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SMR.1402 - 10

SPRING SCHOOL ON SUPERSTRINGS AND RELATED MATTERS

18 - 26 March 2002

LECTURES ON MIRROR SYMMETRY

Lectures 3 and 4

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Please note: These are preliminary notes intended for internal distribution only.

MIRROR SYMMETRY

K. HORI

C. Vafa & KH 2000

A. Kapustin & KH 2001
2002

Mirror Symmetry

T-duality (d=2)

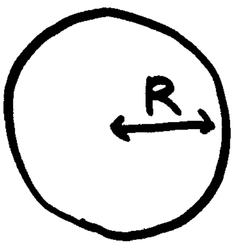
$$\left.\begin{array}{ll} \varphi \equiv \varphi + 2\pi & \text{scalar}, \\ \mathcal{B} : \text{aux. 1-form} & \end{array}\right\} \text{on } \Sigma^2$$

$$S' = \frac{1}{2R^2}\int_\Sigma |\mathcal{B}|^2 + \int_\Sigma \mathcal{B} \wedge d\varphi$$

$\int \mathcal{D}\mathcal{B}$ $\swarrow$ $\qquad\qquad$ $\searrow$ $\int \mathcal{D}\varphi$

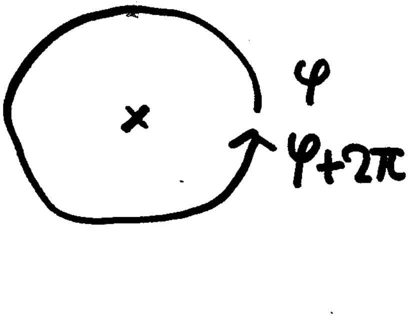
$$\mathcal{B} = R^2 * d\varphi \qquad\qquad \mathcal{B} = d\tilde{\varphi} \quad \tilde{\varphi} \equiv \tilde{\varphi} + 2\pi$$

$$S = \frac{R^2}{2}\int_\Sigma |d\varphi|^2 \qquad\qquad \tilde{S} = \frac{1}{2R^2}\int_\Sigma |d\tilde{\varphi}|^2$$



$$\boxed{R \leftrightarrow \frac{1}{R}}$$

<u>winding #</u> $\longleftrightarrow$ <u>momentum</u>



$\leftrightarrow$ $\quad \times\, \underline{\underline{e^{i\tilde{\varphi}}}}$

(2,2) SUSY

Roček - Verlinde

$$L' = \int d^4\theta \left(\frac{1}{4R^2} B^2 - \frac{1}{2} B (\Phi + \bar{\Phi}) \right)$$

B : real superfield

Φ : chiral

$\Phi \equiv \Phi + 2\pi i$

$\int \mathcal{D}B$

$$B = R^2 (\Phi + \bar{\Phi})$$

$$L = \frac{R^2}{2} \int d^4\theta \, |\Phi|^2$$

$$\boxed{\begin{array}{l} \sigma\text{-model on} \\ \mathbb{C}^\times = \mathbb{R} \times S^1_R \end{array}}$$

$\int \mathcal{D}\Phi \mathcal{D}\bar{\Phi}$

$$\bar{D}_+ \bar{D}_- B = D_+ D_- B = 0$$

$$B = \Theta + \bar{\Theta}$$

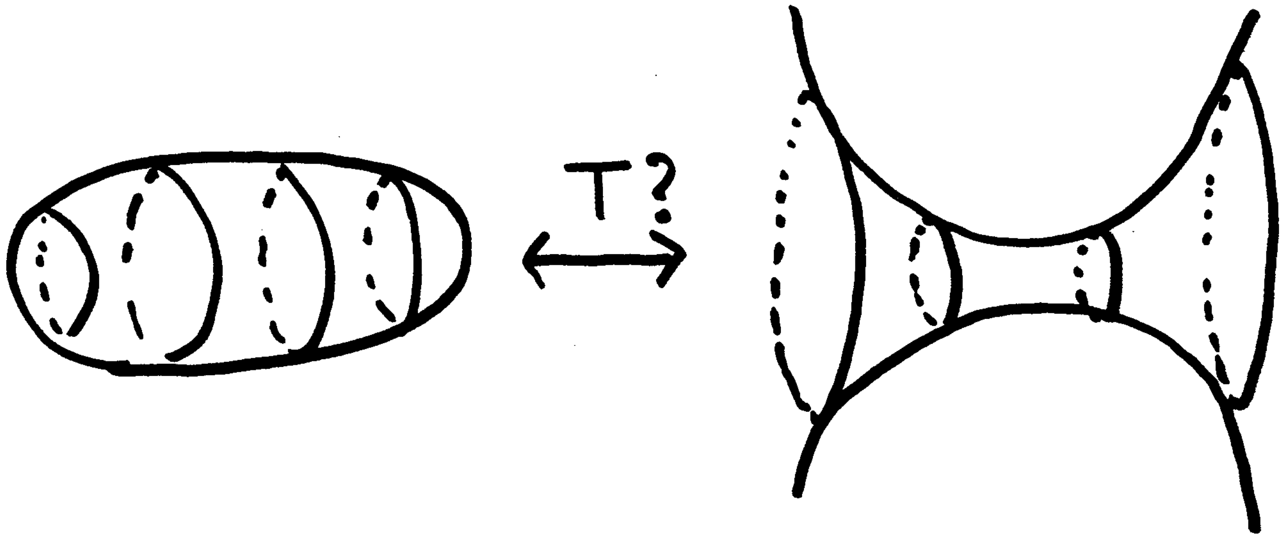
Θ : twisted chiral

$\Theta \equiv \Theta + 2\pi i$

$$\tilde{L} = \frac{1}{2R^2} \int d^4\theta \left(-|\Theta|^2 \right)$$

$$\boxed{\sigma\text{-model on } \tilde{\mathbb{C}}^\times = \mathbb{R} \times \tilde{S}^1_{1/R}}$$

$\therefore$ T-duality is Mirror Symmetry



momentum $\longleftrightarrow$ winding number

O.K.

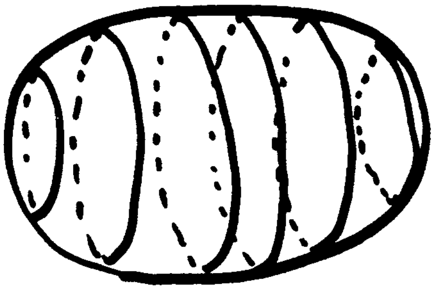
No winding $\longleftrightarrow$ No momentum ?

?

For supersymmetric non-linear sigma models on toric manifolds, we find ...

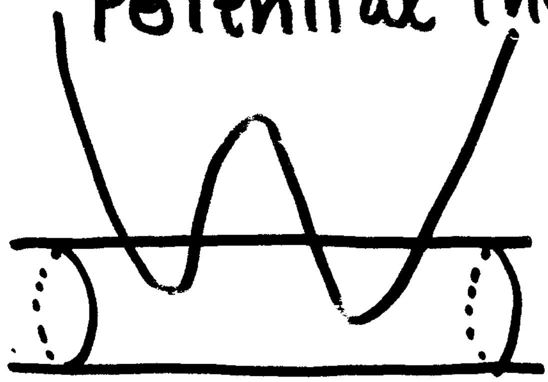
(2,2) SUSY

$\mathbb{C}P^1$



$\longleftrightarrow$

Potential Theory



size r

B-field θ

$$Y \equiv Y + 2\pi i$$

$$W = e^{-Y} + e^{-r+i\theta} \cdot e^{Y}$$

"Sine-Gordon"

$\mathbb{C}P^{N-1}$

$$= \frac{|\phi_1|^2 + \cdots + |\phi_N|^2 = r}{U(1)}$$

B-field θ

$\longleftrightarrow$

<u>affine-Toda</u>

$$Y_1 + \cdots + Y_N = r - i\theta$$

$$W = e^{-Y_1} + \cdots + e^{-Y_N}$$

General Toric

$$\sum_{i=1}^{N} Q_i^a |\phi_i|^2 = r^a \Big/ U(1)^k$$

$a = 1, \cdots, k$

B-field θ^a

$\longleftrightarrow$

$\left\{ \begin{array}{l} \text{SG-type} \\ \text{Liouville} \end{array} \right.$ <u>Landau-Ginzburg model</u>

$$\sum_{i=1}^{N} Q_i^a Y_i = r^a - i\theta^a$$

$$W = \sum_{i=1}^{N} e^{-Y_i}$$

Derivation

Set-up: Linear Sigma Model

U(1) gauge theory

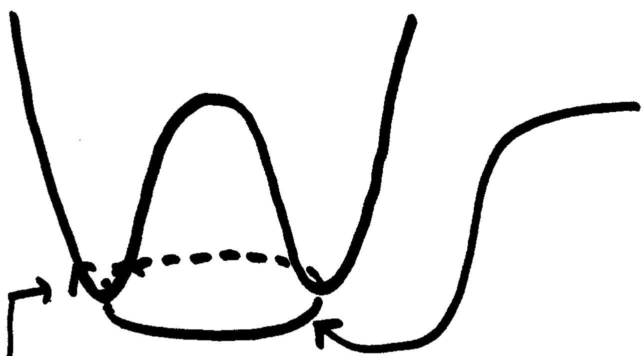
$\Phi_1, \dots, \Phi_N$ charge 1 scalars (+fermions) chiral

V gauge potential (+partners)

$\Sigma = \bar{D}_+ D_- V$ field strength (+partners) twisted chiral

$$L = \int d^4\theta \left[\sum_{i=1}^{N} \bar{\Phi}_i e^V \Phi_i - \frac{1}{2e^2} |\Sigma|^2 \right] + \mathrm{Re} \int d^2\tilde{\theta} \left(-t\Sigma \right)$$

$$= \sum_{i=1}^{N} |D_\mu \phi_i|^2 - \underline{\frac{e^2}{2} \left(\sum_{i=1}^{N} |\phi_i|^2 - r \right)^2} + \underline{\underline{\theta F_{01}}}$$



$$M_{vac} = \frac{\sum_{i=1}^{N} |\phi_i|^2 = r}{U(1)}$$

$$= \mathbb{C}P^{N-1} \circlearrowleft (S^1)^{N-1}$$

mass $\sim e\sqrt{r}$

$e^2 \to \infty$: Supersymmetric NLSM on $\mathbb{C}P^{N-1}$

θ : B-field

Dualise $\arg(\Phi_i)$ $\phi_i = \rho_i e^{i\varphi_i}$

$\boxed{\text{bosonic}}$ $$|D_\mu \phi_i|^2 = (\partial_\mu \rho_i)^2 + \rho_i^2 (\partial_\mu \varphi_i + A_\mu)^2$$

$$S' = \frac{1}{2\rho_i^2} \int |B_i|^2 + \int B_i \wedge (d\varphi_i + A)$$

$\int \mathcal{D}B_i$ ↙   ↘ $\int \mathcal{D}\varphi_i$

$$S = \frac{1}{2} \int \rho_i^2 |d\varphi_i + A|^2$$

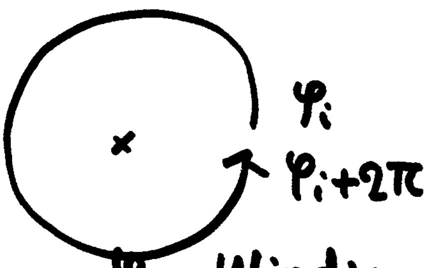
$$B_i = d\tilde{\varphi}_i$$

$$\tilde{S} = \frac{1}{2} \int \frac{1}{\rho_i^2} |d\tilde{\varphi}_i|^2 + \int d\tilde{\varphi}_i \wedge A$$

$$\int d\tilde{\varphi}_i \wedge A = \boxed{-\int \tilde{\varphi}_i F_A}$$

① Dynamical Theta Angle

↳ $\exists$ ϕ_i-vortex



φ_i-Winding

$\longleftrightarrow$

momentum operator

$\times\, e^{i\tilde{\varphi}_i}$

② $e^{i\tilde{\varphi}_i}$-insertion

$\boxed{\text{SUSY}}$

$$\Phi_i = \phi_i + \cdots \quad \longleftrightarrow \quad Y_i = \rho_i^2 - i\tilde{\varphi}_i + \cdots \qquad (Y_i \equiv Y_i + 2\pi i)$$

chiral — twisted chiral

Φ_i charge 1 $\longleftrightarrow$ ① $Y_i \Sigma$

SUSY Φ_i-vortex $\longleftrightarrow$ ② e^{-Y_i}

} in superpotential

$$\therefore \quad \underline{W = -t\Sigma + \sum_{i=1}^{N} Y_i \Sigma + \sum_{i=1}^{N} e^{-Y_i}} \quad \text{exact}$$

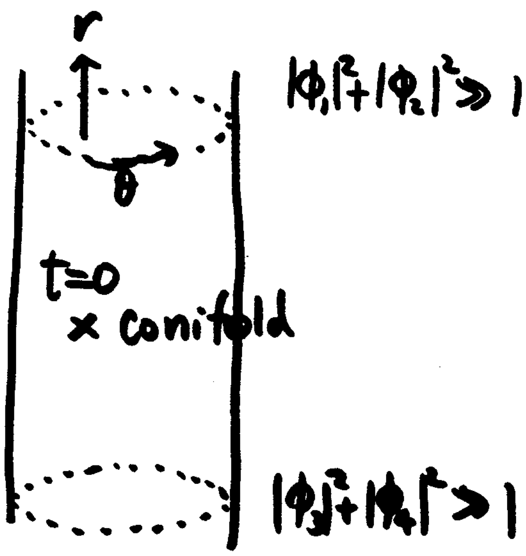
NLSM Limit $e^2 \to \infty$: Σ heavy $\leadsto$ integrate out

$$\leadsto \quad \underline{Y_1 + Y_2 + \cdots + Y_N = t}$$

$$\underline{W = e^{-Y_1} + e^{-Y_2} + \cdots + e^{-Y_N}}$$

This is the Mirror LG Model.
(for $\mathbb{C}P^{N-1}$)

non-compact CY



U(1)	Φ_1	Φ_2	Φ_3	Φ_4
	1	1	−1	−1

M_{vac} : $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ resolved conifold

MIRROR $\Longleftrightarrow$

$$W = e^{-Y_1} + e^{-Y_2} + e^{-Y_3} + e^{-Y_4} \Big/ Y_1+Y_2-Y_3-Y_4 = t$$

$$= e^{-Y_0}\left(e^{-t} + e^{-\Theta_1-\Theta_2} + e^{-\Theta_1} + e^{-\Theta_2}\right)$$

IR equivalent $\dashrightarrow$

$$W' = e^{-Y_0}\left(e^{-t} + e^{-\Theta_1-\Theta_2} + e^{-\Theta_1} + e^{-\Theta_2}\right) + UV$$

<u>U, V : C-valued</u>

<u>mass of BPS D-branes</u>

$$\Pi = \int dY_0\, d\Theta_1\, d\Theta_2\, dU\, dV\; e^{-iW'}$$

change of variables $\underline{U = e^{-Y_0}\tilde{U}\ ,\ V = \tilde{V}}$:

$$\Pi = \int e^{-Y_0} dY_0\, d\Theta_1 d\Theta_2\, d\tilde{U}\, d\tilde{V}$$
$$\exp\left[-ie^{-Y_0}\left(e^{-t} + e^{-\Theta_1-\Theta_2} + e^{-\Theta_1} + e^{-\Theta_2} + \tilde{U}\tilde{V}\right)\right]$$

$$\int de^{-Y_0}$$
$$= \int d\Theta_1 d\Theta_2\, d\tilde{U}\, d\tilde{V}\ \delta\left[e^{-t} + e^{-\Theta_1-\Theta_2} + e^{-\Theta_1} + e^{-\Theta_2} + \tilde{U}\tilde{V}\right]$$

.... Period Integral of

$$e^{-t} + e^{-\Theta_1-\Theta_2} + e^{-\Theta_1} + e^{-\Theta_2} + \tilde{U}\tilde{V} = 0$$

— deformed conifold

$\left(\begin{array}{l} t = 0: \\ (1+e^{-\Theta_1})(1+e^{-\Theta_2}) + \tilde{U}\tilde{V} = 0 \\ \text{conifold singularity at} \\ \qquad \Theta_1 = \Theta_2 = \pi i,\ \tilde{U} = \tilde{V} = 0 \end{array}\right)$

Degree d hypersurface in $\mathbb{C}P^{N-1}$

$G(\Phi_1, \cdots, \Phi_N)$ Polynomial of degree d

like $a_1 \Phi_1^d + \cdots + a_N \Phi_N^d + b_{i_1 \cdots i_d} \Phi_{i_1} \cdots \Phi_{i_d} + \cdots$

$U(1)$ gauge theory $\Phi_1, \cdots, \Phi_N, P$ <u>chirals</u>

charge $1, \cdots, 1, -d$

- Superpotential $W = P\,G(\Phi_1, \cdots, \Phi_N)$

$$\Rightarrow \quad M_{vac} \overset{r>0}{=\!=} \left\{ \begin{array}{l} P = 0 \\ G(\phi_1, \cdots \phi_N) \end{array} \quad \sum_i |\phi_i|^2 = r \right\} \Big/ U(1) \subset \mathbb{C}P^{N-1}$$

$$\underline{\text{degree } d \text{ hypersurface } X_d}$$

$$\left[\begin{array}{lll} c_1(X_d) \propto (N-d) & > 0 & d = 1, \cdots, N-1 \\ & = 0 & d = N \end{array} \right]$$

$X_1 = \mathbb{C}P^{N-2}$

If $W = 0$, $M_{vac} \overset{r>0}{=} \left\{ \sum_{i=1}^{N} |\phi_i|^2 - d|P|^2 = r \right\} \Big/ U(1) =: V_{toric}^N$

The mirror is LG on $(\mathbb{C}^\times)^N$

- $Y_1 + \cdots + Y_N - d\,Y_P = t$
- $\widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_N} + e^{-Y_P}$

Having $W = P\,G(\Phi_1, \dots, \Phi_N)$

$\Rightarrow$ Change of variables (later)

$$e^{-Y_1} = X_1^d, \dots, e^{-Y_N} = X_N^d, \quad e^{-Y_P} = e^{t/d} X_1 \cdots X_N$$

$$\{(Y_i)\} \xleftrightarrow{1 : \mathbb{Z}_d^{N-1}} \{(X_i)\}$$

- $\mathbb{Z}_d^{N-1}$ orbifold of LG on $\mathbb{C}^N = \{(X_1, \dots, X_N)\}$

$$\widetilde{W} = X_1^d + \cdots + X_N^d + e^{t/d} X_1 \cdots X_N$$

- vacua $\Leftrightarrow d\widetilde{W} = 0$

  * $(N-d)$ massive vacua at $X_i^d = S$, $S^{N-d} = (-d)^d e^{-t}$
  * One vacuum at $X_1 = \cdots = X_N = 0$ $\quad (d \geq 2)$

  $\xrightarrow{IR}$ $\underline{\text{non-trivial SCFT}}$ $\quad 2 < d \leq N$

  $$c/3 = N(d-2)/d$$

  (× $d = N$: $c/3 = N-2 = \dim_{\mathbb{C}} X_d$, $e^{t/d} X_1 \cdots X_N$ marginal

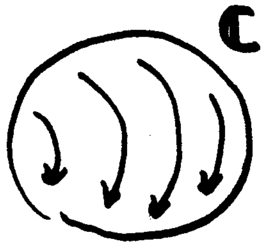
  × $2 < d < N$: $e^{t/d} X_1 \cdots X_N$ irrelevant

  $$W_{IR} = X_1^d + \cdots + X_N^d \,/\, \mathbb{Z}_d^{N-1}$$)

- $\mathrm{Tr}(-1)^F = (N-d) + \mathrm{Tr}_{SCFT}(-1)^F = \dfrac{(1-d)^N + Nd - 1}{d} = \chi(X_d)$

- $U(1)_A \xrightarrow[\text{anomaly}]{} \mathbb{Z}_{2(N-d)} \xrightarrow[IR]{} U(1)_A$

Kähler Potential



CP^1 $\longleftrightarrow$ Sine·Gordon $W = e^{-Y} + e^{-t+Y}$

U(1) isometry $\longleftrightarrow$ Winding # in $Y \to Y + 2\pi i$

$\cap$

SU(2) $\longleftrightarrow$?

Kobayashi-Uematsu

$\left.\begin{array}{l} W = e^{-Y} + e^{-t+Y} \\ K = |Y|^2/2k \end{array}\right\}$ $SU(2)_q$ quantum group symmetry

$\downarrow$

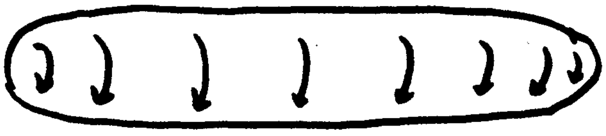
SU(2) $q \to 1$ as $k \to \infty$

× S-matrix ✓ Fendley-Intriligator

× We have seen $K \sim |Y|^2/2r_0$

<u>What is the mirror of S.G. with $k < \infty$?</u>

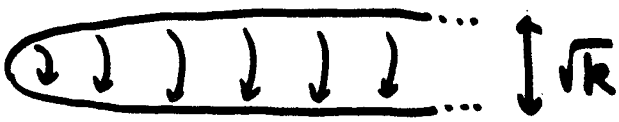
F.I.



$\updownarrow \sqrt{k}$

<u>sausage</u> Can we show this?

A simpler model : Cut in half



cigar

$\leftrightarrow$ $\begin{cases} K = |Y|^2/2k \\ W = e^{-Y} \end{cases}$

Liouville

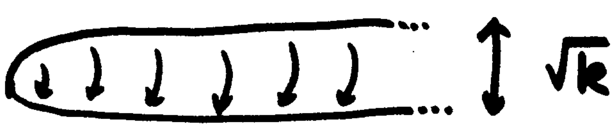
Linear σ-Model

U(1) gauge theory, Φ, $P \equiv P + 2\pi i$ chiral

gauge transf: $\begin{cases} \Phi \to e^{i\Lambda}\Phi & \text{(charge 1)} \\ P \to P + i\Lambda \end{cases}$

$$L = \int d^4\theta \left[\bar{\Phi} e^V \Phi + \frac{k}{4}(P + \bar{P} + V)^2 - \frac{1}{2e^2}|\Sigma|^2 \right]$$

- Space of classical vacua is a cigar



It flows under renormalization group to 2d Black Hole

2d BH: $SL(2,\mathbb{R})_{k+2}$ mod $U(1)$ supercoset

$$c = 3\left(1+\frac{2}{k}\right)$$

The flow can be shown in 3 steps

1. One-loop β-function (Valid at $k \gg 1$)

2. Computation of $c = 3(1+2/k)$

   identify $\mathcal{N}=2$ Superconformal currents in $\bar{Q}_+$ cohomology ring

   $(\bar{Q}_+^2 = 0,\ \{\bar{Q}_+, Q_+\} = H+P)$

   Witten

   Silverstein-Witten

   $$J = \text{classical} + \underline{\frac{1}{2}[\bar{D}_-, D_-](P+\bar{P}+V)}$$

   linear dilaton

   induced by Chiral/Konishi anomaly

3. Rigidity of $SL(2,\mathbb{R})$ mod $U(1)$:

   No SUSY, Parity even marginal deformation dualization + vortex-instanton effect

$$\Rightarrow \begin{cases} \widetilde{W} = \Sigma(Y+Y_P) + \mu e^{-Y} \\ K = -\frac{1}{2e^2}|\Sigma|^2 - \frac{1}{2k}|Y_P|^2 + \cdots \end{cases}$$

$\xrightarrow{e\to\infty}$ int. out Σ : $Y+Y_P=0$

$$\begin{cases} \widetilde{W} = \mu e^{-Y} \\ K = -\frac{1}{2k}|Y|^2 + \cdots \to 0 \end{cases}$$

Thus,

$$L = \int d^4\theta \left[\bar{\Phi} e^V \Phi + \frac{k}{4}(P+\bar{P}+V)^2 - \frac{1}{2e^2}|\Sigma|^2 \right]$$

$SL(2,\mathbb{R})_{k+2}$ mod $U(1)$
super coset
(2d BH)

$\mathcal{N}=2$ Liouville theory
$K = |Y|^2/2k$
$W = \mu e^{-Y}$

Mirror Symmetry

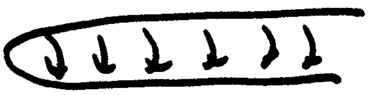
There is an $\mathcal{N}=0$ (No SUSY) version

Fateev
Zamolodchikov
Zamolodchikov

$SL(2,\mathbb{R})_k$ mod $U(1)$ $\leftrightarrow$ sine-Liouville theory

bosonic coset

$$U = \mu^2 e^{-\phi} \cos\vartheta$$

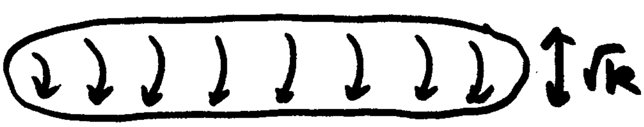


Generalization

LSM with charged fields (like Φ)
& inhomogeneous fields (like P)

NL σ-M on "squashed" toric manifold

LG model of (affine) Toda type with finite Kähler potential $\sim \frac{1}{k}$



Application : NS5-branes

$$\frac{SL_2(\mathbb{R})_{m+2}}{U(1)} \Big/ \underline{\mathbb{Z}_m} \xleftrightarrow{\text{MIRROR}} W = \mu e^{-\underline{m}Y}$$

$$\frac{SU(2)_{m-2}}{U(1)} \Big/ \mathbb{Z}_m \xleftrightarrow{\text{MIRROR}} W = X^m$$

$$\left[\frac{SL_2(\mathbb{R})_{m+2}}{U(1)} \times \frac{SU(2)_{m-2}}{U(1)}\right] \Big/ \mathbb{Z}_m \sim \mathbb{R}_\phi \times \left[U(1)_m \times \frac{SU(2)_{m-2}}{U(1)}\right] \Big/ \mathbb{Z}_m$$

$$\xleftrightarrow{T} \mathbb{R}_\phi \times SU(2)_{m-2}$$

$\underline{m \text{ NS5-branes}}$

$$\|$$

$$\left[\frac{SL_2(\mathbb{R})_{m+2}}{U(1)} \Big/ \mathbb{Z}_m \times \frac{SU(2)_{m-2}}{U(1)} \Big/ \mathbb{Z}_m\right] \Big/ \mathbb{Z}_m \xleftrightarrow{\text{Mirror}} \left[W = \mu e^{-mY} + X^m \right] \Big/ \mathbb{Z}_m$$

$$\xleftrightarrow{\text{IR}} \left[W = e^{-mY}(\mu + \tilde{X}^m + \tilde{U}\tilde{V}) \right] \Big/ \mathbb{Z}_m$$

$$\longleftrightarrow \underline{\{ \mu + \tilde{X}^m + \tilde{U}\tilde{V} = 0 \} \ \sigma\text{-model}}$$

deformed $ALE(A_{m-1})$

× $\underline{\text{move NS5's}} \longleftrightarrow \mu + \tilde{X}^m + \underline{\sum_{\ell=0}^{m-2} u_\ell \tilde{X}^\ell} + \tilde{U}\tilde{V} = 0$

× $NS5(D_n, E_{6,7,8}) \longleftrightarrow ALE(D_n, E_{6,7,8})$

$\mathrm{LSM}_k \quad U(1) \quad \begin{matrix} \Phi_1 & \Phi_2 & P \\ 1 & 1 & \text{shift} \end{matrix}$

$$\mathrm{LSM}_k / \mathbb{Z}_{2m} \xleftrightarrow{\text{MIRROR}} W = e^{-mZ}(e^{-Y} + e^{Y})$$

$$\left[\frac{\mathrm{LSM}_{4m}}{\mathbb{Z}_2} \times \frac{SU(2)_{m-2}}{U(1)}\right] \Big/ \mathbb{Z}_m \sim \mathbb{R}_\phi \times \mathbb{CP}^1 \times \left[U(1)_m \times \frac{SU(2)_{m-2}}{U(1)}\right] \Big/ \mathbb{Z}_m$$

$$\xleftrightarrow{T} \mathbb{R}_\phi \times \mathbb{CP}^1 \times SU(2)_{m-2}$$

$\underline{m \text{ NS5 wrapped on } \mathbb{CP}^1}$

$\hookrightarrow$ 4d $\mathcal{N}=2$ $SU(m)$ Super-YM

$\Vert$

$$\left[\frac{\mathrm{LSM}_{4m}}{\mathbb{Z}_{2m}} \times \frac{SU(2)_{m-2}}{U(1)} \Big/ \mathbb{Z}_m\right] \Big/ \mathbb{Z}_m \xleftrightarrow{\text{MIR}} \left[W = e^{-mZ}(e^{-Y} + e^{Y}) + X^m\right] \Big/ \mathbb{Z}_m$$

$$\dashleftarrow\dashrightarrow \underline{\left\{ e^{-Y} + e^{Y} + \tilde{X}^m + \tilde{U}\tilde{V} = 0 \right\} \text{ CY 3fold}}$$

$\underline{\text{move NS5's}} \leftrightarrow e^{-Y} + e^{Y} + \tilde{X}^m + \underline{\sum_{\ell=0}^{m-2} u_\ell \tilde{X}^\ell} + \tilde{U}\tilde{V} = 0$

$\rightsquigarrow$ Seiberg-Witten solution of $\mathcal{N}=2$ super-YM

$\times$ $\left.\begin{matrix}\text{zero size} \\ \text{zero B-field}\end{matrix}\right\}$ $\mathbb{CP}^1$ $\leftrightarrow$ $e^{-Y} + e^{Y} + \tilde{X}^m \pm 2 + \tilde{U}\tilde{V} = 0$

ADE

Argyres-Douglas SCFT ! ADE

D-BRANES AND MIRROR SYMMETRY

KH.

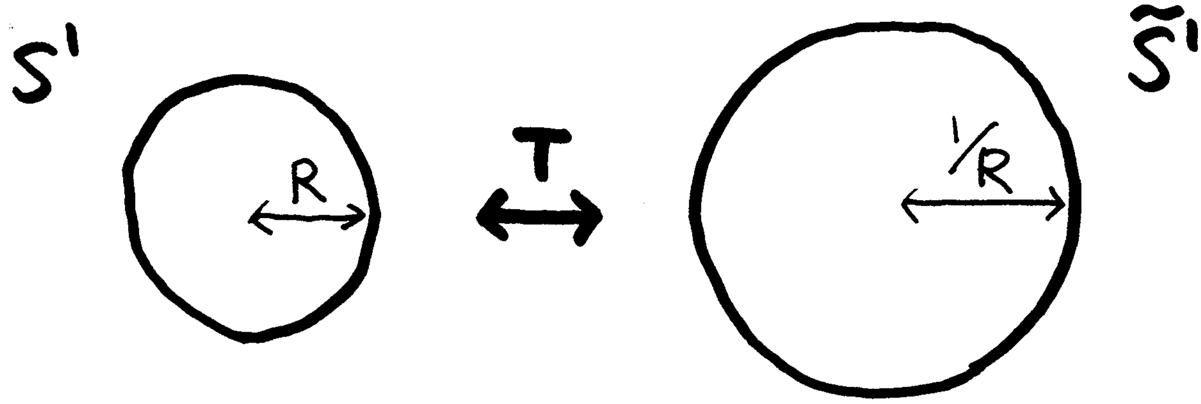
— A. Iqbal, C. Vafa & K.H May 2000

— KH Dec 2000

— "Ch.40" of a book to appear 2002 (?)
by S. Katz, A. Klemm, R. Pandharipande, R. Thomas
R. Vakil, C. Vafa, E. Zaslow & KH

⋮

D-branes & T-duality

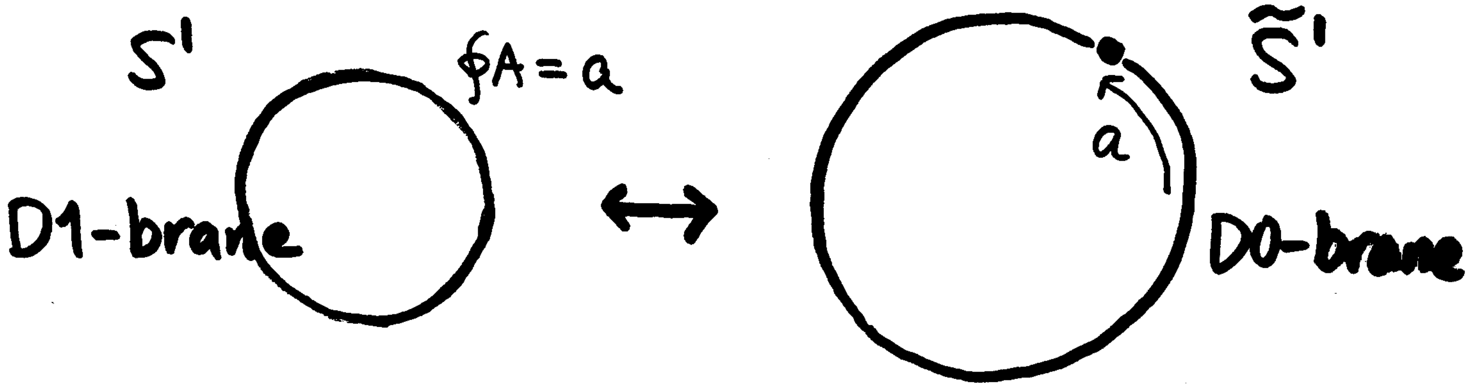


$$*dX = d\tilde{X}$$

Σ : World sheet with boundary

Neumann B.C. Dirichlet B.C.

$$*dX\big|_{\partial\Sigma}=0 \longleftrightarrow d\tilde{X}\big|_{\partial\Sigma}=0$$



Wilson line $\longleftrightarrow$ Position

Dai-Leigh-Polchinski
Horava

Q: How are D-branes transfored under Mirror Symmetry?

We restrict our attention to D-branes that preserve a $\frac{1}{2}$ of the supersymmetry of the bulk of the world sheet

↳ (2,2) supersymmetry
$Q_\pm, \overline{Q}_\pm$ $\qquad \{Q_\pm, \overline{Q}_\pm\} = H \pm P$

(Mirror Symmesry : $Q_- \leftrightarrow \overline{Q}_-$)

two kinds of "$\frac{1}{2}$" Ooguri Oz Yin

$$\left.\begin{aligned} Q_A &= \overline{Q}_+ + Q_- \\ Q_A^\dagger &= Q_+ + \overline{Q}_- \end{aligned}\right\} \text{ A-branes}$$

mirror

$$\left.\begin{aligned} Q_B &= \overline{Q}_+ + \overline{Q}_- \\ Q_B^\dagger &= Q_+ + Q_- \end{aligned}\right\} \text{ B-branes}$$

Derivation

$\mathcal{B}$ 1-form on Σ

$\tilde{\varphi} \equiv \tilde{\varphi} + 2\pi$ on Σ

$u \equiv u + 2\pi$ on $\partial\Sigma$

$$S' = \int_\Sigma \frac{R^2}{2} |\mathcal{B}|^2 + \int_\Sigma d\tilde{\varphi} \wedge \mathcal{B} + \int_{\partial\Sigma} (a - \tilde{\varphi})\, du$$

$\int \mathcal{D}\tilde{\varphi}$ $\qquad$ $\int \mathcal{D}\mathcal{B}\, \mathcal{D}u$

$\mathcal{B} = d\varphi$ on Σ

$\mathcal{B}|_{\partial\Sigma} = du$

$$S = \int_\Sigma \frac{R^2}{2} |d\varphi|^2 + \underbrace{\int_{\partial\Sigma} a\, d\varphi}_{\textbf{Wilson line}}$$

$\mathcal{B} = \frac{1}{R^2} * d\tilde{\varphi}$ on Σ

$\tilde{\varphi}|_{\partial\Sigma} = a$] **Position**

$$\tilde{S} = \int_\Sigma \frac{1}{2R^2} |d\tilde{\varphi}|^2$$

X Kähler manifold $\begin{cases} \omega & \text{symplectic structure} \\ J & \text{complex structure} \end{cases}$

[LG model
$W : X \to \mathbb{C}$ superpotential]

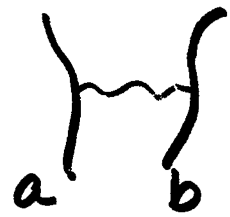
A D-brane wrapped on $\gamma \subset X$
supporting a $U(1)$ gauge potential A

is

an A-brane if $\gamma \subset (X, \omega)$ Lagrangian
A : flat $(F_A = 0)$
$[\, \mathrm{Im}\, W = \text{constant on } \gamma \,]$

a B-brane if $\gamma \subset (X, J)$ complex submanifold
A : holomorphic $(F_A^{0,2} = 0)$
$[\, W = \text{constant on } \gamma \,]$

$\mathbf{a} = (\gamma_a, A_a)$, $\mathbf{b} = (\gamma_b, A_b)$ both A-branes (or both B-branes)

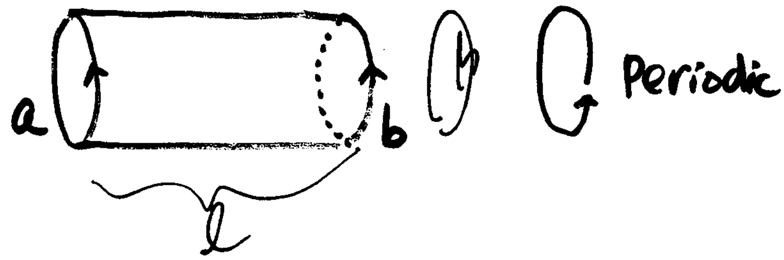


open string

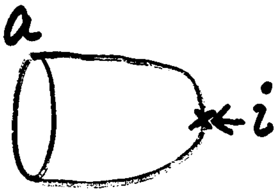
Hilbert space $\mathcal{H}_{a,b}$

$\{ Q, Q^{+} \} = 2H \qquad (Q = Q_A \text{ (or } Q_B))$

Witten index $\mathbf{I(a,b)} = \mathrm{Tr}_{\mathcal{H}_{a,b(\ell)}} (-1)^F e^{-\beta H(\ell)}$

$=$ 

$=$ 

$\Pi_i^a =$ 

.... RR charge of the brane

($\propto$ mass if BPS in space-time)

A


- Independent of Kähler class
- $D_i \Pi_j^a = \underbrace{C_{ij}{}^{k}}_{\text{chiral ring}} \Pi_k^a$ for cplx deformation

A

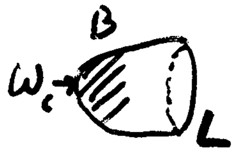

.... topological disc amplitudes

★ $L \subset X$ Lagrangian

$$I(L_1, L_2) = \#(L_1 \cap L_2)$$

$X = CY$: $\Pi_i^L =$ 

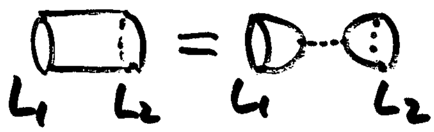
$= \pm i^* \int_L \omega_i$ Period

$\tilde{\Pi}_i^L =$ 

$= \pm i^* \int_L \omega_i$

$\eta_{ij} = \int_X \omega_i \wedge \omega_j$

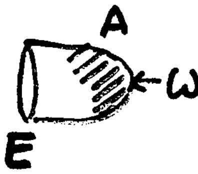
inverse



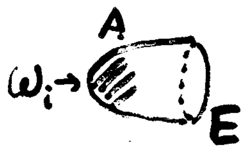
: $\#(L_1 \cap L_2) = \int_{L_1} \omega_i \, \eta^{ij} \int_{L_2} \omega_j$

★ $E \subset X$ holomorphic bundle

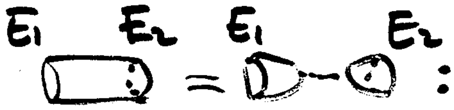
$$I(E_1, E_2) = \chi(E_1, E_2) = \int_X ch(E_1^\vee) ch(E_2)\, Td(X)$$

$\Pi_i^E =$ 

$= \int_X e^{B+i\omega} \omega_i \, ch(E^\vee) \sqrt{Td(X)} + \cdots$

$\tilde{\Pi}_i^E =$ 

$= \int_X e^{-B-i\omega} \omega_i \, ch(E) \sqrt{Td(X)} + \cdots$



:

$$\chi(E_1, E_2) = \int_X e^{B+i\omega} \omega_i \, ch(E_1^\vee) \sqrt{Td(X)} \; \eta^{ij} \int_X e^{-B-i\omega} \omega_j \, ch(E_2) \sqrt{Td(X)}$$

MIRROR SYMMETRY

non-linear σ-model on X^n_{toric} $\longleftrightarrow$ Landau-Ginzburg model $W : (\mathbb{C}^\times)^n \to \mathbb{C}$

e.g. $X = \mathbb{CP}^{N-1} \longleftrightarrow W = e^{-Y_1} + \cdots + e^{-Y_{N-1}} + e^{-t+Y_1+\cdots+Y_N}$

We will study the maps

B-branes in X $\longleftrightarrow$ A-branes in LG

holomorphic bundles — Lagrangians with $\operatorname{Im} W = \text{const}$

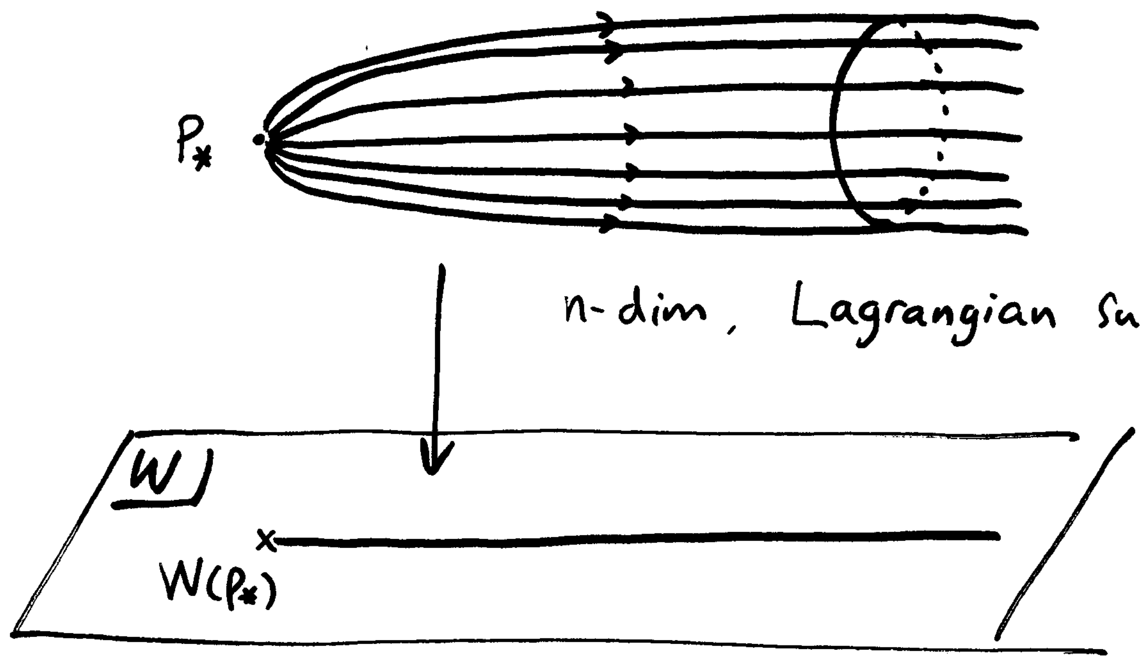
A-branes in X $\longleftrightarrow$ B-branes in LG

Lagrangian submfds — cplx submfds with $W = \text{const}$

A-branes in LG $W : Y^n \to \mathbb{C}$

p_* a non-degenerate critical point of W.

Gradient flow lines of $\underline{\mathrm{Re}(W)}$ starting from p_*



n-dim, Lagrangian submfd γ_{p_*}

$\therefore$ A-brane

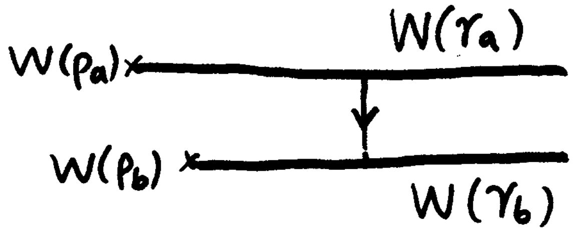
two such branes

Open string quantum mechanics

$$Q = Q_R = \int \bar{\psi}_+ (\partial_t \phi + \partial_\sigma \phi + i\bar{W}') + \psi_- (\partial_t \bar{\phi} - \partial_\sigma \bar{\phi} + iW')$$

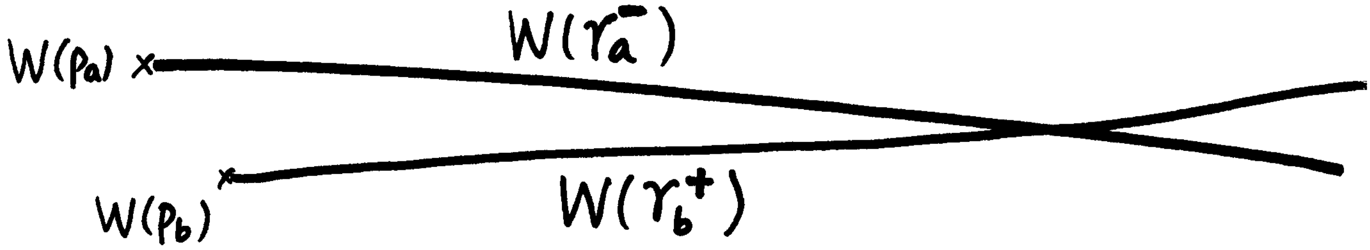
$$Q = Q^\dagger = 0 \quad \Rightarrow \quad \begin{aligned} &\partial_t \phi = 0 \\ &\partial_\sigma \phi = -\,\mathrm{grad}\, \underline{\mathrm{Im}(W)} \end{aligned}$$

<u>SUSY ground states</u> : Grad flows of $-\mathrm{Im}(W)$



$\left(\nexists \text{ if } \mathrm{Im}W(p_a) < \mathrm{Im}W(p_b)\right)$

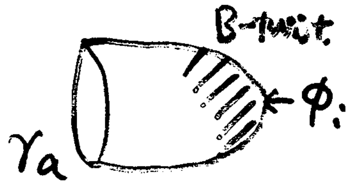
How many ?



- $\boxed{I(a,b) = \#(\gamma_a^- \cap \gamma_b^+)} = \#(\text{BPS solitons in a-b sector})$

- $\mathcal{H}_{SUSY} = HF^{\bullet}_{W}(\gamma_a, \gamma_b)$ dimension $|I(a,b)|$

<u>Overlap with RR ground states</u>

$\Pi_i^a =$  $= \int_{\gamma_a^-} e^{-iW} \phi_i \Omega$

$\widetilde{\Pi}_{\bar{i}}^a =$  $= \int_{\gamma_a^+} e^{-i\overline{W}} * \phi_{\bar{i}} \overline{\Omega}$

$I(a,b) = \Pi_i^a g^{i\bar{j}} \widetilde{\Pi}_{\bar{j}}^b$: Riemann's bilinear identity

B-branes in $X \leftrightarrow$ A-branes in LG

$$X = \mathbb{C}P^{N-1} \quad \longleftrightarrow \quad W = e^{-Y_1} + \cdots + e^{-Y_{N-1}} + e^{-t+Y_1+\cdots+Y_N}$$

$\dim H^{\cdot}(X) = N$ $\qquad$ N : crit pts $e^{-Y_1} = \cdots = e^{-Y_{N-1}} = e^{-t/N} e^{2\pi i l/N}$

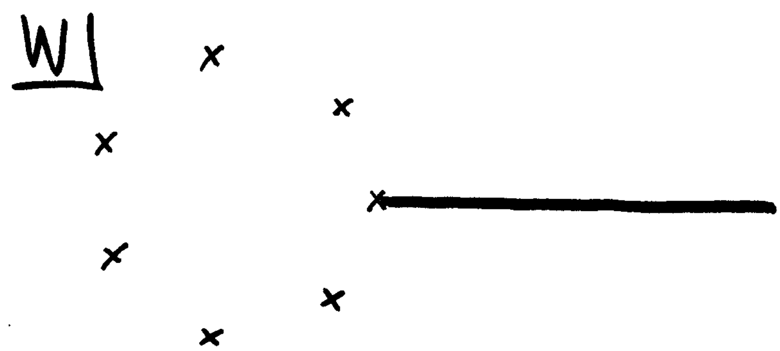
$l = 0, 1, \cdots, N-1$

set $B = 0$ $\qquad$ $\theta = 0 \quad (t \in \mathbb{R}_+)$

$D(2(N-1))$ brane wrapped on X

trivial gauge field

… pure Neumann B.C. $\longleftrightarrow$ $Y_1, \cdots, Y_{N-1}$ all real



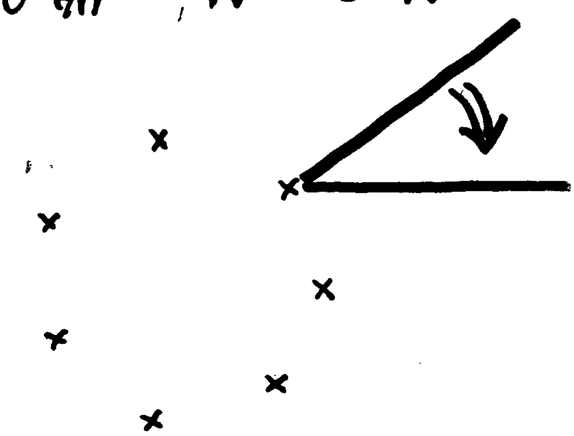
axial R-rotation

$$B \to B + Nr \quad \longleftrightarrow \quad \theta \to \theta + Nr, \quad W \to e^{ir} W$$

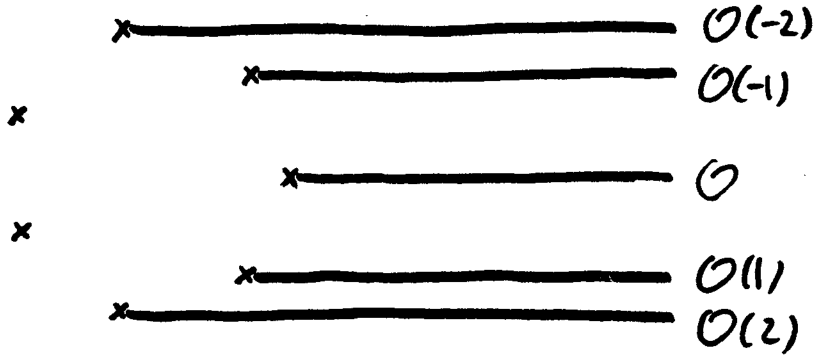
$B = 2\pi$, D-brane on X
trivial gauge field

$\longleftrightarrow$

$\equiv$ $B = 0$, D-brane on X
supporting $\mathcal{O}(-1)$



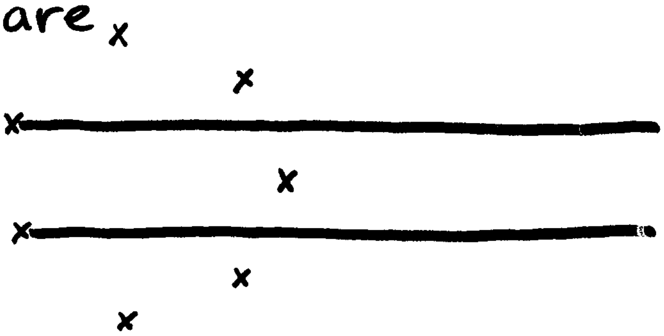
preserves $\bar{Q}_+ + \underbrace{e^{\frac{2\pi i}{N}}}_{\to 1} \bar{Q}_-$



indeed

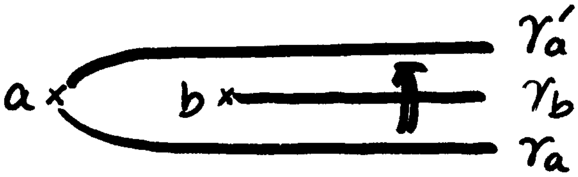
$$\chi(\mathcal{O}(i),\mathcal{O}(j)) = \delta_{i<j}\binom{N+j-i-1}{j-i} = I(\gamma_i,\gamma_j)$$

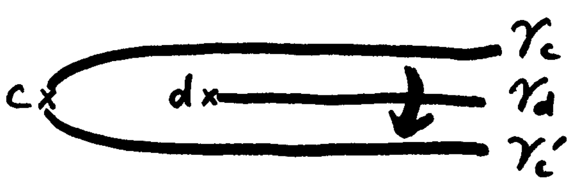
- What about $\mathcal{O}(\pm 3)$, $\mathcal{O}(\pm 4)$, ... ?
- What are



mirror to ?

Picard-Lefshetz :





$$\gamma_a \longrightarrow \gamma_a' + \gamma_b\, I(b,a) \qquad \gamma_c \longrightarrow \gamma_c' + \gamma_d\, I(c,d)$$

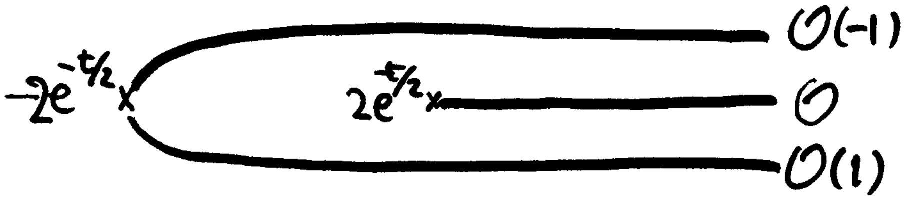
$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$E_a \longrightarrow \underset{\substack{\uparrow \\ \pm L_{E_b}E_a}}{E_a'} + E_b\,\chi(E_b,E_a) \qquad E_c \longrightarrow \underset{\substack{\uparrow \\ \pm R_{E_d}E_c}}{E_c'} + E_d\,\chi(E_c,E_d)$$

<u>*Mutation*</u>

e.g. $X = \mathbb{CP}^1 \longleftrightarrow W = e^{-Y} + e^{-t+Y}$

crit. pts : $e^{-Y} = \pm e^{-t/2}$



$$0 \to \mathcal{O}(-1) \to \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}(1)) \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$$

$$\begin{array}{ll} \| & (f, g) \longmapsto fX_0 + gX_1 \\ L_{\mathcal{O}}\mathcal{O}(1) & \sigma \longmapsto (X_1\sigma, -X_0\sigma) \end{array}$$

$$0 \to \mathcal{O}(-1) \to \mathrm{Ext}^0(\mathcal{O}(-1), \mathcal{O})^* \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$$

$$\mathcal{O}(1) = R_{\mathcal{O}}\mathcal{O}(-1)$$

$$\cdots \; \mathcal{O}(-2) \;\; \mathcal{O}(-1) \;\; \mathcal{O} \;\; \mathcal{O}(1) \;\; \mathcal{O}(2) \; \cdots$$

L, R

Helix of Period 2

$X = \mathbb{CP}^{N-1}$

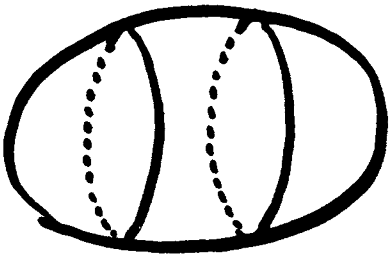
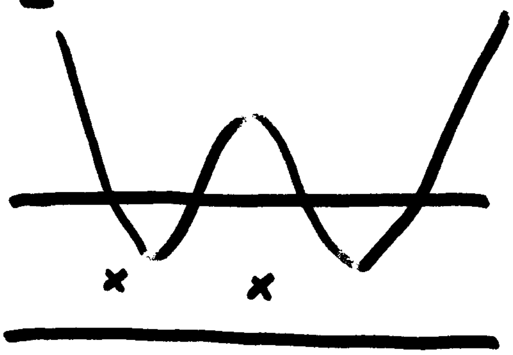
$$\cdots \; \mathcal{O} \;\; \mathcal{O}(1) \cdots \mathcal{O}(N) \cdots$$

Helix of Period N

A-branes in X $\longleftrightarrow$ B-branes in LG, $W: Y \to \mathbb{C}$

$L \subset X$ Lag $\qquad$ $Z \subset Y$ cplx

$\qquad\qquad\qquad\quad$ $W|_Z = \text{const}$

①  $\longleftrightarrow$ 

torus fibers $\qquad\qquad$ points

$\sigma : X \to X$ $\longleftrightarrow$ $\hat{\sigma} : Y \to Y$ holo. inv.

anti-holo. involution $\qquad$ s.t. $W(\hat{\sigma} y) = -W(y)$

$\left[\begin{array}{c} \exists \text{ Parity anomaly} \\ \text{if } w_2(X) \neq 0 \end{array} \right]$ I. Brunner & KH $\qquad$ [$\nexists$ such map]

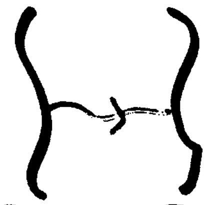
② $\{(x, \sigma x)\} \in X \times X$ $\longleftrightarrow$ $\{(y, \hat{\sigma} y)\} \in Y \times Y$

extra B-field if $w_2(X) \neq 0$ $\qquad$ $W = W(y) + W_o(\hat{\sigma} y) = 0$

③ $X^{\sigma} \subset X$ $\longleftrightarrow$ $Y^{\hat{\sigma}} \subset Y$

B-branes in LG $\quad W : Y \to \mathbb{C}$

$$Q = Q_B = \frac{1}{2\pi}\int_0^{\pi} dx^1 \Big\{ (\bar{\psi}_- + \bar{\psi}_+)\partial_0\phi - (\bar{\psi}_- - \bar{\psi}_+)\partial_1\phi + \frac{i}{2}(\psi_- - \psi_+)W'_j$$



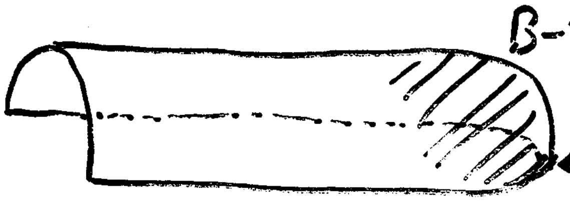
$$\underline{Q^2 = \frac{1}{2\pi i}\left(W|_{Z_b} - W|_{Z_a}\right)}$$

$$Q^2 = 0 \quad \text{iff} \quad W|_{Z_b} = W|_{Z_a}$$

This is the mirror statement of "$m_1^2 \neq 0$"

$Z_a = Z_b =: Z \Rightarrow Q^2 = 0$

What is the $\mathcal{H}_{SUSY}$?



B-twist $\quad \eta^{\bar{i}} = -(\bar{\psi}_-^{\bar{i}} + \bar{\psi}_+^{\bar{i}})$, $\theta_i = g_{i\bar{j}}(\bar{\psi}_-^{\bar{j}} - \bar{\psi}_+^{\bar{j}})$, $\rho_z^i = \psi_-^i \quad \rho_{\bar{z}}^i = \psi_+^i$

$\delta = \bar{\epsilon} Q$

$$\boxed{\begin{array}{ll} \delta\phi^i = 0 & \delta\bar{\phi}^{\bar{i}} = \bar{\epsilon}\eta^{\bar{i}} \\ \delta\theta^i = \bar{\epsilon}\,\partial_i W & \delta\eta^{\bar{i}} = 0 \\ \delta\rho_\mu^i = -2\bar{\epsilon}\, J^\nu{}_\mu \partial_\nu\phi^i & \end{array}}$$

fixed pt : Constant map to $\underline{\mathrm{Crit}(W)}$

Boundary Condition

$$\left.\begin{array}{ll} \phi \in Z & \\ \eta^{\bar{i}} : \text{tan to } Z & \theta_{i} : \text{normal to } Z \\ \rho_n^{i} : \text{tan to } Z & \rho_t^{i} : \text{normal to } Z \end{array}\right\} \text{ on } \partial\Sigma$$

η in $\Lambda^{0,\bullet} TZ$ $\quad$ θ in $N_{Z/X} = T_X / T_Z$

$Q = \bar{\partial} + \partial W \cdot$ $\quad$; $\partial W \cdot : \Lambda^q N \to \Lambda^{q-1} N$ contraction

$$\underline{\mathcal{H}_{SUSY} = H\left(\Omega^{0,\bullet}(Z, \Lambda^{\bullet} N_{Z/X}),\ \bar{\partial} + \partial W \cdot\right)}$$

Examples

(i) $\quad Z \cap \mathrm{Crit}(W) = \phi$

$$\mathcal{H}_{SUSY} = 0 \qquad I(Z, Z) = 0$$

(ii) $\quad Z = \{p_*\}$ a critical pt of W

$$\mathcal{H}_{SUSY} = \Lambda^{\bullet} N_{Z/X} = \Lambda^{\bullet} \mathbb{C}^n \cong \mathbb{C}^{2^n}$$ Powers of $\theta_1 \cdots \theta_n$

$$\mathcal{H}_{SUSY}^{B} = \Lambda^{even} \mathbb{C}^n, \quad \mathcal{H}_{SUSY}^{F} = \Lambda^{odd} \mathbb{C}^n$$

$$I(p_*, p_*) = 0$$

$$Y = \mathbb{C}^2 = \{(U,V)\}$$

(iii) $W = UV$ crit pt at $U=V=0$

$$Z_0 = \{U=V=0\} \quad \text{done } \checkmark$$

$$Z_2 = \{V=0\} = \{U\}$$

$$Z_2' = \{U=0\} = \{V\}$$

$\underline{Z = Z_2}$ $\quad Q = \bar\partial + U\,dV\cdot$

$$f = e^{-|U|^2}\left(1 - d\bar U \otimes \frac{\partial}{\partial V}\right) \quad \text{solves } Q = Q^\dagger = 0$$

$$\mathcal{H}_{SUSY} \cong \mathbb{C}$$

$$I(Z_2, Z_2) = 1$$

Explicit quantization of other pairs $\Rightarrow$

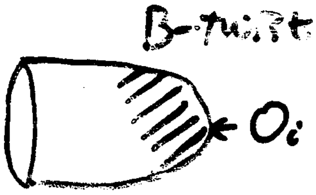
$$\mathcal{H}_{SUSY}^{Z_0 - Z_2} = \underset{B}{\mathbb{C}} \oplus \underset{F}{\mathbb{C}} \qquad I(Z_0, Z_2) = 0$$

$$\mathcal{H}_{SUSY}^{Z_2 - Z_2'} = \mathbb{C} \qquad I(Z_2, Z_2') = \pm 1$$

(iv) General Z

$$I(Z,Z) = \begin{cases} 0 & \dim Z \neq \frac{1}{2}\dim Y \\ \#(Z \cap \mathrm{Crit}(W)) & \dim Z = \frac{1}{2}\dim Y \end{cases}$$

$$\Pi_i^Z = Z$$



- sum over crit(W)

- zero modes .... $\phi^{\tan}$, $\eta^{\overline{\tan}}$, θ_{normal}

at $p_* \in \mathrm{Crit}(W)$

$$\int d\phi^t \, d\eta^{\bar t} \, d\theta_n \exp\left(-|dW|^2 - \partial_{\bar i}\partial_{\bar j}\overline{W}\, \eta^{\bar i} g^{\bar j i}\theta_i\right) \mathcal{O}_i(p_*)$$

$$= \begin{cases} 0 & \text{if} \quad \dim Z \neq \frac{1}{2}\dim Y \\ |\det \partial_t \partial_n W|^{-2} \cdot \det \partial_{\bar t}\partial_{\bar n}\overline{W} \cdot \mathcal{O}_i(p_*) & \text{if } = \end{cases}$$

$$= \frac{\mathcal{O}(p_*)}{\det \partial_t \partial_n W} = \frac{\mathcal{O}(p_*)}{\mathrm{Pf}^Z_{p_*} \partial\partial W}$$

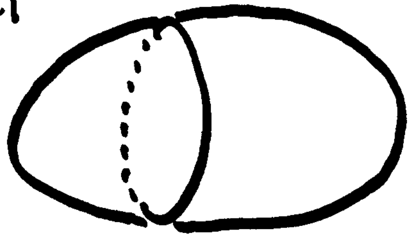
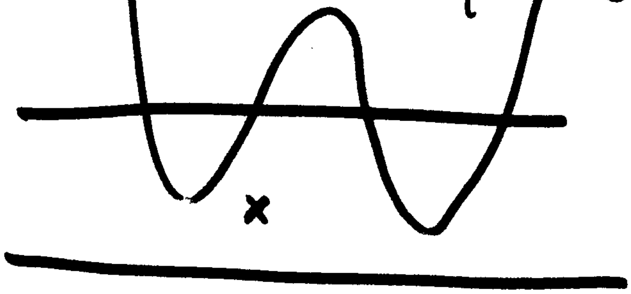
$$\Omega^{t \cdots t\, n \cdots n} \partial_t \partial_n W \cdots \partial_t \partial_n W$$

$$\boxed{\begin{aligned} &\Pi_i^Z = \sum_{p_* \in Z \cap \mathrm{Crit}(W)} \frac{\mathcal{O}_i(p_*)}{\mathrm{Pf}^Z_{p_*} \partial\partial W} = (-1)^{\frac{n}{2}} \widetilde{\Pi}_i^Z \\ &I(Z_1, Z_2) = \Pi_i^{Z_1} \eta^{ij} \widetilde{\Pi}_j^{Z_2} \qquad \left(\eta_{ij} = \sum_{\mathrm{Crit}(W)} \frac{\mathcal{O}_i \mathcal{O}_j}{\det \partial\partial W}\right) \end{aligned}}$$

Back to Mirror

$W = e^{-Y} + e^{-t+Y}$ $\quad$ $\mathrm{Crit} = \{e^{-Y} = \pm e^{-t/2}\}$

$X = \mathbb{CP}^1$

 $\longleftrightarrow$ 

$$\mathrm{D1}\left[\begin{array}{l} |z|^2 = \dfrac{c}{r-c} \\ A = (a - \frac{\theta}{r}c)\dfrac{dz}{z} \end{array}\right] \leftrightarrow \left[e^{-Y} = e^{-c+ia} \right]\mathrm{D0}$$

at critical

$$\underbrace{\begin{cases} |z|^2 = 1 \\ A = 0 \text{ or } \pi \dfrac{dz}{z} \end{cases}}_{L_+ \text{ or } L_-} \longleftarrow \begin{cases} c = \dfrac{r}{2} \\ a = \dfrac{\theta}{2} \text{ or } \dfrac{\theta}{2} + \pi \end{cases}$$

SUSY ground states

$$HF^\bullet(L) = \begin{cases} 0 & L \neq L_\pm \\ \underset{B}{\mathbb{C}} \oplus \underset{F}{\mathbb{C}} & L = L_\pm \end{cases}$$

General toric X^n

$$HF^\bullet(\underset{\text{torus fibre}}{L}) = \begin{cases} 0 & \text{mirror D0} \notin \mathrm{Crit}\, W \\ H^\bullet(T^n) & \text{mirror D0} \in \mathrm{Crit}\, W \end{cases}$$

Correlation functions

bulk

$1 \in H^0(\mathbb{CP}^1) \leftrightarrow 1$

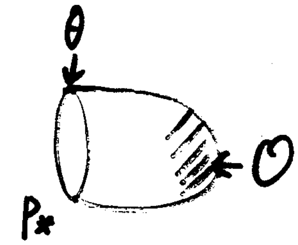
$\omega \in H^2(\mathbb{CP}^1) \leftrightarrow e^{-Y}$

$\langle O_1 O_2 O_3 \rangle = \sum_{\mathrm{Crit}\, W} \frac{O_1 O_2 O_3}{\det \partial\partial W}$

$\langle 1\, 1\, \omega \rangle = 1 = \int_{\mathbb{P}^1} \omega$

$\langle \omega\omega\omega \rangle = e^{-t} \Leftrightarrow \# \left[\begin{array}{c} \overset{\Sigma}{\overset{\|}{\mathbb{P}^1}} \to \mathbb{P}^1 \ \text{degree } 1 \\ 0,1,\infty \to 0,1,\infty \end{array} \right] = 1$

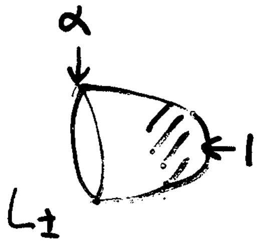
bulk-boundary



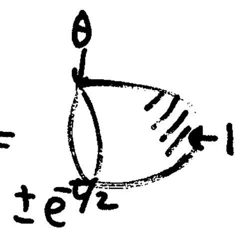
$= \mathcal{O}(p_*)$ [θ is the only zero mode]

$1 \leftrightarrow 1$

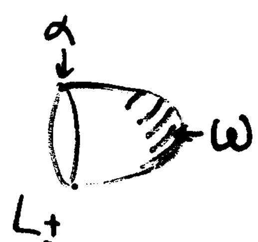
$\alpha = \frac{dz}{z} \leftrightarrow \theta$



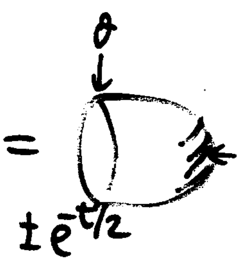
$=$



$= 1 \Leftrightarrow \# \left[\overset{\text{const}}{(\underset{\substack{\cup \\ p}}{D^2}, \partial D^2) \to (\mathbb{P}^1, \underset{\substack{\cup \\ p}}{L_\pm})} \right] = 1$



$=$



$= \pm e^{-t/2} \leftarrow e^{\oint A} = \pm 1$

$\Leftrightarrow \# \left[\begin{array}{c} \text{area } t/2 \text{ map} \\ (\underset{\substack{\cup \\ 0}}{D^2}, \underset{\substack{\cup \\ 1}}{\partial D^2}) \to (\underset{\substack{\cup \\ 0}}{\mathbb{P}^1}, \underset{\substack{\cup \\ 1}}{L_\pm}) \end{array} \right] = 1$