i : \text{tan to } Z & \rho_t^i : \text{normal to } Z \end{array}\right\} \text{ on } \partial\Sigma$$

$$\eta \text{ in } \Lambda^{0,\bullet} TZ \qquad \theta \text{ in } N_{Z/X} = T_X / T_Z$$

$$Q = \bar{\partial} + \partial W \cdot \quad ; \quad \partial W \cdot : \Lambda^q N \to \Lambda^{q-1} N \text{ contraction}$$

$$\underline{\mathcal{H}_{SUSY} = H\left(\Omega^{0,\bullet}(Z, \Lambda^\bullet N_{Z/X}),\ \bar{\partial} + \partial W \cdot\right)}$$

Examples

(i) $Z \cap \mathrm{Crit}(W) = \phi$

$$\mathcal{H}_{SUSY} = 0 \qquad I(Z, Z) = 0$$

(ii) $Z = \{p_*\}$ a critical pt of W

$$\mathcal{H}_{SUSY} = \Lambda^\bullet N_{Z/X} = \Lambda^\bullet \mathbb{C}^n \cong \mathbb{C}^{2^n} \quad \begin{array}{c}\text{Powers of} \\ \theta_1 \cdots \theta_n\end{array}$$

$$\mathcal{H}^B_{SUSY} = \Lambda^{even} \mathbb{C}^n, \quad \mathcal{H}^F_{SUSY} = \Lambda^{odd} \mathbb{C}^n$$

$$I(p_*, p_*) = 0$$

$Y = \mathbb{C}^2 = \{(U,V)\}$

(iii) $W = UV$ crit pt at $U=V=0$

$Z_0 = \{U=V=0\}$ done ✓

$Z_2 = \{V=0\} = \{U\}$

$Z_2' = \{U=0\} = \{V\}$

$\underline{Z = Z_2}$ $Q = \bar{\partial} + U\,dV\cdot$

$f = e^{-|U|^2}\left(1 - d\bar{U} \otimes \frac{\partial}{\partial V}\right)$ solves $Q = Q^\dagger = 0$

$\mathcal{H}_{SUSY} \cong \mathbb{C}$

$I(Z_2, Z_2) = 1$

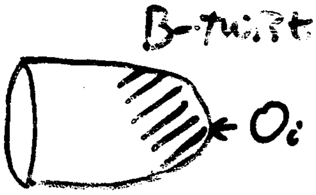
Explicit quantization of other pairs $\Rightarrow$

$\mathcal{H}_{SUSY}^{Z_0 - Z_2} = \underset{B}{\mathbb{C}} \oplus \underset{F}{\mathbb{C}}$ $I(Z_0, Z_2) = 0$

$\mathcal{H}_{SUSY}^{Z_2 - Z_2'} = \mathbb{C}$ $I(Z_2, Z_2') = \pm 1$

(iv) General Z

$$I(Z,Z) = \begin{cases} 0 & \dim Z \neq \frac{1}{2}\dim Y \\ \#(Z \cap \mathrm{Crit}(W)) & \dim Z = \frac{1}{2}\dim Y \end{cases}$$

$$\Pi_i^Z = Z$$ 

- sum over $\mathrm{crit}(W)$

- zero modes .... ϕ^{tan}, $\eta^{\overline{tan}}$, θ_{normal}

at $p_* \in \mathrm{Crit}(W)$

$$\int d\phi^t \, d\eta^{\bar t} \, d\theta_n \exp\left(-|dW|^2 - \partial_{\bar i}\partial_{\bar j}\overline{W}\,\eta^{\bar i} g^{\bar j i}\theta_i\right) O_i(p_*)$$

$$= \begin{cases} 0 & \text{if } \dim Z \neq \frac{1}{2}\dim Y \\ |\det \partial_t\partial_n W|^{-2} \cdot \det \partial_{\bar t}\partial_{\bar n}\overline{W} \cdot O_i(p_*) & \text{if } = \end{cases}$$

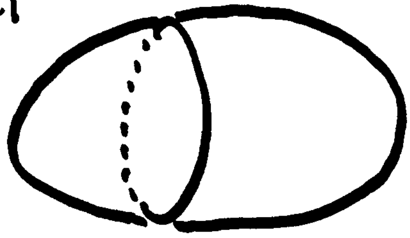
$$= \frac{O(p_*)}{\det \partial_t\partial_n W} = \frac{O(p_*)}{\mathrm{Pf}^Z_{p_*}\partial\partial W}$$

$$\Omega^{t\cdots t\, n\cdots n}\,\partial_t\partial_n W \cdots \partial_t\partial_n W$$

$$\boxed{\begin{gathered} \Pi_i^Z = \sum_{p_* \in Z\cap \mathrm{Crit}(W)} \frac{O_i(p_*)}{\mathrm{Pf}^Z_{p_*}\partial\partial W} = (-1)^{\frac{n}{2}}\tilde{\Pi}_i^Z \\ I(Z_1, Z_2) = \Pi_i^{Z_1}\eta^{ij}\tilde{\Pi}_j^{Z_2} \qquad \left(\eta_{ij} = \sum_{\mathrm{Crit}(W)} \frac{O_i O_j}{\det \partial\partial W}\right) \end{gathered}}$$

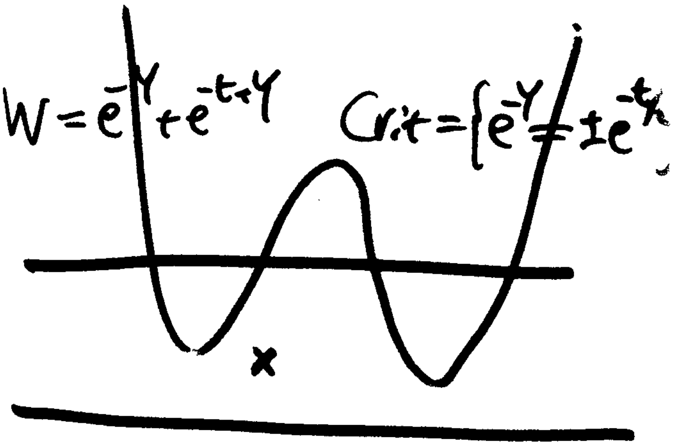
Back to Mirror

$X = \mathbb{CP}^1$



$\longleftrightarrow$

$W = e^{-Y} + e^{-t+Y}$ $\quad$ $\mathrm{Crit} = \{ e^{-Y} = \pm e^{-t/2} \}$



$$\mathrm{D1}\left[\begin{array}{l} |z|^2 = \dfrac{c}{r-c} \\ A = (a - \frac{\theta}{r} c)\dfrac{dz}{z} \end{array}\right] \leftrightarrow \left[e^{-Y} = e^{-c+ia} \right] \mathrm{D0}$$

at critical

$$\left\{\begin{array}{l} |z|^2 = 1 \\ A = 0 \text{ or } \pi \dfrac{dz}{z} \end{array}\right. \longleftarrow \left\{\begin{array}{l} c = \dfrac{r}{2} \\ a = \dfrac{\theta}{2} \text{ or } \dfrac{\theta}{2} + \pi \end{array}\right.$$

L_+ or L_-

SUSY ground states

$$HF^\bullet(L) = \begin{cases} 0 & L \neq L_\pm \\ \underset{B}{\mathbb{C}} \oplus \underset{F}{\mathbb{C}} & L = L_\pm \end{cases}$$

General toric X^n

$$HF^\bullet(\underset{\text{torus fibre}}{L}) = \begin{cases} 0 & \text{mirror D0} \notin \mathrm{Crit}\, W \\ H^\bullet(T^n) & \text{mirror D0} \in \mathrm{Crit}\, W \end{cases}$$

Correlation functions

bulk

$$1 \in H^0(\mathbb{CP}^1) \leftrightarrow 1$$

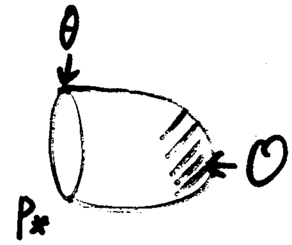
$$\omega \in H^2(\mathbb{CP}^1) \leftrightarrow e^{-Y}$$

$$\langle O_1 O_2 O_3 \rangle = \sum_{\mathrm{Crit}\,W} \frac{O_1 O_2 O_3}{\det \partial\partial W}$$

$$\langle 1\, 1\, \omega \rangle = 1 = \int_{\mathbb{P}^1} \omega$$

$$\langle \omega\omega\omega \rangle = e^{-t} \Leftrightarrow \# \left[\begin{matrix} \overset{\Sigma}{\overset{\shortparallel}{\mathbb{P}^1}} \to \mathbb{P}^1 \ \text{degree } 1 \\ 0,1,\infty \to 0,1,\infty \end{matrix} \right] = 1$$

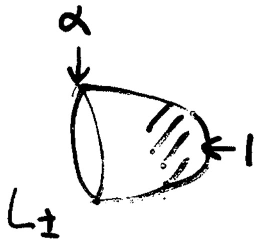
bulk-boundary



$= \mathcal{O}(p_*)$ [⊙ θ is the only zero mode]

$$1 \leftrightarrow 1$$

$$\alpha = \frac{dz}{z} \leftrightarrow \theta$$



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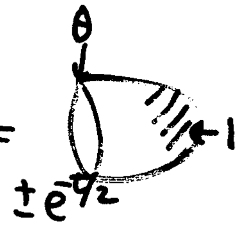
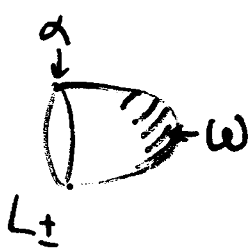


$= 1 \Leftrightarrow \# \left[\overset{\text{const}}{(\underset{\bullet}{D^2}, \partial D^2) \to (\mathbb{P}^1, \underset{\bullet}{L_\pm})} \right] = 1$



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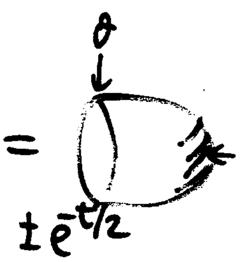

$= \pm e^{-t/2} \qquad e^{\oint A} = \pm 1$

$$\Leftrightarrow \# \left[\begin{matrix} \text{area } t/2 \text{ map} \\ (\underset{0}{D^2}, \underset{1}{\partial D^2}) \to (\underset{0}{\mathbb{P}^1}, \underset{1}{L_\pm}) \end{matrix} \right] = 1$$