A Simple Micro-Model of Market Dynamics
Part I :The "Homogenous Agents" Deterministic Limit

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May 15, 2002

Abstract

In this paper we present a simple agent based model aimed to the qualitative description of some "stylized facts" typical of financial markets. The framework is a simple two assets model: a riskless bond, with a constant riskless return and a risky stock, paying constant dividends. Both the riskless rate of return and the dividends process are assumed known to the agents. Starting from aggregate excess demand the risky asset price is fixed via Walrasian auction. The market participants are speculators described as myopic utility maximizer and are embedded with limited forecasting ability. The exact expressions of the utility function and the forecasting procedure are chosen in order to admit a simple analytic treatment of the market dynamics in the deterministic limit of homogeneous agents. However, a short discussion of the effect of different choices is proposed. We find that in the deterministic limit the model posses many "phases". In particular, the no-arbitrage "fundamental" price can emerge as a stable fixed point, while for different parameterizations the market shows chaotic dynamics with speculative bubbles and crashes.

PRELIMINARY DRAFT

1 Introduction

This paper, the first in a series of two, is devoted to the formulation of an agent-based model intended to describe the dynamics of a bare-bone financial market. The model, both in the description of the agents behavior and in the implementation of the market structure is inspired to the highest simplicity. Concerning the latter, we consider the minimalistic assumption of just two assets, one riskless bond and one risky equity whose price is determined using Walrasian auction.

The "minimality" assumption about the market structure has been quite common in the agent-based literature, starting from its early contributions, and is mainly dictated by the idea that the intrinsic difficulties in the implementation of multi-asset trading behaviors does not pay back in term of an increased richness in the model emerging features and, more importantly, does not reduce the possible market instabilities (some recent investigations seem indeed to strength this general belief, see e.g. Brock and Hommes (2001)).

On the other hand, the "minimality" in the description of agents strategies and behaviors cannot be claimed by many investigations. Indeed one could say that an entire branch of this literature has born from the "complex system" paradigm Arthur et al. (1997) and has modeled very rich settings where not only different trading strategies were competing, but also the same number and structure of the competing strategies did change over time. These models can show, under certain conditions, interesting features, nevertheless their systematic study is made impossible by their enormous number of degree of freedoms.

More recently a different literature has emerged, focusing on more "sober" and treatable settings, in some sense reverting to the pioneering investigations dating back to the '70s (see LeBaron (2000) for a review of early contributions), the main novelty being constituted by the introduction of some form of "bounded rationality" (or better "inductive rationality" as in Arthur (1994)) that account for the agents decisions. We cannot give here even a cursory description of the many studies belonging to this rapidly increasing literature (the reader should refer to the recent books published on this topic, for instance Levy et al (2000)) but we simply want to point out that, in general, heterogeneity in agents behavior and a strong dynamics in agents believes play prominent roles in shaping the dynamics of these models. The basic idea is that only an heterogeneous population of traders, characterized by a switching dynamics between different "trading strategies" (or, more generally, "visions of the world") induced by an (apparent) difference in their relative "rewardingness" (Brock and Hommes, 1998; Chiaromonte et al., 1999) or by imitative behavior (Kirman and Teyssiere, 2002) can actually lead to interesting market dynamics.

In this paper we follow a different path and we tray to analyze the aggregate market dynamics emerging from a very "non-complex" framework of (quasi-) homogeneous agents. In this respect, we think that our work can be considered builded upon the recent investigations of the effect of "simple trading rules" on the market aggregate dynamics (Farmer, 1998; Farmer and Joshi , 1999; Farmer and Lo, 1998; Levy et al, 2000), even if stealing shamelessly many building blocks from various important contributions (Brock and Hommes, 1998; Hommes, 2001; LeBaron, 2001).

Our agents chose their portfolio composition starting from forecasted price dynamics and following a myopic utility maximization procedure. The agent forecasting abilities are limited to "fundamentalist" rules and simple "econometric" procedures.

The choice to consider such an extremely "simplistic" approach is dictated by the observation that there are few "standard arguments" that almost always appear in the discussion about the agent-based models and that heave never undergone precise analysis. In particular

- the idea that there are "stabilizing" and "destabilizing" behaviors and that without some explicit constraints the price of the market is likely to diverge under the pressure of speculative activities.

- the presumed necessity of having heterogeneous strategies and high-frequency switching of agents between them in order to generate non trivial (i.e. constant or explosive) aggregate outcomes.

In what follows we will shows that these assumptions are not always true, that seemingly "destabilizing" behaviors can actually leave untouched the market stability and that, on the other hand, it is not at all necessary to suppose time variations in the agent's trading strategies and believes to generate very volatile bubble-crash dynamics. In particular, concerning the same idea of "efficiency"of the market, our "homogenous agents" limit can be used to shows that, under very natural assumption, an infinitesimal deviation from perfect rationality can generate finite-size effect in the market dynamics.

In this paper we begin with the development of the model and we will limit the analysis of its dynamics to a very special case, the "homogeneous limit", obtained when the "degree of heterogeneity" goes to zero. A slightly more formal discussion of the meaning of this limit will be given below.

The outline of the paper is as follows: in Sec. 2 the model is introduced and the various assumptions discussed. We also describe various possible modification and interpretation of the model parameters. In Sec. 4 the analytical and numerical study of the system is performed while in Sec. 5 some final remarks and some future possible developments are discussed.

2 The basic framework

As said before, this work is focused on the description of market dynamics emerging from the interaction of speculative agents. Roughly speaking, we can say that our agents try to maximize they future wealth without incurring in to much risk. We want to keep our model as simple as possible: we consider only two assets, a risky stock paying a divided d and a bond with riskless rate of return r. We also suppose that agents shape their trading activity only on the basis of their possible wealth one step in the future. The procedure is straightforward: at the beginning of time t the agent forms his personal demand function, deciding the amount of risky asset he want to buy or sell for any possible value of the hypothetical transaction price p_h. His personal demand function is based on his estimate of his own wealth following his post-trading position in the next time step, i.e. on his forecast of the stock price at time $t+1$.

To be more precise, suppose that at the end of period t, after its participation to the market, the agent possesses $B(t)$ riskless assets and $A(t)$ risky assets. The agent wealth is

$$W(t) = B(t) + A(t)p_h \tag{1}$$

where p_h is the stock price (for now "hypothetical") fixed by the market at time t.

Let x be the fraction of agent wealth invested in the risky asset. The future wealth of the agent portfolio (i.e. its wealth at the beginning of the next time step) depend on the hypothetical return on the stock price $h(t) = p(t+1)/p_h - 1$ and reads

$$W(t+1; h(t)) = x\, W(t)\, (h(t) - r + d/p_h) + W(t)\, (1+r) \tag{2}$$

where the dividend d is payed at the end of time t, after the payment of the riskless interest. Obviously, the future value of the portfolio depends on the future price dividend. To keep the model simple, let suppose that both the payed dividends d and the riskless return r are constant and known to each agent. Concerning the stock price dynamics, suppose that the agent possesses some forecasting ability and is able to formulate expectations on the future return h.

Having expectation, the problem of the agent becomes to maximize its utility U consistently with his expectations. In this framework, a natural idea would be to refer to the "expected utility theory" EUT (see, for instance, Fama and Miller (1972)) and to pretend that the agent behavior is obtained by the maximization of its expected utility with respect to his forecast on the probability distribution of the portfolio future wealth. In this contest different expression can be devised for the exact form of the agent utility (For a recent discussion and critical review on the various choices found in literature see Levy et al (2000)). However, a generic choice of the utility function would easily lead to difficult analytical expression and, more important, in its generality this theory require the agent to forecast a whole probability distribution for future wealth, which is a rather strong requirement compared to the portfolio management techniques today applied by the majority of traders on the real financial markets.

In the present paper we try to follow a different direction since first, we want to be able to perform some analytical investigations of the model and, second, we want to model an agent who takes in consideration a "finite" (possibly small) amount of information in his decision processes. In this respect, we propose an expression for the agent utility inspired by the "mean-variance portfolio theory" (see, for instance, Elton and Gruber (1981)) that can be considered a "standard" (even if rather minimalistic) procedure to compare different investment possibilities. It is interesting, however, to observe that, as some empirical investigations have shown (see Kroll et al. (1984) and Levy and Markowitz (1979)), in real application the use of a mean-variance approach with respect to a more demanding utility maximization leads to a reduction of efficiency in the portfolio of less then 5%.

Reassured by this "practical equivalence" between the two approach, we choose as the expression of the agent utility the simplest function of the expected return and variance

$$U(t) = E_{t-1}[W(t+1)] - \frac{\beta}{2} V_{t-1}[W(t+1)] \tag{3}$$

where $E_{t-1}[.]$ and $V_{t-1}[.]$ stand respectively for the expected return and variance computed at the beginning of time t, i.e. with the information available at time $t-1$, and where β is the "risk-aversion" parameter[1].

Using the expression for W in (1) one obtains

$$E_{t-1}[W(t+1)] = x\, W(t)\, (E_{t-1}[h(t)] - r + d/p_h) + W(t)\, (1+r) \tag{4}$$

and

$$V_{t-1}[W(t+1)] = x^2\, W(t)^2\, V_{t-1}[h(t)] \quad . \tag{5}$$

The portfolio position chosen by the trader is the one that maximize its utility, and is obtained equating to 0 the derivative of (3) with respect to x. Using (4) and (5) the personal demand curve then reads:

$$W(t)x = \frac{E_{t-1}[h(t)] - r + d/p}{\beta\, V_{t-1}[h(t)]} \tag{6}$$

or, remembering the definition of x

$$\Delta A(t) = -A(t-1) + \frac{E_{t-1}[h(t)] - r + d/p_h}{\beta\, V_{t-1}[h(t)]\, p_h} \tag{7}$$

that relates the quantity of stock $\Delta A(t)$ the agent is willing to trade (i.e. to buy if it is positive or to sell if it is negative) at time t if the price would be p_h.

[1]Note that (3) is radically different from the (in)famous quadratic utility function (Elton and Gruber, 1981). Indeed (3) cannot be derived via the EUT procedure, i.e. its not possible to write it as the expected value of a smooth function of the wealth. However, it's immediate to seen that it verifies the "nonsatiation" properties, if expressed in probabilistic terms. The same choice for the utility function is also present in other financial market agent-based simulations as Brock and Hommes (1998),Hommes (2001) or Kirman and Teyssiere (2002)

3 Market structure and homogeneous agents dynamics

To model the market trading procedure we choose probably the simplest among all the market structure, namely the Walrasian auction: the demand curve of each agent $\Delta A_i(p)$ is "aggregated" in a global demand curve and the asset present price $p(t)$ is then found putting the aggregate excess demand to 0.

This choice is in part dictated by a requirement of (at least partial) analytical tractability and in part by the fact that this structure perfectly integrate with the agent description we introduced in the previous section. Notice that, obviously, no real market can display this kind of trading procedure since it implies a flow of an infinite amount of information from the traders to the auctioneer. However, the auction phases implemented in many stock exchanges (both during the opening phase or for special situations during the trading session), allowing a constant real-time updating of traders orders, does liken, at least when the market liquidity is sufficiently high, such an approximation (see Bottazzi et al. (2002)). In general, the Walrasian auction is believed to be a reasonable approximations for the description of low frequency dynamics (LeBaron, 2001).

The demand curve computed in the previous Section represent the model of market participation for a single agent. Even keeping constant the form of the utility function, one can vary the parameters from agent to agent to obtain an heterogeneous population. In fact this is exactly our final aim[2], nevertheless it is interesting to start our analysis from the simplest case, i.e. the case in which the behaviors (i.e. the various parameters) of the agents are identical. This is a rather standard analytical tool for the study of noisy dynamical systems, where the resulting deterministic limit is called the *deterministic skeleton* of the model. Maybe it is useful, however, to shortly clarify what we have in mind. Indeed one can argue that in the "homogeneous" case, assuming that agents have different initial endowments, the actual trading can stand only for one step, then the market position of each agent is identical and no more trading can take place.

In fact the studied situation is different: consider a population of agents whose demand curve are a noisy perturbation around the demand curve defined in (7). For the i-th agent it reads

$$\Delta A(t)_i = \Delta A(t) + \epsilon_i(t) \tag{8}$$

where ϵ are independent stochastic terms [3]. Now suppose to have completely specified the model (we have not did it yet and will fill this gap in next Section) and to describe the evolution of a market in terms of a bunch of variables that completely specify its state (for instance prices, instantaneous volatility, etc.) $X(t, \vec{\epsilon})$. In general, the dynamics of these variables will depend on the realizations of all the stochastic processes involved $\vec{\epsilon}$. This noisy terms keep the market constantly "out of equilibrium" so that the trading can indefinitely go on. The homogeneous approximation we want to consider is in fact analogous to taking the no-noise limit of the dynamics

$$X(t) = \lim_{|\epsilon| -> 0} X(t, \vec{\epsilon}) \tag{9}$$

in some sense "after" the dynamics did evolve. This must be understood as an instrumental simplification, introduced for the purpose of obtaining analytically tractable results. Its validity resides in the ability of providing a reliable description of the market dynamics that is

[2] This kind of analysis will be pursued in the forthcoming Bottazzi (2002)

[3] A complete specification is not useful here, for more details see Bottazzi (2002). Notice that this is actually the kind of noisy perturbations used in Levy et al (1994, 2000)

preserved, at least qualitatively, when differences among agents are introduced. The analysis of this issue constitutes the topic of a forthcoming work (Bottazzi, 2002) but we can anticipate that this is actually the case and the following analysis will guide us in the understanding of more complex situations.

Let us assume the problem of the "homogeneous" limit understood and complete the description of the model. Consider a market composed of N agents, whose demand curve follows (6). Any agent i possesses a personal demand function $\Delta A_i(p_h)$ and from the market balance condition $\sum_{i=1}^{N} \Delta A_i(t) = 0$ the equation for the clearing price is

$$\gamma V_{t-1}[h(t)] p_h^2 + (r - E_{t-1}[h(t)]) p_h - d = 0 \tag{10}$$

where $\gamma = \beta A_{TOT}/N$ with A_{TOT} total number of assets. The previous equation tell us how the price at time t is fixed starting from the agent expectations about the average return and its variance.

In order to "close the system" we should provide a description of the agent procedure to obtain forecasted values. Again we want to stick with the most "standard" assumptions and we choose to model our agent as "naive econometricians" that obtain forecasted variables using EWMA (exponentially weighted moving averages) predictors. The expression for the expected returns and variance then becomes:

$$\begin{array}{rcl} E_t[h(t+1)] & = & (1-\lambda) \sum_{\tau=0} \lambda^\tau h(t-\tau) \\ V_t[h(t+1)] & = & (1-\lambda) \sum_{\tau=0} \lambda^\tau h(t-\tau)^2 - E_{t-1}[h]^2 \end{array} \tag{11}$$

where $\lambda \in [0,1]$ is a weighting coefficient setting the "time scale" on which the averaging procedure is performed. Notice that the expression for $V_{t-1}[h]$ is analogous to the one proposed by RiskMetrics group (see the RiskMetrix Technical Manual), and widely applied by the real operators in their forecasting activity[4].

Using the positive root of (10) and (11) we can finally write the dynamical equations governing the evolution of the market in this case

$$\begin{array}{rcl} p(t) & = & \left(E_{t-1}[h] - r + \sqrt{(E_{t-1}[h] - r)^2 + 4\gamma V_{t-1}[h] c} \right) / \left(2\gamma V_{t-1}[h]\right) \\ E_t[h(t+1)] & = & \lambda E_{t-1}[h(t)] + (1-\lambda) h(t-1) \\ V_t[h(t+1)] & = & \lambda V_{t-1}[h(t)] + \lambda(1-\lambda)(h(t-1) - E_{t-1}[h(t)])^2 \end{array} \tag{12}$$

where $h(t) = p(t+1)/p(t) - 1$ stand for the realized return at time t.

This model, notwithstanding the simplicity of its hypotheses, is able to show a remarkable set of different features. In the next Section, we will try to perform a rigorous description of some of its properties. For now, however, it is useful to mention a couple of interesting feature.

First of all, notice that the dynamics described in (12) is bounded. Indeed if the forecasted return tend toward a constant value, the variance is progressively reduced and the price increases. This behavior rules out the possibility of an indefinite steady increase of the price. On the other hand, if the forecasted return $E_{t-1}[h]$, after a period of explosively increasing prices, scales of a factor a the forecasted variance $V_{t-1}[h]$ scales of a factor a^2 but then the price scales down of a factor $1/a$.

It is interesting to see how this bounded behavior is generated. A typical[5] price dynamics is shown in Fig. 1: with the chosen parameters (see caption) the dynamics is stuck in a periodic

[4]The RiskMetrics group actually propose an EWMA estimator of the volatility, defined as the second moment of the returns distribution. The expression above represent its natural extension to central moment

[5]In the next Section we will see that in fact the model possesses many phases and, depending on the parameters values, shows quite different trajectories. In this respect, here "typical" has to be intended as both "not strange" and "not trivial".

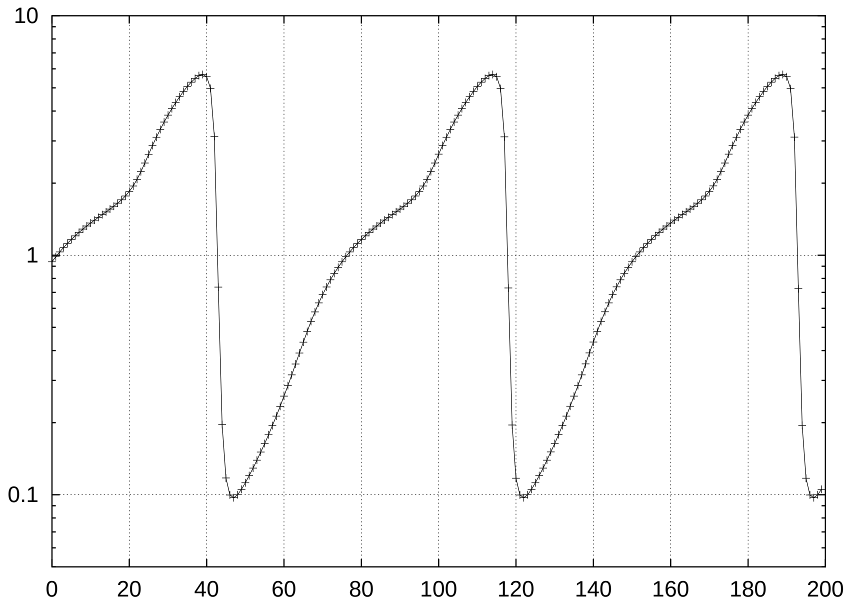

Figure 1: The price history computed with $g = 2.5$, $d = 0.01$, $\lambda = .95$, $r = .01$ and with initial condition $p(0) = 100.$, $E_{t-1}[h]_0 = .01$ and $V_{t-1}[h^2]_0 = .0001$ after a transient of 1000 time steps.

cycle. The boundedness of the dynamics manifests itself as a relative slow (but "explosive" i.e. more then exponential) rise in price followed by a sudden fall, that remembers the "crashes after speculative bubbles" dynamics which seems to be one of the characteristic feature of financial markets.

To see how this "crashes" are generated, we can inspect the few steps that precede one of them. In Fig. 1 are reported the price, forecasted return and forecasted variance of the same simulation as in Fig. 1 for the time interval $30 - 40$ that precedes one price crash at around $41 - 42$. As can be seen from the first steps, the constant increase in price comes both from an increase in forecasted returns and a decrease in the forecasted variance. Indeed in the computation of the forecasted variance the high contribution from the last price crash is progressively discounted. Nevertheless, the contribution from the progressively increasing returns keeps $V_t[h(t+1)]$ bounded away from zero so that, at a given point, the progressive decrease in the forecasted variance starts to slow down. This slowing down, in turn, decreases the price growth rate and consequently the value of $E_t[h(t+1)]$ so generating a feedback effect on the same forecasted variance, strengthening its slowing trend. After few steps, the reversed trend in returns is so high that the same variance starts to increase. At this point, the price starts to go down. This generates a big jump in the forecasted return value and, consequently, on the forecasted variance and, thanks to the feedback mechanism, a sudden price change is generated in a very short time.

Incidentally, notice that in previous works (Brock and Hommes, 1998; Hommes, 2001) the same form (3) for the agents utility function has been used but the authors did not introduce an agent forecasting rule for the price variance, simply assuming that all the agents equated it to a given constant value. From the discussion above, it is clear that this approximation, apart of being inconsistent with the same time series generated by these models, which topically show strong volatility dynamics, would essentially change the nature of the dynamics. For a

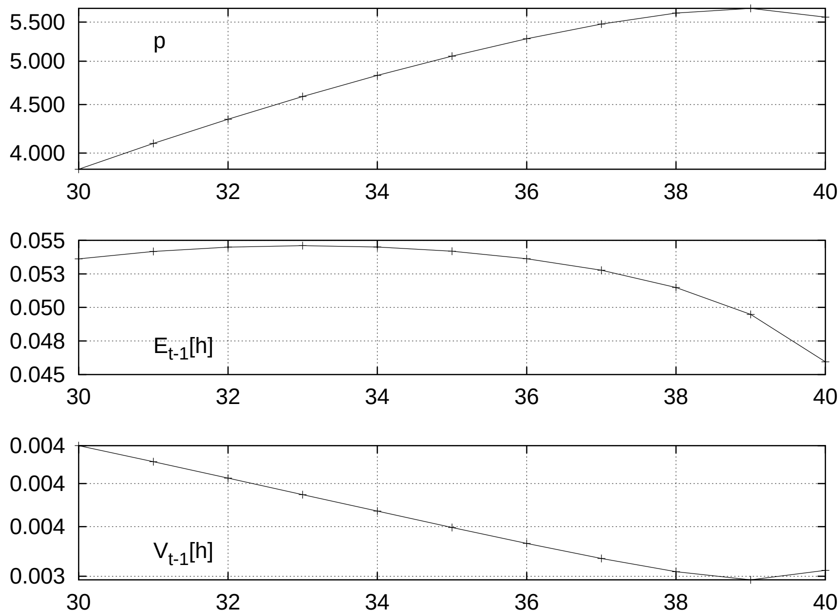

Figure 2: The same simulation as in Fig. 1. Here price (top), forecasted return (middle) and forecasted variance (bottom) are shown for few time steps preceding a sudden crash.

discussion of this approximation and a comparison with (12) see Appendix A.

Before proceed to an analytical and numerical investigation of (12) in the rest of this Section we want to discuss some generalizations and limitations of the present model whose analysis will result useful in what follows.

We will start by briefly discussing what should be the behavior of our simple model under the hypothesis of perfect rational agents trying to evaluate the "fundamental" price of the stock under the Efficient Market Hypothesis

3.1 The EMH "equilibrium" price

In the previous section we described a market participation model for a population of risk-averse speculators in a Walrasian setting. The agents considered were exposing a technical trading behavior: they used simple techniques to extract predictions from the past market outcomes, trying to "guess" the market future and exploit speculation opportunities.

A completely different approach would be the one chosen by a "fundamentalist" trader, i.e. a trader that build his portfolio management starting from his supposed knowledge of the fundamental value of the involved assets and under the hypothesis that the market would eventually stabilize the prices around these values. Of course this behavior is self-fulfilling in an homogeneous word, but the assumption on asymptotic convergence to the fundamental value can be, in general, wrong when different agents (with different believes) participate to the market.

It is immediate to see that under the no arbitrage hypothesis, the future $p(t+1)$ and present $p(t)$ value of the risky asset must satisfies the relation

$$p(t+1) + d = p(t)(1+r) \tag{13}$$

Indeed the left hand side is the value, at time $t+1$, of a portfolio made up of a single asset bought at time t, while the right hand side is the value, always at time $t+1$ of an equivalent (equally valued) portfolio made up of riskless assets. (13) leads to the "equilibrium" price $\bar{p} = d/r$[6] As we will see in the next Section, for certain value of the parameters, the present model, even if characterized by purely speculative agents, can actually converge toward such a kind of "fundamentalist" behavior.

3.2 Random dividends

If one want to repeat the previous analysis in presence of a non constant stream of dividends, things become more complicated. In order to evaluate its portfolio, the agent must posses forecasts not only for the future stock returns, but also for the stock dividend and for the covariance between dividend and returns. To be clearer, consider expression (2) where the dividend d is replaced by a random variable $d(t)$. The expression for the portfolio value expectation and variance reported in (4) and (5) now become respectively

$$E_{t-1}[W(t+1)] = x\,W(t)\,(E_{t-1}[h(t)] - r + E_{t-1}[d(t)]/p_h) + W(t)\,(1+r) \tag{14}$$

and

$$V_{t-1}[W(t+1)] = x^2\,W(t)^2\,(V_{t-1}[h(t)] + V_{t-1}[d(t)]/p_h^2 + 2C_{t-1}[h(t), d(t)]/p_h) \tag{15}$$

where $C_{t-1}[.,.]$ stands for the covariance of the two variables. The personal demand curve then reads

$$\Delta A(t) = -A(t-1) + \frac{E_{t-1}[d(t)] + (E_{t-1}[h(t)] - r)p_h}{V_{t-1}[d(t)] + 2p_h C_{t-1}[h(t), d(t)] + p_h^2 V_{t-1}[h(t)]} \quad . \tag{16}$$

Notice that now the agent demand for stock is bounded even when $p_h \to 0$. Moreover the demand curve is not assured to be monotonic everywhere inside the agent budget constraints. The aggregate price dynamics obtained from the previous equation is obtained from

$$p_h^2\,\gamma V_{t-1}[h(t)] + p_h\,\left(\gamma C_{t-1}[h(t), d(t)] - E_{t-1}[h(t)] + r\right) + \gamma V_{t-1}[d(t)] - E_{t-1}[d(t)] = 0 \tag{17}$$

which is written in implicit from since its explicit form depends from the sign of the various coefficients. This expression can be strongly simplified if one assumes that $d(t)$ is a random variable independently extracted from a constant distribution at each time step. In this case $C_{t-1}[h(t), d(t)] = 0$, since $d(t)$ is by definition independent from any previous realization, while $E_{t-1}[d(t)]$ and $V_{t-1}[d(t)]$ are constant values. The agent forecasting is a noisy prediction of these constants but now one is able to obtain an expression similar to (10) where d is replaced by

$$d_1 = \gamma V_{t-1}[d(t)] - E_{t-1}[d(t)] \tag{18}$$

This quantity must be positive in order to assure the existence of a real price for any value of r and of the forecasted return $E_{t-1}[h(t)]$

As the previous equations show, the inclusion in our model of a dividend dynamics can pose some problem, since now the ability of the market to express a price, and then the existence of a transaction, depend on the agent forecasts. As a first approximation, we can say the agent are actually able to perfectly forecast dividend, i.e. they have perfect knowledge about dividend distribution so that d_1 becomes a constant parameter in the model. If one follows this approach, one can consider the d parameter in the following discussion as representing not a constant dividend but an expression as in (18).

[6] Of course (13) posses also a non-stationary divergent solution with an exponentially increasing price which is sometimes referred as "rational expectation bubble". Since, as we discussed before, the dynamics described in (12) is bounded, the present model can never shows this kind of behavior and we can ignore this solution in the present discussion.

3.3 Mixing chartists and fundamentalists

In the discussion leading to (12) the whole population of agent forecast future returns using a "trend-following" chartist rule. The "equilibrium price" previously discussed allows one to consider also fundamentalist rules, i.e. rules where the future position of the market is evaluated with respect to the asset "fundamental" price. This kind of fundamentalists can be more precisely described, following Hommes (2001), as "Efficient Market Believers" (EMB).

We suppose that the EMB traders too are myopic utility-maximizer so that the determination of their personal demand curve is analogous to the chartist case and leads to (7). The difference is introduced by different expressions for expectations are: we consider that the forecasted price for an EMB trader is "somewhere in between" the present price and the fundamental price $p_h + \theta(\bar{p} - p_h)$ where the parameter $\theta \in (0, 1)$ measures the (supposed) market strength in recovering the equilibrium price if moved away from it. The forecasted return then reads

$$E_{t-1}[h(t)] = \theta(\frac{\bar{p}}{p_h} - 1) \ . \tag{19}$$

Moreover we suppose that the fundamentalist estimation of volatility is equal to the "chartist" estimation[7].

Now consider a market composed of N_1 chartists and N_2 fundamentalists. The market clearing condition $\sum_{i=1}^{N_1} \Delta A_i(t) + \sum_{i=1}^{N_2} \Delta A_i(t) = 0$ can be written:

$$\gamma V_{t-1}[h(t)] p_h^2 = d + f_1 \, (y - r) \, p_h + f_2 \, (\theta \bar{p} - (\theta + r) p_h) \tag{20}$$

were the definition of γ is as in (10) and where f_1 and f_2 are respectively the share of chartists and fundamentalists traders. Quite interestingly, the resulting price dynamic can be reduce after some algebraic manipulation to a form analogous to (10)

$$p_h = \frac{E_{t-1}[h(t)] - r^\star + \sqrt{(E_{t-1}[h(t)] - r^\star)^2 + 4\gamma^\star V_{t-1}[h(t)] d^\star}}{2\gamma^\star V_{t-1}[h(t)]} \tag{21}$$

where

$$\begin{aligned} \gamma^\star &= \gamma / f_1 && (22) \\ r^\star &= r(1 + f_2\theta/r)/f_1 && (23) \\ d^\star &= d(1 + f_2\theta/r)/f_1 && (24) \end{aligned}$$

All three parameters increase when a part of the market follows the fundamentalist trading procedure. Notice that the equilibrium price, as should be, is left invariant.

Then, the introduction of a share of traders that follow the simple fundamentalist procedure described in (19) simply translates in a redefinition of the original parameters, leaving the functional form invariant.

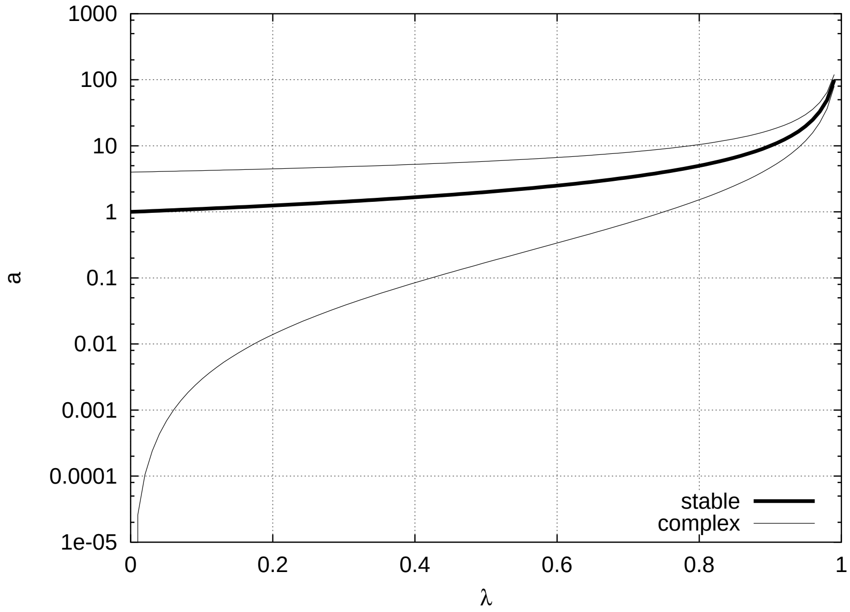

Figure 3: (λ, a) parameter space. The fixed point stable region is the bottom right region delimited by the thickest (denoted with "stable" in the legend) line. The region inside the two other lines (denoted with "complex" in the legend) is where the eigenvalues are complex. Notice that both these region are unbounded from above.

4 The deterministic dynamical system

In order to simplify the analysis of the market dynamics it is convenient to rewrite the system in (12) as

$$\begin{aligned} x(t+1) &= f(y(t), z(t)) = \frac{y(t)-r+\sqrt{(y(t)-r)^2+4sz(t)}}{2z(t)} \\ y(t+1) &= \lambda y(t) + (1-\lambda)\left(\frac{f(y(t),z(t))}{x(t)} - 1\right) \\ z(t+1) &= \lambda z(t) + \lambda(1-\lambda)\left(\frac{f(y(t),z(t))}{x(t)} - 1 - y(t)\right)^2 \end{aligned} \tag{25}$$

where $x(t) = \gamma p(t)$, $y(t) = E_{t-1}[h(t)]$, $z(t) = V_{t-1}[h(t)]$ and $s = d\gamma$. This system depends on three parameters r, s and λ. Notice that the "risk-aversion" parameter γ has been absorbed in a rescaling of both the prices and the payed dividends.

In these new variables, the "fundamental" value defined in (13) becomes $\bar{x} = s/r$. First of all, we can ask if the system state associated with this fundamental value $x = \bar{x}, y = 0, z = 0$[8] is in fact a fixed point of our dynamics. We immediately meet a problem since the right hand part of the first equation in (25) is only defined for $z > 0$ and $y \geq 0$. It is however easy to shows that it can be extended continously to $z = 0$ when $y < r$ using its alternative expression

$$f(y,z) = \frac{2s}{\sqrt{(y-r^2)+4sz}+r-y} \quad y < r \tag{26}$$

[7]This assumptions is in accordance with the observed behavior of real traders among which the risk estimation based on historical data seems largely adopted even when "fundamentalist" approach is recommended for future returns estimation.

[8]If the price is constant, then the returns are constantly zero, from which $y = 0$ and $z = 0$

such that $f(0,0) = s/r$ and the point $(\bar{x}, 0, 0)$ is a fixed point for the dynamics. Trivial is to check that the system doesn't posses any other fixed point. But what about its stability? There exists a region in parameters space where the system evolve constantly toward this equilibrium price? This would imply that it is possible to recover a market equilibrium around the "fundamental" price with an ecology made exclusively of "technical" traders.

To this purpose, notice that using (26) one can check immediately that the partial derivatives

$$\begin{aligned} f_y(y,z) &= f(y,z)/\sqrt{(y-r)^2 + 4sz} \\ f_z(y,z) &= (s/\sqrt{(y-r)^2+4sz} - f(y,z))/z \quad . \end{aligned} \tag{27}$$

are continuous for the domain $\mathcal{D} = \{y \geq 0, z > 0\} \bigcup \{y < r, z = 0\}$. In particular $\partial_y f(0,0) = s/r^2$ and $f_z(0,0) = -s^2/r^3$. Since the dynamics described in (25) is bounded in $\{x > 0, z > 0\}$ one can conclude that there is a neighborhood of the fixed point $(\bar{x}, 0, 0)$ such that in its intersection with the largest invariant set of the dynamics, the system is differentiable with continuous derivatives. This is enough to use the following theorem, which can be applied to slightly more general cases the the one at hand.

Theorem Suppose a system dynamics is described by a set of equations analogous to (25) with a generic function f continuous in $0, 0$ with continuous first derivatives. Then if $a = \partial_y \ln(f(0,0))$ the point $(f(0,0), 0, 0)$ is stable when

$$a < \frac{1}{1-\lambda} \tag{28}$$

Moreover, the stability of the fixed point is lost by an Hopf bifurcation (Eckmann, 1981; Ruelle and Takens, 1971) (i.e. by two complex conjugate eigenvalues that cross the unit circle) when $a = 1/(1-\lambda)$.

Proof See Appendix B.

The curve defined in (28) is plotted in Fig. 3 together with the region in which the Jacobian of the system computed in the fixed point possesses two conjugated complex eigenvalues.

The result above suggests some considerations:

- Notice that the validity of the theorem for a "generic" function f unties the obtained result from our choice for the utility function in (3). The existence of a stability region for the fixed point, when the agent evaluate future prices starting from forecasted return and variance, is thus guaranteed whatever expression one chooses for utility as long as $\partial_y f(0,0) > 0$. This is a rather general assumption in a speculative trading framework.
- A market can be perfectly stable, i.e. at "equilibrium", even when only chartist traders are present. This suggest that the general idea of technical trading as a destabilizing force of the market is not always true or , at least, is not enough to generate highly volatile dynamics.
- Relatively long memory agents, i.e. agents smoothing their forecasts on time scale that are large if compared to a, behave like fundamentalist, even if they follows, in they choices, only the price trend.
- With the expression of f as in (25) and following (27), $a = 1/r$ such that (28) becomes $\lambda > 1 - r$. Thus the market tend toward the equilibrium point when the riskless return is relatively high and the agent forecasting behavior sufficiently "smooth".

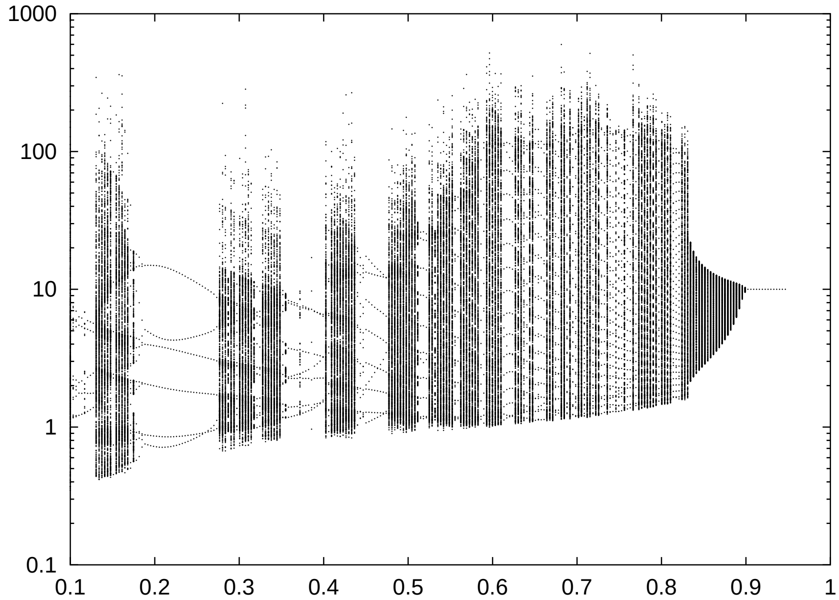

Figure 4: Bifurcation diagram. The x support of a 500 steps orbit (after a 1000 steps transient) is shown for 800 distinct values of λ in $[0,1]$ ($r = .1$ and $s = 1$). The initial condition is $(1, .01, .0001)$.

- Quite surprisingly, for the expression of f defined in (25) , the s parameter doesn't play any role in the stability of the fixed points. This means that the existence of a stable fixed point does not depend neither on the dividend d nor on the value of the aggregate risk aversion γ.

The next natural two questions are what is the basin of attraction of the fixed point and what happens when the fixed point is no more stable. We are not able to discuss these points in general terms and, in the following parts of the present papers, we will refer to the expression of f defined in (25). Moreover, since from now one we will tray a global analysis of this system, we will mainly relay on numerical method, the simplest and more general tools in this kind of investigation[9]

Let us postpone the discussion of the fixed point attraction domain and proceed with a straightforward inspection of the system behavior when one leaves the stability region. Keeping fixed $r = .1$ and $s = 1$ we plot the support for the x values (after a suitable "transient" period) when the λ parameter is varied, to obtain a bifurcation plot. The result is reported in Fig. 4. As can be seen, for $\lambda > .9$ the system is stationary in the stable fixed point. This is is fact our expectation, following the previous analysis and the chosen value for r.

As the nature of the bifurcation suggest, when the level of λ cross the .9 boundary, the system move toward quasi-periodic, multi frequency orbits (this cannot directly see on Fig. 4 due to its coarse grain, but can be directly checked). Moreover, when λ keeps moving toward lower values, we can see the appearance of region in which the system show clear periodic

[9]The theory of global analysis of dynamical system (for a rather gentle introduction see e.g. Katok and Hasselblatt (1995)) is both a well established and growing field. Nevertheless these techniques, even if of great relevance in theory, are of difficult (sometimes very difficult) applicability and rarely provides more hints on the qualitative system behavior then an extensive numerical analysis.

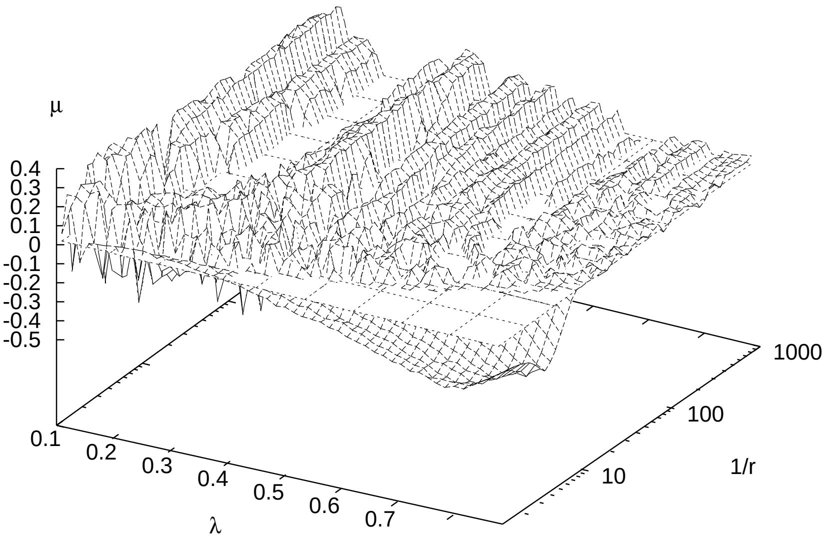

Figure 5: The system largest Lyapunov exponent as a function of λ and $a = 1/r$. The value are obtained with a simulation length of 3000 steps, after discarding the first 1000 as transient ($s = 1$).

behavior and other regions where the density of the support suggest the presence of strange attractors (i.e. attractor whose dimension is not integer) and chaos.

This can be confirmed studying the values of the system Lyapunov exponents for different parameterizations. In Fig. 5 the largest Lyapunov exponent[10] is shown as a function of both λ and r. This plot confirm again the presence of "periodic" region (with below 0 largest exponent) and "chaotic" regions, heavily intermixed. Even if the Jacobian is a smooth function of λ, the Lyapunov exponents show, at least as a first inspection, non-smooth behavior with respect to this parameter (this fact is reported as typical in Eckmann and Ruelle (1985))[11].

A second fact to be noted is the role played by the parameter r. Indeed the "mountains landscape" of Fig. 5 seems to show rather stable valleys or hills along the r direction. This would suggest that the central role in the determination of the attractor structure is played by λ much more than by r.

As an extensive numerical investigation show, this is actually the case. The two parameters mainly shaping the global structure of the system are s and λ. It turns out that even if the parameter s does not play any role in the stability of the fixed points, its role is of major relevance in the characterization of its domain of attraction.

Let start with the discussion of the effect of s in the region associated to fixed point stability in the parameters space, i.e. for $l > 1 - r$. In general, if one takes not too large values for s, the stable fixed points is a global attractor. When the parameter s increases, however, a new attractor constituted by a periodic orbit appears and the domain of attraction of the fixed point shrinks rapidly to a small neighborhood of $\bar{x}$. This can be directly checked considering

[10]Notice that since we posses the explicit expression for the Jacobian matrix (31) the computation of the whole Lyapunov spectrum can be efficiently performed via QR factorization (see for instance Eckmann and Ruelle (1985)) and is more easily obtained than the correlation D_2 or information D_1 dimension of the attractor.

[11]A typical shape of a strange attractor is shown in Fig. 10.

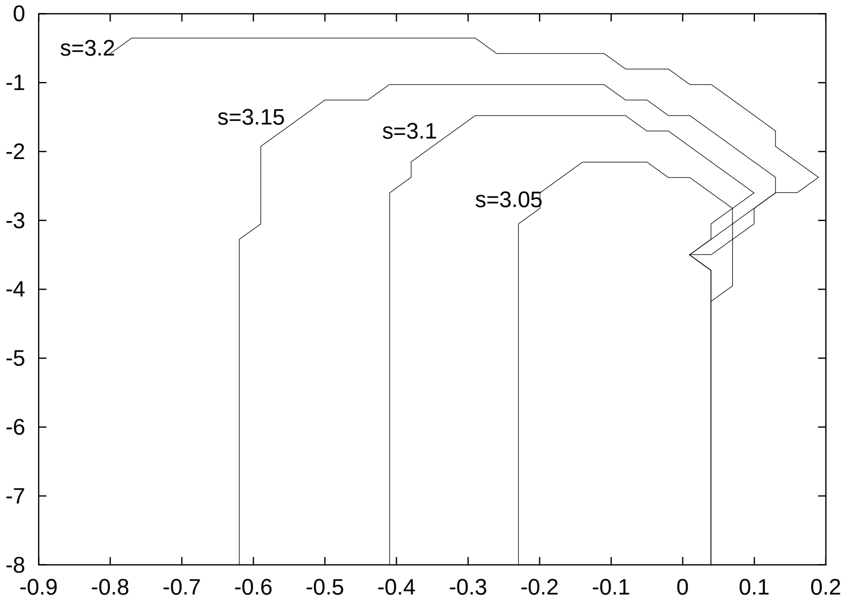

Figure 6: The boundaries of the fixed point domain of attraction in the $y - z$ space. Each points represent an initial condition for the y and z values. The initial condition for x is chosen equal to $\bar{x}$. the system is then iterated for 20.0000 steps and the initial condition is assumed belonging to the domain of attraction if $|x - \bar{x}| + |y| + |z| < .00001$. The chosen values for the parameters are $r = .01$ and $\lambda = .91$. Please notice that the choice of the threshold value and the form of the distance function are asymptotical irrelevant but introduce noticeable effect at finite time lengths. Thus, the lines in the present plot must be read as a qualitative guess of the real boundaries. No attempt has been made to obtain any estimate of the error.

simulations with different initial condition $(\bar{x}, y(0), z(0))$ and plotting the trajectory average distance from $(\bar{x}, 0, 0)$ after a sufficiently high number of steps. The results of this analysis for $r = .1$ and $\lambda = .91$ have been reported in Fig. 6. The boundaries reported there delimit the attracting region for different values of s. As can be seen, when s increases above a given threshold, the attraction domain rapidly shrink. For $s > 3.3$ it becomes a small neighborhood of the fixed point while for $s < 3.03$ the fixed point was a global attractor. This threshold values is an increasing function of λ and diverges for $\lambda \to 1$ (where the system dynamics is definitely frozen). Two attractors coexist even for low values of s when λ is slightly higher then r, so that a quite complex picture emerge. We will try a general description in what follows.

In the following discussion we will refer generically as "orbit" to the various structures appearing in the analysis since the actual topological nature of these object, i.e. if periodic orbit, quasi periodic orbit or strange sets, depends generally in a non smooth way on the parameter values as suggested by Fig. 5.

The qualitative behavior of the system for $\lambda \sim 1-r$ is depicted in Fig. 7. For $\lambda > 1-r$ and moderate values of s, only the global attractor constituted by the stable fixed pointe exists (region E in the plot). When s is relatively high two attractor coexist: the fixed points and an orbit (region D). The fixed point, in the $x - y$ plane, is external to the orbit (characterized by prices constantly lower than the equilibrium one). When s is low and *lambda* near to the threshold value, a new orbit appears now containing the fixed point in the interior, so that

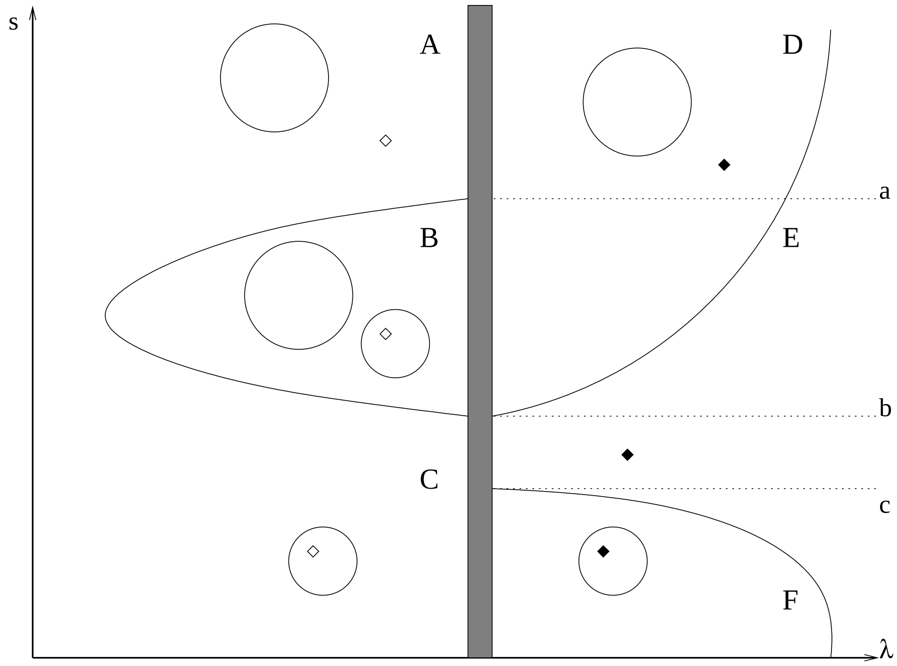

Figure 7: A summary description of the system behavior. See the text for comment. The various regions are indicated with capitals letter and separated by continuous lines. This picture has been obtained with the use of numerical simulations. In particular we have chosen $r = .1$ and we have studied the system near the bifurcation point $\lambda = .9$ along the lines $\lambda = .8991$ and $\lambda = .9001$ (being nearer to the bifurcation generally implies waiting for longer transients). With these choices the values for the boundaries of the various regions are $a = 2.39$, $b = 1.88$ for higher λ, $b = 1.758$ for lower λ and $c = .33$. The region B disappears for $\lambda \sim .89$ and region F for $\lambda \sim 9.05$.

along this orbit the price oscillates around the equilibrium price (region F). For large enough values of λ this region does not appear.

When λ cross the $1 - r$ boundary, the fixed point looses its stability and for large (region A) or small (region C) value of s the orbits keep the same characteristics. Interestingly, for moderate values of s and for λ near the boundary (region B) two stable orbits coexist.

It is also interesting to look at the average price generated by the dynamics in the various regions of the parameters space. In Fig. 8 and Fig. 9 we report the average price computed after a suitable transient as a function of λ and s for $r = .1$ and for values of λ respectively above and below the fixed point stability threshold. In both these plots the price are rescaled so that the equilibrium vale is 1. As can be seen in Fig. 8 both region D and F of Fig. 7 clearly show up and are associated respectively to lower and higher average (rescaled) price. In the second case, even if the price move "around" the equilibrium price as mentioned above, its value is on average much higher. Another interesting feature is the appearance, for quite low values of λ and moderate values of s of a region in which the price dynamics becomes "extreme": the big mountains in the average prices signals the presence of very large cycles in the $x - y$ plane. The typical trajectory is analogous to the one in Fig. 1 but with price varying over many orders of magnitude.

For different values of r the stability region boundary moves, so that all the regions in Fig. 7 shift but their qualitative shape remains unchanged.

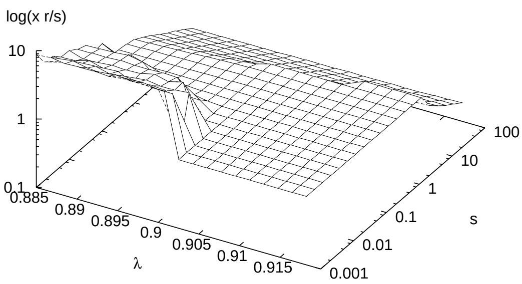

Figure 8: Rescaled average price $x/\bar{x}$ as a function of λ and s in the fixed point stability region with $r = .1$ and initial condition $(\bar{x}, -.1, 0)$. Averages are computed for 1.000 steps after a transient of 50000.

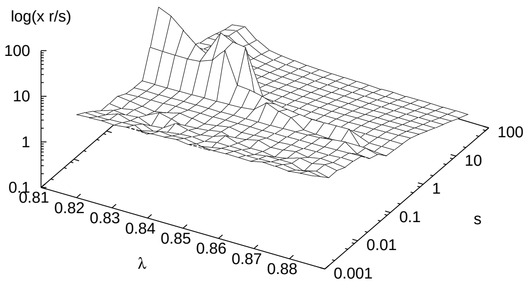

Figure 9: Rescaled average price $x/\bar{x}$ as a function of λ and s outside of fixed point stability region with $r = .1$ and initial condition $(\bar{x}, -.1, 0)$. Averages are computed for 1.000 steps after a transient of 50000.

Finally, and mainly for esthetic purposes, let us plot in Fig. 10 a “typical” strange attractor. It is from the C region of Fig. 7 and, if plotted on the x-y planes, the fixed point $(.1, 0, 0)$ clearly appears in its “interior”, while the average price is almost 10 times larger.

5 Conclusions and Outlook

The interesting part of the foregoing analysis is constituted by the richness of the dynamic scenario one has been able to generate starting from very simple assumption about the agents behavior and the structure of the market. In standard book of finance the description of the agents as risk-averse utility maximizers is in general considered a good approximation of real traders behavior[12]. At the same time, when the problem of obtaining forecasts for the future values of the expected returns and (co)variances is discussed, as always is since it is a central point for the practical application of the Markowitz mean-variance portfolio theory, the possibility of obtaining forecasts from past prices history is discussed as a rather neutral, if not too effective, possibility. Nothing in these discussions seems suggest that these apparently “harmless” hypotheses can lead to the destruction of one of the pillar of the same theory: the Efficient Market Hypothesis. In the previous analysis we have seen that it is actually the case. We can draw two lessons from this discovery:

- first, that the notion of equilibrium expressed by the Efficient Market Hypothesis is in fact extremely weak and can be easily made instable with very mild assumption about the agents behavior (in some sense, this conclusion is analogous to Akerlof and Yellen (1985) where the more general idea of economic equilibrium is analyzed)
- second, that in order to destroy EMH stability is not necessary to suppose the existence of a complex ecologies of strategies together with an high-frequency switching dynamics of agents behaviors.

[12] Or, when a more “normative approach” is chosen, of what the traders behavior should be

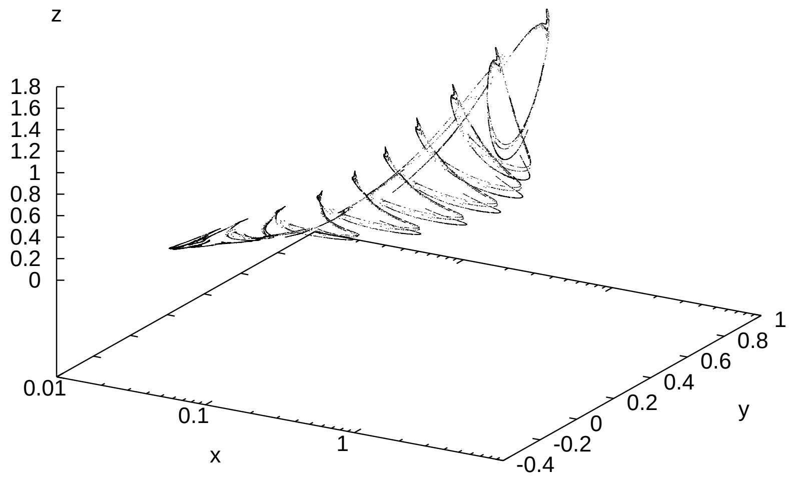

Figure 10: The shape of the strange attractor reconstructed with 20.000 points for $\lambda = .8$,$r = .1$ and $s = .01$. The associated Lyapunov exponents are $2.1e-02$ $-6.3e-02$ $-2.8e-01$.

The present paper provides however, by itself, a weak amount of information, since, in the author view, the more interesting part of this kind of studies reside in the analysis of the effect of heterogeneity on the dynamics of the system. Indeed the choice of analyzing the simplest among all the model found in the literature[13] is dictated by the desire of clearly understanding what effects are generated by the chosen strategies and market structure and what are, on the contrary, introduced by the presence of heterogeneity in the traders population.

We will try to pursue this analysis in the forthcoming Bottazzi (2002). In particular, we will investigate two cases, among the many possible way of introducing heterogeneity in the agent populations. First, following the approach in Levy et al (2000) we will consider randomly perturbed agents, i.e. agents that act according to some uncorrelated random perturbation of a baseline behavior. In this approach both the population average that the time average of a single agent behavior tend to the baseline model and from the point of view of the aggregated dynamics, this reduce to an addition of a “noise term” to the dynamics obtained using homogenous agent. Second, and maybe more interestingly, we will consider “truly” heterogeneous agents characterized by different parameters such that only the population average, and not the time average for each single agent, can be approximated by some homogeneous limit.

APPENDIX

A The constant volatility approximation

Many of the aspects characterizing the framework described in Sec. 2 are quite common in the literature on agent base simulations of financial market. In particular, great similarity

[13] We refer here to model having the same of auction structure that the present one. For notable studies on the effect of simple trading model when the market mechanism is captured by an “impact function” see Farmer (1998); Farmer and Lo (1998)

exists with a series of works (see e.g. Brock and Hommes (1998); Gaunersdorfer (2000); Hommes (2001) and the home page of the Center for Nonlinear Dynamics in Economic and Finance, University of Amsterdam, http://www.fee.uva.nl/cendef/) where, instead of having just one typology of agent, the market dynamics is generated from the aggregate outcome of a population of heterogeneous agents dynamically changing their trading strategies. In these works, one further approximation is however made, with respect to the model described in Section 3: the agents forecasted volatility is assumed constant and homogeneous. In other words, in these models the dynamic of volatility is ignored by the agents when they choose their trading behavior. Given the strong similarity between the present model and the ones referred above, it is maybe interesting to check what happen to our model when the same assumption is made.

Let v be the constant value of the forecasted stock return variance. This value will replace $V_{t-1}[h(t)]$ in (7) for any agent. One can repeat the same analysis performed in Sec. 3, and will obtain the following expression for the aggregate dynamics:

$$\begin{array}{rcl} x(t+1) & = & f(y(t),z(t)) = \frac{y(t)-r+\sqrt{(y(t)-r)^2+4sv}}{2v} \\ y(t+1) & = & \lambda y(t) + (1-\lambda)\left(\frac{f(y(t),z(t))}{x(t)} - 1\right) \end{array} \qquad (29)$$

The system described in (25) reduces to a two variables system. It is immediate to see that this system posses a single fixed point:

$$\begin{array}{rcl} x^\star & = & \frac{\sqrt{r^2+4sv}-r}{2v} \\ y^\star & = & 0 \end{array} \qquad (30)$$

which is notably different from the equilibrium price (even if the latter is recovered in the $v \to 0$ limit). This can be easily understood, since the agents discount the asset price by an amount proportional to their evaluation of risk, which is constant.

This implies that some consistent evaluation of the risk is required in order for a group of agents characterized by a speculative behavior as modeled in Sec. 2 to stabilize the market around the equilibrium price. From a modeling point of view, the assumption of a constant variance forecast generates an "exogenous" differentiation between speculative and "fundamental" behavior, since the two groups evaluation of price, even when the market is stable, fluctuate around two distinct point.

B About the stability of the fixed point

In what follows the proof of the Theorem in Section 4 is outlined. It is a straightforward application of the stability theorem for dynamical system (see e.g. Hirsh and Smale (1970)).

Let us consider the general expression for the Jacobian matrix of the dynamical system defined in (25). It reads:

$$\left[\begin{array}{lll} 0 & f_y & f_z \\ -(1-\lambda)f/x^2 & \lambda + (1-\lambda)f_y/x & (1-\lambda)f_z/x \\ -2\lambda(1-\lambda)hf/x & 2l(1-\lambda)h\left(-1+f_y/x\right) & l + 2\lambda(1-\lambda)hf_z/x \end{array}\right] \qquad (31)$$

where $h(x,y,z) = -1-y+f(y,z)/x$ and from (25) and $f_y = \partial_y f$ and $f_z = \partial_z f$. The functional Dependance has been dropped for readability.

The Jacobian computed in the fixed point reads

$$\begin{bmatrix} 0 & f_y(0,0) & f_z(0,0) \\ -(1-\lambda)/\bar{x} & \lambda + (1-\lambda) f_y(0,0)/\bar{x} & (1-\lambda) f_z(0,0)/\bar{x} \\ 0 & 0 & \lambda \end{bmatrix} \tag{32}$$

From this expression it's clear that the $J(\bar{x}, 0, 0)$ eigenvalues do not depend on $f_z(0,0)$. Setting $a = \partial_y \ln(f(0,0))$ the three eigenvalues read

$$\begin{aligned} \mu_0 &= \lambda \\ \mu_+ &= (\lambda + (1-\lambda)a + \sqrt{(\lambda + (1-\lambda)a)^2 - 4(1-\lambda)a})/2 \\ \mu_- &= (\lambda + (1-\lambda)a - \sqrt{(\lambda + (1-\lambda)a)^2 - 4(1-\lambda)a})/2 \end{aligned} \tag{33}$$

The fixed point $(\bar{x}, 0, 0)$ is stable when $||\mu_i|| < 1$ for $i \in 0, +, -$.

After a little algebra[14] one obtains the boundary of the "stable" domain in the parameters space as an explicit equation of the form $a = a(\lambda)$. Its simple expression reads

$$a(\lambda) = \frac{1}{1-\lambda} \tag{34}$$

Moreover, it is immediate to check that the stability of the fixed point is lost when two complex eigenvalues cross the unit circle (see Fig. 3) so that the system display an Hopf bifurcation (Katok and Hasselblatt, 1995).

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[14]It is easy to show that purely real value for the μ_+, μ_- eigenvalues cannot have modulus equal to one and then it's immediate to obtain the explicit equation for the boundaries equating the modulus of the second (or third) eigenvalue in (33) to 1.

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