On continuous-time random walks in finance

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Abstract

This paper summarises the analysis of 29 out of 30 tick-by-tick time series of stocks included in the Dow Jones Industrial Average (DJIA) and traded in October 1999. This study was performed to test a phenomenological model of high-frequency market dynamics based on continuous-time random walks.

Introduction

In financial markets, not only prices vary at random, but also waiting-times between two consecutive trades [1]. It is therefore possible to model the high-frequency market dynamics by means of continuous-time random walks (CTRWs) [2-6].

To this purpose, let $x(t)$ represent the log-price at time t; let x_i denote the log-return $x(t_{i+1})$-$x(t_i)$ and t_i the waiting time t_{i+1}-t_i. The evolution equation for $p(x,t)$, the probability of finding the log-price x at time t, assuming the initial condition $p(x,0) = d(x)$, is the following *master equation*:

$$p(x,t) = \delta(x)\Psi(t) + \int_0^t \int_{-\infty}^{+\infty} \varphi(x - x', t - t')\, p(x', t')\, dx'\, dt'$$

where $\Psi(t)$ is the *survival probability* and $\varphi(\xi, \tau)$ is the *joint probability density* of jumps and of waiting times.

Two marginal probability densities can be defined:

the *waiting-time density*: $\psi(\tau) = \int_{-\infty}^{+\infty} \varphi(\xi, \tau)\, d\xi$;

the *jump density*: $\lambda(\xi) = \int_0^{+\infty} \varphi(\xi, \tau)\, d\tau$.

If jumps and waiting-times are independent, the joint probability density can be factorised: $\varphi(\xi, \tau) = \lambda(\xi)\psi(\tau)$.

In its turn, the survival probability is given by:

$$\Psi(t) = 1 - \int_0^t y(t')\, dt'$$

In principle, if the joint probability density function is known, it is possible to generate a realisation of the CTRW, for instance by means of a Monte Carlo simulation.

The purpose of this paper is to present some relevant empirical statistical properties of Dow Jones Industrial Average stocks, based on the above outlined theory. In particular results will be presented on the following points:

- The independence between log-returns and waiting-times;
- The autocorrelation of log-returns and its scaling properties;
- The autocorrelation of waiting times and the presence of seasonalities;
- The survival probability.

In the following, the analysis of 29 stocks of the Dow Jones Industrial Average (DJIA) is performed. In fig. 1, a histogram is shown with the number of trades for each stock. The total number of trades analysed is 762,044.

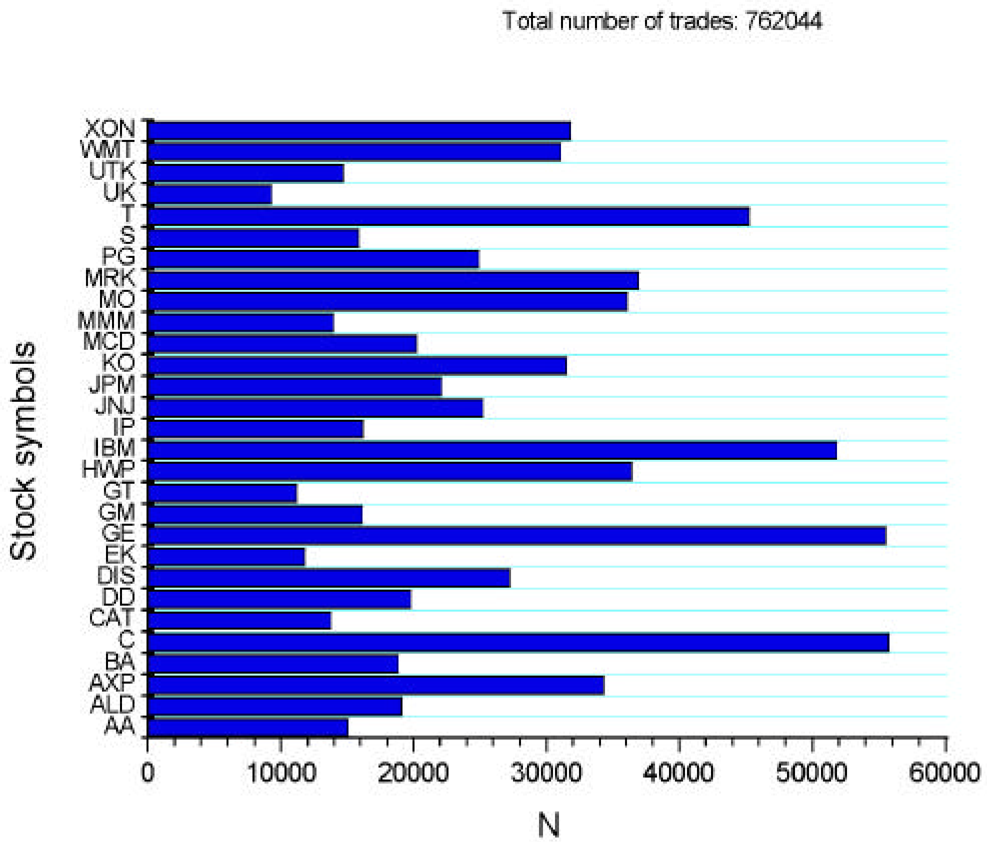

Fig. 1. Number of trades N for each stock.

Test of independence between log-returns and waiting times

In fig. 2, waiting times t_i are plotted as a function of the corresponding log-returns $?_i$.

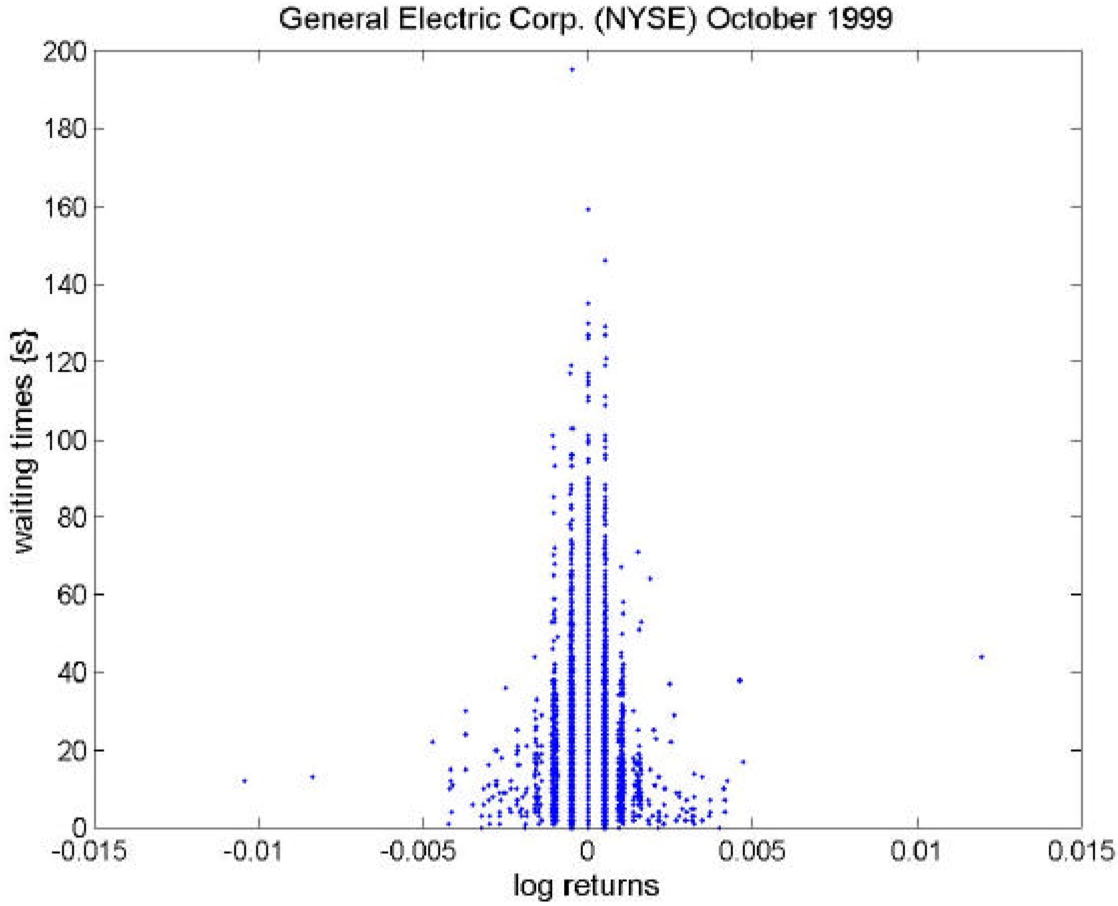

Fig. 2 Waiting times as a function of the corresponding log-returns.

The independence of the two random variables can be studied by means of the contingency table method. A direct inspection of the above figure indicates a possible correlation. Actually, a hypothesis test performed on the empirical joint frequency density function for the 29 stocks shows that the independence hypothesis can be rejected. As an example, in tab. 1 a contingency table is shown for GE stocks and a particular choice of intervals. The value of the reduced chi-square is 27.2, yielding the rejection of the null hypothesis.

		t_i		
		0 ÷ 10	10 ÷ 20	> 20
x_i	< -0.002	25 (38.9)	21 (10.1)	9 (6.0)
	-0.002 ÷ -0.001	516 (613.6)	230 (159.5)	122 (94.9)
	-0.001÷ 0	6641 (7114.3)	2085 (1849.1)	1338 (1100.6)
	0 ÷ 0.001	31661 (31008.0)	7683 (8059.2)	4520 (4797.0)
	0.001 ÷ 0.002	398 (464.4)	179 (120.7)	80 (71.9)
	> 0.002	34 (36.1)	10 (9.4)	7 (5.6)

Tab. 1. Contingency table for log-returns and waiting times. Every cell contains the observed frequency as well as (in brackets) the theoretical frequency computed under the null hypothesis of independence.

In fig. 3, as a further example, the value of the reduced chi-square is shown for three more stocks and as a function of the number of degrees of freedom. The values of chi-square decrease as the number of degrees of freedom increases due to a different choice of intervals in the contingency table. However, they are well above the values for the acceptance of the null hypothesis. For all the 29 time series, the two stochastic variables are not independent.

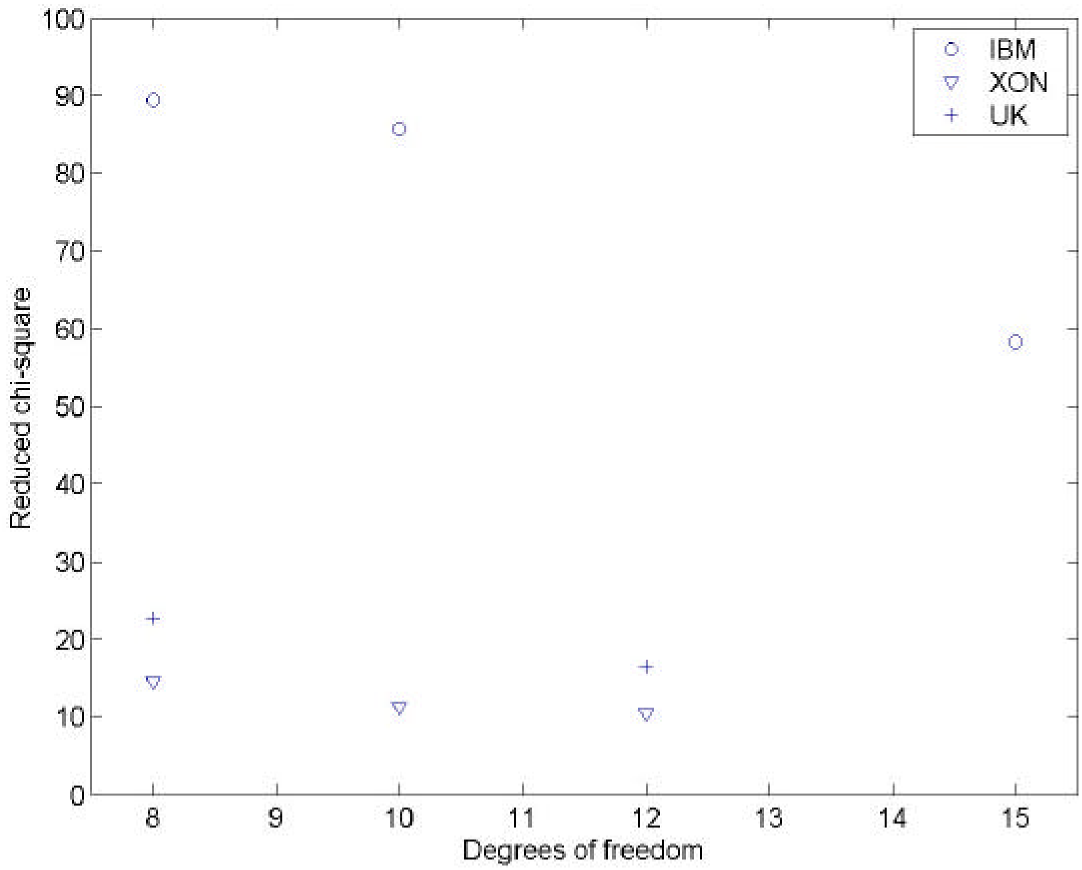

Fig. 3. Chi-square as a function of degrees of freedom for three stocks.

Autocorrelation of log-returns and waiting times

In fig. 4, the autocorrelation function is plotted for the absolute value of the log-returns of General Electric stocks. The following estimator was used:

$$C(m) = \frac{1}{N-m} \sum_{n=0}^{N-m-1} \left(\left|\xi_{n+m}\right| - \overline{\left|\xi\right|}\right)\left(\left|\xi_n\right| - \overline{\left|\xi\right|}\right)$$

where $\overline{\left|\xi\right|}$ is the average value of the random variable $\left|\xi_i\right|$.
Due to scale persistence, the autocorrelation exhibits a power-law decay and reaches the noise level after a 16-tick lag, corresponding to about 3 minutes. A similar behaviour is found for the other stocks, whose autocorrelations reach the noise level for times between 2 and 20 minutes. Indeed this is a well-known stylised fact in high-frequency financial time series [7-9]. However, due to a possible unreliability of autocorrelation estimators for heavy-tailed distribution, caution is necessary to correctly interpret these results [10-12].

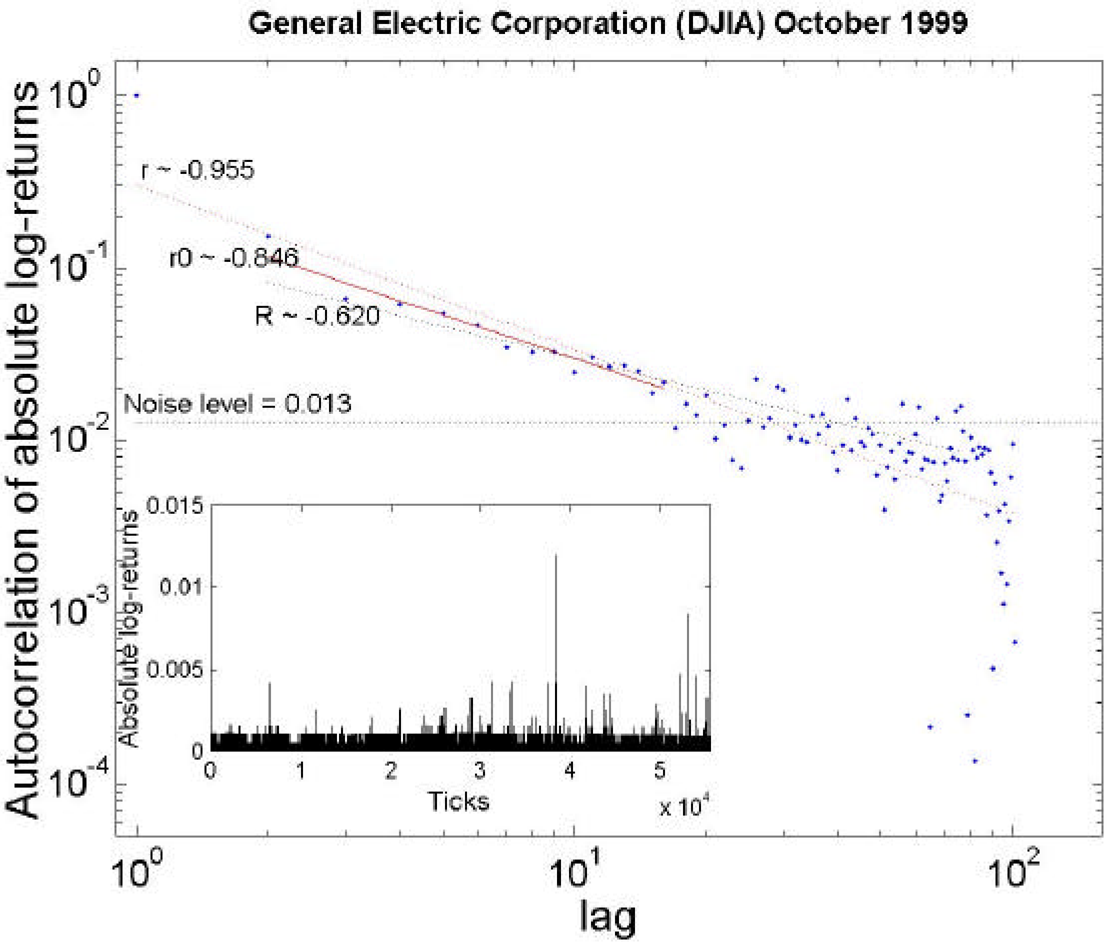

Fig. 4. Autocorrelation of absolute log-returns. The original time series is in the inset. The straight lines are different fits of the power-law decay. The solid line only takes into account points above the noise level.

In fig. 5, the autocorrelation function for waiting times is shown. A one-day periodicity emerges, by direct inspection, by the analysis of the autocorrelation function as well as by means of Fourier analysis (not discussed here). Such a periodicity corresponds to the daily stock-market activity [13]. The waiting-time autocorrelations of other stocks exhibit a similar behaviour, but, sometimes, the daily seasonality is less evident.

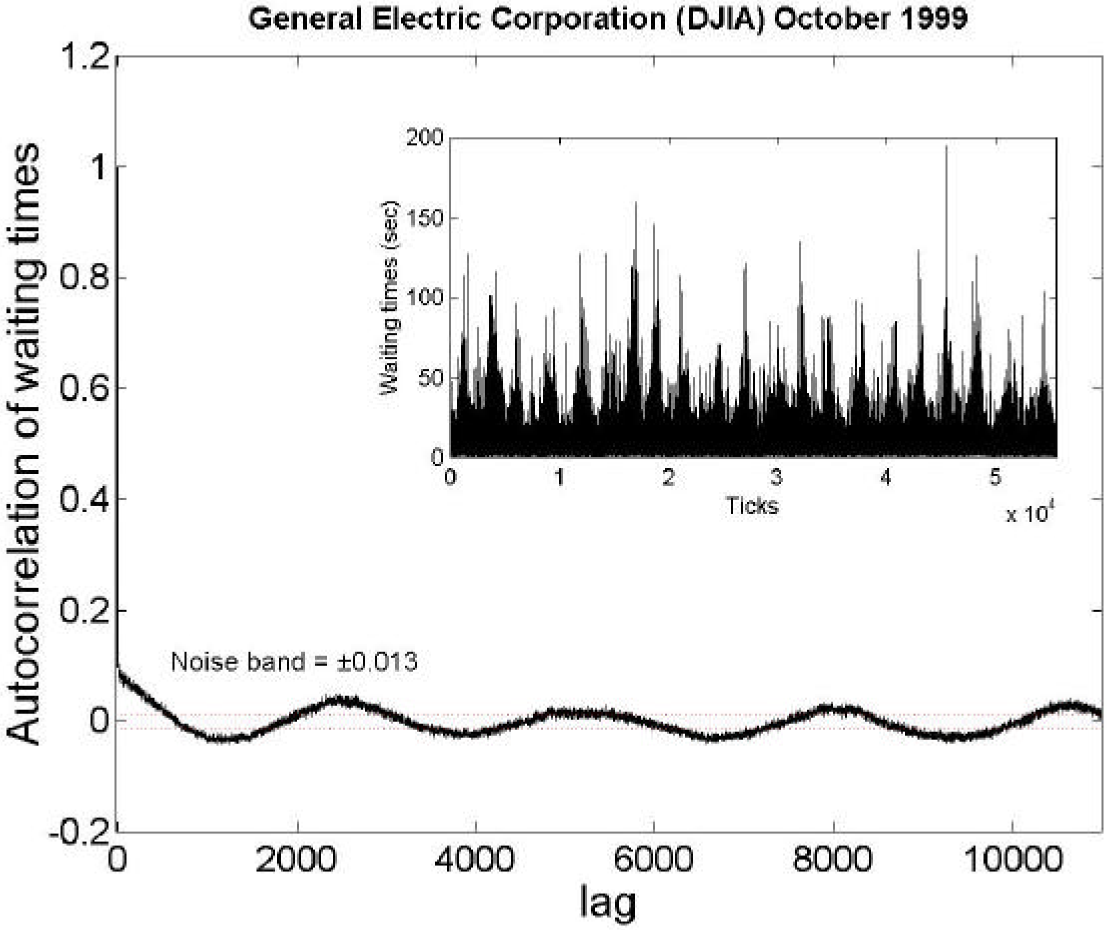

Fig. 5. Autocorrelation of waiting times. As in fig. 4, the original time series is shown in the inset.

The survival probability

In a previous paper, the Mittag-Leffler function has been suggested as a suitable model for the empirical survival probability [4]. Such a hypothesis was tested for bond future prices in refs. [4,6]. For small waiting times, the Mittag-Leffler function is well approximated by a stretched exponential of the following form:

$$\Psi(t) = \exp\left[-(gt)^{b}/\Gamma(1+b)\right]$$

In fig. 6, a fit of the empirical survival probability is shown using a stretched exponential. In fig. 7 and fig 8, plots of ? and its inverse, the time scale 1/? are presented for the 29 stocks. It turns out that 1/? varies linearly with the number of trades N. On the other side, the exponent ß fluctuates around the average value of 0.81 with a standard deviation of 0.05. It is worth noting that the stocks with the larger number of trades (here called liquid stocks for the sake of simplicity) exhibit values of ß systematically greater than the average value.

These findings suggest the possibility of a scaling transformation of the time variable ($u = (gt)^b / \Gamma(1+b)$) leading to the collapse of all the 29 survival probability into the single curve shown in fig. 9.

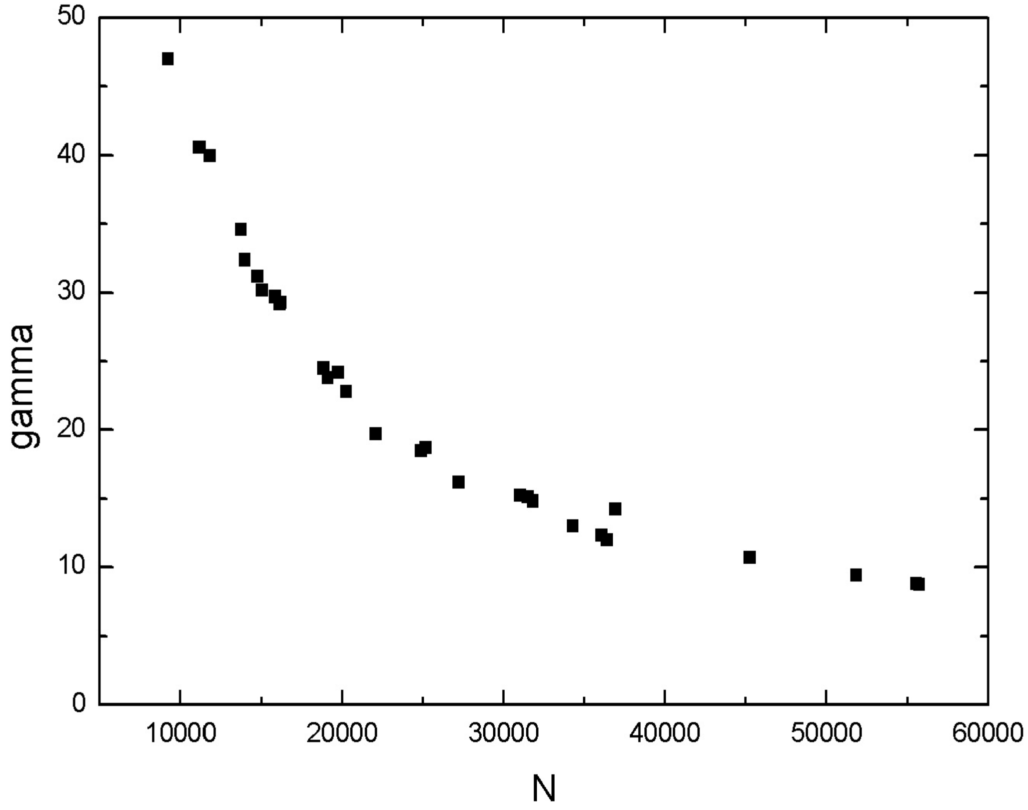

Fig. 6. g as a function of the number of trades.

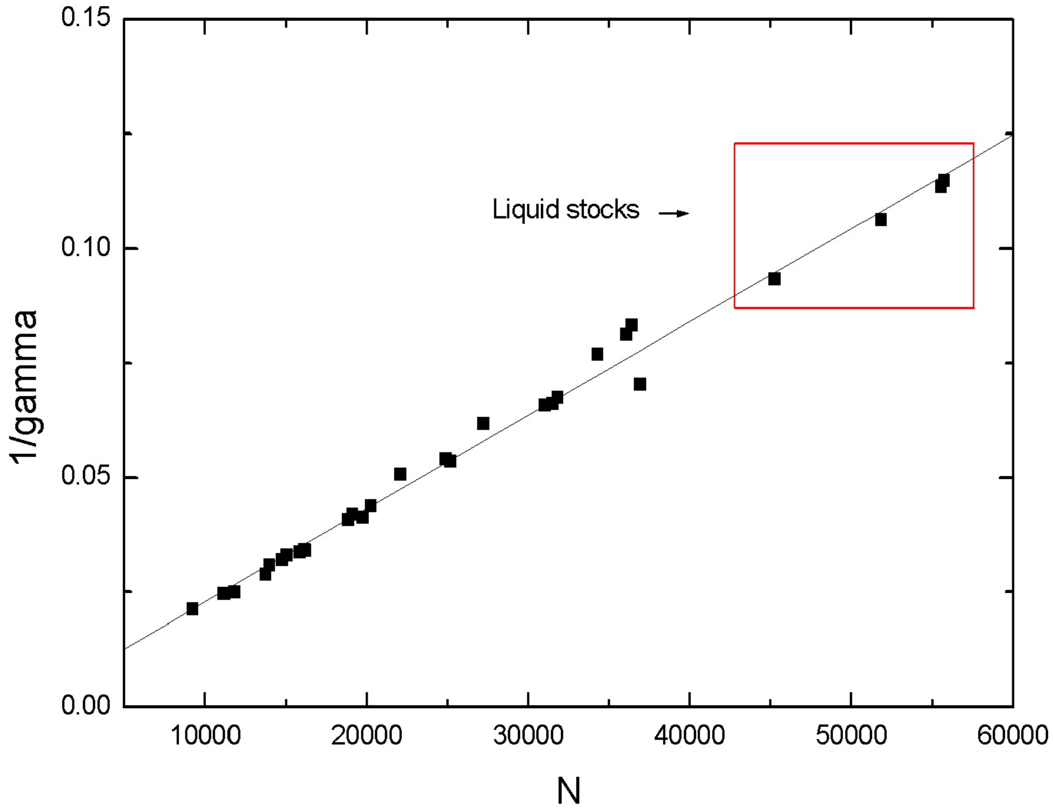

Fig. 7. 1/g as a function of the number of trades.

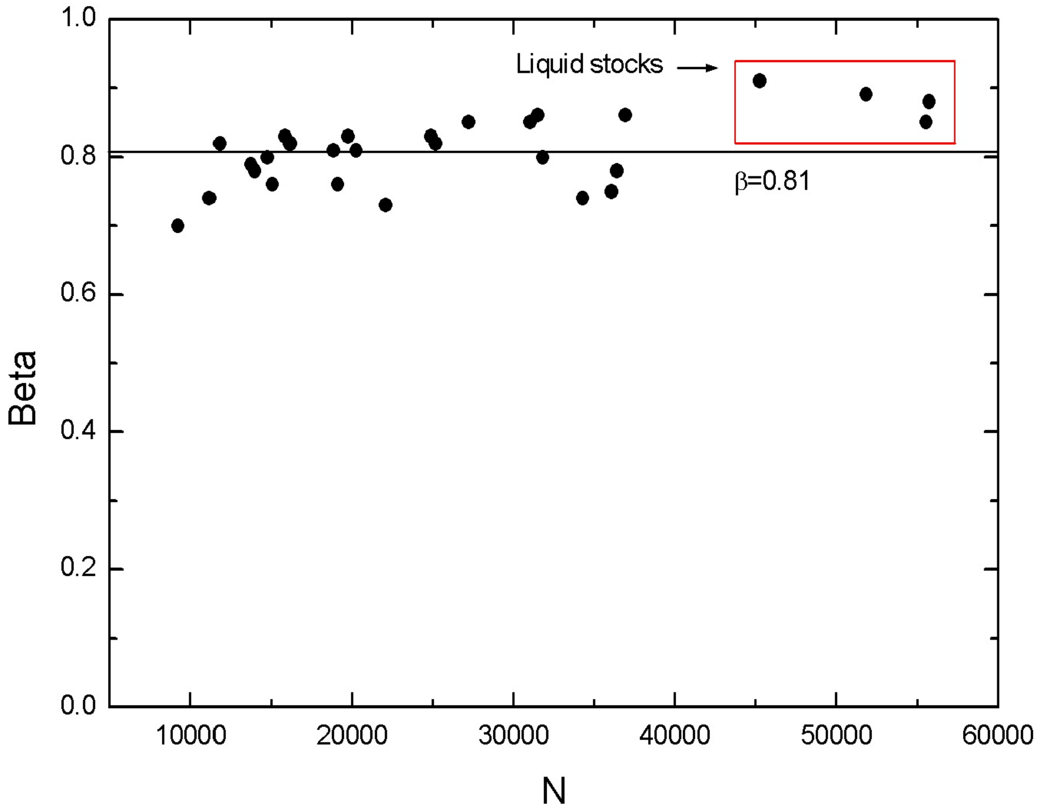

Fig. 8. b as a function of the number of trades.

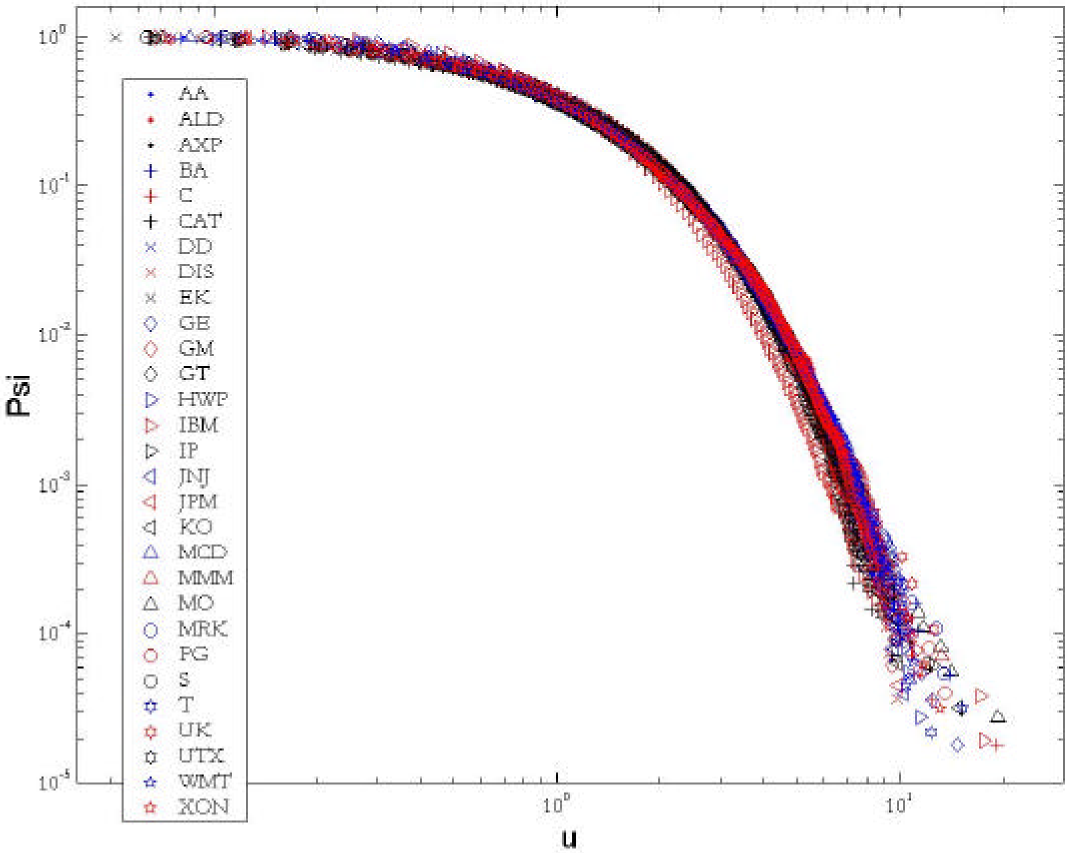

Fig. 9. Scaling of the waiting-time survival distributions.

Summary and conclusions

In the present paper, some statistical properties of 29 out of 30 time series of DJIA stocks traded in October 1999 were investigated. The purpose was to assess the limits of validity of the continuous-time random walk phenomenological model of tick-by-tick dynamics in financial markets.
It turns out that the two basic random variables of the model, log-returns and waiting times cannot be considered independent from each other. In particular, large log-returns are almost never associated to long waiting times.
The autocorrelation of log-returns exhibits a power-law behaviour with non-universal exponent. It reaches the noise level for times between 2 and 20 minutes. On the other hand, the autocorrelation of waiting times has a one-day period, corresponding to the daily market activity.
The complementary distribution function for waiting times, that is the survival probability, is well fitted by a stretched exponential function with two parameters. This leads to a simple scaling transformation after which all the 29 survival probabilities collapse into a single curve. These results seem to corroborate a previous theoretical prediction of the authors.

Acknowledgements

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References

[1] P.K. Clark, *A subordinated stochastic process model with finite variance for speculative prices*, Econometrica **41** (1973) 135-156.

[2] E. W. Montroll and B.J. West, *Random walks on lattices II*, J. Math. Phys. **6** (1965) 167-181.

[3] E. Scalas, R. Gorenflo and F. Mainardi, *Fractional calculus and continuous-time finance*, Physica A **284** (2000) 376-384.

[4] F. Mainardi, M. Raberto, R. Gorenflo and E. Scalas, *Fractional calculus and continuous-time finance II: the waiting-time distribution,* Physica A **287** (2000) 376-384.

[5] R. Gorenflo, F. Mainardi, E. Scalas and M. Raberto, *Fractional calculus and continuous-time finance III: the diffusion limit*, in M. Kohlmann and S. Tang (Eds.), *Mathematical finance*, Birkhäuser Verlag, Basel, 2001, pp. 120-145.

[6] F. Mainardi, M. Raberto, E. Scalas, and R. Gorenflo, *Survival probability of LIFFE bond futures via the Mittag-Leffler function,* in H. Takayasu (Ed.), *Empirical Science of Financial Fluctuations - The Advent of Econophysics,* Springer Verlag, Tokyo, 2002, pp. 195-206.

[7] Y. Liu, P. Cizeau, M. Meyer, C.K. Peng and H.E. Stanley, *Correlations in economic time series*, Physica A **245** (1997) 437-440.

[8] R. Cont, *Statistical finance: empirical and theoretical approaches to the statistical modeling of price variations in speculative markets*, PhD thesis, 1998, Université de Paris XI.

[9] P. Gopikrishnan, V. Plerou, Y. Liu, L. Amaral, X. Gabaix, and H.E. Stanley, *Scaling and correlation in financial time-series,* Physica A **287** (2000) 362-373.

[10] R.A. Davis and T. Mikosch, *Limit theory for the sample ACF of stationary process with heavy tails with applications to ARCH*, Ann. Stat. **26** (1998) 2049-2080.

[11] C. Gourieroux, *ARCH models and financial applications,* Springer Verlag, Berlin, 1997.

[12] T. Mikosch and C. Starica, *Limit theory for the sample autocorrelation and extremes of a GARCH(1,1) process*, Ann. Stat. **28** (2000) 1427-1451.

[13] M. Dacorogna, R. Genay, U.A. Miller, R.B. Olsen, O.V. Pictet, *An Introduction to High-Frequency Finance*, Academic Press, 2001.