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M THEORY, G_2-MANIFOLDS AND FOUR DIMENSIONAL PHYSICS

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ABSTRACT

M theory on a manifold of G_2-holonomy is a natural framework for obtaining vacua with four large spacetime dimensions and $\mathcal{N} = 1$ supersymmetry. In order to obtain, within this framework, the standard features of particle physics, namely non-Abelian gauge groups and chiral fermions, we consider G_2-manifolds with certain kinds of singularities at which these features reside. The aim of these lectures is to describe in detail how the above picture emerges. Along the way we will see how interesting aspects of strongly coupled gauge theories, such as confinement, receive relatively simple explanations within the context of M theory.

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1. Introduction.

Supersymmetry is one of our best candidates for physics beyond the Standard Model. M theory goes further in the sense that it is supersymmetric, contains gravity and is quantum mechanically consistent. Since the theory is formulated on spacetimes with eleven dimensions, a natural question to ask is are there vacua of M theory with four macroscopic spacetime dimensions *and* a realistic particle physics spectrum? Since supersymmetry is intrinsic to M theory, it is perhaps more natural to look for vacua with supersymmetric particle physics in four dimensions. Since non-minimal or extended supersymmetry in four dimensions cannot accommodate chiral fermions, to answer this question we should really be studying M theory vacua with $\mathcal{N} = 1$ supersymmetry.

There are two (to date) natural looking ways to obtain four large spacetime dimensions with $\mathcal{N} = 1$ supersymmetry from M theory. Both of these require the seven extra dimensions to form a manifold X whose metric obeys certain properties. The first consists of taking X to be a manifold whose boundary ∂X is a Calabi-Yau threefold [1]. The second possibility, which will be the subject of these lectures is to take X to be a manifold of G_2-holonomy[2]. We will explain what this means in section two.

If X is large compared to the Planck scale (the only scale in M theory) and smooth, then at low energies a good approximation is provided by eleven dimensional supergravity. Compactifications of the latter have been studied for several decades. See [2] for a review. Unfortunately, Witten proved that none of these could give rise to chiral fermions [3]. However, this does not mean that G_2-manifolds are useless for obtaining models of particle physics from a fundamental theory. This is because we have learned in recent years that additional light degrees of freedom can be "hidden" at *singularities* of X. These are typically branes wrapped on submanifolds of X which have shrunk at the singularity[3]. In M theory these are either the $M2$-brane or the $M5$-brane. This can provide a novel picture of conventional field theory dynamics and can even lead to new theories. The supergravity approximation breaks down at such singularities and the analysis of [3] no longer applies.

Within the past couple of years there has been a tremendous amount of progress in understanding M theory physics near singularities in manifolds of G_2-holonomy [4 – 10]. In particular we now understand at which kinds of singularities in G_2-manifolds the basic requisites of the standard model -

[2]We will often refer to X as a G_2-manifold.

[3]A more conventional example of light states at a singularity is provided by string theory on an orbifold. Typically one finds extra light states confined to the singularity. These arise in the so-called twisted sectors.

non-Abelian gauge groups and chiral fermions - are to be found. The purpose of these lectures will be to explain how this picture was developed in detail. Along the way we will how important properties of strongly coupled gauge theories such as confinement can receive a semi-classical description in M theory on G_2-manifolds.

At the beginning of each main section we will offer a section summary. In section two we derive the basic properties of M theory when X is large and smooth. We also derive some basic properties of G_2-manifolds. Section three explains how classical supersymmetric Yang-Mills theory can be obtained from M theory on a singular G_2-manifold X. We describe these singularities in detail. Section four describes how quantum properties of the Yang-Mills theory, confinement and a mass gap, can be understood from M theory. The reason that this can be done successfully is that M theory contains semi-classical limits which are not present in the quantum gauge theory. Having understood how non-Abelian gauge groups emerge, section five goes on to describe how additional singularities of X give rise to chiral fermions.

2. Supersymmetry, G_2-holonomy and Kaluza Klein spectrum.

In this section we will describe in detail why G_2-holonomy manifolds naturally emerge in the context of supersymmetric M theory compactification. We will describe some of the basic properties of G_2-manifolds. We will then discuss the Kaluza-Klein spectrum of M theory on a large and smooth G_2-manifold.

2.1 Supersymmetry and G_2-holonomy.

At low energies M theory admits a description in terms of eleven dimensional supergravity. This description is valid on smooth spacetimes whose smallest length scale is much larger than the eleven dimensional Planck length. The supergravity contains three fields, a metric g, a three-form potential C and a gravitino Ψ. In addition to being generally covariant and supersymmetric, theory has a gauge invariance under which

$$\delta C = d\lambda \tag{1}$$

with λ a 2-form, so the gauge invariant field is the derivative of C, G. The action for the bosonic fields is of the form

$$S = \int \sqrt{g}R - \frac{1}{2}G \wedge *G - \frac{1}{6}C \wedge G \wedge G \tag{2}$$

The equations of motion for C and g are of the form,

$$d * G = G \wedge G \tag{3}$$

and

$$R_{MN} = T_{MN}(C) \tag{4}$$

where T is the energy-momentum tensor for the C field.

Since the theory is supersymmetric, it is natural to look for supersymmetric vacua. In the classical theory these are just the conditions that the supersymmetry variations of the three fields vanish. In a Lorentz invariant background the expectation value of Ψ is zero, in which case the variations of g and C vanish automatically. In order to find classically supersymmetric field configurations we must find values of C and g for which the variation of Ψ is zero:

$$\delta_\eta \Psi_M \equiv \nabla_M \eta + \frac{1}{288}(\Gamma_M^{PQRS} G_{PQRS} - 6\Gamma^{PQR} G_{MPQR})\eta = 0 \tag{5}$$

The simplest way to solve these equations is to take $G = 0$ in which case we are looking for 11-manifolds with metric g which admit a covariantly constant or parallel spinor:

$$\nabla_M \eta = 0 \tag{6}$$

We will re-write this equation in the more symbolic form,

$$\nabla_g \eta = 0 \tag{7}$$

where by ∇_g we mean the Levi-Cevita connection constructed from g. Solutions to these conditions can be classified via the holonomy group of the connection ∇_g.

The holonomy group of a connection acting on a field like η or a vector field can be understood in terms of parallel transport. One takes a closed loop on the manifold and literally transports the field around it. In the case of the Levi-Cevita connection, the field comes back to itself up to a rotation in $SO(n)$ in the case of Riemannian n-manifolds or $SO(10,1)$ in the case of M theory. The set of all such rotations based at some point on the manifold generates a group, the holonomy group of ∇_g, $Hol(g)$. If g is sufficiently *generic* then $Hol(g) = SO(10,1)$. However, if we take special choices of g, then $Hol(g)$ can be a proper subgroup of $SO(10,1)$. For instance, if g is such that parallel spinors (supersymmetries) can be found, then η is a field which undergoes no parallel transport atall and therefore $Hol(g)$ must be a subgroup $SO(10,1)$ for which there are spinors in the trivial representation.

We will concern ourselves with compactifications of M theory to four dimensions on a 7-manifold X. More precisely we will take the eleven manifold to be a product $X \times \mathbb{R}^{3,1}$ with g a product metric of a metric $g(X)$ and the Minkowski metric on $\mathbb{R}^{3,1}$. With this choice of 11-metric, we have explicitly broken $SO(10,1)$ to $SO(7) \times SO(3,1)$. The second factor now plays the role of the Lorentz group of the compactified theory. The conditions for supersymmetry can be satisfied by taking $g(X)$ to be such that it admits a spinor θ obeying

$$\nabla_{g(X)} \theta = 0 \tag{8}$$

and choosing

$$\eta = \theta \otimes \epsilon \tag{9}$$

with ϵ a basis of constant spinors in Minkowski space.

The condition

$$\nabla_{g(X)} \theta = 0 \tag{10}$$

implies that $Hol(g(X))$ is G_2 or a subgroup. This is because G_2 is the maximal proper subgroup of $SO(7)$ under which the spinor representation contains a singlet. Specifically, a spinor of $SO(7)$ can be regarded as a fundamental of G_2 and a singlet:

$$\mathbf{8} \to \mathbf{7} + \mathbf{1} \tag{11}$$

Therefore if X is a compact manifold of precisely G_2-holonomy, the effective theory in four dimensions will be minimally $\mathcal{N} = 1$ supersymmetric. We get precisely $\mathcal{N} = 1$ and no more because there is only one singlet spinor according to the above group theory.

2.2. Properties of G_2-manifolds.

From the parallel spinor θ we can construct other covariantly constant fields on X. More precisely, any p-form with components,

$$\theta^T \Gamma_{i_1 i_2 \ldots i_p} \theta \tag{12}$$

is obviously parallel with respect to $\nabla_{g(X)}$. In fact, since the antisymmetric representations of $SO(7)$, when decomposed as representations of G_2, contain singlets only when p is $0, 3, 4, 7$ the above p-forms are non-zero precisely for these values. The 0-form is just a constant on X. The 7-form is the volume form. Locally the three-form, which we will conventionally denote by φ, can be regarded as a set of structure constants for the octonion algebra. This stems from the fact that G_2 is the automorphism group of the octonion algebra, where we regard the tangent space at a point on X as a copy of $\mathrm{Im}\mathbb{O}$, the imaginary octonions. A specific representation of φ locally is

$$\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{356} \tag{13}$$

where the subscript refers to the fact that we are considering a local model. The covariantly constant 4-form is the Hodge dual of φ, which in the local model is given by

$$*\varphi_0 = dx_{4567} + dx_{2367} + dx_{2345} + dx_{1357} - dx_{1346} - dx_{1256} - dx_{1247} \tag{14}$$

In addition to implying that there are other parallel fields on X, the existence of a parallel spinor (or a G_2-holonomy metric) also has other implications. One of these is that the metric of G_2-holonomy, $g(X)$, is Ricci flat. To see this, observe that the commutator of the covariant derivative is the Riemann curvature. Acting on θ this implies,

$$[\nabla_{g(X)}, \nabla_{g(X)}]_{mn} \theta = \frac{1}{4} R_{mnpq} \Gamma^{pq} \theta = 0 \tag{15}$$

Now, contract again with a Γ-matrix to obtain,

$$\Gamma^n \Gamma^{pq} R_{mnpq} \theta = 0 \quad (16)$$

The Bianchi identity for R_{mnpq} which asserts that the components totally antisymmetric in $[npq]$ are zero then implies that

$$\Gamma^j R_{ij} \theta = 0 \quad (17)$$

which implies the Ricci tensor vanishes. This shows that G_2-manifolds obey the equations of motion of d=11 supergravity when the 4-form field strength G and the gravitino is zero: these equations are simply the vacuum Einstein equations.

A final implication - whose proof goes beyond the scope of these lecture but which can be found in [11] - is that compact manifolds with $Hol(g) = G_2$ have a finite fundamental group. This implies that the first Betti number vanishes.

2.3 Kaluza-Klein Reduction.

At low energies, the eleven-dimensional supergravity approximation is valid when spacetime is smooth and large compared to the eleven dimensional Planck length. So, when X is smooth and large enough, we can obtain an effective four dimensional description by considering a Kaluza- Klein analysis of the fields on X. This analysis was first carried out in [12]

In compactification of eleven dimensional supergravity, massless scalars in four dimensions can originate from either the metric or the C-field. If $g(X)$ contains k parameters ie there is a k-dimensional family of G_2-holonomy metrics on X, then there will be correspondingly k massless scalars in four dimensions.

The scalars in four dimensions which originate from C arise via the Kaluza-Klein ansatz,

$$C = \Sigma_I \omega^I(x) \phi_I(y) + ... \quad (18)$$

where ω^I form a basis for the harmonic 3-forms on X. These are zero modes of the Laplacian on X and are also closed. There are $b_3(X)$ linearly independent such forms. The dots refer to further terms in the Kaluza-Klein ansatz which will be prescribed later. The $\phi_I(y)$ are scalar fields in four dimensional Minkowski space with coordinates y. With this ansatz, these scalars are classically massless in four dimensions. To see this, note that,

$$G = \Sigma_I \omega^I \wedge d\phi_I \quad (19)$$

and $d * G$ is just

$$d * G = *\Sigma_I \omega^I d * d\phi_I \quad (20)$$

Since $G \wedge G$ vanishes identically, the equations of motion actually asert that the scalar fields ϕ_I are all massless in four dimensions. Thus, the C-field gives rise to $b_3(X)$ real massless scalars in four dimensions.

In fact it now follows from $\mathcal{N} = 1$ supersymmetry in four dimensions that the Kaluza-Klein analysis of g will yield an additional $b_3(X)$ scalars in four dimensions. This is because the superpartners of C should come from g as these fields are superpartners in eleven dimensions. We should also add that (up to duality tranformations) all representations of the $\mathcal{N} = 1$ supersymmetry algebra which contain one massless real scalar actually contain two scalars in total which combine into complex scalars. We will now describe how these scalars arise explicitly.

We began with a G_2-holonomy metric $g(X)$ on X. $g(X)$ obeys the vacuum Einstein equations,

$$R_{ij}(g(X)) = 0 \quad (21)$$

To obtain the spectrum of modes originating from g we look for fluctuations in $g(X)$ which also satisfy the vacuum Einstein equations. We take the fluctuations in $g(X)$ to depend on the four dimensional coordinates y in Minkowski space. Writing the fluctuating metric as

$$g_{ij}(x) + \delta g_{ij}(x, y) \quad (22)$$

and expanding to first order in the fluctuation yields the Lichnerowicz equation

$$\Delta_L \delta g_{ij} \equiv -\nabla^2_M \delta g_{ij} - 2R_{ijmn} \delta g^{mn} + 2R^k_{(i} \delta g_{j)k} = 0 \quad (23)$$

Next we make a Kaluza-Klein ansatz for the fluctuations as

$$\delta g_i j = h_{ij}(x) \rho(y) \quad (24)$$

Note that the term ∇^2 is the square of the full d=11 covariant derivative. If we separate this term into two:

$$\nabla^2_M = \nabla^2_\mu + \nabla^2_i \quad (25)$$

then we see that the fluctuations are scalar fields in four dimensions with squared masses given by the eigenvalues of the Lichnerowicz operator acting on the h_{ij}:

$$h_{ij} \nabla^2_\mu \rho(y) = -(\Delta_L h_{ij}) \rho(y) = -\lambda h_{ij} \rho(y) \quad (26)$$

Thus, zero modes of the Lichnerowicz operator give rise to massless scalar fields in four dimensions. We will now show that we have precisely $b_3(X)$ such zero modes.

On a 7-manifold of $SO(7)$ holonomy, the h_{ij} - being symmetric 2-index tensors - transform in the **27** dimensional representation. Under G_2 this representation remains irreducible. On the other hand, the 3-forms on a G_2-manifold, which are usually in the **35** of $SO(7)$ decompose under G_2 as

$$\mathbf{35} \longrightarrow \mathbf{1} + \mathbf{7} + \mathbf{27} \tag{27}$$

Thus, the h_{ij} can also be regarded as 3-forms on X. Explicitly,

$$\varphi_{n[pq}h^{n}_{r]} = \omega_{pqr} \tag{28}$$

The ω's are 3-forms in the same representation as h_{ij} since φ is in the trivial representation. The condition that h is a zero mode of Δ_L is equivalent to ω being a zero mode of the Laplacian:

$$\Delta_L h = 0 \leftrightarrow \Delta\omega = 0 \tag{29}$$

This shows that there are precisely $b_3(X)$ additional massless scalar fields coming from the fluctuations of the G_2-holonomy metric on X.

As we mentioned above, these scalars combine with the ϕ's to give $b_3(X)$ massless scalars, $\Phi^I(y)$, which are the lowest components of massless chiral superfields in four dimensions. There is a very natural formula for the complex scalars $\Phi^I(y)$. Introduce a basis α_I for the third homology group of X, $H_3(X, \mathbf{R})$. This is a basis for the incontractible holes in X of dimension three. We can choose the α_I so that

$$\int_{\alpha_I} \omega^J = \delta_I^J \tag{30}$$

Since the fluctuating G_2-structure is

$$\varphi' = \varphi + \delta\varphi = \varphi + \Sigma_I \rho^I(y)\omega^I(x) \tag{31}$$

we learn that

$$\Phi^I(y) = \int_{\alpha_I} \varphi' + iC \tag{32}$$

The fluctuations of the four dimensional Minkowski metric give us the usual fluctuations of four dimensional gravity, which due to supersymmetry implies that the four dimensional theory is locally supersymmetric.

In addition to the massless chiral multiplets, we also get massless vector multiplets. The bosonic component of such a mulitplet is a massless abelian gauge field which arises from the C-field through the Kaluza-Klein ansatz,

$$C = \Sigma_\alpha \ \beta^\alpha(x) \wedge A_\alpha(y) \tag{33}$$

where the β's are a basis for the harmonic 2-forms and the A's are one-forms in Minkowski space ie Abelian gauge fields. Again, the equations of motion for C imply that the A's are massless in four dimensions. This gives $b_2(X)$ such gauge fields. As with the chiral multiplets above, the fermionic superpartners of the gauge fields arise from the gravitino field. Note that we could have also included an ansatz giving 2-forms in four dimensions by summing over harmonic 1-forms on X. However, since $b_1(X) = 0$, this does not produce any new massless fields in four dimensions.

We are now in a position to summarise the basic effective theory for the massless fields. The low energy effective theory is an $\mathcal{N} = 1$ supergravity theory coupled to $b_2(X)$ abelian vector multiplets and $b_3(X)$ massless, *neutral* chiral multiplets. This theory is relatively interesting physically. In particular, the gauge group is abelian and there are no light charged particles. We will thus have to work harder to obtain the basic requisites of the standard model - non-Abelian gauge fields and chiral fermions - from G_2-compactifications. The basic point of these lectures is to emphasise that these features emerge naturally from singularities in G_2-manifolds.

3. Super Yang-Mills from G_2-manifolds: Classical

In this section we will describe how to obtain non-Abelian gauge groups from singular G_2-manifolds. We have known for some time now that non-Abelian gauge groups emerge from M theory when space has a so-called ADE-singularity. We learned this in the context of the duality between M theory on $K3$ and the heterotic string on a flat three torus, $\mathbf{T}^3$ [13]. So, our basic strategy will be to embed ADE-singularities into G_2-manifolds. After reviewing the basic features of the duality between M theory on $K3$ and heterotic string theory on $\mathbf{T}^3$, we describe ADE-singularities explicitly. We then develop a picture of a G_2-manifold near an embedded ADE-singularity. Based on this picture we analyse what kinds of *four* dimensional gauge theories these singularities give rise to. We then go on to describe local models for such singular G_2-manifolds as finite quotients of smooth ones.

3.1 M theory - Heterotic Duality in Seven Dimensions

M theory compactified on a $K3$ manifold is widely believed to be equivalent to the heterotic string theory compactified on a 3-torus $\mathbf{T}^3$. As with G_2 compactification, both of these are compactifications to flat Minkowski space. Up to diffeomorphisms, $K3$ is the only simply connected, compact 4-manifold admitting metrics of $SU(2)$-holonomy. $SU(2)$ is the analog in four dimensions of G_2 in seven dimensions. Interestingly enough in this case $K3$ is the only simply connected example, whereas there are many G_2-manifolds.

There is a 58-dimensional moduli space of $SU(2)$-holonomy metrics on $K3$ manifolds of fixed volume. This space $\mathcal{M}(K3)$ is locally a coset space:

$$\mathcal{M}(K3) = \mathbf{R}^{+}\times\frac{SO(3,19)}{SO(3)\times SO(19)} \tag{34}$$

An $SU(2)$ holonomy metric also admits two parallel spinors, which when tensored with the 8 constant spinors of 7-dimensional Minkowski space give 16 global supercharges. This corresponds to minimal supersymmetry in seven dimensions (in the same way that G_2-holonomy corresponds to minimal supersymmetry in four dimensions). If we work at a smooth point in $\mathcal{M}$ we can use Kaluza-Klein analysis and we learn immediately that the effective d=7 supergravity has 58 massless scalar fields which parametrise $\mathcal{M}$. These are the fluctuations of the metric on $K3$. Additionally, since $H^2(K3,\mathbf{R}) \cong \mathbf{R}^{22}$ there are twenty-two linearly independent classes of harmonic 2-forms. These may be used a la equation (32) to give a $U(1)^{22}$ gauge group in seven dimensions. We now go on to describe how this spectrum is the same as that of the heterotic string theory on $\mathbf{T}^3$, at generic points in $\mathcal{M}$.

The heterotic string in ten dimensions has a low energy description in terms of a supergravity theory whose massless bosonic fields are a metric, a 2-form B, a dilaton ϕ and non-Abelian gauge fields of structure group $SO(32)$ or $E_8\times E_8$. There are sixteen global supersymetries. Compactification on a flat $\mathbf{T}^3$ preserves all supersymmetries which are all products of constant spinors on both $\mathbf{T}^3$ and Minkowski space. A flat metric on $\mathbf{T}^3$ involves six parameters so the metric gives rise to six massless scalars. and since there are three independent harmonic two forms we obtain from B three more. The condition for the gauge fields to be supersymmetric on $\mathbf{T}^3$ is that their field strengths vanish: these are so called flat connections. They are parametrised by Wilson lines around the three independent circles in $\mathbf{T}^3$. These are representations of the fundamental group of $\mathbf{T}^3$ in the gauge group. Most of the flat connections actually arise from Wilson loops which are actually in the maximal torus of the gauge group, which in this case is $U(1)^{16}$. Clearly, this gives a 48 dimensional moduli space giving 58 scalars altogether. Narain showed that this moduli space is actually also locally the same form as $\mathcal{M}$ [14].

From the point of view of the heterotic string on $\mathbf{T}^3$, the effective gauge group in 7 dimensions (for generic metric and B-field) is the subgroup of $SO(32)$ or $E_8\times E_8$ which commutes with the flat connection on $\mathbf{T}^3$. At generic points in the moduli space of flat connections, this gauge group will be $U(1)^{16}$. This is because the generic flat connection defines three generic elements in $U(1)^{16} \subset G$. We can think of these as diagonal 16 by 16 matrices with all elements on the diagonal non-zero. Clearly, only the diagonal elements of G will commute with these. So, at a generic point in moduli space the gauge group is abelian.

Six more $U(1)$ gauge fields arise as follows from the metric and B-field. $\mathbf{T}^3$ has three harmonic one forms, so Kaluza-Klein reduction of B gives three gauge fields. Additionally, since $\mathbf{T}^3$ has a $U(1)^3$ group of isometrics, the metric gives three more. In fact, the local action for supergravity theories in seven dimensions are actually determined by the number of massless vectors. So, in summary, we have shown that at generic points in $\mathcal{M}$ the low energy supergravity theories arising from M theory on $K3$ or heterotic string on $\mathbf{T}^3$ are the same.

At special points, some of the eigenvalues of the flat connections will vanish. At these points the unbroken gauge group can get enhanced to a non-Abelian group. This is none other than the Higgs mechanism: the Higgs fields are just the Wilson lines. Additionally, because seven dimensional gauge theories are infrared trivial (the gauge coupling has dimension a positive power of length), the low energy quantum theory actually has a non-Abelian gauge symmetry.

If M theory on $K3$ is actually equivalent to the heterotic string in seven dimensions, it too should therefore exhibit non-Abelian symmetry enhancement at special points in the moduli space. These points are precisely the points in moduli space where the $K3$ develops orbifold singularities. We will not provide a detailed proof of this statement, but will instead look at the $K3$ moduli space in a neighbourhood of this singularity, where all the interesting behaviour of the theory is occuring. So, the first question is what do these orbifold singularities look like?

3.1.1 ADE-singularities.

An orbifold singularity in a Riemannian 4-manifold can locally be described as $\mathbb{R}^4/\Gamma$, where Γ is a finite subgroup of $SO(4)$. For generic enough Γ, the only singular point of this orbifold is the origin. These are the points in $\mathbb{R}^4$ left invariant under Γ. A very crucial point is that on the heterotic side, supersymmetry is completely unbroken all over the moduli space, so our orbifold singularities in $K3$ should also preserve supersymmetry. This means that Γ is a finite subgroup of $SU(2) \subset SO(4)$. The particular $SU(2)$ can easily be identified as follows. Choose some set of complex coordinates so that $\mathbb{C}^2 \equiv \mathbb{R}^4$. Then, a point in $\mathbb{C}^2$ is labelled by a 2-component vector. The $SU(2)$ in question acts on this vector in the standard way:

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{35}$$

The finite subgroups of $SU(2)$ have a classification which may be described in terms of the simply laced semi-simple Lie algebras: A_n, D_k, E_6, E_7 and E_8. There are two infinite series corresponding to $SU(n+1) = \mathsf{A}_n$ and $SO(2k) = \mathsf{D}_k$ and three exceptional subgroups corresponding to the three exceptional Lie groups of E-type. The subgroups, which we will denote by Γ_{A_n}, Γ_{D_k}, Γ_{E_l} can be described explicitly.

$\Gamma_{\mathsf{A}_{n-1}}$ is isomorphic to $\mathbb{Z}_n$ - the cyclic group of order n - and is generated by

$$\begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{\frac{-2\pi i}{n}} \end{pmatrix} \tag{36}$$

Γ_{D_k} is isomorphic to $\mathbb{D}_{k-2}$ - the binary dihedral group of order $4k-8$ - and has two generators α and β given by

$$\alpha = \begin{pmatrix} e^{\frac{\pi i}{k-2}} & 0 \\ 0 & e^{\frac{-\pi i}{k-2}} \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{37}$$

Γ_{E_6} is isomorphic to $\mathbb{T}$ - the binary tetrahedral group of order 24 - and has two generators given by

$$\begin{pmatrix} e^{\frac{\pi i}{2}} & 0 \\ 0 & e^{\frac{-\pi i}{2}} \end{pmatrix} \quad and \quad \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{2\pi i7}{8}} & e^{\frac{2\pi i7}{8}} \\ e^{\frac{2\pi i5}{8}} & e^{\frac{2\pi i}{8}} \end{pmatrix} \tag{38}$$

Γ_{E_7} is isomorphic to $\mathbb{O}$ - the binary octohedral group of order 48 - and has three generators. Two of these are the generators of $\mathbb{T}$ and the third is

$$\begin{pmatrix} e^{\frac{2\pi i}{8}} & 0 \\ 0 & e^{\frac{2\pi i7}{8}} \end{pmatrix} \tag{39}$$

Finally, Γ_{E_8} is isomorphic to $\mathbb{I}$ - the icosahedral group of order 120 - and has two generators given by

$$-\begin{pmatrix} e^{\frac{2\pi i3}{5}} & 0 \\ 0 & e^{\frac{2\pi i2}{5}} \end{pmatrix} \quad and \quad \frac{1}{e^{\frac{2\pi i2}{5}} - e^{\frac{2\pi i3}{5}}} \begin{pmatrix} e^{\frac{2\pi i}{5}} + e^{\frac{-2\pi i}{5}} & 1 \\ 1 & -e^{\frac{2\pi i}{5}} - e^{\frac{-2\pi i}{5}} \end{pmatrix} \tag{40}$$

Since all the physics of interest is happening near the orbifold singularities of $K3$, we can replace the $K3$ by $\mathbb{C}^2/\Gamma_{\mathsf{ADE}}$ and study the physics of M theory on $\mathbb{C}^2/\Gamma_{\mathsf{ADE}} \times \mathbb{R}^{6,1}$ near its singular set which is just $0 \times \mathbb{R}^{6,1}$. Since the $K3$ went from smooth to singular as we varied its moduli we expect that the singular orbifolds $\mathbb{C}^2/\Gamma_{\mathsf{ADE}}$ are singular limits of *non-compact* smooth 4-manifolds X^{ADE}. Because of supersymmetry, these should have $SU(2)$-holonomy. This is indeed the case. The metrics of $SU(2)$-holonomy on the X^{ADE} are known as ALE-spaces, since they asymptote to the locally Euclidean metric on $\mathbb{C}^2/\Gamma_{\mathsf{ADE}}$. Their existence was proven by Kronheimer [15] - who constructed a gauge theory whose Higgs branch is precisely the $\mathbb{C}^2/\Gamma_{\mathsf{ADE}}$ with its $SU(2)$-holonomy (or hyper-Kahler) metric.

A physical description of this gauge theory arises in string theory. Consider Type IIA or IIB string theory on $\mathbb{C}^2/\Gamma_{\mathsf{ADE}} \times \mathbb{R}^{5,1}$. Take a flat D$p$-brane (with $p \leq 5$) whose world-volume directions span $\mathbb{R}^{p,1} \subset \mathbb{R}^{5,1}$ ie the D-brane is sitting at a point on the orbifold. Then the world-volume gauge theory, which was first derived in [16]., is given by the Kronheimer gauge theory. This theory has eight supersymmetries which implies that its Higgs branch is a hyper-Kahler manifold. For one D-brane this theory has a gauge group which is a product of unitary groups of ranks given by the Dynkin indices (or dual Kac labels) of affine Dynkin diagram of the corresponding ADE-group. So, for the A_n-case the gauge group is $U(1)^{n+1}$. The matter content is also given by the affine Dynking diagram - each link between a pair of nodes represents a hyper-multiplet transforming in the bi-fundamental representation of the two unitary groups. This is an example of a quiver gauge theory - a gauge theory determined by a quiver diagram.

We will make this explicit in the simplest case of Γ_{A_1}. Γ_{A_1} is isomorphic to $\mathbb{Z}_2$ and is in fact the center of $SU(2)$. Its generator acts on $\mathbb{C}^2$ as

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} -u \\ -v \end{pmatrix} \tag{41}$$

We can parametrise $\mathbb{C}^2/\Gamma_{A_1}$ algebraically in terms of the Γ_{A_1} invariant coordinates on $\mathbb{C}^2$. These are u^2, v^2 and uv. Defining $x = u^2 - v^2$, $y = iu^2 + iv^2$ and $z = 2uv$. gives a map from $\mathbb{C}^2/\Gamma_{A_1}$ to $\mathbb{C}^3$. Clearly however,

$$x^2 + y^2 + z^2 = 0 \tag{42}$$

which means that $\mathbb{C}^2/\Gamma_{A_1}$ is the hypersurface in $\mathbb{C}^3$ defined by this equation.

The orbifold can be deformed by adding a small constant to the right hand side,

$$x^2 + y^2 + z^2 = r^2 \tag{43}$$

If we take x, y and z to all be real and r to be real then it is clear that the deformed 4-manifold contains a 2-sphere of radius r. This 2-sphere contracts to zero size as r goes to zero. The total space of the deformed 4-manifold is in fact the co-tangent bundle of the 2-sphere, $\mathbf{T^*S^2}$. To see this write the real parts of the x, y and z as x_i and their imaginary parts as p_i. Then, since r is real, the x_i are coordinates on the sphere which obey the relation

$$\Sigma_i x_i p_i = 0 \tag{44}$$

This means that the p_i's parametrise tangential directions. The radius r sphere in the center is then the zero section of the tangent bundle. Since the manifold is actually complex it is natural to think of this as the co-tangent bundle of the Riemann sphere, $\mathbf{T^*CP^1}$. In the context of Euclidean quantum gravity, Eguchi and Hanson constructed a metric of $SU(2)$-holonomy on this space, asymptotic to the locally flat metric on $\mathbb{C}^2/\Gamma_{A_1}$.

3.1.2 M theory Physics at The Singularity.

This metric, whose precise form we will not require actually has three parameters which control the size and shape of the two-sphere which desingularises the orbifold. From a distance it looks as though there is an orbifold singularity, but as one looks more closely one sees that the singularity has been smoothened out by a two-sphere. The 2-sphere is dual to a compactly supported harmonic 2-form, α. Thus, Kaluza-Klein reducing the C-field using α gives a $U(1)$ gauge field in seven dimensions. A vector multiplet in

seven dimensions contains precisely one gauge field and three scalars and the latter are the parameters of the $\mathbf{S^2}$. So, when $\mathbf{T^*CP^1}$ is smooth the massless spectrum is an abelian vector multiplet.

From the duality with the heterotic string we expect to see an enhancement in the gauge symmetry when we vary the scalars to zero ie when the sphere shrinks to zero size. In order for this to occur, $W^{\pm}$-bosons must become massless at the singularity. These are electrically charged under the $U(1)$ gauge field which originated from C. From the eleven dimensional point of view the object which is charged under C is the $M2$-brane. If the $M2$-brane wraps around the two-sphere, it appears as a particle from the seven dimensional point of view. This particle is electrically charged under the $U(1)$ and has a mass which is classically given by the volume of the sphere. Since, the $M2$-brane has tension its dynamics will push it to wrap the smallest volume two-sphere in the space. This least mass configuration is in fact invariant under half of the supersymmetries [4] - a fact which means that it lives in a short representation of the supersymmetry algebra. This in turn means that its classical mass is in fact uncorrected quantum mechanically. The $M2$-brane wrapped around this cycle with the opposing orientation has the opposing $U(1)$ charge to the previous one.

Thus, when the two-sphere shrinks to zero size we find two oppositely charged BPS multiplets become massless. These have precisely the right quantum numbers to enhance the gauge symmetry from $U(1)$ to $\mathsf{A}_1 = SU(2)$. Super Yang-Mills theory in seven dimensions depends only on its gauge group. In this case we are asserting that in the absence of gravity, the low energy physics of M theory on $\mathbb{C}^2/\Gamma_{A_1}\times\mathbb{R}^{6,1}$ is described by super Yang-Mills theory on $0\times\mathbb{R}^{6,1}$ with gauge group A_1.

The obvious generalisation also applies: in the absence of gravity, the low energy physics of M theory on $\mathbb{C}^2/\Gamma_{ADE}\times\mathbb{R}^{6,1}$ is described by super Yang-Mills theory on $0\times\mathbb{R}^{6,1}$ with ADE gauge group. To see this, note that the smoothing out of the orbifold singularity in $\mathbb{C}^2/\Gamma_{ADE}$ contains rank(ADE) two-spheres which intersect according to the Cartan matrix of the ADE group. At smooth points in the moduli space the gauge group is thus $U(1)^{\mathrm{rank(ADE)}}$. The corresponding wrapped membranes give rise to massive BPS multiplets with precisely the masses and quantum numbers required to enhance the gauge symmetry to the full ADE-group at the origin of the moduli space.

[4]This is because the least volume two-sphere is an example of a calibrated or supersymmetric cycle.

3.2 ADE-singularities in G_2-manifolds.

We have thus far restricted our attention to the ADE singularities in $K3\times\mathbb{R}^{6,1}$. However, the ADE singularity is a much more local concept. We can consider more complicated spacetimes $X^{10,1}$ with ADE singularities along more general seven-dimensional spacetimes, $Y^{6,1}$. Then, if X has a modulus which allows us to scale up the volume of Y, the large volume limit is a semi-classical limit in which X approaches the previous maximally symmetric situation discussed above. Thus, for large enough volumes we can assert that the description of the classical physics of M theory near Y is in terms of seven dimensional super Yang-Mills theory on Y - again with gauge group determined by which ADE singularity lives along Y.

In the context of G_2-compactification on $X\times\mathbb{R}^{3,1}$, we want Y to be of the form $W\times\mathbb{R}^{3,1}$, with W the locus of ADE singularities inside X. Near $W\times\mathbb{R}^{3,1}$, $X\times\mathbb{R}^{3,1}$ looks like $\mathbb{C}^2/\Gamma_{ADE}\times W\times\mathbb{R}^{3,1}$. In order to study the gauge theory dynamics without gravity, we can again focus on the physics near the singularity itself. So, we want to focus on seven-dimensional super Yang-Mills theory on $W\times\mathbb{R}^{3,1}$.

3.2.1 M theory Spectrum Near The Singularity

In flat space the super Yang-Mills theory has a global symmetry group which is $SO(3)\times SO(6,1)$. The second factor is the Lorentz group, the first is the R-symmetry. The theory has gauge fields transforming as $(\mathbf{1},\mathbf{7})$, scalars in the $(\mathbf{3},\mathbf{1})$ and fermions in the $(\mathbf{2},\mathbf{8})$ of the universal cover. All fields transform in the adjoint representation of the gauge group. Moreover the sixteen supersymmetries also transform as $(\mathbf{2},\mathbf{8})$.

On $W\times\mathbb{R}^{3,1}$ - with an arbitrary W. the symmetry group gets broken to $SO(3)\times SO(3)'\times SO(3,1)$. Since $SO(3)'$ is the structure group of the tangent bundle on W, covariance requires that the theory is coupled to a background $SO(3)'$ gauge field - the spin connection on W. Similarly, though perhaps less intuitively, $SO(3)$ acts on the normal bundle to W inside X, hence there is a background $SO(3)$ gauge field also.

The supersymmetries transform as $(\mathbf{2},\mathbf{2},\mathbf{2})+(\mathbf{2},\mathbf{2},\bar{\mathbf{2}})$. For large enough W and at energy scales below the inverse size of W, we can describe the physics in terms of a *four* dimensional gauge theory. But this theory as we have described it is not supersymmetric as this requires that we have covariantly constant spinors on W. Because W is curved, there are none. However, we actually want to consider the case in which W is embedded inside a G_2-manifold X. In other words we require that our local model - $\mathbb{C}^2/\Gamma_{ADE}\times W$ - admits a G_2-holonomy metric. When W is curved this metric

cannot be the product of the locally flat metric on $\mathbb{C}^2/\Gamma_{ADE}$ and a metric on W. Instead the metric is warped and is more like the metric on a fiber bundle in which the metric on $\mathbb{C}^2$ varies as we move around in W. Since the space has G_2-holonomy we should expect the four dimensional gauge theory to be supersymmetric. We will now demonstrate that this is indeed the case by examining the G_2-structure more closely. In order to do this however, we need to examine the $SU(2)$ structure on $\mathbb{C}^2/\Gamma$ as well.

4-dimensional spaces of $SU(2)$-holonomy are actually examples of hyper-Kahler manifolds. They admit three parallel 2-forms ω_i. These are analogous to the parallel forms on G_2-manifolds. These three forms transform locally under $SO(3)$ which locally rotates the complex structures. On $\mathbb{C}^2$ these forms can be given explicitly as

$$\omega_1+i\omega_2 = du\wedge dv \tag{45}$$

$$\omega_3 = \frac{i}{2}du\wedge d\bar{u}+dv\wedge d\bar{v} \tag{46}$$

Γ_{ADE} is defined so that it preserves all three of these forms. The $SO(3)$ which rotates these three forms is identified with the $SO(3)$ factor in our seven dimensional gauge theory picture. This is because the moduli space of $SU(2)$-holonomy metrics is the moduli space of the gauge theory and this has an action of $SO(3)$.

In a locally flat frame we can write down a formula for the G_2-structure on $\mathbb{C}^2/\Gamma_{ADE}\times W$,

$$\varphi=\frac{1}{6}\omega_i\wedge e_j\delta^{ij}+e_1\wedge e_2\wedge e_3 \tag{47}$$

where e_i are a flat frame on W. Note that this formula is manifestly invariant under the $SO(3)$ which rotates the w_i *provided* that it also acts on the e_i in the same way.

The key point is that when the $SO(3)$ of the gauge theory acts, in order for the G_2-structure to be well defined , the e_i's must transform in precisely the same way as the ω_i. But $SO(3)'$ acts on the e_i, because it is the structure group of the tangent bundle to W. Therefore, if $\mathbb{C}^2/\Gamma_{ADE}\times W$, admits a G_2-holonomy metric, we must identify $SO(3)$ with $SO(3)'$. In other words, the connection on the tangent bundle is identified with the connection on the normal $SO(3)'$ bundle. This breaks the symmetries to the diagonal subgroup of the two $SO(3)$'s and implies that the effective four dimensional field theory is classically supersymmetric. Identifying the two groups breaks the symmetry group down to $SO(3)''\times SO(3,1)$ under which the supercharges transform as $(\mathbf{1},\mathbf{2})+(\mathbf{3},\mathbf{2})+\mathbf{cc}$. We now have supersymmetries since the $(\mathbf{1},\mathbf{2})$ and its conjugate can be taken to be constants on W.

An important point which we will not actually prove here, but will require in the sequel is that the locus of ADE-singularities - namely the copy of W at the center of $\mathbb{C}^2/\Gamma$ is actually a supersymmetric cycle (also known as a calibrated cycle). This follows essentially from the fact that Γ_{ADE} fixes W and therefore the φ restricts to be the volume form on W. This is the condition for W to be supersymmetric.

Supposing we could find a G_2-manifold of this type, what exactly is the four dimensional supersymmetric gauge theory it corresponds to? This we can answer also by Kaluza-Klein analysis [4, 5], since W will be assumed to be smooth and 'large'. Under $SO(3)''\times SO(3,1)$, the seven dimensional gauge fields transform as $(\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{4})$, the three scalars give $(\mathbf{3},\mathbf{1})$ and the fermions give $(\mathbf{1},\mathbf{2}) + (\mathbf{3},\mathbf{2}) + \mathbf{cc}$. Thus the fields which are scalars under the four dimensional Lorentz group are two copies of the $\mathbf{3}$ of $SO(3)''$. These may be interpreted as two one forms on W. These will be massless if they are zero modes of the Laplacian on W (wrt its induced metric from the G_2-manifold). There will be precisely $b_1(W)$ of these ie one for every harmonic one form. Their superpartners are clearly the $(\mathbf{3},\mathbf{2}) + \mathbf{cc}$ fermions, which will be massless by supersymmetry. This is precisely the field content of $b_1(W)$ chiral supermultiplets of the supersymmetry algebra in four dimensions.

The $(\mathbf{1},\mathbf{4})$ field is massless if it is constant on W and this gives one gauge field in four dimensions. The requisite superpartners are the remaining fermions which transform as $(\mathbf{1},\mathbf{2}) + \mathbf{cc}$.

All of these fields transform in the adjoint representation of the seven dimensional gauge group. Thus the final answer for the massless fields is that they are described by $\mathcal{N} = 1$ super Yang-Mills theory with $b_1(W)$ massless adjoint chiral supermultiplets. The case with pure "superglue" ie $b_1(W) = 0$ is a particularly interesting gauge theory at the quantum level: in the infrared the theory is believed to confine colour, undergo chiral symmetry breaking and have a mass gap. We will actually exhibit some of these very interesting properties semi-classically in M theory ! Much of the sequel will be devoted to explaining this. But before we can do that we must first describe concrete examples of G_2-manifolds with the properties we desire.

One idea is to simply look for *smooth* G_2-manifolds which are topologically $\mathbb{C}^2\times W$ but admit an action of $SU(2)$ which leaves W invariant but acts of $\mathbb{C}^2$ in the natural way. Then we simply pick a $\Gamma_{ADE} \subset SU(2)$ and form the quotient space $\mathbb{C}^2/\Gamma_{ADE}\times W$.

Luckily, such non-compact G_2-manifolds were constructed some time ago [17] !! Moreover, in these examples, $W = S^3$, the simplest possible compact 3-manifold with $b_1(W) = 0$. Perfect.

3.2.2. Examples of G_2-manifolds with ADE-singularities.

The G_2-holonomy metrics on $\mathbb{C}^2\times S^3$ were constructed in [17]. These metrics are smooth and complete and depend on one parameter a. There is a radial coordinate r and the metric shows that there is a finite size S^3 (of size a) in the center which grows as we move out along the r direction. They asymptote at infinity in a radial coordinate r to a conical form

$$ds^2 = dr^2 + r^2 d\Sigma^2 \tag{48}$$

where $d\Sigma^2$ is an Einstein metric on $S^3\times S^3$. Taking r large is equivalent to taking a to zero, so the finite volume S^3 shrinks to zero size. This Einstein metric $d\Sigma^2$ is not the standard product metric on the product of two spheres, although it is the homogeneous metric on G/H with $G = SU(2)^3$ and $H = SU(2)$. H acts on G as the 'diagonal' subgroup of the three $SU(2)$'s. G/H is obviously isomorphic as a manifold to $S^3\times S^3$ since S^3 is isomorphic to $SU(2)$. This description of the conical G_2-metric obviously has an asymptotic $SU(2)^3\times\Sigma_3$ group of isometries with Σ_3 the group of permutations of the three $SU(2)$ factors.

The conical metric is obviously incomplete, since the base of the cone goes to zero size at $r = 0$. The complete G_2 metrics can be thus regarded as completions of the cones obtained by smoothing out the singularity at its apex. Topologically the conical manifold is $\mathbb{R}^+\times S^3\times S^3$, which gets smoothened out to $\mathbb{R}^4\times S^3$. Concretely this amounts to choosing an $SU(2)$ factor in G and 'filling it in' to form $\mathbb{R}^4$. We remind the reader that $\mathbb{R}^4 - 0$ is the same as $\mathbb{R}^+\times S^3$ and filling in the origin gives back $\mathbb{R}^4$.

Clearly there are three natural ways to to carry out this procedure since G consists of three copies of $SU(2)$. Obviously each of these gives the same topological manifold but it is very important for what follows that we realise that there are actually three G_2 manifolds that we can make this way. The point is that the classical moduli space of G_2-holonomy metrics consists of three real lines which intersect at one point - the conical singularity. Moving off of the conical manifold in the three different directions amounts to choosing an S^3 in G and filling it in. Along these three directions three different S^3's develop a finite volume. Another way to say more or less the same thing in a perhaps more physical way is that there are three smooth G_2-manifolds with the prescribed behaviour at infinity: the metric on G/H.

We can give a simple algebraic description of the phenomenon of collapsing one sphere and growing another with the following model taken from [6]. Consider the hypersurface in $\mathbb{R}^4\times\mathbb{R}^4$ cut out by the following equation

$$\Sigma_i x_i^2 - y_i^2 = a \tag{49}$$

where the x's and y' s are linear coordinates on the two $\mathbb{R}^4$'s. For a positive, we have a radius a S^3 at the origin in the second $\mathbb{R}^4$. This manifold is clearly topologically $S^3\times\mathbb{R}^4$. For a negative, its again the same manifold topologically, but the roles of x and y have been interchanged. Therefore as a passes from positive to negative an S^3 shrinks to zero volume, the manifold becomes singular at 0 and *another* S^3 grows and the space remains smooth. This is obviously a much cruder description of the space as a function of a, since as we have seen above there are actually three directions in the moduli space and not two, but it has the advantage that it makes the basic picture transparent.

3.2.3 M theory Physics on $X = \mathbb{R}^4\times S^3$

We saw earlier that the moduli of the G_2-metric get complexified in M theory by the addition of the C-field. This is necessary for supersymmetry. We observed that on a compact G_2-manifold the low energy four dimensional theory contains one massless scalar for every parameter in the G_2-metric. The situation on a non-compact manifold X can in general be quite different, since the metric fluctuations need not be localised on X. The more delocalised these fields are, the more difficult it is to excite them. Indeed to obtain a four dimensional action we have to integrate over X, and this integral will diverge if the fluctuations are not L^2-normalisable. If this is indeed the case then we should not regard the corresponding four dimensional fields as fluctuating: rather they are background parameters, coupling constants and we should study the four dimensional physics as a function of them. We refer the reader to [7] for the simple calculation which shows this explicitly.

In the case at hand, by examining the first order fluctation in the G_2-metric one can readily see that a should indeed be treated as a coupling constant. It follows from supersymmetry that its complex partner should also. Our formula (32) can now be applied to write this complex coupling constant as

$$\tau = \int_{S^3} \varphi + iC = Vol(S^3) + i\int C \tag{50}$$

where we integrate over the minimal volume three sphere in the center of X. This sphere generates the third homology. Note that there is no prime here, since the field is not flucutating.

So, we arrive at the conclusion that M theory on our G_2-manifold X, is actually a one complex dimensional family of theories parameterised by τ. There are three semi-classical regimes corresponding to the three regions in which the spheres are large, X is smooth and thus supergravity is valid. The four dimensional spectrum is massive in each of these semi-classical regimes

since $b_2(X)$ is zero and the zero mode of the Lichnerowicz operator does not fluctuate. The physics in each of these three regions is clearly the same since the metrics are the 'same'. What about the physics on the bulk of the τ-plane?

Our intuition asserts that there is only one further interesting point, $\tau = 0$ where the manifold develops a conical singularity. In minimally supersymmetric field theories in four dimensions which do have massless complex scalars which parameterise a moduli space M, singularities in the physics typically only occur at subloci in M which are also complex manifolds. In our case, we dont have a moduli space, but rather a parameter space, but we can think of τ as a background superfield.

In any case at $\tau = 0$, we have zero size S^3's and instanton effects can become important here, since the action of an $M2$-brane instanton is τ. These effects could generate a non-zero quantum value of the C-field period and remove this potential singularity in which case we would be in the situation that there are no physical phase transitions as a function of τ. Of course, physical quantities will depend on τ, but the qualitative nature of the physics will remain the same for any value of τ.

This was first suggested in [6] and proven rigorously in [7].

4. Super Yang-Mills from G_2-manifolds: Quantum.

We now move on to study the interplay between quantum super Yang-Mills theory and M theory on G_2-manifolds. The results of this section are based upon [5, 6, 7, 8]. We will be studying the physics of M theory on the G_2-manifolds with ADE-singularities whose construction we described at the end of section 3.2.1. We begin by reviewing the basic properties of super Yang-Mills theory. We then go on to describe how these features are reflected in M theory. We first show how membrane instantons can be seen to generate the superpotential of the theory. Second we go on to exhibit confinement and a mass gap semi-classically in M theory.

The G_2-manifolds that actually interest us are obtained as quotients of $\mathbb{R}^4 \times S^3$ by Γ_{ADE}. We saw previously that for large volume and low energies, four dimensional super Yang-Mills theory is a good description of the M theory physics. We will thus begin this section with a review of the basic properties of the gauge theory.

4.1 Super Yang-Mills Theory.

For completeness and in order to compare easily with M theory results obtained later we briefly give a review of $\mathcal{N}=1$ pure super Yang-Mills theory. We begin with gauge group $SU(n)$. $\mathcal{N}=1$ $SU(n)$ super Yang-Mills theory in four dimensions is an extensively studied quantum field theory. The classical Lagrangian for the theory is

$$\mathcal{L} = -\frac{1}{4g^2}(F^a_{\mu\nu})^2 + \frac{1}{g^2}\bar{\lambda}^a i \not{D} \lambda^a + i\frac{\theta}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \quad (51)$$

F is the gauge field strength and λ is the gaugino field.

It is widely believed that this theory exhibits dynamics very similar to those of ordinary QCD: confinement, chiral symmetry breaking, a mass gap. There are n supersymmetric vacua. Supersymmetry constrains the dynamics of the theory so strongly, that the values of the low energy effective superpotential in the n vacua is known. These are of the form

$$W_{eff} = c\mu^3 e^{2\pi i\tau/n} \quad (52)$$

here τ is the complex coupling constant,

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \quad (53)$$

and μ the mass scale. Shifting θ by 2π gives n different values for W.

In particular, the form of this potential suggests that it is generated by dynamics associated with "fractional instantons", ie instantonic objects in the theory whose quantum numbers are formally of instanton number $\frac{1}{n}$. Such states are also closely related to the spontaneously broken chiral symmetry of the theory. Let us briefly also review some of these issues here.

Under the $U(1)$ R-symmetry of the theory, the gauginos transform as

$$\lambda \to e^{i\alpha}\lambda \quad (54)$$

This is a symmetry of the classical action but not of the quantum theory (as can easily be seen by considering the transformation of the fermion determinant in the path integral). However, if the above transformation is *combined* with a shift in the theta angle of the form

$$\tau \to \tau + \frac{2n}{2\pi}\alpha \quad (55)$$

then this cancels the change in the path integral measure. This shift symmetry is a bona fide symmetry of the physics if $\alpha = \frac{2\pi}{2n}$, so that even in the quantum theory a $\mathbf{Z}_{2n}$ symmetry remains. Associated with this symmetry is the presence of a non-zero value for the following correlation function,

$$\langle \lambda\lambda(x_1)\lambda\lambda(x_2)...\lambda\lambda(x_n)\rangle \quad (56)$$

which is clearly invariant under the $\mathbf{Z}_{2n}$ symmetry. This correlation function is generated in the 1-instanton sector and the fact that $2n$ gauginos enter is due to the fact that an instanton of charge 1 generates $2n$ chiral fermion zero modes.

Cluster decomposition implies that the above correlation function decomposes into 'n constituents' and therefore there exists a non-zero value for the gaugino condensate:

$$\langle \lambda\lambda \rangle \neq 0 \quad (57)$$

Such a non-zero expectation value is only invariant under a $\mathbf{Z}_2$ subgroup of $\mathbf{Z}_{2n}$ implying that the discrete chiral symmetry has been spontaneously broken. Consequently this implies the existence of n vacua in the theory.

In fact, it can be shown that

$$\langle \lambda\lambda \rangle = 16\pi i \frac{\partial}{\partial\tau} W_{eff} = -\frac{32\pi^2}{n} c\mu^3 e^{2\pi i\tau/n} \quad (58)$$

In view of the above facts it is certainly tempting to propose that 'fractional instantons' generate the non-zero gaugino condensate directly. But this is difficult to see directly in super Yang-Mills on $\mathbb{R}^{3,1}$. We will return to this point later.

More generally, if we replace the $SU(n)$ gauge group by some other gauge group H, then the above statements are also correct but with n replaced *everywhere* with $c_2(H)$ the dual Coxeter number of H. For A-D-E gauge groups $c_2(H) = \Sigma_{i=1}^{r+1} a_i$, where r is the rank of the gauge group and the a_i are the Dynkin indices of the affine Dynkin diagram associated to H. For A_n, all the $a_i = 1$; for D_n groups the four 'outer' nodes have index 1 whilst the rest have $a_i = 2$. E_6 has indices $(1,1,1,2,2,2,3)$, E_7 has $(1,1,2,2,2,3,3,4)$ whilst E_8 has indices $(1,2,2,3,3,4,4,5,6)$.

4.2. Theta angle and Coupling Constant in M theory.

The physics of M theory supported near the singularities of $\mathbb{C}^2/\Gamma \times \mathbb{R}^{6,1}$ is described by super Yang-Mills theory on $\mathbb{R}^{6,1}$. The gauge coupling constant of the theory is given by

$$\frac{1}{g_{7d}^2} \sim \frac{1}{l_p^3} \tag{59}$$

where l_p is the eleven dimensional Planck length. In seven dimensions, one analog of the theta angle in four dimensions is actually a three-form Θ. The reason for this is the seven dimensional interaction

$$L_I \sim \Theta \wedge F \wedge F \tag{60}$$

(with F the Yang-Mills field strength). In M theory Θ is given by C, the three-form potential for the theory.

If we now take M theory on our G_2-manifold $\mathbb{R}^4/\Gamma \times W$ we have essentially compactified the seven dimensional theory on W and the four dimensional gauge coupling constant is roughly given by

$$\frac{1}{g_{4d}^2} \sim \frac{V_W}{l_p^3} \tag{61}$$

where V_W is the volume of W. The four dimensional theta angle can be identified as

$$\theta = \int_W C \tag{62}$$

The above equation is correct because under a global gauge transformation of C which shifts the above period by 2π times an integer - a transformation which is a symmetry of M theory - θ changes by 2π times an integer. Such shifts in the theta angle are also global symmetries of the field theory.

Thus the complex gauge coupling constant of the effective four dimensional theory may be identified as the τ parameter of M theory

$$\tau = \int_W i\frac{C}{2\pi} + \frac{\varphi}{l_p^3} \tag{63}$$

This is of course entirely natural, since τ is the only parameter in M theory on this space!

4.3. Superpotential in M theory.

There is a very elegant way to calculate the superpotential of super Yang-Mills theory on $\mathbb{R}^{3,1}$ by first compactifying it on a circle to three dimensions [18]. The three dimensional theory has a perturbative expansion since the Wilson lines on the circle behave as Higgs fields whose vev's break the gauge symmetry to the maximal torus. The theory has a perturbative expansion in the Higss vevs, which can be used to compute the superpotential of the compactified theory. One then takes the four dimensional limit. In order to compute the field theory superpotential we will mimick this idea in M theory [4]. Compactifying the theory on a small circle is equivalent to studying pertubative Type IIA string theory on our G_2-manifold.

4.3.1. Type IIA theory on $X = \mathbb{R}^4/\Gamma_{ADE} \times S^3$.

Consisder Type IIA string theory compactified to three dimensions on a seven manifold X with holonomy G_2. If X is smooth we can determine the massless spectrum of the effective supergravity theory in three dimensions as follows. Compactification on X preserves four of the 32 supersymmetries in ten dimensions, so the supergravity theory has three dimensional $\mathcal{N} = 2$ local supersymmetry. The relevant bosonic fields of the ten dimensional supergravity theory are the metric, B-field, dilaton plus the Ramond-Ramond one- and three-forms. These we will denote by g, B, ϕ, A_1, A_3 respectively. Upon Kaluza-Klein reduction the metric gives rise to a three-metric and $b_3(X)$ massless scalars. The latter parametrise the moduli space of G_2-holonomy metrics on X. B gives rise to $b_2(X)$ periodic scalars φ_i. ϕ gives a three dimensional dilaton. A_1 reduces to a massless vector, while A_3 gives $b_2(X)$ vectors and $b_3(X)$ massless scalars. In three dimensions a vector is dual to a periodic scalar, so at a point in moduli space where the vectors are free we can dualise them. The dual of the vector field originating from A_1 is the period of the RR 7-form on X, whereas the duals of the vector fields coming from A_3 are given by the periods of the RR 5-form A_5 over a basis of 5-cycles which span the fifth homology group of X. Denote these by scalars by σ_i. All in all, in the dualised theory we have in addition to the supergravity multiplet, $b_2(X) + b_3(X)$ scalar multiplets. Notice that $b_2(X)$ of the scalar multiplets contain two real scalar fields, both of which are periodic.

Now we come to studying the Type IIA theory on $X = \mathbb{R}^4/\Gamma_{ADE} \times S^3$. Recall that $X = \mathbb{R}^4/\Gamma_{ADE} \times S^3$ is defined as an orbifold of the standard spin bundle of S^3. To determine the massless spectrum of IIA string theory on X

we can use standard orbifold techniques. However, the answer can be phrased in a simple way. X is topologically $\mathbb{R}^4/\Gamma_{ADE}\times S^3$. This manifold can be desingularised to give a smooth seven manifold $M^{\Gamma_{ADE}}$ which is topologically $X^{\Gamma_{ADE}}\times S^3$, where $X^{\Gamma_{ADE}}$ is homeomorphic to an ALE space. The string theoretic cohomology groups of X are isomorphic to the usual cohomology groups of $M^{\Gamma_{ADE}}$. The reason for this is simple: X is a global orbifold of $S(S^3)$. The string theoretic cohomology groups count massless string states in the orbifold CFT. The massless string states in the twisted sectors are localised near the fixed points of the action of Γ_{ADE} on the spin bundle. Near the fixed points we can work on the tangent space of $S(S^3)$ near a fixed point and the action of Γ_{ADE} there is just its natural action on $\mathbb{R}^4\times\mathbb{R}^3$.

Note that blowing up X to give $M^{\Gamma_{ADE}}$ cannot give a metric with G_2-holonomy which is continuosly connected to the singular G_2-holonomy metric on X, since this would require that the addition to homology in passing from X to $M^{\Gamma_{ADE}}$ receives contributions from four-cycles. This is necessary since these are dual to elements of $H^3(M)$ which generate metric deformations preserving the G_2-structure. This argument does not rule out the possibility that $M^{\Gamma_{ADE}}$ admits 'disconnected' G_2-holonomy metrics, but is consistent with the fact that pure super Yang-Mills theory in four dimensions does not have a Coulomb branch.

The important points to note are that the twisted sectors contain massless states consisting of r scalars and r vectors where r is the rank of the corresponding ADE group associated to Γ. The r scalars can intuitively thought of as the periods of the B field through r two cycles. In fact, for a generic point in the moduli space of the orbifold conformal field theory the spectrum contains massive particles charged under the r twisted vectors. These can be interpreted as wrapped D2-branes whose quantum numbers are precisely those of W-bosons associated with the breaking of an ADE gauge group to $U(1)^r$. This confirms our interpretation of the origin of this model from M theory: the values of the r B-field scalars can be interpreted as the expectation values of Wilson lines around the eleventh dimension associated with this symmetry breaking. At weak string coupling and large S^3 volume these states are very massive and the extreme low energy effective dynamics of the twisted sector states is described by $\mathcal{N}=2$ $U(1)^r$ super Yang-Mills in three dimensions. Clearly however, the underlying conformal field theory is not valid when the W-bosons become massless. The appropriate description is then the pure super Yang-Mills theory on $\mathbb{R}^{2,1}\times S^1$ which corresponds to a sector of M theory on $X\times S^1$. In this section however, our strategy will be to work at a generic point in the CFT moduli space which corresponds to being far out along the Coulomb branch of the gauge theory. We will attempt to calculate the superpotential there and then continue the result to four dimensions. This exactly mimics the strategy of [18] in field theory. Note that we are implicitly ignoring gravity here. More precisely, we are assuming that in the absence of gravitational interactions with the twisted sector, the low energy physics of the twisted sectors of the CFT is described by the Coulomb branch of the gauge theory. This is natural since the twisted sector states are localised at the singularities of $J\times\mathbb{R}^{2,1}$ whereas the gravity propagates in bulk.

In this approximation, we can dualise the photons to obtain a theory of r chiral multiplets, each of whose bosonic components (φ and σ) is periodic. But remembering that this theory arose from a non-Abelian one we learn that the moduli space of classical vacua is

$$\mathcal{M}_{cl} = \frac{\mathbb{C}^r}{\Lambda_W^{\mathbb{C}} \rtimes W_g} \tag{64}$$

where $\Lambda_W^{\mathbb{C}}$ is the complexified weight lattice of the ADE group and W_g is the Weyl group.

We can now ask about quantum effects. In particular is there a non-trivial superpotential for these chiral multiplets? In a theory with four supercharges BPS instantons with only two chiral fermion zero modes can generate a superpotential. Are there instantons in Type IIA theory on J ? BPS instantons come from branes wrapping supersymmetric cycles and Type IIA theory on a G_2-holonomy space can have instantons corresponding to D6-branes wrapping the space itself or D2-brane instantons which wrap supersymmetric 3-cycles. For smooth G_2-holonomy manifolds these were studied in [19]. In the case at hand the D6-branes would generate a superpotential for the dual of the graviphoton multiplet which lives in the gravity multiplet but since we wish to ignore gravitational physics for the moment, we will ignore these. In any case, since X is non-compact, these configurations have infinite action. The D2-branes on the other hand are much more interesting. They can wrap the supersymmetric S^3 over which the singularities of X are fibered. We can describe the dynamics of a wrapped D2-brane as follows. At large volume, where the sphere becomes flatter and flatter the world-volume action is just the so called 'quiver gauge theory' described in [16]. Here we should describe this theory not just on S^3 but on a supersymmetric S^3 embedded in a space with a non-trivial G_2-holonomy structure. The upshot is that the world-volume theory is in fact a cohomological field theory [20] so we can trust it for any volume as long as the ambient space has G_2-holonomy. This is because the supersymmetries on the world-volume are actually scalars on S^3 and so must square to zero.

Note that, since we are ignoring gravity, we are implicitly ignoring higher derivative corrections which could potentially also affect this claim. Another crucial point is that the S^3 which sits at the origin in $\mathbb{R}^4$ in the covering space of X is the supersymmetric cycle, and the spheres away from the origin are not supersymmetric, so that the BPS wrapped D2-brane is constrained to live on the singularities of X. In the quiver gauge theory, the origin is precisely the locus in moduli space at which the single D2-brane can fractionate (according to the quiver diagram) and this occurs by giving expectation values to the scalar fields which parametrise the Coulomb branch which corresponds to the position of our D2-brane in the dimensions normal to X.

What contribution to the superpotential do the fractional D2-branes make? To answer this we need to identify the configurations which possess only two fermionic zero modes. We will not give a precise string theory argument for this, but using the correspondence between this string theory and field theory will identify exactly which D-brane instantons we think are responsible for generating the superpotential. This may sound like a strong assumption, but as we hope will become clear, the fact that the fractional D2-branes are wrapped D4-branes is actually anticipated by the field theory! This makes this assumption, in our opinion, somewhat weaker and adds credence to the overall picture being presented here.

In [21], it was shown that the fractionally charged D2-branes are actually D4-branes which wrap the 'vanishing' 2-cycles at the origin in $\mathbb{R}^4/\Gamma$. More precisely, each individual fractional D2-brane, which originates from a single D2-brane possesses D4-brane charge, but the total configuration, since it began life as a single D2-brane has zero D4-brane charge. The possible contributions to the superpotential are constrained by supersymmetry and must be given by a holomorphic function of the r chiral superfields and also of the holomorphic gauge coupling constant τ which corresponds to the complexified volume of the S^3 in eleven dimensional M theory. We have identified above the bosonic components of the chiral superfields above. τ is given by

$$\tau = \int \varphi + iC \tag{65}$$

where φ is the G_2-structure defining 3-form on X. The period of the M theory 3-form through S^3 plays the role of the theta angle.

The world-volume action of a D4-brane contains the couplings

$$L = B \wedge A_3 + A_5 \tag{66}$$

Holomorphy dictates that there is also a term

$$B \wedge \varphi \tag{67}$$

so that the combined terms are written as

$$B \wedge \tau + A_5 \tag{68}$$

Since the fourbranes wrap the 'vanishing cycles' and the S^3 we see that the contribution of the D4-brane corresponding to the k-th fractional D2 takes the form

$$S = -\boldsymbol{\beta}_k . \boldsymbol{z} \tag{69}$$

where we have defined

$$\boldsymbol{z} = \tau \boldsymbol{\varphi} + \boldsymbol{\sigma} \tag{70}$$

and the $\boldsymbol{\beta}_k$ are charge vectors. The r complex fields $\boldsymbol{z}$ are the natural holomorphic functions upon which the superpotential will depend.

The wrapped D4-branes are the magnetic duals of the massive D2-branes which we identified above as massive W-bosons. As such they are magnetic monopoles for the original ADE gauge symmetry. Their charges are therefore given by an element of the co-root lattice of the Lie algebra and thus each of the $r+1$ $\boldsymbol{\beta}$'s is a rank r vector in this space. Choosing a basis for this space corresponds to choosing a basis for the massless states in the twisted sector Hilbert space which intuitively we can think of as a basis for the cohomology groups Poincare dual to the 'vanishing' 2-cycles. A natural basis is provided by the simple co-roots of the Lie algebra of ADE, which we denote by $\boldsymbol{\alpha}_k^*$ for $k = 1, ..., r$. This choice is natural, since these, from the field theory point of view are the fundamental monopole charges.

At this point it is useful to mention that the r wrapped D4-branes whose magnetic charges are given by the simple co-roots of the Lie algebra correspond in field theory to monopoles with charges $\boldsymbol{\alpha}_k^*$ and each of these is known to possess precisely the right number of zero modes to contribute to the superpotential. Since we have argued that in a limit of the Type IIA theory on X, the dynamics at low energies is governed by the field theory studied in [18] it is natural to expect that these wrapped fourbranes also contribute to the superpotential. Another striking feature of the field theory is that these monopoles also possess a fractional instanton number - the second Chern number of the gauge field on $\mathbb{R}^{2,1} \times S^1$. These are precisely in correspondence with the fractional D2-brane charges. Thus, in this sense, the field theory anticipates that fractional branes are wrapped branes.

In the field theory on $\mathbb{R}^{2,1} \times S^1$ it is also important to realise that there is precisely one additional BPS state which contributes to the superpotential. The key point is that this state, unlike the previously discussed monopoles have dependence on the periodic direction in spacetime. This state is associated with the affine node of the Dynkin diagram. Its monopole charge is given by

$$-\Sigma_{k=1}^{r} \boldsymbol{\alpha}_k^* \tag{71}$$

and it also carries one unit of instanton number.

The action for this state is

$$S = \Sigma_{k=1}^{r} \alpha_k^{*}.z - 2\pi i\tau \tag{72}$$

Together, these $r+1$ BPS states can be regarded as fundamental in the sense that all the other finite action BPS configurations can be thought of as bound states of them.

Thus, in the correspondence with string theory it is also natural in the same sense as alluded to above that a state with these corresponding quantum numbers also contributes to the superpotential. It may be regarded as a bound state of anti-D4-branes with a charge one D2-brane. In the case of $SU(n)$ this is extremely natural, since the total D4/D2-brane charge of the $r+1$ states is zero/one, and this is precisely the charge of the D2-brane configuration on S^3 whose world-volume action is the quiver gauge theory for the affine Dynkin diagram for $SU(n)$. In other words, the entire superpotential is generated by a single D2-brane which has fractionated.

In summary, we have seen that the correspondence between the Type IIA string theory on X and the super Yang-Mills theory on $\mathbb{R}^{2,1}\times S^1$ is quite striking. Within the context of this correspondence we considered a smooth point in the moduli space of the perturbative Type IIA CFT, where the spectrum matches that of the Yang-Mills theory along its Coulomb branch. On the string theory side we concluded that the possible instanton contributions to the superpotential are from wrapped D2-branes. Their world volume theory is essentially topological, from which we concluded that they can fractionate. As is well known, the fractional D2-branes are really wrapped fourbranes. In the correspondence with field theory, the wrapped fourbranes are magnetic monopoles, whereas the D2-branes are instantons. Thus if, these branes generate a superpotential they correspond, in field theory to monopole-instantons. This is exactly how the field theory potential is known to be generated. We thus expect that the same occurs in the string theory on X.

Finally, the superpotential generated by these instantons is of affine-Toda type and is known to possess $c_2(\mathsf{ADE})$ minima corresponding to the vacua of the ADE super Yang-Mills theory on $\mathbb{R}^{3,1}$. The value of the superpotential in each of these vacua is of the form $e^{\frac{2\pi i\tau}{c_2}}$. As such it formally looks as though it was generated by fractional instantons, and in this context fractional $M2$-brane instantons. This result holds in the four dimensional M theory limit because of holomorphy.

Let us demonstrate the vacuum structure in the simple case when the gauge group is $SU(2)$. Then there is only one scalar field, z. There are two

fractional D2-brane instantons whose actions are

$$S_1 = -z \quad and \quad S_2 = z - 2\pi i\tau \tag{73}$$

Both of these contribute to the superpotential as

$$W = e^{-S_1} + e^{-S_2} \tag{74}$$

Defining $z = lnY$ we have

$$W = Y + \frac{e^{2\pi i\tau}}{Y} \tag{75}$$

The critical points of W are

$$Y = \pm e^{\frac{2\pi i\tau}{2}} \tag{76}$$

This result about the superpotential suggests strongly that there is a limit of M theory near an ADE singularity in a G_2-manifold which is precisely super Yang-Mills theory. We will now go on to explore other limits of this M theory background.

4.4. M theory Physics on ADE-singular G_2-manifolds.

We saw previously that before taking the quotient by Γ, that the M theory physics on $\mathbb{R}^4\times S^3$, with its G_2-metric was smoothly varying as a function of τ. In fact the same is true in the case with ADE-singularities. One hint for this was that we explicitly saw just now that the superpotential is non-zero in the various vacua and this implies that the C-field is non-zero. This suggestion was concretely proven in [7].

Before orbifolding by Γ we saw there were three semiclassical limits of M theory in the space parameterised by τ. These were described by M theory on three large and smooth G_2-manifolds X_i, all three of which were of the form $\mathbb{R}^4\times S^3$. There are also three semiclassical ie large volume G_2-manifolds when we orbifold by Γ. These are simply the quotients by Γ of the X_i. One of these is the G_2-manifold $\mathbb{R}^4/\Gamma_{\mathsf{ADE}}\times S^3$. The other two are both of the form $S^3/\Gamma_{\mathsf{ADE}}\times\mathbb{R}^4$. To see this, note that the three S^3's in the three G_2-manifolds X_i of the form $\mathbb{R}^4\times S^3$ correspond to the three S^3 factors in $G = S^3\times S^3\times S^3$. Γ_{ADE} is a subgroup of one of these S^3's . If Γ_{ADE} acts on the $\mathbb{R}^4$ factor of X_1 in the standard way, then it must act on S^3 in X_2 - since X_2 can be thought of as the same manifold but with the two S^3's at infinity exchanged. Then, because of the permutation symmetry it also acts on the S^3 in X_3.

In the simple, crude, algebraic description in section 3.2.2, let X_1 be the manifold with a negative. Then define Γ_{ADE} to act on the $\mathbb{R}^4$ parametrised by

x_i. Then $x_i = 0$ is an S^3 of fixed points parametrised by y_i. Thus X_1/Γ_{ADE} is isomorphic to our G_2-manifold with ADE-singularities. Consider now what happens when a is taken positive. This is our manifold X_2. Then, because $x_i = 0$ is not a point on X_2, there is no fixed point and Γ_{ADE} acts freely on the S^3 surrounding the origin in the x-space. Thus, X_2/Γ_{ADE} is isomorphic to $S^3/\Gamma_{ADE}\times\mathbb{R}^4$, as is X_3.

On X_1/Γ_{ADE} in the large volume limit, we have a semi-classical description of the four dimensional physics in terms of perturbative super Yang-Mills theory. But, at extremely low energies, this theory becomes strongly coupled, and is believed to confine and get a mass gap. So, apart from calculate the superpotential in each vacuum, as we did in section 4.3 we cant actually calculate the spectrum here.

What about the physics in the other two semiclassical limits, namely large $X_{2,3}/\Gamma_{ADE}$?? These G_2-manifolds are completely smooth. So supergravity is a good approximation to the M theory physics. What do we learn about the M theory physics in this approximation?

4.5. Confinement from G_2-manifolds.

If it is true that the qualitative physics of M theory on X_2/Γ_{ADE} and X_3/Γ_{ADE} is the same as that of M theory on X_1/Γ_{ADE}, then some of the properties of super Yang-Mills theory at low energies ought to be visible. The gauge theory is believed to confine ADE-charge at low energies. If a gauge theory confines in four dimensions, electrically charged confining flux tubes (confining strings) should be present. If the classical fields of the gauge theory contain only fields in the adjoint representation of the gauge group G, then these strings are charged with respect to the center of G, $Z(G)$. Can we see these strings in M theory on X_2/Γ_{ADE}? As we described in [8], the answer is a resounding yes.

The natural candidates for such strings are $M2$-branes which wrap around 1-cycles in X_2/Γ_{ADE} or $M5$-branes which wrap 4-cycles in X_2/Γ_{ADE}. Since X_2/Γ_{ADE} is homeomorphic to $S^3/\Gamma_{ADE}\times\mathbb{R}^4$ which is contractible to S^3/Γ_{ADE}, the homology groups of X_2/Γ_{ADE} are the same as those of the three-manifold S^3/Γ_{ADE}. Thus, our space has no four cycles to speak of, so the confining strings can only come from $M2$-branes wrapping one-cycles in S^3/Γ_{ADE}. The string charges are classified by the first homology group $H_1(S^3/\Gamma_{ADE}, \mathbf{Z})$. For any manifold, the first homology group is isomorphic to the abelianisation of its fundamental group, Π_1. The abelianisation is obtained by setting all commutators in Π_1 to be trivial ie

$$H_1(M,\mathbf{Z}) \cong \frac{\Pi_1(M)}{[\Pi_1(M),\Pi_1(M)]} \tag{77}$$

The fundamental group of S^3/Γ_{ADE} is Γ_{ADE}. Hence, in order to calculate the charges of our candidate confining strings we to compute the abelianisations of all of the finite subgroups of $SU(2)$.

$\Gamma_{A_{n-1}} \cong \mathbf{Z}_n$. The gauge group is locally $SU(n)$. Since $\mathbf{Z}_n$ is abelian, its commutator subgroup is trivial and hence the charges of our strings take values in $\mathbf{Z}_n$. Since this is isomorphic to the center of $SU(n)$ this is the expected answer for a confining $SU(n)$ theory.

For $\Gamma \cong \mathbb{D}_{k-2}$, the binary dihedral group of order $4k-8$, the local gauge group of the Yang-Mills theory is $SO(2k)$. The binary dihedral group is generated by two elements α and β (see section 3.1.1) which obey the relations

$$\alpha^2 = \beta^{k-2} \tag{78}$$

$$\alpha\beta = \beta^{-1}\alpha \tag{79}$$

$$\alpha^4 = \beta^{2k-4} = 1 \tag{80}$$

To compute the abelianisation of $\mathbb{D}_{k-2}$, we simply take these relations and impose that the commutators are trivial. From the second relation this implies that

$$\beta = \beta^{-1} \tag{81}$$

which in turn implies that

$$\alpha^2 = 1 \quad for \quad k = 2p \tag{82}$$

and

$$\alpha^2 = \beta \quad for \quad k = 2p+1 \tag{83}$$

Thus, for $k = 2p$ we learn that the abelianisation of $\mathbb{D}_{k-2}$ is isomorphic to $\mathbf{Z}_2\times\mathbf{Z}_2$, whereas for $k = 2p+1$ it is isomorphic to $\mathbf{Z}_4$. These groups are respectively the centers of $Spin(4p)$ and $Spin(4p+2)$. This is the expected answer for the confining strings in $SO(2k)$ super Yang-Mills which can be coupled to spinorial charges.

To compute the abelianisations of the binary tetrahedral (denoted $\mathbb{T}$), octahedral ($\mathbb{O}$) and icosahedral ($\mathbb{I}$) groups which correspond respectively to E_6, E_7 and E_8 super Yang-Mills theory, we utilise the fact that the order of $F/[F,F]$ - with F a finite group - is the number of inequivalent one dimensional representations of G. The representation theory of the finite subgroups of $SU(2)$ is described through the Mckay correspondence by the representation theory of the corresponding Lie algebras. In particular the dimensions of the irreducible representations of $\mathbb{T}, \mathbb{O}$ and $\mathbb{I}$ are given by the coroot integers (or dual Kac labels) of the affine Lie algebrae associated to E_6, E_7 or E_8 respectively. From this we learn that the respective orders of

$\mathbb{T}/[\mathbb{T},\mathbb{T}]$, $\mathbb{O}/[\mathbb{O},\mathbb{O}]$ and $\mathbb{I}/[\mathbb{I},\mathbb{I}]$ are three, two and one. Moreover, one can easily check that $\mathbb{T}/[\mathbb{T},\mathbb{T}]$ and $\mathbb{O}/[\mathbb{O},\mathbb{O}]$ are $\mathbf{Z}_3$ and $\mathbf{Z}_2$ respectively, by examining their group relations. Thus we learn that $\mathbb{T}/[\mathbb{T},\mathbb{T}]$, $\mathbb{O}/[\mathbb{O},\mathbb{O}]$ and $\mathbb{I}/[\mathbb{I},\mathbb{I}]$ are, respectively isomorphic to the centers $Z(E_6)$, $Z(E_7)$ and $Z(E_8)$ in perfect agreement with the expectation that the super Yang-Mills theory confines. Note that the E_8-theory does not confine, since the strings are uncharged.

This result is also natural from the following point of view. In the singular $X_1/\Gamma_{\mathsf{ADE}}$ (where the actual gauge theory dynamics is) the gauge bosons correspond to $M2$-branes wrapped around zero-size cycles. When we vary τ away from the actual gauge theory limit until we reach M theory on a large and smooth $X_2/\Gamma_{\mathsf{ADE}}$ the confining strings are also wrapped $M2$-branes. In the gauge theory we expect the confining strings to be "built" from the excitations of the gauge fields themselves. In M theory, the central role played by the gauge fields is actually played by the $M2$-brane.

4.6. Mass Gap from G_2-manifolds.

We can also see the mass gap expected of the gauge theory, by studying the spectrum of M theory on the smooth G_2-manifolds $X_2/\Gamma_{\mathsf{ADE}}$ and $X_3/\Gamma_{\mathsf{ADE}}$. We already noted previously that the four dimensional spectrum of M theory on the X_i was massive. For precisely the same reasons the spectrum of M theory on $X_{2,3}/\Gamma_{\mathsf{ADE}}$ is also massive.

5. Chiral Fermions from G_2-manifolds.

Thus far, we have seen that the simplest possible singularities of a G_2-manifold, namely ADE orbifold singularities produce a convincing picture of how non-Abelian gauge groups emerge. However, for the purpose of obtaining a realistic model of particle physics this is not enough. To this end, a basic requisite is the presence of chiral fermions charged under these gauge symmetries. Chiral fermions are important in nature since they are massless as long as the gauge symmetries they are charged under are unbroken. This enables us to understand the lightness of the electron in terms of the Higgs vev.

What sort of singularity in a G_2-manifold X might we expect to give rise to a chiral fermion in M theory? If the singularity is "bigger than a point" then we dont expect chiral fermions. This is because if the codimension of the singularity is less than seven, the local structure of the singularity can actually be regarded as a singularity in a Calabi-Yau threefold or $K3$ and these singularities give rise to a spectrum of particles which form representations of $\mathcal{N} > 1$ supersymmetry. Such spectra are $\mathcal{CPT}$ self-conjugate. For instance, real codimension four singularities in G_2-manifolds are the ADE-singularities we discuused above and the corresponding four dimensional spectra were not chiral. Similarly, if the singularity is codimension six ie is along a one-dimensional curve Σ in X. Then everywhere near Σ, the tangent spaces of X naturally split into tangent and normal directions to X. Hence, the holonomy of X near Σ actually reduces to $SU(3)$ acting normally to Σ.

So we want to consider point like singularities of X. The simplest such singularities are conical, for which the metric looks locally like

$$ds^2 = dr^2 + r^2 g(Y) \tag{84}$$

for some six-dimensional metric $g(Y)$ on a 6-manifold Y. This has a singularity at $r = 0$. We will argue that for many different choices of Y that chiral fermions are part of the M theory spectrum.

5.1 Hints from Anomaly-Inflow.

The basic strategy of this subsection will be to assume there is a G_2-manifold with a conical singularity of the above type and consider the variation of bulk terms in the M theory effective action under various gauge symmetries. These will be shown to be non-zero if Y obeys certain conditions. If the theory is to be consistent, these anomalous variations must be cancelled and this suggests the presence of chiral fermions in the spectrum. This is based upon [22] who showed that when X is compact all these variations add up to zero!

The gauge symmetries we will consider are the ones we have focussed on in these lectures: the $U(1)$ gauge symmetries from Kaluza-Klein reducing the C-field and the ADE symmetries from the ADE-singularities.

We begin with the former case. We take M theory on $X\times\mathbb{R}^{3,1}$ with X a cone on Y so that $X = \mathbb{R}\times Y$. The Kaluza-Klein ansatz for C which gives gauge fields in four dimensions is

$$C = \Sigma_\alpha\ \beta^\alpha(x) \wedge A_\alpha(y) \tag{85}$$

where the β's are harmonic 2-forms on X. With this ansatz, consider the eleven dimensional Chern-Simons interaction

$$S = \int_{X\times\mathbb{R}^{3,1}} \frac{q}{6} C \wedge G \wedge G \tag{86}$$

Under a gauge transformation of C under which

$$C \longrightarrow C + d\epsilon \tag{87}$$

S changes by something of the form[5]

$$\delta S \sim \int_{X\times\mathbb{R}^{3,1}} d(\epsilon \wedge\ G \wedge G) \tag{88}$$

We can regard X as a manifold with boundary $\partial X = Y$ and hence

$$\delta S \sim \int_{Y\times\mathbb{R}^{3,1}} \epsilon \wedge\ G \wedge G \tag{89}$$

If we now make the Kaluza-Klein ansatz for the 2-form ϵ

$$\epsilon = \Sigma_i \epsilon^\alpha \beta^\alpha \tag{90}$$

and use our ansatz for C, we find

$$\delta S \sim \int_Y \beta^\rho \wedge \beta^\sigma \wedge \beta^\delta \int_{\mathbb{R}^{3,1}} \epsilon^\rho dA^\sigma \wedge dA^\delta \tag{91}$$

Thus if the integrals over Y (which are topological) are non-zero we obtain a non-zero four dimensional interaction characteristic of an anomaly in an abelian gauge theory. Thus, if the theory is to be consistent, it is natural to expect a spectrum of chiral fermions at the conical singularity which exactly cancels δS.

[5] We will not be too careful about factors in this section.

We now turn to non-Abelian gauge anomalies. We have seen that ADE gauge symmetries in M theory on a G_2-manifold X are supported along a three-manifold W in X. If additional conical singularities of X are to support chiral fermions charges under the ADE-gauge group, then these singularities should surely also be points P_i on W. So let us assume that near such a point, the metric on X assumes the conical form. In four dimensional ADE gauge theories the triangle anomaly is only non-trivial for A_n-gauge groups. So, we restrict ourselves to this case. In this situation, there is a seven dimensional interaction of the form

$$S = \int_{W\times\mathbb{R}^{3,1}} K \wedge \Omega_5(A) \tag{92}$$

where A is the $SU(n)$ gauge field and

$$d\Omega_5(A) = trF \wedge F \wedge F \tag{93}$$

K is a two-form which is the field strength of a $U(1)$ gauge field which is part of the normal bundle to W. K measures how the A_n-singularity twists around W. The $U(1)$ gauge group is the subgroup of $SU(2)$ which commutes with Γ_{A_n}.

Under a gauge transformation,

$$A \longrightarrow A + D_A\lambda \tag{94}$$

and

$$\delta S \sim \int_{W\times\mathbb{R}^{3,1}} (K \wedge dtr\lambda F \wedge F) \tag{95}$$

so if K is closed, $\delta S = 0$. This will be the case if the A_n-singularity is no worse at the conical singularity P than at any other point on W. If however, the A_n-singularity actually increases rank at P, then

$$dK = 2\pi q \delta_P \tag{96}$$

and we have locally a Dirac monopole of charge q at P. The charge is an integer because of obvious quantisation conditions. In this situation we have that

$$\delta S \sim \int_{W\times\mathbb{R}^{3,1}} d(K \wedge tr\lambda F \wedge F) = -q \int_{\mathbb{R}^{3,1}} tr\lambda F \wedge F \tag{97}$$

which is precisely the triangle anomaly in an $SU(n)$ gauge theory. Thus, if we have this sort of situation in which the ADE-singularity along W degernates further at P we also expect chiral fermions to be present.

We now go on to utilise the M theory heterotic duality of subsection (3.1) to construct explicitly conically singular manifolds at which we know the existence of chiral fermions.

5.2 Chiral Fermions via Duality With The Heterotic String.

In section three we utilised duality with the heterotic string on $\mathbf{T}^3$ to learn about enhanced gauge symmetry in M theory. We applied this to G_2-manifolds quite successfully. In this section we will take a similar approach. The following is based upon [10].

We start by considering duality with the heterotic string. The heterotic string compactified on a Calabi-Yau three-fold Z can readily give chiral fermions. On the other hand, most Calabi-Yau manifolds participate in mirror symmetry. For Z to participate in mirror symmetry means, according to Strominger, Yau and Zaslow its moduli space, it is a $\mathbf{T}^3$ fibration (with singularities and monodromies) over a base W. Then, taking the $\mathbf{T}^3$'s to be small and using on each fiber the equivalence of the heterotic string on $\mathbf{T}^3$ with M theory on $K3$, it follows that the heterotic string on Z is dual to M theory on a seven-manifold X that is K3-fibered over W (again with singularities and monodromies). X depends on the gauge bundle on Z. Since the heterotic string on Z is supersymmetric, M-theory on X is likewise supersymmetric, and hence X is a manifold of G_2 holonomy.

The heterotic string on Z will typically have a four dimensional spectrum of chiral fermions. Since there are many Z's that could be used in this construction (with many possible classes of gauge bundles) it follows that there are many manifolds of G_2 holonomy with suitable singularities to give nonabelian gauge symmetry with chiral fermions. The same conclusion can be reached using duality with Type IIA, as many six-dimensional Type IIA orientifolds that give chiral fermions are dual to M theory on a G_2 manifold [23]

Let us try to use this construction to determine what kind of singularity X must have. (The reasoning and the result are very similar to that given in [24] for engineering matter from Type II singularities. In [24] the Dirac equation is derived directly rather than being motivated – as we will – by using duality with the heterotic string.) Suppose that the heterotic string on Z has an unbroken gauge symmetry G, which we will suppose to be simply-laced (in other words, an A, D, or E group) and at level one. This means that each $K3$ fiber of X will have a singularity of type G. As one moves around in X one will get a family of G-singularities parameterized by W. If W is smooth and the normal space to W is a smoothly varying family of

G-singularities, the low energy theory will be G gauge theory on $W \times \mathbb{R}^{3,1}$ without chiral multiplets. This was the situation studied in sections three and four. So chiral fermions will have to come from singularities of W or points where W passes through a worse-than-orbifold singularity of X.

We can use the duality with the heterotic string to determine what kind of singularities are required. The argument will probably be easier to follow if we begin with a specific example, so we will consider the case of the $E_8 \times E_8$ heterotic string with $G = SU(5)$ a subgroup of one of the E_8's. Such a model can very easily get chiral **5**'s and **10**'s of $SU(5)$; we want to see how this comes about, in the region of moduli space in which Z is $\mathbf{T}^3$-fibered over W with small fibers, and then we will translate this description to M theory on X.

Let us consider, for example, the **5**'s. The commutant of $SU(5)$ in E_8 is a second copy of $SU(5)$, which we will denote as $SU(5)'$. Since $SU(5)$ is unbroken, the structure group of the gauge bundle E on Z reduces from E_8 to $SU(5)'$. Massless fermions in the heterotic string transform in the adjoint representation of E_8. The part of the adjoint representation of E_8 that transforms as **5** under $SU(5)$ transforms as **10** under $SU(5)'$. So to get massless chiral **5**'s of $SU(5)$, we must look at the Dirac equation $\mathcal{D}$ on Z with values in the **10** of $SU(5)'$; the zero modes of that Dirac equation will give us the massless **5**'s of the unbroken $SU(5)$.

We denote the generic radius of the $\mathbf{T}^3$ fibers as α, and we suppose that α is much less than the characteristic radius of W. This is the regime of validity of the argument for duality with M theory on X (and the analysis of mirror symmetry syz). For small α, we can solve the Dirac equation on Z by first solving it along the fiber, and then along the base. In other words, we write $\mathcal{D} = \mathcal{D}_T + \mathcal{D}_W$, where $\mathcal{D}_T$ is the Dirac operator along the fiber and $\mathcal{D}_W$ is the Dirac operator along the base. The eigenvalue of $\mathcal{D}_T$ gives an effective "mass" term in the Dirac equation on W. For generic fibers of $Z \to W$, as we explain momentarily, the eigenvalues of $\mathcal{D}_T$ are all nonzero and of order $1/\alpha$. This is much too large to be canceled by the behavior of $\mathcal{D}_W$. So zero modes of $\mathcal{D}$ are localized near points in W above which $\mathcal{D}_T$ has a zero mode.

When restricted to a $\mathbf{T}^3$ fiber, the $SU(5)'$ bundle E can be described as a flat bundle with monodromies around the three directions in $\mathbf{T}^3$. In other words, as in section three, we have three Wilson lines on each fiber. For generic Wilson lines, every vector in the **10** of $SU(5)'$ undergoes non-trivial "twists" in going around some (or all) of the three directions in $\mathbf{T}^3$. When this is the case, the minimum eigenvalue of $\mathcal{D}_T$ is of order $1/\alpha$. This is simply because for a generic flat gauge field on the $\mathbf{T}^3$-fiber there will be no zero mode.

A zero mode of $\mathcal{D}_T$ above some point $P \in W$ arises precisely if for some vector in the **10**, the monodromies in the fiber are all trivial.

This means that the monodromies lie in the subgroup of $SU(5)'$ that leaves fixed that vector. If we represent the **10** by an antisymmetric 5×5 matrix B^{ij}, $i,j = 1,\ldots,5$, then the monodromy-invariant vector corresponds to an antisymmetric matrix B that has some nonzero matrix element, say B^{12}; the subgroup of $SU(5)'$ that leaves B invariant is clearly then a subgroup of $SU(2) \times SU(3)$ (where in these coordinates, $SU(2)$ acts on $i,j = 1,2$ and $SU(3)$ on $i,j = 3,4,5$). Let us consider the case (which we will soon show to be generic) that B^{12} is the only nonzero matrix element of B. If so, the subgroup of $SU(5)'$ that leaves B fixed is precisely $SU(2) \times SU(3)$. There is actually a distinguished basis in this problem – the one that diagonalizes the monodromies near P – and it is in this basis that B has only one nonzero matrix element.

The commutant of $SU(2) \times SU(3)$ in E_8 is $SU(6)$. So over the point P, the monodromies commute not just with $SU(5)$ but with $SU(6)$. Everything of interest will happen inside this $SU(6)$. The reason for this is that the monodromies at P give large masses to all E_8 modes except those in the adjoint of $SU(6)$. So we will formulate the rest of the discussion as if the heterotic string gauge group were just $SU(6)$, rather than E_8. Away from P, the monodromies break $SU(6)$ to $SU(5) \times U(1)$ (the global structure is actually $U(5)$). Restricting the discussion from E_8 to $SU(6)$ will mean treating the vacuum gauge bundle as a $U(1)$ bundle (the $U(1)$ being the second factor in $SU(5) \times U(1) \subset SU(6)$) rather than an $SU(5)'$ bundle.

The fact that, over P, the heterotic string has unbroken $SU(6)$ means that, in the M theory description, the fiber over P has an $SU(6)$ singularity. Likewise, the fact that away from P, the heterotic string has only $SU(5) \times U(1)$ unbroken means that the generic fiber, in the M theory description, must contain an $SU(5)$ singularity only, rather than an $SU(6)$ singularity. As for the unbroken $U(1)$, in the M theory description it must be carried by the C-field. Indeed, over generic points on W there is a non-zero size S^2 which shrinks to zero size at P in order that the gauge symmetry at that point increases. Kaluza-Klein reducing C along this S^2 gives a $U(1)$.

If we move away from the point P in the base, the vector B in the **10** of $SU(5)'$ is no longer invariant under the monodromies. Under parallel transport around the three directions in $\mathbf{T}^3$, it is transformed by phases $e^{2\pi i\theta_j}$, $j = 1,2,3$. Thus, the three θ_j must all vanish to make B invariant. As W is three-dimensional, we should expect generically that the point P above which the monodromies are trivial is isolated. (Now we can see why it is natural to consider the case that, in the basis given by the monodromies near P, only one matrix element of B is nonzero. Otherwise, the monodromies could act separately on the different matrix elements, and it would be necessary

to adjust more than three parameters to make B invariant. This would be a less generic situation.) We will only consider the (presumably generic) case that P is disjoint from the singularities of the fibration $Z \to W$. Thus, the $\mathbf{T}^3$ fiber over P is smooth (as we have implicitly assumed in introducing the monodromies on $\mathbf{T}^3$).

In [10] we explicitly solved the Dirac equation in a local model for this situation. We found that the net number of chiral zero modes was one. We will not have time to describe the details of the solution here.

In summary, before we translate into the M theory language, the chiral fermions in the heterotic string theory on Z are localised at points on W over which the Wilson lines in the $\mathbf{T}^3$-fibers are trivial. In M theory this translates into the statement that the chiral fermions are localised at points in W over which the ADE-singularity "worsens". This is also consistent with what we found in the previous section.

5.2.2 M theory Description.

So we have found a local structure in the heterotic string that gives a net chirality – the number of massless left-handed **5**'s minus right-handed **5**'s – of one. Let us see in more detail what it corresponds to in terms of M-theory on a manifold of G_2 holonomy.

Here it may help to review the case considered in [24] where the goal was geometric engineering of charged matter on a Calabi-Yau threefold in Type IIA. What was considered there was a Calabi-Yau three-fold R, fibered by $K3$'s with a base W', such that over a distinguished point $P \in W'$ there is a singularity of type $\hat{G}$, and over the generic point in W' this singularity is replaced by one of type G – the rank of $\hat{G}$ being one greater than that of G. In our example, $\hat{G} = SU(6)$ and $G = SU(5)$. In the application to Type IIA, although R also has a Kahler metric, the focus is on the complex structure. For $\hat{G} = SU(6)$, $G = SU(5)$, let us describe the complex structure of R near the singularities. The $SU(6)$ singularity is described by an equation $xy = z^6$ - cf section three. Its "unfolding" depends on five complex parameters and can be written $zy = z^6 + P_4(z)$, where $P_4(z)$ is a quartic polynomial in z. If – as in the present problem – we want to deform the $SU(6)$ singularity while maintaining an $SU(5)$ singularity, then we must pick P_4 so that the polynomial $z^6 + P_4$ has a fifth order root. This determines the deformation to be

$$xy = (z + 5\epsilon)(z - \epsilon)^5, \tag{98}$$

where we interpret ϵ as a complex parameter on the base W'. Thus, the above equation gives the complex structure of the total space R.

What is described above is the partial unfolding of the $SU(6)$ singularity, keeping an $SU(5)$ singularity. In our G_2 problem, we need a similar construction, but we must view the $SU(6)$ singularity as a hyper-Kahler manifold, not just a complex manifold. In unfolding the $SU(6)$ singularity as a hyper-Kahler manifold, each complex parameter in P_4 is accompanied by a real parameter that controls the area of an exceptional divisor in the resolution/deformation of the singularity. The parameters are thus not five complex parameters but five triplets of real parameters. (There is an $SO(3)$ symmetry that rotates each triplet. This is the $SO(3)$ rotating the three kahler forms in section three.)

To get a G_2-manifold, we must combine the complex parameter seen in with a corresponding real parameter. Altogether, this will give a three-parameter family of deformations of the $SU(6)$ singularity (understood as a hyper-Kahler manifold) to a hyper-Kahler manifold with an $SU(5)$ singularity. The parameter space of this deformation is what we have called W, and the total space is a seven-manifold that is our desired singular G_2-manifold X, with a singularity that produces the chiral fermions that we analyzed above in the heterotic string language.

To find the hyper-Kahler unfolding of the $SU(6)$ singularity that preserves an $SU(5)$ singularity is not difficult, using Kronheimer's description of the general unfolding via a hyper-Kahler quotient [15] At this stage, we might as well generalize to $SU(N)$, so we consider a hyper-Kahler unfolding of the $SU(N+1)$ singularity to give an $SU(N)$ singularity. The unfolding of the $SU(N+1)$ singularity is obtained by taking a system of $N+1$ hypermultiplets $\Phi_0, \Phi_1, \ldots \Phi_N$ with an action of $K = U(1)^N$. Under the i^{th} $U(1)$ for $i = 1, \ldots, N$, Φ_i has charge 1, Φ_{i-1} has charge -1, and the others are neutral. This configuration of hypermultiplets and gauge fields is known as the quiver diagram of $SU(N+1)$ and appears in studying D-branes near the $SU(N+1)$ singularity We let $\mathbb{H}$ denote $\mathbb{R}^4$, so the hypermultiplets parameterize $\mathbb{H}^{N+1}$, the product of $N+1$ copies of $\mathbb{R}^4$. The hyper-Kahler quotient of $\mathbb{H}^{N+1}$ by K is obtained by setting the $\vec{D}$-field (or components of the hyper-Kahler moment map) to zero and dividing by K. It is denoted $\mathbb{H}^{N+1}//K$, and is isomorphic to the $SU(N+1)$ singularity $\mathbb{R}^4/\mathbf{Z}_{N+1}$. Its unfolding is described by setting the $\vec{D}$-fields equal to arbitrary constants, not necessarily zero. In all, there are $3N$ parameters in this unfolding – three times the dimension of K – since for each $U(1)$, $\vec{D}$ has three components, rotated by an $SO(3)$ group of R-symmetries.

We want a partial unfolding keeping an $SU(N)$ singularity. To describe this, we keep $3(N-1)$ of the parameters equal to zero and let only the remaining three vary; these three will be simply the values of $\vec{D}$ for one of

the $U(1)$'s. To carry out this procedure, we first write $K = K' \times U(1)'$ (where $U(1)'$ denotes a chosen $U(1)$ factor of $K = U(1)^N$). Then we take the hyper-Kahler quotient of $\mathbb{H}^{N+1}$ by K' to get a hyper-Kahler eight-manifold $\hat{X} = \mathbb{H}^{N+1}//K'$, after which we take the *ordinary* quotient, not the hyper-Kahler quotient, by $U(1)'$ to get a seven-manifold $X = \hat{X}/U(1)'$ that should admit a metric of G_2-holonomy. X has a natural map to $W = \mathbb{R}^3$ given by the value of the $\vec{D}$-field of $U(1)'$ – which was not set to zero – and this map gives the fibration of X by hyper-Kahler manifolds.

In the present example, we can easily make this explicit. We take $U(1)'$ to be the "last" $U(1)$ in $K = U(1)^N$, so $U(1)'$ only acts on Φ_{N-1} and Φ_N. K' is therefore the product of the first $N-1$ $U(1)$'s; it acts trivially on Φ_N, and acts on $\Phi_0, \ldots, \Phi_{N-1}$ according to the standard quiver diagram of $SU(N)$. So the hyper-Kahler quotient $\mathbb{H}^{N+1}//K'$ is just $(\mathbb{H}^N//K') \times \mathbb{H}'$, where $\mathbb{H}^N//K'$ is the $SU(N)$ singularity, isomorphic to $\mathbb{H}/\mathbf{Z}_N$, and $\mathbb{H}'$ is parameterized by Φ_N. So finally, X will be $(\mathbb{H}/\mathbf{Z}_N \times \mathbb{H}')/U(1)'$. To make this completely explicit, we just need to identify the group actions on $\mathbb{H}$ and $\mathbb{H}'$. If we parameterize $\mathbb{H}$ and $\mathbb{H}'$ respectively by pairs of complex variables (a, b) and (a', b') then the $\mathbf{Z}_N$ action on $\mathbb{H}$, such that the quotient $\mathbb{H}/\mathbf{Z}_N$ is the $SU(N)$ singularity, is given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} e^{2\pi ik/N} a \\ e^{-2\pi ik/N} b \end{pmatrix} \tag{99}$$

while the $U(1)'$ action that commutes with this (and preserves the hyper-Kahler structure) is

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} e^{i\psi/N} a \\ e^{-i\psi/N} b \end{pmatrix} \tag{100}$$

The $U(1)'$ action on $\mathbb{H}'$ is similarly

$$\begin{pmatrix} a' \\ b' \end{pmatrix} \to \begin{pmatrix} e^{i\psi/N} a' \\ e^{-i\psi/N} b' \end{pmatrix} \tag{101}$$

In all, if we set $\lambda = e^{i\psi/N}$, $w_1 = \bar{a}'$, $w_2 = b'$, $w_3 = a$, $w_4 = \bar{b}$, then the quotient $(\mathbb{H}/\mathbf{Z}_N \times \mathbb{H}')/U(1)$ can be described with four complex variables $w_1, \ldots, w_4$ modulo the equivalence

$$(w_1, w_2, w_3, w_4) \to (\lambda^N w_1, \lambda^N w_2, \lambda w_3, \lambda w_4), \quad |\lambda| = 1 \tag{102}$$

This quotient is a cone on a weighted projective space $\mathbf{WCP}^3_{N,N,1,1}$. In fact, if we impose the above equivalence relation for all nonzero complex λ, we would get the weighted projective space itself; by imposing this relation only for $|\lambda| = 1$, we get a cone on the weighted projective space. Note, that the conical metric of G_2-holonomy on this space does not use usual Kahler metric on weighted projective space.

$\mathbb{WCP}^3_{N,N,1,1}$ has a family of A_{N-1}-singularities at points $(w_1, w_2, 0, 0)$. This is easily seen by setting λ to $e^{2\pi i/N}$. This set of points is a copy of $\mathbb{CP}^1 = S^2$. Our proposed G_2-manifold is a cone over weighted projective space, so it has a family of A_{N-1}-singularities which are a cone over this S^2. This is of course a copy of $\mathbb{R}^3$. Away from the origin in $\mathbb{R}^3$ the only singularities are these orbifold singularities. At the origin however, the whole manifold develops a conical singularity. There, the 2-sphere, which is incontractible in the bulk of the manifold, shrinks to zero size. This is in keeping with the anomaly inflow arguments of the previous section. There we learned that an ADE-singularity which worsens over a point in W is a good candidate for the appearance of chiral fermions. Here, via duality with the heterotic string, we find that the conical singularity in this example supports one chiral fermion in the N of the $SU(N)$ gauge symmetry coming from the A_{N-1}-singularity. In fact, the $U(1)$ gauge symmetry from the C-field in this example, combines with the $SU(N)$ to give a gauge group which is globally $U(N)$ and the fermion is in the fundamental representation.

Some extensions of this can be worked out in a similar fashion. Consider the case that away from P, the monodromies break $SU(N+1)$ to $SU(p) \times SU(q) \times U(1)$, where $p+q = N+1$. Analysis of the Dirac equation along the above lines shows that such a model will give chiral fermions transforming as $(\mathbf{p}, \bar{\mathbf{q}})$ under $SU(p) \times SU(q)$ (and charged under the $U(1)$). To describe a dual in M theory on a manifold of G_2 holonomy, we let $K = K' \times U(1)'$, where now $K' = K_1 \times K_2$, K_1 being the product of the first $p-1$ $U(1)$'s in K and K_2 the product of the last $q-1$, while $U(1)'$ is the p^{th} $U(1)$. Now we must define $\hat{X} = \mathbb{H}^{N+1}//K'$, and the manifold admitting a metric of G_2 holonomy should be $\hat{X}/U(1)'$.

We can compute $\hat{X}$ easily, since K_1 acts only on $\Phi_1, \ldots, \Phi_p$ and K_2 only on $\Phi_{p+1}, \ldots, \Phi_{N+1}$. The hyper-Kahler quotients by K_1 and K_2 thus simply construct the $SU(p)$ and $SU(q)$ singularities, and hence $\hat{X} = \mathbb{H}/\mathbf{Z}_p \times \mathbb{H}/\mathbf{Z}_q$. $\hat{X}$ has planes of $\mathbf{Z}_p$ and $\mathbf{Z}_q$ singularities, which will persist in $X = \hat{X}/U(1)'$, which will also have a more severe singularity at the origin. So the model describes a theory with $SU(p) \times SU(q)$ gauge theory and chiral fermions supported at the origin. $U(1)'$ acts on $\mathbb{H}/\mathbf{Z}_p$ and $\mathbb{H}/\mathbf{Z}_q$ as the familiar global symmetry that preserves the hyper-Kahler structure of the $SU(p)$ and $SU(q)$ singularities. Representing those singularities by pairs (a, b) and (a', b') modulo the usual action of $\mathbf{Z}_p$ and $\mathbf{Z}_q$, $U(1)'$ acts by

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\psi/p}a \\ e^{-i\psi/p}b \end{pmatrix} \quad and \quad \begin{pmatrix} a' \\ b' \end{pmatrix} \rightarrow \begin{pmatrix} e^{-i\psi/q}a' \\ e^{i\psi/q}b' \end{pmatrix} \tag{103}$$

Now if p and q are relatively prime, we set $\lambda = e^{i\psi/pq}$, and we find that the $U(1)'$ action on the complex coordinates $w_1, \ldots, w_4$ (which are defined in terms of a, b, a', b' by the same formulas as before) is

$$(w_1, w_2, w_3, w_4) \rightarrow (\lambda^p w_1, \lambda^p w_2, \lambda^q w_3, \lambda^q w_4). \tag{104}$$

If p and q are relatively prime, then the $U(1)'$ action, upon taking λ to be a p^{th} or q^{th} root of 1, generates the $\mathbf{Z}_p \times \mathbf{Z}_q$ orbifolding that is part of the original definition of $\hat{X}$. Hence in forming the quotient $\hat{X}/U(1)'$, we need only to act on the w's by the equivalence relation. The quotient is therefore a cone on a weighted projective space $\mathbb{WCP}^3_{p,p,q,q}$. If p and q are not relatively prime, we let $(p, q) = r(n, m)$ where r is the greatest common divisor and n and m are relatively prime. Then we let $\lambda = \exp(ir\psi/pq)$, so the equivalence relation above is replaced with

$$(w_1, w_2, w_3, w_4) \rightarrow (\lambda^n w_1, \lambda^n w_2, \lambda^m w_3, \lambda^m w_4) \tag{105}$$

To reproduce $\hat{X}/U(1)$ we must now also divide by $\mathbf{Z}_r$, acting by

$$(w_1, w_2, w_3, w_4) \rightarrow (\zeta w_1, \zeta w_2, w_3, w_4), \tag{106}$$

where $\zeta^r = 1$. So X is a cone on $\mathbb{WCP}^3_{n,n,m,m}/\mathbf{Z}_r$.

5.3 Other Gauge Groups and Matter Representations.

We now explain how to generalise the above construction to obtain singularities with more general gauge groups and chiral fermion representations. Suppose that we want to get chiral fermions in the representation R of a simply-laced group G. This can be achieved for certain representations. We find a simply-laced group $\hat{G}$ of rank one more than the rank of G, such that $\hat{G}$ contains $G \times U(1)$ and the Lie algebra of $\hat{G}$ decomposes as $\mathbf{g} \oplus \mathbf{o} \oplus \mathbf{r} \oplus \bar{\mathbf{r}}$, where $\mathbf{g}$ and $\mathbf{o}$ are the Lie algebras of G and $U(1)$, $\mathbf{r}$ transforms as R under G and of charge 1 under $U(1)$, and $\bar{\mathbf{r}}$ transforms as the complex conjugate. Such a $\hat{G}$ exists only for special R's, and these are the R's that we will generate from G_2 singularities.

Given $\hat{G}$, we proceed as above on the heterotic string side. We consider a family of $\mathbf{T}^3$'s, parameterized by W, with monodromy that at a special point $P \in W$ leaves unbroken $\hat{G}$, and at a generic point breaks $\hat{G}$ to $G \times U(1)$.

We moreover assume that near P, the monodromies have the same sort of generic behavior assumed above. Then the same computation as above will show that the heterotic string has, in this situation, a single multiplet of fermion zero modes (the actual chirality depends on the solving the Dirac equation) in the representation R, with $U(1)$ charge 1.

Dualizing this to an M theory description, over P we want a $\hat{G}$ singularity, while over a generic point in W, we should have a G singularity. Thus, we want to consider the unfolding of the $\hat{G}$ singularity (as a hyper-Kahler manifold) that preserves a G singularity. To do this is quite simple. We start with the Dynkin diagram of $\hat{G}$. The vertices are labeled with integers n_i, the Dynkin indices. In Kronheimer's construction, the $\hat{G}$ singularity is obtained as the hyper-Kahler quotient of $\mathbb{H}^k$ (for some k) by the action of a group $K = \prod_i U(n_i)$. Its unfolding is obtained by allowing the $\vec{D}$-fields of the $U(1)$ factors (the centers of the $U(n_i)$) to vary.

The G Dynkin diagram is obtained from that of $\hat{G}$ by omitting one node, corresponding to one of the $U(n_i)$ groups; we write the center of this group as $U(1)'$. Then we write K (locally) as $K = K' \times U(1)'$, where K' is defined by replacing the relevant $U(n_i)$ by $SU(n_i)$. We get a hyper-Kahler eight-manifold as the hyper-Kahler quotient $\hat{X} = \mathbb{H}^k//K'$, and then we get a seven-manifold X by taking the *ordinary* quotient $X = \hat{X}/U(1)'$. This maps to $W = \mathbb{R}^3$ by taking the value of the $U(1)'$ $\vec{D}$-field, which was not set to zero. The fiber over the origin is obtained by setting this $\vec{D}$-field to zero after all, and gives the original $\hat{G}$ singularity, while the generic fiber has a singularity of type G.

One can readily work out examples of pairs G, $\hat{G}$. We will just consider the cases most relevant for grand unification. For $G = SU(N)$, to get chiral fields in the antisymmetric tensor representation, $\hat{G}$ should be $SO(2N)$. For $G = SO(10)$, to get chiral fields in the **16**, $\hat{G}$ should be E_6. For $G = SO(2k)$, to get chiral fields in the **2k**, $\hat{G}$ should be $SO(2k+2)$. (Note in this case that **2k** is a real representation. However, the monodromies in the above construction break $SO(2k+2)$ to $SO(2k) \times U(1)$, and the massless **2k**'s obtained from the construction are charged under the $U(1)$; under $SO(2k) \times U(1)$ the representation is complex.) For $2k = 10$, this example might be used in constructing $SO(10)$ GUT's. For $G = E_6$, to get **27**'s, $\hat{G}$ should be E_7. A useful way to describe the topology of X in these examples is not clear.

In this construction, we emphasized, on the heterotic string side, the most generic special monodromies that give enhanced gauge symmetry, which corresponds on the M theory side to omitting from the hyper-Kahler quotient a rather special $U(1)$ that is related to a single node of the Dynkin diagram.

We could also consider more general heterotic string monodromies; this would correspond in M theory to omitting a more general linear combination of the $U(1)$'s.

6. Outlook.

Having gathered all the necessary ingredients we can now briefly describe how one goes about building a model of particle physics from M theory on a G_2-manifold, X. First it is natural that X admits a map to a three manifold W. The generic fibers of the map are all $K3$-surfaces which have an ADE-singularity of some fixed type. $A_4 = SU(5)$ is a promising possibility for particle physics. This plays the role of the GUT gauge group.

At a finite number of points on X which are also on W, there are conical singularities of the kind discussed in section five. These support chiral fermions in various representations of the ADE-gauge group. For instance, in the case of $SU(5)$ we would like to obtain three **5**'s and three $\bar{\mathbf{10}}$'s. The singularities of X should be of the required type.

We then take W to non-simply connected (eg W might be $S^3/\mathbb{Z}_n$). Wilson lines (or flat connections) of the $SU(5)$ gauge fields can then be used to break $SU(5)$ to $SU(3)\times SU(2)\times U(1)$ - the gauge group of the standard model.

An analysis of some of the basic properties of such models (assuming suitable X exists) was carried out in [25]. It was found that one of the basic physical tests of such a model - namely the stability of the proton - was not problematic. This is because the various families of chiral fermions originate from different points on X so it is natural for them to be charged under different discrete symmetries. These symmetries prevent the existence of operators which would otherwise mediate the decay of the proton too quickly.

In another direction - namely cosmology - a recent study of string and M theory compactifications with $\mathcal{N} = 1$ has argued (with some assumptions) that G_2-compactifications could give measurable predictions for forthcoming experiments to measure fluctuations in the cosmic microwave background [26]. Moreover, these authors also claimed that the other classes of compactifications with supersymmetry gave no foreseeable predictions for these experiments atall. If correct, these issues deserve a much more detailed exploration, since then G_2-compactifcations are apparently the only context in which we have any testable predictions coming from string theory or M theory[6].

[6] Models in the context of 'Braneworlds' and 'large extra dimensions' also offer an alternative scenario which we are not assessing here.

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