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Concept of Scaling

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LECTURE NOTES

Concept of Scaling

Extended text of the textbook
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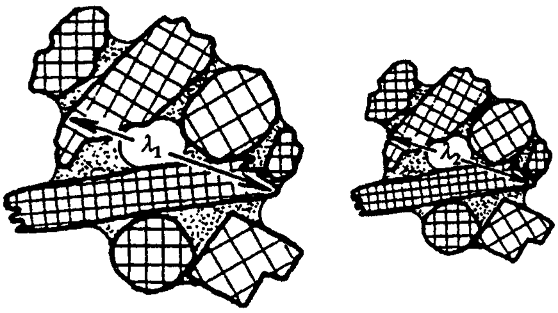

Figure 8.12. A self similar microscopic soil particle structure is the principle of Millers' scaling (Miller and Miller, 1956).

8.4 SCALING

Scaling theories are based upon the assumption that the continuously heterogeneous field is an ensemble of mutually similar homogeneous domains. We assume that each of the domains can be characterized by the SWRC $h(\theta)$ which is related to the porous system through (4.11), or more generally by

$$h = f\left(\frac{1}{r}\right). \tag{8.12}$$

Two porous media of equal porosity are similar according to Miller and Miller (1956) when a scale factor λ exists which will transform one of the porous media to the other. Such similar media have identical microscopic structures except for scale, see Fig. 8.12. This kind of similarity leads to the constancy $r_1/\lambda_1 = r_2/\lambda_2 = r_i/\lambda_i$ and to the formulation of a scaled, invariant pressure head h^* such that

$$\lambda_1 h_1 = \lambda_2 h_2 = \lambda_i h_i = \lambda^* h^* \tag{8.13}$$

where h^* can also be called an average pressure head and λ^* is an average scaling factor. Alternatively, we can denote the parameters of a reference soil with an asterisk.

Invariant microscopic flow in pores leads first to the formulation of an equation for the mean pore water velocity, see Hagen-Poiseuille (5.10) for laminar flow. Hence, the saturated hydraulic conductivity is

$$K_S = f(r^2) \tag{8.14}$$

and with the dimensionless term r/λ, we obtain

$$K_S^*/(\lambda^*)^2 = K_{Si}/\lambda_i^2 \tag{8.15}$$

and analogously for $K(\theta)$

$$K^*(\theta)/(\lambda^*)^2 = K_i(\theta)/\lambda_i^2 \tag{8.16}$$

where K_S^* and $K^*(\theta)$ are either average values or values of a reference soil. The original scaling of Miller and Miller (1956) is restrictive in two aspects. First, a microscopic length is physically interpreted, and second, the requirement of a constant porosity is rarely applicable.

Warrick et al. (1977) extended the use of the Millers' single scaling factor by introducing the degree of saturation (equivalent to the earlier relative soil water content θ_R) and eliminating the assumption of identical porosities. Thus, they scaled θ with the scaling factor θ_S. Additionally, their derivation of λ does not require a search for a microscopic physical length. For the derivation of λ, the sum of squares

$$\sum_{r,i}\left(h^* - \lambda_{ri}h_i\right)^2$$

was minimized for r locations. Using this scaling procedure, a large dispersion of experimental data of $h(\theta/\theta_S)$ and $K(\theta/\theta_S)$ was nicely coalesced into unique functions, see Fig. 8.13. Hence, soil heterogeneity is approximated by the stochastic character of λ which retains its universal meaning with relations

$$\lambda = \frac{h^*}{h} \qquad \lambda = \left(\frac{K}{K^*}\right)^{1/2} \qquad \theta_R = \frac{\theta}{\theta_S} \tag{8.17}$$

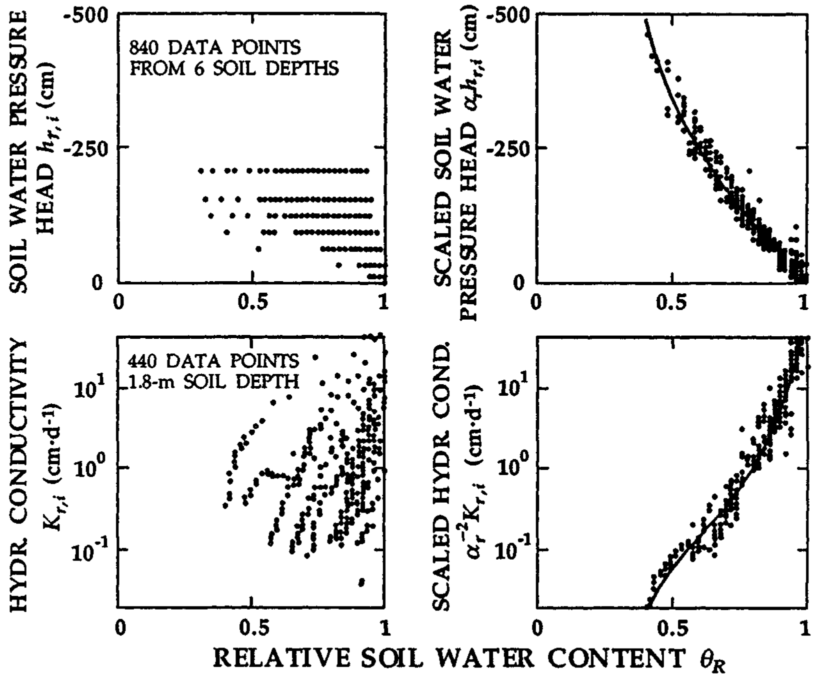

Figure 8.13. The scattering of experimental $h_r(\theta_R)$ and $K_r(\theta_R)$ data (left) is substantially reduced by Warrick's scaling (Warrick et al., 1977). The lines given in the upper and lower right hand graphs are the equations $\{h = -6020\ \theta_R^{-1}[(1-\theta_R) - 2.14(1-\theta_R^2) + 2.04(1-\theta_R^3) - 0.694(1-\theta_R^4)]\}$ and $[\ln K = -20.5 + 75.0\,\theta_R - 109\,\theta_R^2 + 59.7\,\theta_R^3]$, respectively.

The modification for estimating λ indicated in the Warrick et al. (1977) procedure was fully developed by Simmons et al. (1979). They rejected the assumption of microscopic geometrical similarity and based their method on the similarity between soil hydraulic functions.

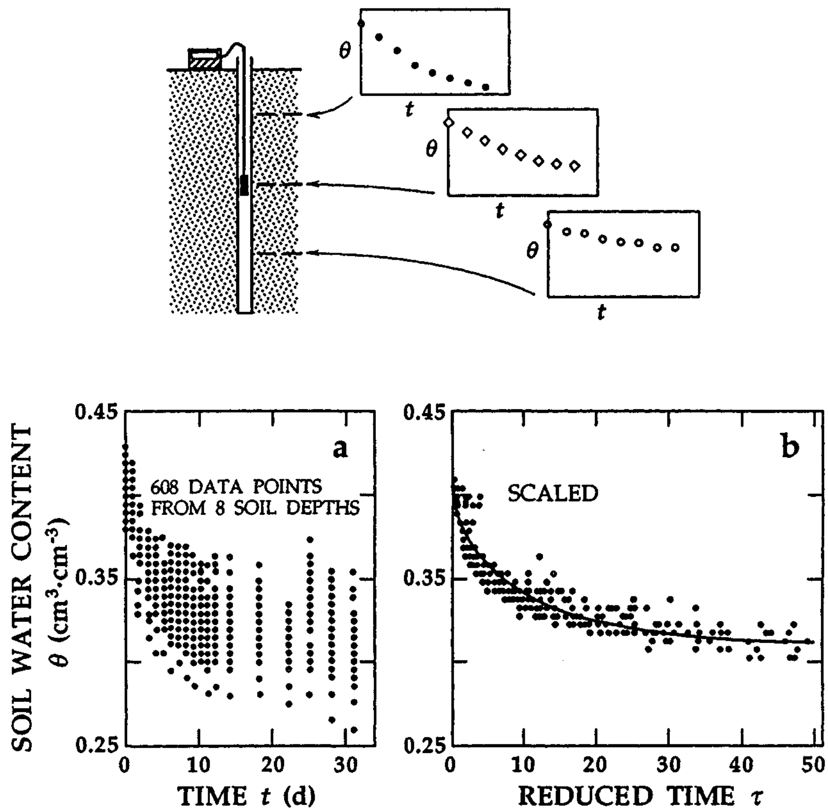

Figure 8.14. Neutron probe soil water contents measured at different soil depths and spatial locations during redistribution. Equation (8.18) is the solid line describing the scaled data.

Simmons et al. (1979) as well as others (see authors and those cited in Hillel and Elrick, 1990) have derived scaling relations that have not yet been sufficiently examined under field conditions to define criteria for their acceptance or rejection. Some of the formulations are physically based while others are mathematical techniques of inspectional analysis. Sposito and Jury (1990) showed that Richards' equation will be invariant under scaling transformations only if $K(\theta)$ is a power or exponential function. If that is the case, the solution of Richards' equation obtained for one location can then be scaled to other locations in the same field or domain. Assuming $K(\theta)$ is an exponential function and a unit hydraulic gradient exists during redistribution in the absence of evapotranspiration, Simmons et al. (1979) recommended that (7.28) be scaled with a common value of β in (7.27) for all locations within a field. Fig. 8.14a shows soil water content versus redistribution time for a total of 608 measurements (19 times at 32 locations) within 4 plots covered by plastic sheeting to prevent evaporation after steady state infiltration had ceased. Fig. 8.14b shows the data from Fig. 8.14a. scaled with reduced time τ and a common initial value of $\overline{\theta}_o$. The solid line in Fig. 8.14b is

$$\theta=\overline{\theta}_o-\frac{1}{\beta}\ln\left[1+\frac{\beta K^*\tau}{z^*}\right] \tag{8.18}$$

where $\overline{\theta}_o$ = 0.408 $cm^3 \cdot cm^{-3}$, β = 50, K^* = 5.29 $cm \cdot d^{-1}$, z^* = 120 cm and $\tau=\omega^2 z^*(az)^{-1}$ [a is defined in (7.26) and ω is the scale factor for each location defined by $K_o=\omega^2 K^*$ where K^* is the scale mean of all K_o.]. The measured θ deviate from the solid line with a pooled standard deviation is 0.008 $cm^3 \cdot cm^{-3}$ – a value comparable to the neutron probe measurement error.

Methods based upon regression analysis are described as functional normalization techniques (Tillotson and Nielsen, 1984). With this approach being only macroscopic, geometrical similarity of the porous system is therefore not the condition for scaling. The idea of a universal λ for all hydraulic functions was abandoned as it was found that λ for scaling $h(\theta_R)$ is not necessarily identical with that of $K(\theta_R)$. Even if the two scaling factors are different, they are generally correlated. In order to differentiate from the previous universal single set of scaling factors λ_i, we shall now use symbols α_{1i} and α_{2i} for the two sets of scaling factors, the first one denoting the scaling of h and the second denoting the scaling of K. Hence, we have

$$\alpha_1=\frac{h^*}{h} \qquad \alpha_2=\left(\frac{K}{K^*}\right)^{1/2} \qquad \theta_R=\frac{\theta}{\theta_S} \tag{8.19}$$

A set of sampling locations is similar if the soil hydraulic functions can be scaled. Warrick (1990) reviewed the application of this scaling in three different regions, while Clausnitzer et al. (1992) demonstrated that simultaneous scaling with α_1 and α_2 is not always as successful as independent scaling using a single scaling factor λ (our notation here).

Still yet another scaling proposal of Vogel et al. (1991) is based upon the assumption that the spatial variability of soil hydraulic functions has two components, one being linear and the other being non-linear. Supposing that the linear component is dominant, he proposed linear scaling with

$$\alpha_h=\frac{h}{h^*} \qquad \alpha_K=\frac{K(h)}{K^*(h^*)} \qquad \alpha_\theta=\frac{\theta(h)-\theta_r}{\theta^*(h^*)-\theta_r^*} \tag{8.20}$$

Any of the above types of scaling should be tested for the measured set of functions. For the selected type of scaling the invariant form of Richards' equation is applied together with the scaled boundary conditions. We can denote the soils as Warrick, or Simmons, or Vogel similar. Once successfully scaled, computed fluxes can be "descaled" for any given sampling point. Of greater importance is the study of the variability of the scaling parameters by stochastic and geostatistical methods. Scaling yields higher quality data or more useful information when soil samples within one soil type are scaled independently from other soil types (Clausnitzer et al., 1992). The variability of soil hydraulic functions is appropriately expressed by scaling factors within each soil type where their *pdf* and correlation structure can be easily determined. Additional details on numerical procedures of scaling are described by Clausnitzer et al. (1992). Appropriate developments of dimensionless variables and scaled basic equations for various types of boundary conditions and soil hydrological problems have been assembled by Hillel and Elrick (1990).

Up to now we have demonstrated the scaling of soil hydraulic functions of field soils in order to ascertain reference soil parameters and the statistical character of the scaling factors. However, scaling techniques offer still a greater opportunity to formulate Richards' equation in an invariant form for the solution of elementary hydrological processes. Once a solution is known for a defined soil or boundary condition and is expressed in scaled variables, it is valid for all soils or boundary conditions within the given class of flow problem. Two different procedures are available. (i) Variables and soil hydraulic functions are scaled by the boundary condition with these scaled variables usually not being dimensionless. (ii) Variables are scaled to dimensionless forms using soil hydraulic functions. These solutions are similar to traditional expressions of solutions of flow problems in dimensionless variables.

The first procedure was used in the study of two scaling classes. (i) Infiltration with a constant flux at the soil surface (Neuman's boundary condition) and infiltration into a crust-topped soil (Kutílek et al., 1991). Variables z, t and θ scaled by the boundary condition, i.e. either using the flux density q_o (left-hand column) or using the resistance R (right-hand column) are

$$t = q_o^{\alpha} T^* \qquad t = R^a T^* \tag{8.21}$$

$$z = q_o^{\beta} Z^* \qquad z = R^b Z^* \tag{8.22}$$

$$\theta - \theta_r = q_o^{\gamma} \theta^* \qquad \theta - \theta_r = R^c \theta^* \tag{8.23}$$

The soil hydraulic functions expressed in a power form $D(\theta) = D_o(\theta - \theta_r)^n$, $h(\theta) = -p(\theta - \theta_r)^{-m}$ and $K(\theta) = K_o(\theta - \theta_r)^v$ similarly scaled are

$$D(\theta) = q_o^{n\gamma} D^*(\theta^*) \qquad D(\theta) = R^{cn} D^*(\theta^*) \tag{8.24}$$

$$h(\theta) = q_o^{-m\gamma} h^*(\theta^*) \qquad h(\theta) = R^{-cm} h^*(\theta^*) \tag{8.25}$$

$$K(\theta) = q_o^{v\gamma} K^*(\theta^*) \qquad K(\theta) = R^{cv} K^*(\theta^*) \tag{8.26}$$

And, Richards' equation in the diffusive form (5.68) transcribed into scaled variables invariant to q_o is

$$q_o^{(\gamma-\alpha)} \frac{\partial \theta^*}{\partial T^*} = q_o^{(n\gamma+\gamma-2\beta)} \frac{\partial}{\partial Z^*}\left[D^*(\theta^*)\frac{\partial \theta^*}{\partial Z^*}\right] - q_o^{(\gamma v-\beta)} \frac{\partial K^*(\theta^*)}{\partial Z^*}. \tag{8.27}$$

Similarly, (5.68) transcribed into scaled variables invariant to R is

$$R^{(c-a)} \frac{\partial \theta^*}{\partial T^*} = R^{(cn+c-2b)} \frac{\partial}{\partial Z^*}\left[D^*(\theta^*)\frac{\partial \theta^*}{\partial Z^*}\right] - R^{(cv-b)} \frac{\partial K^*(\theta^*)}{\partial Z^*}. \tag{8.28}$$

Boundary conditions are expressed in a similar manner. The exponents for the above conditions are

$$\alpha = \frac{-2m-n}{m+n+1} \qquad a = \frac{2m+n}{2m+n+1} \tag{8.29}$$

$$\beta = \frac{-m}{m+n+1} \qquad b = \frac{m}{2m+n+1} \tag{8.30}$$

$$\gamma = \frac{1}{m+n+1} \qquad c = \frac{-1}{2m+n+1} \tag{8.31}$$

The solution of the infiltration problem plotted in scaled variables $\theta^*(Z^*, T^*)$ is valid for the whole class, i.e. either for all variations of constant flux at the soil surface, see Fig. 8.15, or for all variations of resistance (except for $R = 0$), see Fig. 6.25.

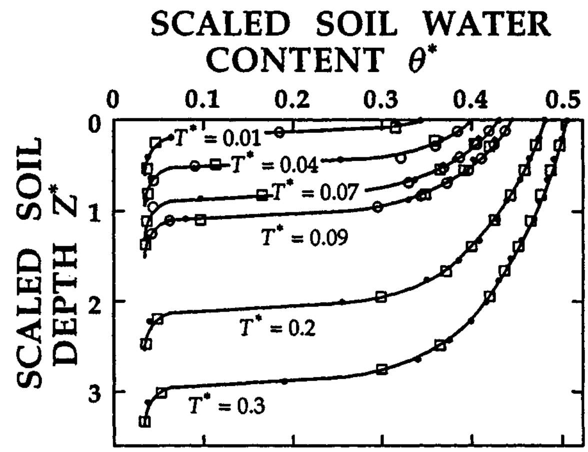

Figure 8.15. Scaling of Richards' equation through the boundary flux for NBC infiltration offers unique soil water profiles for all fluxes. Scaled variables are θ^*, Z^* and time T^* (Kutílek et al., 1991). The symbols □, • and o designate values of q_0 = 0.05, 0.15 and 0.25 cm·h^{-1}, respectively. The value of K_S is 0.196 cm·h^{-1}.

The second scaling procedure elaborated by Warrick and Hussen (1993) is applicable to a broader family of flow classes, e.g. redistribution and upward flow. They defined the dimensionless variables

$$Z^* = z/z_0 \qquad T^* = t/t_0 \tag{8.32}$$

and functions

$$\theta^* = \frac{\theta - \theta_r}{\theta_0 - \theta_r}, \quad K^* = K/K_0, \quad h^* = h/z_0 \tag{8.33}$$

where $\theta_0 \le \theta_S$ and $K_0 = K(\theta_0)$. When they considered the soil hydraulic characteristics

$$\frac{\theta-\theta_r}{\theta_S-\theta_r}=\left(\frac{h_A}{h}\right)^{\lambda} \qquad \frac{K}{K_r}=\left(\frac{\theta-\theta_r}{\theta_S-\theta_r}\right)^{n} \tag{8.34}$$

they defined

$$z_o=|h_A|\left(\frac{\theta_o-\theta_r}{\theta_S-\theta_r}\right)^{-1/\lambda} \tag{8.35}$$

and

$$t_o=\frac{(\theta_o-\theta_r)z_o}{K_o}. \tag{8.36}$$

Hence, in Richards' equation all variables and soil hydraulic functions are scaled with each equation being invariant and dimensionless. An example of a scaled soil water profile $\theta^*(Z^*, T^*)$ for infiltration with DBC is given in Fig. 8.16.

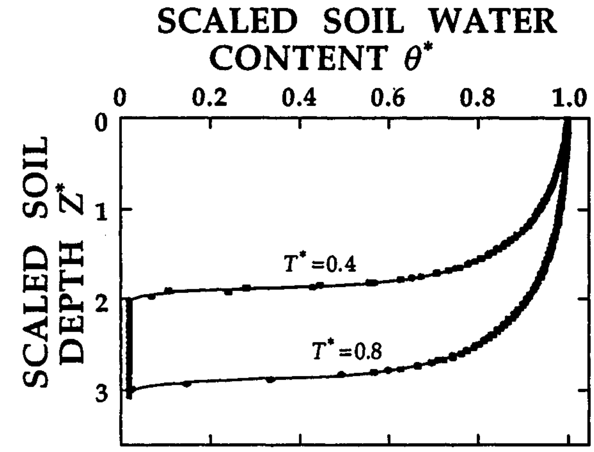

Figure 8. 16. Soil water profile for Guelph loam (Warrick and Hussen, 1993) in scaled variables $\theta^*(Z^*,T^*)$ for DBC infiltration is unique in $\theta^*(Z^*)$ for T^* = 0.4 and 0.8. Data points represent the following combinations of saturated hydraulic conductivity (cm·h^{-1}) and relative soil water content [K_S, θ_o/θ_S]: [1.332, 1.0], [2.664, 1.0], [1.332, 0.9] and [2.664, 0.9].

A scaling procedure is also advantageous for the solution of the inverse problem of infiltration (Warrick, 1993). With the soil hydraulic functions being scaled together with variables, the procedure is conveniently reduced to simple algebraic computations instead of repetitive numerical simulations.