Stylized Facts, finite size effects and control parameters in minority games

Damien Challet[1] and Matteo Marsili[2]
[1] *Theoretical Physics, Oxford University, 1 Keble Road, Oxford OX1 3NP, United Kingdom*
[2] *Istituto Nazionale per la Fisica della Materia (INFM), Trieste-SISSA Unit, Via Beirut 2-4, Trieste 34014, Italy*

We introduce and study the simplest Minority Game able to reproduce some chararectisics of financial markets known as stylized facts. The exact solution of our model in the usual thermodynamic limit does not contain any stylized facts: we show that the usual control parameter is not appropriate in this case. A general mechanism, valid for any minority game with naive agents, can predict for any system size the onset of anomalous dynamics responsible for stylized facts, and gives a new control parameter such that the latter survive in the correct thermodynamic limit. It also explains the origin of finite size effects near the critical point of the original Minority Game.

I. INTRODUCTION

Financial markets are a quite special type of socio-economic system. Apart from their intrinsic interest for the services they provide, they have attracted a great deal of interest also from the academic world. Seen as complex systems of many interacting degrees of freedom, financial markets are unique because a great deal of information on their microscopic dynamics has been recently made available by the information technology revolution. Such a level of detailed information – down to each transaction – is definitely unique for a socio-economic system, but it is even much more than what experimental techniques can offer for many systems in natural sciences.

Empirical analysis of high frequency financial data has shown a number of non-trivial properties of price fluctuations, known as *stylized facts* [1], which are remarkably stable across different markets and periods. These observations have prompted a number of theoretical investigation aimed at understanding the microscopic origin of these stylized facts. As a first step, a number of agents based models [2–12] have been proposed in order to understand what features of the interactions at the micro level are responsible for the emergence of the stylized facts. As a result, several simple mechanisms acting in agent based models of financial markets have been found to reproduce realistic statistical properties: agents switching between fundamentalists and trend followers [4], percolation of herds [5], multiplicative processes [6], grand canonical minority games [7–10] or minority games with evolving capitals [7,11,9,12].

Some of these models have a very complex dynamics which is hard to analyze beyond the numerical approach. Other models which are accessible to analytic investigation are often based on over-simplifying and unrealistic assumptions on the behavior of agents. Can stylized facts be captured by a simple, analytically tractable model?

Grand canonical minority games (GCMG) offer such a possibility. Indeed a quite successful analytic approch to Minority Games has been developed recently [13,14] and we shall see that it can easily be extended to the grand canonic case. We shall introduce, in the following the simplest possible GCMG which reproduces the main stylized facts, i.e. fat tails and volatility clustering (see below). This model is characterized by a phase transition and numerical simulations show that stylized facts emerge at the critical point.

Indeed we find that away from the critical line the behavior of GCMG in the usual thermodynamic limit is characterized by normal fluctuation properties. Numerical simulation confirm these conclusion: stylized facts are observed close to the phase transition for finite systems. But as the system size increases, they disappear. We show that anomalous fluctuations occur for finite systems in a critical window around the transition, which shrinks as the system size increases. This is exactly the same behavior that one observes in critical phenomena. Hence, unless one sits exactly at the phase transition, stylized facts are finite size effects. Remarkably, ref [15] has shown that this is what happens in a variety of models.

Thanks to the simplicity of our model, we are able to analyze finite size effects in detail. As a result, we uncover a general mechanism leading to anomalous fluctuations responsible of the stylized facts. This same mechanism also allows us for a detailed quantitative understanding of finite size effects in the original Minority Game. The analytic approach cannot be extended, in a simple way, to the anomalous fluctuations in the critical window. This poses a new challenge for theoretical investigation and calls for a renormalization group approach to the critical behavior.

The plan of the paper is the following: The next section introduces the model and Section III analyzes its behavior. There we first outline the analytic solution, then we discuss the emergent picture and finally compare it with numerical results. The analytic theory predicts Gaussian fluctuations and no stylized facts. Numerical results show that stylized facts emerge, in the region of the phase diagram where the market is close to being efficient, for small system sizes. The crossover that takes place between this regime of anomalous fluctuations and the Gaussian behavior is analyzed in section V. The crossover arises from a condition requiring that anomalous fluctuations be sustained by agent-agent interaction.

Conclusions are drawn in the final section. Technical details and most of the mathematics involved in the theoretic approach are relegated in the appendix.

II. THE MODEL

In the market described by the minority game, N agents submit a positive or negative bid $b_i(t)$ to the market in every period $t = 1, 2, \ldots$. Agents whose bid has the opposite sign of the total bid $\sum_i b_i(t)$, win whereas the others lose. In any period, the value $\mu(t)$ of a public information variable is drawn uniformly from the integers $1, \ldots, P$ and communicated to agents. Agents have a pool of *trading strategies*, each of which prescribes a bid $b^{\mu}_{s,i}$ for each information μ. Here i labels the agents whereas s labels the strategies of each agent.

The MG was introduced with S strategies taking values $b^{\mu}_{s,i} = \pm 1$ drawn at random for each i, s and μ. Here we consider the case in which each agents has two strategies and $b^{\mu}_{s,i} = s a^{\mu}_i$ with $s = 0, 1$ and $a^{\mu}_i = \pm 1$ drawn at random. In other words, each agent has one active strategy ($s = 1$) and a passive one ($s = 0$) which prescribes not to trade whatever μ is. This a simplification of a more complex model with two active strategies that was recently studied [10]: here we only keep the grand canonical mechanism. Agents learn and adapt their behavior by computing scores $U_{s,i}(t)$ for each of their strategies. These scores are updated as

$$U_{s,i}(t+1) = U_{s,i} - s a^{\mu(t)}_i A(t) + \epsilon_i \delta_{s,0}, \qquad s = 0, 1 \quad (1)$$

where

$$A(t) = \sum_{i=1}^{N} b_i(t) = \sum_{i=1}^{N} s_i(t) a^{\mu(t)}_i \quad (2)$$

is a sort of excess demand [12] and $s_i(t)$ is the strategy chosen by agent i at time t. The choice of agents depends on the scores: The larger the score of a strategy the more likely it is that the agent will adopt it. In particular we take the Logit model [16,30]:

$$\text{Prob}\{s_i(t) = 1\} \equiv \pi_i(t) = \frac{1}{1 + e^{\Gamma[U_{0,i}(t) - U_{1,i}(t)]}}. \quad (3)$$

The term $\epsilon_i \delta_{s,0}$ models the incentives of agents for trading in the market ($\epsilon_i < 0$) or his/her outside opportunities ($\epsilon_i > 0$). For example investors who need the market for exchanging goods or assets have $\epsilon_i < 0$ whereas speculators who only trade for profiting of arbitrage opportunities have $\epsilon_i > 0$, representing for example a fixed interest rate. Of course there may be prudent investors with $\epsilon_i > 0$ or risk-lover speculators with $\epsilon_i < 0$ and a whole range of other type of traders. In the following we focus, for simplicity*, on the case

$$\epsilon_i = \epsilon \qquad \text{for } i \le N_s$$
$$\epsilon_i = -\infty \qquad \text{for } N_s < i \le N$$

The $N_p = N - N_s$ agents who have $\epsilon_i = -\infty$ — we call them *producers* after Refs. [17,18] — trade no matter what, whereas the remaining N_s — the *speculators* — trade only if the score $U_{1,i}(t)$ of their active strategy increases more rapidly than ϵt.

The usual thermodynamic limit is defined as the limit $N_s, N_p, P \to \infty$, keeping constant the reduced number of speculators and producers

$$n_s = \frac{N_s}{P}, \qquad n_p = \frac{N_p}{P}.$$

The collective behavior of the model is best understood in terms of the market's predictability and of the volatility. The market is predictable if the conditional time average $\langle A|\mu\rangle$ of $A(t)$ given $\mu(t) = \mu$, is non-zero (here and in the following $\langle \ldots \rangle$ stands for time average in the stationary state). This means indeed that knowledge of $\mu(t)$ allows a statistical prediction of the sign of $A(t)$ which will be more likely be equal to the sign of $\langle A|\mu\rangle$ than to its opposite. Hence predictability is defined as

$$H_0 = \overline{\langle A\rangle^2} = \frac{1}{P}\sum_{\mu=1}^{P} \langle A|\mu\rangle^2$$

where we introduced the notation $\overline{(\ldots)}$ for averages over μ. When $H_0 = 0$ the market is unpredictable or *informationally efficient*. Volatility is instead defined as

$$\sigma^2 = \overline{\langle A^2\rangle} = \frac{1}{P}\sum_{\mu=1}^{P} \langle A^2|\mu\rangle$$

and it measures market's fluctuations. There are two sources of fluctuations in the model: the choice of the states $\mu(t)$ and the choice of strategies $s_i(t)$. Accordingly σ^2 can be decomposed in fluctuations due to the state $\mu(t)$ – which accounts for a contribution H_0 to σ^2 – and fluctuations induced by the stochastic strategic choice of agents. The contribution to volatility due to the behavior of agents is $\sigma^2 - H_0$.

Both σ^2 and H_0 are extensive, i.e. diverge with the system size. Indeed we expect that $A(t) \sim \sqrt{N}$ because it is a sum of N finite contributions whose average is zero.

In our case, it is then convenient to study predictability or volatility per state of the world† H_0/P or σ^2/P respectively which attain a finite limit when $P \to \infty$. There is another important quantity that is most relevant to grand canonical models, namely the reduced number of speculators in the market: $n_{\text{act}}(t) = \sum_i \langle s_i(t)\rangle / P$.

*Our results can be extended to any reasonable distribution of ϵ_i.

†This constrasts with the usual scaling H_0/N_s and σ^2/N_s.

III. PROPERTIES OF THE MODEL IN THE THERMODYNAMIC LIMIT

This section is devoted to the description of the behavior of the model. First we present the analytic approach, then we discuss the behavior of the solution and compare it to numerical results. The limits $\epsilon \to 0^{\pm}$ and $n_p \to 0$, which have been studied numerically elsewhere [19], deserve special attention and we shall devote a subsection to this. The reader interested in the behavior of the model can skip the next subsection.

A. Analytic solution

In order to study the dynamics in the thermodynamics limit, it is convenient to introduce a rescaled continuous time $\tau = \Gamma t/N$. Indeed characteristic times, which are proportional to N/Γ time steps [20,14,21], remain finite when measured in units of τ when $N \to \infty$[‡]. Following Ref. [14] we derive an Ito stochastic differential equations for the strategy scores

$$y_i(\tau) = \Gamma[U_{1,i}(t) - U_{0,i}(t)], \qquad \tau = \Gamma t/N$$

which read

$$\frac{\mathrm{d}y_i}{\mathrm{d}\tau} = -\overline{a_i \langle A \rangle_\pi} - \epsilon + \eta_i. \tag{4}$$

Here η_i is a Gaussian noise term characterized by zero average and covariance

$$\langle \eta_i(\tau)\eta_j(\tau')\rangle = \frac{\Gamma}{N}\overline{a_i a_j \langle A^2\rangle_\pi}\delta(\tau - \tau'). \tag{5}$$

In Eqs. (4,5) averages $\langle \ldots \rangle_\pi$ are taken on the distribution of Eq. (3) of $s_i(t)$. Unlabeled averages $\langle \ldots \rangle$ stand for averages on the stationary state. Note that $\pi_i(t) = 1/[1+e^{-y_i(t)}]$ depends on the dynamical variables $y_i(t)$. Eq. (4) is a quite complex system of non-linear equations. Note in particular that the noise strength is proportional to the time dependent volatility $\overline{\langle A^2\rangle_\pi}$. This feedback is responsible for the emergence of volatility clustering: In periods of large volatility agents are subject to strong noise which, in turn contributes to volatility build-ups.

Following the approach of Ref. [14] we find that for $\epsilon > 0$:

1. the fraction $n_i \equiv \langle s_i(t)\rangle = \langle \pi_i(t)\rangle$ of times that agent i plays his active strategy in the stationary state is the solution of the minimization problem of the function

$$H_\epsilon\{n_i\} = H_0 + 2\epsilon \sum_{i=1}^{N_s} n_i \tag{6}$$

where

$$H_0 = \overline{\langle A \rangle^2} = \frac{1}{P}\sum_{\mu=1}^{P}\left[\sum_{i=1}^{N} n_i a_i^\mu + \sum_{i=N_s+1}^{N_s+N_p} a_i^\mu\right]^2 \tag{7}$$

is the market's predictability. This conclusion holds also for $\epsilon \le 0$.

2. The solution to this problem, and hence the stationary state, is unique for all parameters.[§]

3. An exact statistical mechanics description of the solution to the minimization of H_ϵ can be carried out with the replica method. The solution is indeed replica symmetric. This allows one to compute exaclty all thermodynamic quantities which depend on n_i as well as the distribution of n_i. The calculation is sketched in appendix A.

4. The solution to the Fokker-Planck equation corresponding to Eq. (4) can be well approximated by[**]:

$$P_0(\vec{\pi}) = \prod_{i=1}^{N_s} \frac{\Gamma(\beta)\pi_i^{\beta n_i - 1}(1-\pi_i)^{\beta(1-n_i)-1}}{\Gamma(\beta n_i)\Gamma(\beta - \beta n_i)} \tag{8}$$

where

$$\beta = \frac{N}{\Gamma \sigma^2} \tag{9}$$

depends on the volatility $\sigma^2 = \overline{\langle A^2\rangle}$. Note that the distribution of π_i factorizes, which means that the fluctuations of the scores are independent across agents in the stationary state. Details are reported in appendix B.

5. The volatility is given by

$$\sigma^2 = \overline{\langle A^2\rangle} = H_0 + \sum_{i=1}^{N_s} n_i(1-n_i) \tag{10}$$

and it is independent of Γ. This is a consequence of the fact that, by Eq. (8), $\langle s_i s_j\rangle = \langle s_i\rangle\langle s_j\rangle = n_i n_j$ for all $i \ne j$. Then σ^2/P can be computed exactly as a function of the parameters n_s, n_p and ϵ.

[‡]This also means that the simulation times must be rescaled accordingly.

[§]This contrasts with the original MG.

[**]Here $\Gamma(x)$ stands for Euler's Gamma function and it has nothing to do with the constant Γ introduced earlier. For *frozen* agents who have $n_i = 0$ or $n_i = 1$ we assume that $x^{z-1}/\Gamma(z) \to \delta(x)$ as $z \to 0$ in the above formula. In other words $\pi_i = n_i$ for frozen agents.

6. The variable

$$X(t) = \frac{A(t)}{\sigma}$$

converges in distribution to a Gaussian variable with zero average and unit variance. No long range correlation arises. This is just a consequence of the previous results and of the central limit theorem.

B. Characteristic behavior

The behavior of H_0/P, σ^2/P and $n_{\rm act}$ as a function of n_s is reported in Fig. 1 for $\epsilon = 0.1$ and $n_p = 1$. These graphs illustrate the generic behavior of the model. With a fixed number n_p of producers, as the number n_s of speculators increases the market becomes more and more unpredictable, i.e. H_0 decreases. Naively speaking, agents are attracted by the profit opportunities of a predictable market. But as they join the market and start exploiting these opportunities, they reduce market's predictability. At the same time also the volatility σ^2 decreases. For a market with few speculators ($n_s < 1$ in Fig. 1), most of the fluctuations in $A(t)$ are due to the random choice of $\mu(t)$ (i.e. $\sigma^2 \simeq H_0$). In this region the number $n_{\rm act}$ of active speculators is roughly a constant fraction of the total n_s.

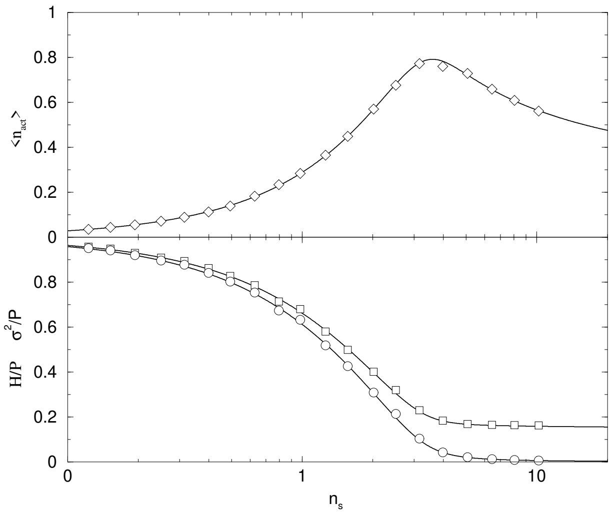

FIG. 1. Left panel: σ^2/P and H/P; right panel: $\langle n_s \rangle$. Continous lines are predictions of the replica theory whereas point are results of numerical simulations done keeping $N_s P = 10000$ fixed (averages over 200 samples).

When the number of speculators increases further, the market reaches a point where it is barely predictable. Then adding new speculators the number of active traders starts decreasing and it finally converges to a constant. This means that the market becomes highly selective: Only a negligible fraction of speculators trade ($s_i(t) = 1$) whereas the majority is inactive ($s_i(t) = 0$). The volatility σ^2 also remains constant in this limit.

Summarizing, there are two regimes according to whether $n_s \le n_s^\star$ or $n_s \ge n_s^\star$ where $n_s^\star$ is the point where $n_{\rm act}$ attains its maximum. The crossover between these two turns into a phase transition when $\epsilon \to 0^+$. We shall analyze in detail the nature of this limit. For the moment being, let it suffice to say that crowded markets $n_s \ge n_s^\star$ become efficient, i.e. $H_0 \propto \epsilon^2 \to 0$, as $\epsilon \to 0$ whereas H_0 reaches a finite limit as $\epsilon \to 0$ in uncrowded markets ($n_s < n_s^\star$); when $\epsilon = 0$ the transition at $n_s^\star$ is the usual transition between an efficient and an inefficient market phase which occurs in the Minority Game with naive agents. As shown below the line $\epsilon = 0$ for $n_s \ge n_s^\star$ is the locus of a first order phase transition.

It is exactly close to this line that stylized facts are observed in numerical simulations. Before coming to that we observe that *i)* numerical simulation fully confirm the picture outlined here and *ii)* this same picture applies to a wider range of GCMG models such as that of Ref. [10].

C. The limits $\epsilon \to 0^\pm$ and $n_p \to 0$

The limit $\epsilon \to 0$ requires a particular discussion. There exists a critical number $n_s^\star(n_p)$ of speculators such that for $n_s > n_s^\star(n_p)$

$$\lim_{\epsilon \to 0^+} n_{\rm act}(\epsilon, n_s, n_p) \neq \lim_{\epsilon \to 0^-} n_{\rm act}(\epsilon, n_s, n_p). \qquad (11)$$

For $\epsilon > 0$ the reduced number of active speculators $n_{\rm act}$ remains constant in the limit $n_s \to \infty$. On the contrary, when $\epsilon < 0$, $n_{\rm act}$ increases linearly with n_s for $n_s > n_s^\star$. In other words, $\epsilon = 0$ marks the location of a first order phase transition for $n_s > n_s^\star$ (see Fig 2). At the transition point $\epsilon = 0$ any fraction of active agents between $n_{\rm act}(0^+, n_s, n_p)$ and $n_{\rm act}(0^-, n_s, n_p)$ can be observed depending on the initial conditions. This feature is reminiscent of the phase coexistence which accompanies first order phase transitions. Fig. 2 plots $n_{\rm act}$, σ^2/P and H_0/P as a function of ϵ.

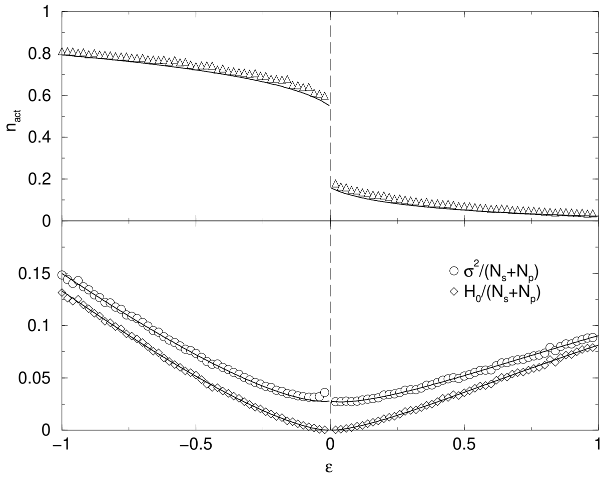

FIG. 2. Average relative number of active speculators, fluctuations and available information as a function of ϵ ($P = 40$, $n_s = 5$, $n_p = 1$, average over 100 samples). Continous lines are theoretical predictions.

It is also of interest to discuss the behavior of the solution for $n_p \to 0$ at fixed n_s, which is the case which has been studied in Ref. [19]. For $\epsilon > 0$, we find that $n_{\rm act} \to 0$ in this limit for all values of n_s: when there are no producers speculators refrain from entering the game. In addition, when ϵ and n_p are small and n_s is large, we find $n_{\rm act} \sim \sqrt{n_p}$, as observed in Ref. [10].

For $\epsilon < 0$ we find instead that $n_{\rm act}$ vanishes only for $n_s \leq n_s^\star$ as $n_p \to 0$. In crowded markets, i.e. when $n_s > n_s^\star$, we find that $n_{\rm act}$ remains positive. Hence for $\epsilon < 0$ and $n_p = 0$ we find a sort of phase transition to an absorbing state as n_s gets larger than $n_s^\star$.

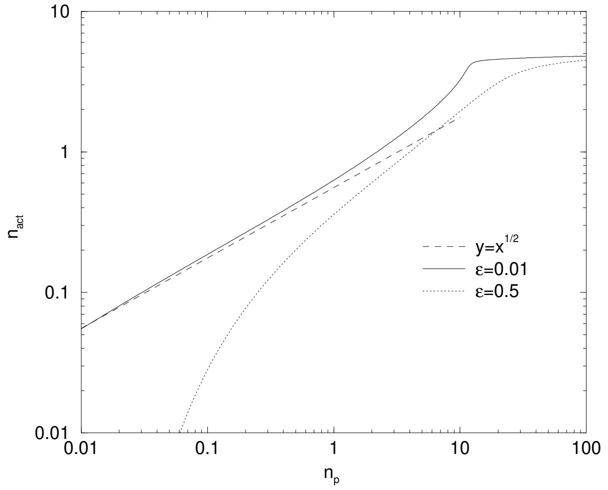

FIG. 3. Average fraction of active speculators $n_{\rm act}$ as a function of n_p for various values of ϵ at fixed $n_s = 5$. For small ϵ and n_p, $n_{\rm act} \propto \sqrt{n_p}$.

We conclude this section observing that the model of Ref. [19] was rather peculiarly placed *on* a first order phase transition line ($\epsilon = n_p = 0$).

IV. STYLIZED FACTS: NUMERICAL EVIDENCE

Numerical simulations of the model for finite sizes reproduce anomalous fluctuations similar to the stylized facts close to the phase transition line. Typically, at fixed P or PN and for any $\epsilon > 0$, the deeper one moves in the crowded regime the wilder the fluctuations which the market exhibits. For $n_s \gg n_s^\star$ volatility clustering becomes so dramatic that bursts of activity become real market crashes, as observed in Ref. [10].

A. Finite size effects

However, stylized facts disappear in the thermodynamic limit $P \to \infty$ as long as $\epsilon > 0$. This is clearly shown in Fig. 4, which analyzes the (excess) kurtosis $K = \langle A^4 \rangle / \langle A^2 \rangle^2 - 3$ of the probability distribution of $A(t)$ as a function of P for fixed values of ϵ, n_s and n_p. Large values of K are indicative of a fat tailed distribution whereas a Gaussian $A(t)$ would have $K = 0$. The kurtosis generally vanishes as the system size P increases.

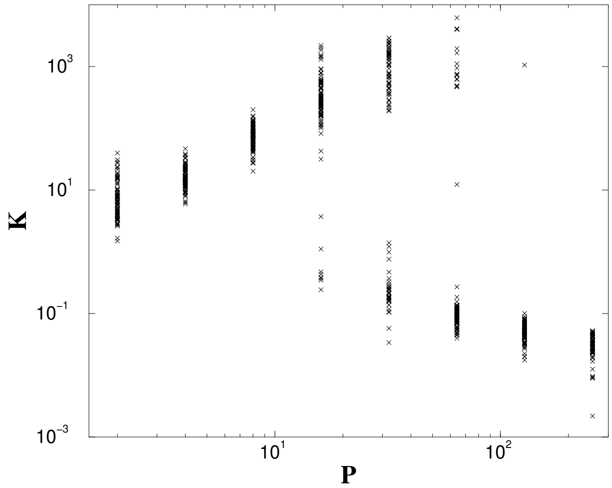

FIG. 4. Kurtosis of $A(t)$ in simulations with $\epsilon = 0.01$, $n_s = 70$, $n_p = 1$ and several different system sizes P.

For a given system size, some realizations exhibit anomalous dynamics and stylized facts (high kurtosis) whereas others follow statistics closer to Gaussian. For example, Fig. 5 shows three realizations, for the same values of the parameters and system size, with a totally different statistics of fluctuations and returns. The coexistence of anomalous and normal realizations of market's dynamics for the same parameters is reminiscent of the discontinuous nature of the phase transition at $\epsilon = 0$.

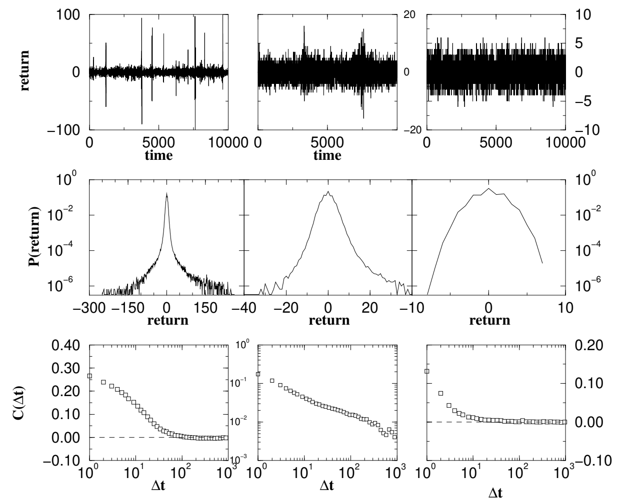

FIG. 5. Three runs of the model with the same parameters ($P=16$, $n_s=62.5625$, $n_p=1$, $\epsilon=0.1$) but different realizations of the strategies a_i^μ of the agents. Volatility clustering and/or fat tails appear in some realizations but not in others.

Fig. 4 shows that the realizations with anomalous statistics – which have a relatively large Kurtosis – become less and less probable as the system size gets large. This is a strong evidence that stylized facts are finite size effects in these market models.

B. Initial conditions, grand grand canonical MG

All previous numerical simulations were done by taking $y_i(t)=0$ for all speculators, that is, they had no *a priori* bias towards playing or not playing. The cause of stylized facts in GCMG is the same as that in the MG for large volatility (see section V). But the latter is known to strongly depend on the initial conditions [29,?]. The question is what initial conditions are relevant to markets. We argue here that in reality, new agents interested in a given stock will typically look at the time serie of the stock, and try to learn something about it. In GCMG, the simplest corresponding experiment consists in defining two groups of speculators, one which is allowed to play from the beginning, and the other one which is allowed to play only after a given time T_0 (a kind of grand grand canonical MG). At time T_0, a fraction of the latter will enter the market, provoking a shock. The region where stylized facts can be observed has typically a very large susceptibility, hence the shock can destabilize the market. We observed that if the initial market has stylized facts, so will the resulting market. What is more, if the initial market has no stylized facts, the second group sometime provoques stylized facts. This clearly shows that the emergence stylized facts is easier when the model is closer to real markets, and will be explored in more details elsewhere.

Finally, note that if $\epsilon=0$ and $n_s>n_s^*$, the susceptibility is infinite and the system is very sensitive to initial conditions.

V. STYLIZED FACTS: FINITE SIZE SCALING THEORY

If stylized facts are finite size effects which arise only close to the phase transition it is natural to expect that they can be related to the critical properties at the transition point by finite size scaling theory. A quite similar behavior occurs in statistical mechanics, e.g. in the Ising model. If we study a finite system at temperature $T=T_c+\varepsilon$ we shall see critical fluctuations as long as the system size L is shorter than the correlation length $\sim\varepsilon^{1/\nu}$. Only for $L\gg\varepsilon^{1/\nu}$ we shall start to see the correlations of a normal paramagnetic system. There are several differences, notably the long-range interaction of the GCMG and the discontinuous nature of the transition. This makes it non-trivial to apply the theory of critical phenomena to study the critical region of the GCMG. In what follows, we shall move the first step by identifying the size of the critical region and the relevant scaling limit.

From the perspective of Eqs. (4,5), volatility clustering arises because the noise strength is proportional to the time dependent volatility $\overline{\langle A^2\rangle_\pi}$. Time dependence of the volatility can be sustained by collective fluctuations of the agents scores y_i. In order for this feedback effect to be self-sustained, the contribution to agent-agent interaction due to the noise should be larger than that due to the deterministic dynamics. By Eqs. (4,5) this requires

$$(\overline{a_i\langle A\rangle_\pi}+\epsilon)(\overline{a_j\langle A\rangle_\pi}+\epsilon)\ll\overline{a_ia_j\langle A^2\rangle_\pi}$$

For a rough estimate of this condition we note that

$$\overline{a_ia_j\langle A^2\rangle_\pi}\sim\frac{\sigma^2}{P}\frac{1}{\sqrt{P}}$$

where we factorize the average over μ and we used used $\overline{a_ia_j}\sim 1/\sqrt{P}$. Similarly

$$\overline{a_i\langle A\rangle_\pi}\sim\sqrt{\frac{H_0}{P}}$$

which comes from Cauchy-Schwartz inequality. This suggests that volatility clustering is possible when

$$\frac{H_0}{\sigma^2}+2\epsilon\sqrt{\frac{H_0}{P}}\frac{P}{\sigma^2}+\epsilon^2\,\frac{P}{\sigma^2}=\frac{K}{\sqrt{P}}\tag{12}$$

where K is a constant; we made here some very crude estimations that we be refined elsewhere. Note that $\frac{H_0}{\sigma^2}$ is roughly equal to the signal-to-noise ratio $\frac{H_0}{\sigma^2-H_0}$ when $H_0/\sigma^2\ll 1$ [12]. Hence, when $\epsilon\to 0$, anomalous dynamics arises when the signal to noise ratio is smaller than $K/\sqrt{P}$ where K is a constant. This general mechanism also explains finite size effects close to the phase transition of the standard MG (see section VI), and in fact is universal in MG with this type of learning. Note also that this mechanisms does not depend at all on Γ.

This prediction is remarkably confirmed by Fig. 6: In the lower panel we plot $F=\frac{H_0}{\sigma^2}+2\epsilon\sqrt{\frac{H_0}{P}}\frac{P}{\sigma^2}+\epsilon^2\,\frac{P}{\sigma^2}$ and $K/\sqrt{P}$ as a function of n_s. These two curves intersect in a point $n_s^c(P)$ and the upper panel shows that the volatility σ^2/N starts deviating from the value it attains in the thermodynamic limit exactly beyond $n_s^c(P)$ where condition (12) starts holding. Furthermore Fig. 7 shows that the region $n_s>n_s^c(P)$ is described by a novel type of scaling limit. Indeed the curves of Fig. 6 collapse one on top of the other when plotted against $n_s/n_s^c(P)$.

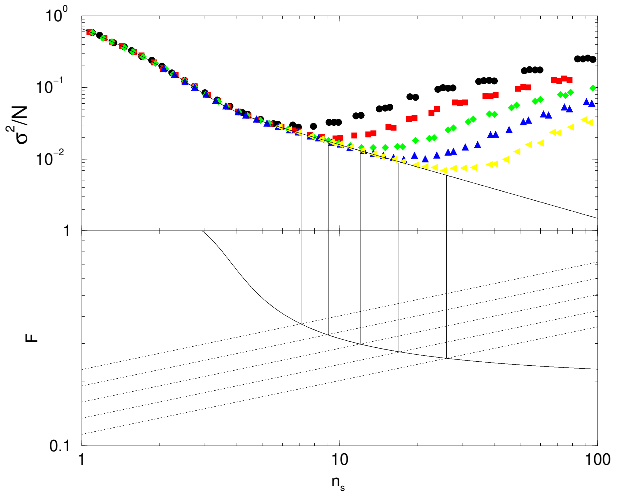

FIG. 6. Onset of the anomalous dynamics for different system sizes. Top: σ^2/N for different series of simulations with $L \equiv PN_s$ constant: $PN_s = 1000$ (circles), 2000 (squares), 4000 (diamonds), 8000 (up triangles) and 16000 (left triangles). In all simulations $n_p = 1$ and $\epsilon = 0.1$ Bottom: $F = \frac{H_0}{\sigma^2} + 2\epsilon\sqrt{\frac{H_0}{P}\frac{P}{\sigma^2}} + \epsilon^2 \frac{P}{\sigma^2}$ (full line) given by the exact solution and $K/\sqrt{P} = K(n_s/L)^{1/4}$ (parallel dashed lines) as a function of n_s ($K \simeq 1.1132$ in this plot). The intersection defines $n_s^c(P)$.

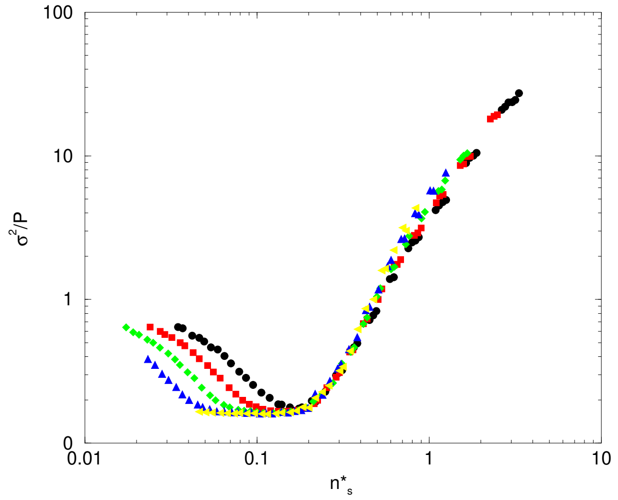

FIG. 7. Collapse plot of finite size effects.

VI. ANOMALOUS DYNAMICS IN MINORITY GAMES

The same mechanism can explain anomalous dynamics, i.e. deviations from the replica solution in all Minority Games with the learning defined above. For instance, consider finite size effects near the critical point of the original Minority Game [22]. We shall refer the reader to Ref. [13] for a precise definition of the model (see also Ref. [17] for the model with producers). In brief there are only speculators ($n_p = 0$) with two randomly drawn active strategies. The behavior is characterized again by H_0 and σ^2 as a function of $\alpha = 1/n_s$ [31](now $n_{\mathrm{act}} = 1$ by definition). The original MG undergoes a phase transition, with $H_0 = 0$ for $\alpha \leq \alpha_c = 0.3374\ldots$, and $H_0 > 0$ otherwise. Here the condition for the onset of the anomalous dynamics is given by

$$\frac{H_0}{\sigma^2} = \frac{K}{\sqrt{P}}. \tag{13}$$

Fig. 8 shows that the same graphical construction of the previous section predicts the onset of critical fluctuations for a finite size system. In particular, given that $H_0 \propto (\alpha - \alpha_c)^2$ and σ^2 is finite, we find that the critical region shrinks as $\alpha - \alpha_c \sim P^{-1/4}$.

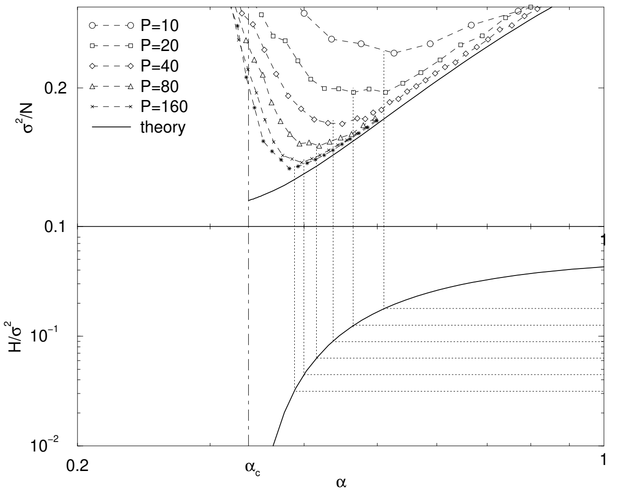

FIG. 8. Top panel: σ^2/N for increasing P and $\Gamma = \infty$. Bottom panel: ratio H_0/σ^2 (continuous line) and $y = C/\sqrt{P}$ (dotted lines). The intersection between these curves identify the critical region for finite sizes.

Therefore, in the thermodynamic limit, σ^2/N converges to the theoretical value given by the replica symmetric ansatz for $\alpha > \alpha_c$ and it is independent of Γ for all $\Gamma > 0$. This is fully supported by numerical simulations. Note that the time to equilibration is very large near the critical point, and special care has to be taken in numerical simulations. For example, we found that, at $\alpha = 0.345$, numerical results for $\Gamma = \infty$ converge to the theoretical prediction for $P \simeq 4000$. According to the argument above, convergence would require $P \simeq 40000$ for $\alpha = 0.34$, which is out of reach of our present computing facilities. For $\alpha \leq \alpha_c$, as discussed in Ref. [14], the fluctuations in the scores of agents are no more independent, as in Eq. (8), because the strategy space becomes too "small". This results into a dependence of the stationary state on Γ and on initial conditions.

Finally, this mechanism explains why there is no anomalous dynamics when agents take into account their impact on the game [13,33]: in this case, one can show that the noise strength and the payoff depends on each other, implying that the former cannot mask the latter.

VII. CONCLUSIONS

We have shown that stylized facts are related to the critical properties of the phase transition in minority games, which are not trivial. A deep understanding of volatility clustering and stylized facts in GCMG requires a detailed theory of the critical phenomena taking place at the phase transition.

This constrasts with the universal mechanism for volatility clustering recently proposed [19]. That argument is uniquely based on the fact that the processes $y_i(\tau)$ can be considered as random walks and the statistics of return to the origin is sufficient to explain volatility clustering. If this argument were true, any realization of the process would display stylized facts with a cutoff which would depend on the distance to the critical point. But we observed that stylized facts appear in some realizations but are absent in others (see Fig. 4). We conclude that understanding stylized facts in GCMG remain an open challenge. Further tools of statistical physics, such as the renormalization group, are required to investigate the nature of the critical state and its anomalous fluctuations.

APPENDIX A: SOLUTION OF THE PROBLEM $\min H_\epsilon$ BY REPLICA METHOD

The Hamiltonian to minimize is $H_\epsilon = H_0 + 2\epsilon\sum_i \pi_i$, where $H_0 = \overline{\langle A\rangle^2}$ is the usual hamiltonian. Since we are interested in the average of this quantity over the possible realizations, we have to compute the average free energy $F = \langle \ln Z\rangle$ in the limit $\beta \to \infty$, where $Z = \mathrm{Tr}_s e^{-\beta H_\epsilon}$ is the partition function. The replica trick rests on the identity $\ln Z = \lim_{n\to 0}(Z^n - 1)/n$ and consists in introducing n replica of a given system, that is, the quenched disorder (the a_i^μ's and b_i^μ's) is kept fixed while one replicates n times the π_i's. The calculus itself is of not special interest (see for instance ref for more details [17]). We finally find that the free energy at the saddle point, in the $\beta \to \infty$ limi,t is equal to

$$H_\epsilon = \epsilon^2\,\frac{\rho + G(\zeta, K)}{K^2} + 2\epsilon\phi_{\rm in}(K,\zeta) \tag{A1}$$

where $G(K,\zeta)$ is given by

$$G(K,\zeta) = \frac{1}{\zeta\sqrt{2\pi}}\left[(K-1)\mathrm{e}^{-(1+K)^2\zeta^2/2} - K\mathrm{e}^{-K^2z^2/2}\right] \\ +\frac{1}{2}\left(K^2 + \frac{1}{\zeta^2}\right)\left(\mathrm{erf}\,[(1+K)\zeta/\sqrt{2}] - \mathrm{erf}\,(K\zeta/\sqrt{2})\right) \\ +\frac{1}{2}\,\mathrm{erfc}\,[(1+K)\zeta/\sqrt{2}]. \tag{A2}$$

and where ζ and K take their saddle point values determined by

$$\frac{\alpha}{\zeta^2} = \rho + G(K,\zeta) \tag{A3}$$

and

$$K = \epsilon\left[1 - \frac{\mathrm{erf}\,[(1+K)\zeta/\sqrt{2}] - \mathrm{erf}\,(K\zeta/\sqrt{2})}{2\alpha}\right]^{-1}. \tag{A4}$$

These two last equations form a close non-linear set of equations that can be solved numerically The fluctuations are given by

$$\sigma^2 = \epsilon^2\,\frac{\rho + G}{K^2} + \phi_{\rm in} - G. \tag{A5}$$

The fraction of agents who never enter into the market is $\phi_0 = (1 + \mathrm{erf}\,(\zeta K/\sqrt{2})/2$, and the ones who always participate is $\phi_1 = \mathrm{erfc}\,[(1+K)\zeta/\sqrt{2}]/2$. The pdf of the π_i is given by

$$\mathcal{P}(\pi) = \phi_0\delta(\pi) + \phi_1\delta(\pi - 1) + \frac{\zeta}{\sqrt{2\pi}}e^{-(\zeta(\pi+K))^2/2} \tag{A6}$$

while the average number of agents in the market $\phi_{\rm in} = \langle\pi\rangle$, where the average is over $\mathcal{P}(\pi)$, is

$$\phi_{\rm in} = \phi_1 + \frac{1}{\zeta\sqrt{2\pi}}\left(\mathrm{e}^{-K^2\zeta^2/2} - \mathrm{e}^{-\zeta^2(1+K)^2/2}\right) \\ +\frac{K}{2}(\mathrm{erf}\,(K\zeta/\sqrt{2}) - \mathrm{erf}\,[(1+K)\zeta/\sqrt{2}])$$

One can also show that $H_0 \propto \epsilon^2$ for small ϵ: according to Eq. (A1), this hold if $K \to 0 < K_0 < \infty$, which can be seen by expanding Eq. (A4) in powers of ϵ.

APPENDIX B: SOLUTION OF THE FOKKER-PLANCK EQUATION

As in Ref. [14] we can simplify considerably the problem by using the stationary state conditions

$$\frac{d}{dt}\langle y_i\rangle = 0$$

for all agents i such that $0 < n_i < 1$ strictly, which are called non frozen agents. Note that producers can be considered as frozen agents with $n_i = 1$. Given the producers the solution to the problem $\min H_0$ specifies the set of frozen agents. Let N_u be the number of non-frozen agents and let us relabel agents so that agent $i \le N_u$ is non-frozen while $i > N_u$ is frozen.

The Fokker-Planck equation for the N_u non-frozen agents, after a straightforward calculation, reads

$$\frac{\partial P}{\partial \tau} = \sum_{i=1}^{N_u}\frac{\partial}{\partial y_i}\left[\sum_{j=1}^{N_u}\overline{a_i a_j}(\pi_j - n_j) + \frac{\Gamma}{N}\frac{\partial}{\partial y_j}\overline{a_i a_j\langle A^2\rangle_\pi}\right]P. \tag{B1}$$

The solution of this equation in the stationary state is made problematic by the presence of $\langle A^2|\mu\rangle_\pi$ in the diffusion matrix. Indeed Ref. [14] shows that if one neglects both the dependence on μ and on π of this term, i.e. if one substitutes $\langle A^2|\mu\rangle_\pi$ with $\overline{\langle A^2\rangle}=\sigma^2$, one easily finds the zero probability current solution

$$P_0(\vec{y})=\prod_{i=1}^{N_u}\frac{\Gamma(\beta)}{\Gamma(\beta n_i)\Gamma(\beta-\beta n_i)}\frac{e^{\beta n_i y_i}}{(1+e_i^y)^\beta}. \quad \text{(B2)}$$

This solution is a good approximation to the real solution. In order to show this, let $\langle\ldots\rangle_0$ denote averages taken w.r.t. P_0. Then, using Eq. (B1), it is easy to find that for any function $F(\vec{y})=F(y_1,\ldots,y_{N_u})$

$$\frac{\partial\langle F(\vec{y})\rangle_0}{\partial\tau}=\beta\sum_{i,j}\left\langle\overline{a_i a_j(1-\langle A^2\rangle_\pi/\sigma^2)}F(\vec{y})(\pi_i-n_i)(\pi_j-n_j)\right\rangle_0 \quad \text{(B3)}$$

Then, if we neglect the dependence on π of $\langle A^2|\mu\rangle_\pi$, which corresponds to the time independent volatility approximation, all single agent observables of the form $F(\vec{y})=\sum_k f_k(y_k)$ are stationary: $\frac{\partial}{\partial\tau}\langle F(y_k)\rangle_0=0$. This is because the average on P_0 factorizes and all terms $i\neq j$ vanish. Also the term $i=j$ vanishes because $\overline{a_i a_i(\langle A^2\rangle/\sigma^2-1)}=0$. In particular one such observable is the entropy $S_0=-\langle\ln P_0\rangle_0$.

Furthermore, if we take $F(\vec{y})=(\pi_k-n_k)(\pi_l-n_l)$ we find

$$\frac{\partial}{\partial\tau}\langle(\pi_k-n_k)(\pi_l-n_l)\rangle_0= -\beta\overline{a_k a_l(\langle A^2\rangle/\sigma^2-1)}\frac{n_k(1-n_k)}{1+\beta}\frac{n_l(1-n_l)}{1+\beta} \quad \text{(B4)}$$

This does not vanish identically. However it has a random sign, depending on the realization of the disorder, ant it is of order $1/\sqrt{P}$. Hence in the thermodynamic limit also two agents observables are stationary. In particular, if correlations $\langle(\pi_k-n_k)(\pi_l-n_l)\rangle$ vanish for $k\neq l$ in the state 0, they will not be generated by the dynamics. In addition also $\langle A|\mu\rangle_\pi$, which is a two agents observable, is stationary in the thermodynamic limit. This supports the validity of the time independent volatility approximation above. The same calculation can be redone relaxing this last approximation. This results in more complex expressions for time derivatives of one and two agents observables. However they remain of order $1/\sqrt{P}$ and hence vanish in the thermodynamic limit.

---

[1] For a review, see chapter V of M. M. Dacorogna, R. Gençay, U. Müller, R. B. Olsen, O. V. Pictet, *An Introduction to High-Frequency Finance*, Academic Press, London (2001)

[2] B. LeBaron, W. B. Arthur, R. G. Palmer, J. of Economic Dynamics and Control, Vol. 23, (1999)

[3] M. Marsili and YC Zhang, Europhys. Lett. **40**, 479 (1997)

[4] T. Lux and M. Marchesi, Nature **397**, 498 - 500 (1999)

[5] R. Cont and J.-P. Bouchaud, Macroecon. Dyn. **4**, 170 (2000)

[6] H. Levy, M. Levy, S. Solomon, *From Investor Behavior to Market Phenomena*, Academic Press, London (2000)

[7] F. Slanina and Y.-C. Zhang, Physica A **272**, 257 (1999), e-print cond-mat/9906249

[8] P. Jefferies, M.L. Hart, P.M. Hui, N.F. Johnson, Int. J. Th and Appl. Fin. **3**-3 (2000), e-print cond-mat/9910072.

[9] T. S. Lo, P. M. Hui, N. F. Johnson, e-print cond-mat/0008387

[10] D. Challet, M. Marsili and Y.-C. Zhang, Physica A **294**, 514 (2001), e-print cond-mat/0101326

[11] J. D. Farmer and S. Joshi, Santa Fe Institute working paper 99-10-071.

[12] D. Challet, A. Chessa, M. Marsili, Y.-C. Zhang, Quant. Fin. **1**, 168 (2001), e-print cond-mat/0011042

[13] D. Challet, M. Marsili and R. Zecchina, Phys. Rev. Lett. **84**, 1824 (2000), e-print cond-mat/9904392

[14] M. Marsili, D. Challet, Phys. Rev. E **64**, 056138 (2001), preprint cond-mat/0102257

[15] E. Egenter, T. Lux, D. Stauffer, Physica A **268**, 250 (1999)

[16] The learning process belongs to the class of models described in C. Camerer and T.-H. Ho, Econometrica **67**, 827 (1999) with $\phi=\kappa=\delta=1$.

[17] Challet D., M. Marsili and Y.-C. Zhang, Physica A **276**, 284 (2000) e-print cond-mat/9909265

[18] Y.-C. Zhang, Physica A **269**, 30 (1999). e-print cond-mat/9901243

[19] J.-P. Bouchaud, I. Giardina, M. Mezard, e-print cond-mat/0012156

[20] D. Challet, M. Marsili and R. Zecchina, Phys. Rev. Lett. **85**, 5008 (2000), e-print cond-mat/0004308

[21] A. C. C. Coolen, J. A. F. Heimel, e-print cond-mat/0107600

[22] Challet D. and Zhang Y.-C., Physica A **246**, 407 (1997) e-print adap-org/9708006

[23] See the Minority Game's web page on `http://www.unifr.ch/econophysics`.

[24] M. Marsili, D. Challet and R. Zecchina, Physica A **280**, 522 (2000), e-print cond-mat/9908480

[25] M. Marsili and D. Challet, Adv Compl. Sys. **3**-I (2001), e-print cond-mat/0004376 e-print cond-mat/0004196

[26] J. A. F. Heimel, A. C. C. Coolen, Phys. Rev. E **63**, 056121 (2001), e-print cond-mat/0012045.

[27] M. Hart, P. Jefferies, P. Hui, N. F. Johnson, e-print cond-mat/0008385

[28] D. Challet and M. Marsili, Phys. Rev. E **60**, 6271 (1999), e-print cond-mat/9904071

[29] M. Marsili and D. Challet, Adv Compl. Sys. **3**-I (2001), e-print cond-mat/0004376 e-print cond-mat/0004196

[30] A. Cavagna, J.P. Garrahan, I. Giardina, D. Sherrington, Phys. Rev. Lett. **83**, 4429 (1999) e-print cond-mat/9903415 (1999).
bibitemOxf2 J. P. Garrahan, E. Moro, D. Sherrington, Phys. Rev. E 62, R9 (2000)

[31] Savit R., Manuca R., and Riolo R., Phys. Rev. Lett. **82(10)**, 2203 (1999), e-print adap-org/9712006.
[32] M. Marsili, M. Pia, e-print cond-mat/0202479
[33] M. Marsili, D. Challet and R. Zecchina, Physica A **280**, 522 (2000), e-print cond-mat/9908480