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Introduction to algebraic stacks

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Introduction

These notes are an introduction to the formalism of stacks, for the consumption of those who attend my lectures at the ICTP school on moduli theory. They are still in an extremely preliminary stage, many proof, and even parts of statements, are still missing (the missing parts are marked with **TO BE ADDED**). My justification for handing out such a rough product is that I think they might still be useful. The are also extremely dry, and are only meant to supplement my lectures, where I try to give a more articolated view of the subject, adding some examples. **TO BE ADDED**

The formal prerequisites for reading these notes are few. We make heavy use the categorical language, so I assume that the reader is acquainted with the notions of category, functor and natural transformation, equivalence of categories. On the other hand, I do not use any advanced concepts, nor do I use any real results in category theory, with one single exception: the reader should know that a fully faithful essentially surjective functor is an equivalence.

The reader should also recall that a *groupoid* is a category in which every arrow is invertible.

Also, we will manipulate some cartesion diagrams. In particular the reader will encounter diagrams of the type

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd} ;$$

we will say that this is cartesian when both squares are cartesian. This is equivalent to saying that the right hand square and the square

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & C' \arrow[d] \\ A \arrow[r] & C \end{tikzcd} ,$$

obtained by composing the rows, are cartesian. There will be other statements of the type "there is a cartesian diagram ...". These should all be straightforward to check.

CHAPTER 1

Contravariant functors

1.1. Representable functors and the Yoneda lemma

1.1.1. Representable functors. Let us start by recalling a few basic notions of category theory.

Let $\mathcal{C}$ be a category; we will always assume that $\mathcal{C}$ has both fiber products and products. Consider functors from $\mathcal{C}^{\mathrm{opp}}$ to (Set). These are the objects of a category, denoted by

$$\mathrm{Func}\big(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Set})\big),$$

in which the arrows are the natural transformations. From now on we will refer to natural transformations of contravariant functors on $\mathcal{C}$ as *morphisms*.

Let X be an object of $\mathcal{C}$. There is a contravariant functor

$$\mathrm{h}_X \colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$$

to the category of sets, which sends an object U of $\mathcal{C}$ to the set

$$\mathrm{h}_X U = \mathrm{Hom}_{\mathcal{C}}(U, X).$$

If $\alpha\colon U' \to U$ is an arrow in $\mathcal{C}$, then $\mathrm{h}_X\alpha\colon \mathrm{h}_X U \to \mathrm{h}_X U'$ is defined to be composition with α.

Now, an arrow $f\colon X \to Y$ yields a function $\mathrm{h}_f U\colon \mathrm{h}_X U \to \mathrm{h}_X U$ for each object U of $\mathcal{C}$, obtained by composition with f. The important fact is that this is a morphism $\mathrm{h}_X \to \mathrm{h}_Y$, that is, for all arrows $\alpha\colon U' \to U$ the diagram

$$\begin{tikzcd}
\mathrm{h}_X U \arrow[r, "\mathrm{h}_f U"] \arrow[d, "\mathrm{h}_X\alpha"] & \mathrm{h}_Y U \arrow[d, "\mathrm{h}_Y\alpha"] \\
\mathrm{h}_X U' \arrow[r, "\mathrm{h}_f U'"] & \mathrm{h}_Y U'
\end{tikzcd}$$

commutes.

Sending each object X of $\mathcal{C}$ to h_X, and each arrow $f\colon X \to Y$ of $\mathcal{C}$ to $\mathrm{h}_f\colon \mathrm{h}_X \to \mathrm{h}_Y$ defines a functor $\mathcal{C} \to \mathrm{Func}\big(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Set})\big)$.

YONEDA LEMMA (WEAK VERSION). *Let X and Y be objects of $\mathcal{C}$. The function*

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}(\mathrm{h}_X, \mathrm{h}_Y)$$

that sends $f\colon X \to Y$ to $\mathrm{h}_f\colon \mathrm{h}_X \to \mathrm{h}_Y$ is bijective.

In other words, the functor $\mathcal{C} \to \operatorname{Func}(\mathcal{C}^{\text{opp}}, (\text{Set}))$ is fully faithful. It fails to be an equivalence of categories, because in general it will not be essentially surjective. This means that not every functor $\mathcal{C}^{\text{opp}} \to (\text{Set})$ is isomorphic to a functor of the form h_X. However, if we restrict to the full subcategory of $\operatorname{Func}(\mathcal{C}^{\text{opp}}, (\text{Set}))$ consisting of functors $\mathcal{C}^{\text{opp}} \to (\text{Set})$ which are isomorphic to a functor of the form h_X, we do get a category which is equivalent to $\mathcal{C}$.

DEFINITION 1.1. A *representable functor* on the category $\mathcal{C}$ is a functor

$$F\colon \mathcal{C}^{\text{opp}} \to (\text{Set})$$

which is isomorphic to a functor of the form h_X for some object X of $\mathcal{C}$.

If this happens, we say that F *is represented by* X.

Given two isomorphisms $F \simeq \mathrm{h}_X$ and $F \simeq \mathrm{h}_Y$, we have that the resulting isomorphism $\mathrm{h}_X \simeq \mathrm{h}_Y$ comes from a unique isomorphism $X \simeq Y$ in $\mathcal{C}$, because of the weak form of Yoneda's lemma. Hence *two objects representing the same functor are canonically isomorphic.*

1.1.2. Yoneda's lemma. The condition that a functor be representable can be given a new expression with the more general version of Yoneda's Lemma. Let X be an object of $\mathcal{C}$ and $F\colon \mathcal{C}^{\text{opp}} \to (\text{Set})$ a functor. Given a natural transformation $\tau\colon \mathrm{h}_X \to F$, one gets an element $\xi \in FX$, defined as the image of the identity map $\mathrm{id}_X \in \mathrm{h}_X X$ via the function $\tau_X\colon \mathrm{h}_X X \to FX$. This construction defines a function $\operatorname{Hom}(\mathrm{h}_X, F) \to FX$.

Conversely, given an element $\xi \in FX$, one can define a morphism $\tau\colon \mathrm{h}_X \to F$ as follows. Given an object U of $\mathcal{C}$, an element of $\mathrm{h}_X U$ is an arrow $f\colon U \to X$; this arrow induces a function $Ff\colon FX \to FU$. We define a function $\tau U\colon \mathrm{h}_X U \to FU$ by sending $f \in \mathrm{h}_X U$ to $Ff(\xi) \in FU$. It is straightforward to check that the τ that we have defined is in fact a morphism. In this way we have defined functions

$$\operatorname{Hom}(\mathrm{h}_X, F) \longrightarrow F(X)$$

and

$$F(X) \longrightarrow \operatorname{Hom}(\mathrm{h}_X, F).$$

YONEDA LEMMA. *These two functions are inverse to each other, and therefore establish a bijective correspondence*

$$\operatorname{Hom}(\mathrm{h}_X, F) \simeq FX.$$

The proof is easy and left to the reader. Yoneda's lemma is not a deep fact, but its importance cannot be overestimated.

Let us see how this form of Yoneda's lemma implies the weak form above. Suppose that $F = \mathrm{h}_Y$: the function $\operatorname{Hom}(X, Y) = \mathrm{h}_Y X \to \operatorname{Hom}(\mathrm{h}_X, \mathrm{h}_Y)$ constructed here sends each arrow $f\colon X \to Y$ to

$$\mathrm{h}_Y f(\mathrm{id}_Y) = \mathrm{id}_Y \circ f\colon X \to Y,$$

so it is exactly the function $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathrm{h}_X, \mathrm{h}_Y)$ appearing in the weak form of the result.

One way to think about Yoneda's lemma is as follows. The weak form says that the category $\mathcal{C}$ is embedded in the category $\mathrm{Func}(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Set}))$. The strong version says that, given a functor $F\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$, this can be extended to the representable functor $\mathrm{h}_F\colon \mathrm{Func}(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Set})) \to (\mathrm{Set})$.

We can use Yoneda's lemma to give a very important characterization of representable functors.

DEFINITION 1.2. Let $F\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ be a functor. A *universal object* for F is a pair (X, ξ) consisting of an object X of $\mathcal{C}$, and an element $\xi \in FX$, with the property that for each object U of $\mathcal{C}$ and each $\sigma \in FU$, there is a unique arrow $f\colon U \to X$ such that $Ff(\xi) = \sigma \in FU$.

In other words: the pair (X, ξ) is a universal object if the morphism $\mathrm{h}_X \to F$ defined by ξ is an isomorphism. Since every natural tranformation $\mathrm{h}_X \to F$ is defined by some object $\xi \in FX$, we get the following.

PROPOSITION 1.3. *A functor* $F\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ *is representable if and only if it has a universal object.*

Also, if F has a universal object (X, ξ), then is represented by X.

Yoneda's lemma insures that the natural functor $\mathcal{C} \to \mathrm{Func}(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Set}))$ which sends an object X to the functor h_X is an equivalence of $\mathcal{C}$ with the category of representable functors. From now on we will not distinguish between an object X and the functor h_X it represents. So, if X and U are objects of $\mathcal{C}$, we will write $X(U)$ for the set $\mathrm{h}_X U = \mathrm{Hom}_{\mathcal{C}}(U, X)$ of arrows $U \to X$. Furthemore, if X is an object and $F\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ is a functor, we will also identify the set $\mathrm{Hom}(X, F) = \mathrm{Hom}(\mathrm{h}_X, F)$ of morphisms from h_X to F with FX.

1.1.3. Examples. Here are some examples of representable functors.

(i) Consider the functor $\mathrm{P}\colon (\mathrm{Set})^{\mathrm{opp}} \to (\mathrm{Set})$ that sends each set S into the set $\mathrm{P}(S)$ of subsets of S. If $f\colon S \to T$ is a function, then $\mathrm{P}(f)\colon \mathrm{P}(T) \to \mathrm{P}(S)$ is defined by $\mathrm{P}(f)\tau = f^{-1}\tau$ for all $\tau \subseteq T$.

Given a subset $\sigma \subseteq S$, there is a unique function $\chi_\sigma\colon S \to \{0, 1\}$ such that $\chi_\sigma^{-1}(\{1\}) = \sigma$, namely the *characteristic function*, defined by
$$\chi_\sigma(s) = \begin{cases} 1 & \text{if } s \in \sigma \\ 0 & \text{if } s \notin \sigma. \end{cases}$$
Hence the pair $(\{0, 1\}, \{1\})$ is a universal obejct, and the functor P is represented by $\{0, 1\}$.

(ii) This example is similar to the previous one. Consider the category (Top) of all topological spaces, with the arrows being given by continuous functions. Define a functor $\mathrm{F}\colon (\mathrm{Top})^{\mathrm{opp}} \to (\mathrm{Set})$ sending each topological space S to the collection $\mathrm{F}(S)$ of all its closed subspaces. Endow $\{0, 1\}$ with the coarsest topology in which the subset $\{1\} \subseteq \{0, 1\}$ is closed; the closed subsets in this topology are $\emptyset$, $\{1\}$ and $\{0,1\}$. A function $S \to \{0,1\}$ is continuous if and only if $f^{-1}(\{1\})$ is closed in S, and so one sees that the pair $(\{0,1\},\{1\})$ is a universal object for this functor.

(iii) The next example may look similar, but the conclusion is very different. Let (HausTop) be the category of all Hausdorff topological spaces, and consider the restriction $\mathrm{F}\colon (\mathrm{HausTop}) \to (\mathrm{Set})$ of the functor above. I claim that this functor is not representable.

In fact, assume that (X,ξ) is a universal object. Let S be any set, considered with the discrete topology; by definition, there is a unique function $f\colon S \to X$ with $f^{-1}\xi = S$, that is, a unique function $S \to \xi$. This means that ξ can only have one element. Analogously, there is a unique function $S \to X \setminus \xi$, so $X \setminus \xi$ also has a unique element. But this means that X is a Hausdorff space with two elements, so it must have the discrete topology; hence ξ is also open in X. Hence, if S is any topological space with a closed subset σ that is not open, there is no continuous function $f\colon S \to X$ with $f^{-1}\xi = \sigma$.

(iv) Take (Grp) to be the category of groups, and consider the functor $\mathrm{Sgr}\colon (\mathrm{Grp})^{\mathrm{opp}} \to (\mathrm{Set})$ that associates to each subgroup G the set of all its subgroups. If $f\colon G \to H$ is a group homomorphism, we take $\mathrm{Sgr}\, f\colon \mathrm{Sgr}\, H \to \mathrm{Sgr}\, G$ to be the function associating to each subgroup of H its inverse image in G.

This is not representable: there does not exist a group Γ, together with a subgroup $\Gamma_1 \subseteq \Gamma$, with the property that for all groups G with a subgroup $G_1 \subseteq G$, there is a unique homomorphism $f\colon G \to \Gamma$ such that $f^{-1}\Gamma_1 = G_1$. This can be checked in several ways; for example, if we take the subgroup $\{0\} \subseteq \mathbb{Z}$, there should be a unique homomorphism $f\colon \mathbb{Z} \to \Gamma$ such that $f^{-1}\Gamma_1 = \{0\}$. But given one such f, then the homomorphism $\mathbb{Z} \to \Gamma$ defined by $n \mapsto f(2n)$ also has this property, and is different, so this contradicts unicity.

(v) Here is a much more sophisticated example. Let (Hot) be the category of all finite CW complexes, with the arrows being given by continuous functions modulo homotopy. There is a functor $\mathrm{H}^n\colon (\mathrm{Hot}) \to (\mathrm{Set})$ that sends a CW complex S into its n^{th} cohomology group $\mathrm{H}^n(S,\mathbb{Z})$. Then is a highly nontrivial fact that this functor is represented by a CW complex, known as a Eilenberg–MacLane space, usually denoted by $\mathrm{K}(\mathbb{Z},n)$.

But we are really interested in algebraic geometry, so let's give some examples in this context. Let $S = \operatorname{Spec} R$ (this is only for simplicity of notation, if S is not affine, nothing substantial changes).

EXAMPLE 1.4. Consider the affine line $\mathbb{A}^1_S$ over a base scheme S. We have a functor

$$\mathcal{O}\colon (\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Set})$$

that sends each scheme S to the ring of global sections $\mathcal{O}(S)$. Then $x \in \mathcal{O}(\mathbb{A}^1_S)$, and given a scheme S over S, and an element $f \in \mathcal{O}(S)$, there is a unique morphism $S \to \mathbb{A}^1_S$ such that the pullback of x to S is precisely f. This means that the functor $\mathcal{O}$ is represented by $\mathbb{A}^1_S$.

More generally, the affine space $\mathbb{A}^n_S$ represents the functor $\mathcal{O}^n$ that sends each scheme S into the ring $\mathcal{O}(S)^n$.

EXAMPLE 1.5. Now we look at $\mathbb{G}_{\mathrm{m},S} = \mathbb{A}^1_S \setminus 0_S$. Here by 0_S we mean the image of the zero-section $S \to \mathbb{A}^1_S$. Now, a morphism of S-schemes $\mathbb{G}_{\mathrm{m},S} \to S$ is determined by the image of $x \in \mathcal{O}(\mathbb{G}_{\mathrm{m},S})$ in $\mathcal{O}(S)$; therefore $\mathbb{G}_{\mathrm{m},S}$ represents the functor $\mathcal{O}^*\colon (\mathrm{Sch}/^{\mathrm{opp}}) \to (\mathrm{Set})$ that sends each scheme S to the group $\mathcal{O}^*(S)$ of invertible sections of the structure sheaf.

A much more subtle example is given by projective spaces.

EXAMPLE 1.6. On the projective space $\mathbb{P}^n_S = \operatorname{Proj} R[x_0, \dots, x_n]$ there is a line bundle $\mathcal{O}(1)$, with n sections $x_1, \dots, x_n$ which generate it.

Suppose that S is a scheme, and consider the set of sequences $(\mathcal{L}, s_0, \dots, s_n)$, where $\mathcal{L}$ is an invertible sheaf on S, $s_0, \dots, s_n$ sections of $\mathcal{L}$ that generate it. We say that $(\mathcal{L}, s_0, \dots, s_n)$ is equivalente to $(\mathcal{L}', s'_0, \dots, s'_n)$ if there exists an isomorphism of invertible sheaves $\phi\colon \mathcal{L} \simeq \mathcal{L}'$ carring each s_i into s'_i. Notice that, since the s_i generate $\mathcal{L}$, if ϕ exists than it is unique.

One can consider a function $Q_n\colon (\mathrm{Sch}/ \to)(\mathrm{Set})$ that associates to each scheme S the set of sequences $(\mathcal{L}, s_0, \dots, s_n)$ as above, modulo equivalence. If $f\colon T \to S$ is a morphism, and $(\mathcal{L}, s_0, \dots, s_n) \in F(S)$, then there are sections $f^*s_0, \dots, f^*s_n$ of $f^*\mathcal{L}$ that generate it; this gives the structure of a functor to Q_n.

Another description of the functor Q_n is as follows. Given a scheme Q_n and a sequence $(\mathcal{L}, s_0, \dots, s_n)$ as above, the s_i define a homomorphism $\mathcal{O}_S^{n+1} \to \mathcal{L}$, and the fact that the s_i generate is equivalent to the fact that this homomorphism is surjective. Then two sequences are equivalent if and only if the represent the same quotient of $\mathcal{O}^n_S$.

It is a very well known fact (see [**Har??**, ???],) and, indeed, one of the founding stones of algebraic geometry, that for any sequence $(\mathcal{L}, s_0, \dots, s_n)$ over a scheme S, there is exists a unique morphism $f\colon S \to \mathbb{P}^n_S$ such that $(\mathcal{L}, s_0, \dots, s_n)$ is equivalent to $(f^*\mathcal{O}(1), f^*x_0, \dots, f^*x_n)$. This means precisely that $\mathbb{P}^n_S$ represents the functor Q_n.

EXAMPLE 1.7. A generalization of the previous examples is given by grassmannians. Suppose that $\mathcal{E}$ is a locally free coherent sheaf on S, and fix a non-negative integer r. Here we are not going to assume that S is affine. Consider the functor $\mathbb{G}(r, \mathcal{E})\colon (\mathrm{Sch}/^{\mathrm{opp}}) \to (\mathrm{Set})$ that sends each scheme $s\colon S \to S$ over S to the set of all locally free quotients of rank r of the pullback $s^*\mathcal{E}$. If $f\colon T \to S$ is a morphism from $t\colon T \to S$ to $s\colon S \to S$, and $\phi\colon s^*\mathcal{E} \twoheadrightarrow \mathcal{Q}$ is an object of $\mathbb{G}(r, \mathcal{E})(S)$, then

$$f^*\phi\colon t^*\mathcal{E} = f^*s^*\mathcal{E} \twoheadrightarrow f^*\mathcal{Q}$$

is an object of $\mathbb{G}(r, \mathcal{E})(T)$.

If $\mathcal{E}$ is the trivial locally free sheaf $\mathcal{O}_S^n$, we denote $\mathbb{G}(r, \mathcal{O}_S^n)$ by $\mathbb{G}(r, n)$.

Notice that in the previous example we have that $\mathbb{G}(1, \mathcal{O}^{n+1})$ is the functor Q_n represented by $\mathbb{P}_S^n$.

1.1.4. Group objects and actions. In this section, as usual, the category $\mathcal{C}$ will have both finite products and fiber products; we will also assume that it has a final object pt.

DEFINITION 1.8. A *group object* of $\mathcal{C}$ is an object G of $\mathcal{C}$, together with a functor $\mathcal{C}^{\text{opp}} \to (\text{Grp})$ into the category of groups, whose composition with the forgetful functor $(\text{Grp}) \to (\text{Set})$ equals h_G.

Equivalently: a group object is an object G, together with a group structure on $G(U)$ for each object U of $\mathcal{C}$, so that the function $f^*\colon G(V) \to G(U)$ associated with an arrow $f\colon U \to V$ in $\mathcal{C}$ is always a homomorphism of groups.

This can be restated using Yoneda's lemma.

PROPOSITION 1.9. *To give a group scheme structure on an object G of $\mathcal{C}$ is equivalent to assigning three arrows $\mathrm{m}_G\colon G \times G \to G$ (the multiplication), $\mathrm{i}_G\colon G \to G$ (the inverse), and $\mathrm{e}_G\colon \mathrm{pt} \to G$ (the identity), such that the following diagrams commute.*

(i) *The identity is a left and right identity:*

$$\begin{tikzcd} \mathrm{pt} \times G \arrow[r, "\mathrm{e}_G \times \mathrm{id}_G"] \arrow[dr, equal] & G \times G \arrow[d, "\mathrm{m}_G"] \\ & G \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} G \times \mathrm{pt} \arrow[r, "\mathrm{id}_G \times \mathrm{e}_G"] \arrow[dr, equal] & G \times G \arrow[d, "\mathrm{m}_G"] \\ & G \end{tikzcd}$$

(ii) *Multiplication is associative:*

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{m}_G \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times \mathrm{m}_G"] & G \times G \arrow[d, "\mathrm{m}_G"] \\ G \times G \arrow[r, "\mathrm{m}_G"] & G \end{tikzcd}$$

(iii) *The inverse is a left and right inverse:*

$$\begin{tikzcd} G \arrow[r, "\mathrm{i}_G \times \mathrm{id}_G"] \arrow[d] & G \times G \arrow[d, "\mathrm{m}_G"] \\ \mathrm{pt} \arrow[r, "\mathrm{e}_G"] & G \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} G \arrow[r, "\mathrm{id}_G \times \mathrm{i}_G"] \arrow[d] & G \times G \arrow[d, "\mathrm{m}_G"] \\ \mathrm{pt} \arrow[r, "\mathrm{e}_G"] & G \end{tikzcd}$$

PROOF. It is immediate to check that, if $\mathcal{C}$ is the category of sets, the commutativity of the diagram above gives the usual group axioms. Hence the result follows by evaluating the diagrams above (considered as diagrams of functors) at any object U of $\mathcal{C}$. ♠

Thus, for example, a group object in the category of topological spaces is simply a group, that has a structure of a topological space, such that the multiplication map and the inverse map are continuous (of course the identity map is automatically continuous).

One can also define an action of a group object G on an object X. There are two ways of doing this.

DEFINITION 1.10. A *left action* of a group object G of a category $\mathcal{C}$ on a object X consist of a left action $G(U) \times X(U) \to X(U)$, denoted by $(g, x) \mapsto g \cdot x$ of the group $G(U)$ on the set $X(U)$, in such a way that for any arrow $f \colon U \to V$ in $\mathcal{C}$, and $g \in G(V)$ and any $x \in X(V)$ we have

$$f^*g \cdot f^*x = f^*(g \cdot x) \in X(U).$$

A right action is defined in the analogous way.

Again, we can reformulate this definition in terms of diagrams.

PROPOSITION 1.11. *To give a left action of a group object G on an object X is equivalent to assigning an arrow $\rho \colon G \times X \to X$, such that the following diagrams commute.*

(i) *The identity of G acts like the identity on X:*

$$\begin{tikzcd}
\mathrm{pt} \times X \arrow[r, "\mathrm{e}_G \times \mathrm{id}_X"] \arrow[dr, equal] & G \arrow[d, "\rho"] \\
& X
\end{tikzcd}$$

(ii) *The action is associative with respect to the multiplication on G:*

$$\begin{tikzcd}
G \times G \times X \arrow[r, "\mathrm{m}_G \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \rho"] & G \times X \arrow[d, "\rho"] \\
G \times X \arrow[r, "\rho"] & X
\end{tikzcd}$$

PROOF. It is immediate to check that, if $\mathcal{C}$ is the category of sets, the commutativity of the diagram above gives the usual axioms for a left action. Hence the result follows by evaluating the diagrams above (considered as diagrams of functors) at any object U of $\mathcal{C}$. ♠

1.2. Relative representability

1.2.1. Fiber products of functors. The category $\mathrm{Func}(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Set}))$ has fiber products. These are defined as follows. Suppose that we are given three functors F_1, F_2 and G from $\mathcal{C}^{\mathrm{opp}}$ to (Set), together with two natural tranformations $\alpha_1 \colon F_1 \to G$ and $\alpha_2 \colon F_2 \to G$. The fiber product $F_1 \times_G F_2$ sends each object U of $\mathcal{C}$ into the fiber product of sets $F_1U \times_{GU} F_2U$, where of course the functions $F_1U \to GU$ and $F_1U \to GU$ are induced respectively by α_1 and α_2. The action of $F_1 \times_G F_2$ on arrows is defined in the obvious fashion.

Since the category $\mathrm{Func}(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Set}))$ has terminal object, the functor that sends each object to a fixed set with one element, it also has products, defined by the usual formula $(F_1 \times F_2)U = F_1U \times F_2U$.

If $X_1 \to Y$ and $X_2 \to Y$ are arrows in $\mathcal{C}$, $\mathrm{h}_{X_1} \to \mathrm{h}_Y$ and $\mathrm{h}_{X_2} \to \mathrm{h}_Y$ are the induced morphisms, then the fiber product $X_1 \times_Y X_2$ represents the fiber product $\mathrm{h}_{X_1} \times_{\mathrm{h}_Y} \mathrm{h}_{X_2}$; so we can write $X_1 \times_Y X_2$ to mean either the fiber product as an object of $\mathcal{C}$ or the fiber product of contravariant functors on $\mathcal{C}$.

1.2.2. Representable natural transformations.

DEFINITION 1.12. Let F and G be functors in $\mathrm{Func}(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Set}))$. A morphism of functors $\phi\colon F \to G$ is *representable* if for any object Y of $\mathcal{C}$ and any morphism $Y \to G$, the fiber product $F \times_G Y$ is representable.

Equivalently, the morphism τ is representable if whenever $H \to Y$ is a morphism and H is representable, so is $F \times_G H$.

PROPOSITION 1.13. *If $\tau\colon F \to G$ is a morphism of contravariant functors $\mathcal{C} \to$ (Set) and G is representable, then τ is representable if and only if F is representable.*

PROOF. Since the category $\mathcal{C}$ has fibered products, the fiber products of two representable functors is representable; hence if F is representable so is the morphism τ.

Conversely, if τ is representable, so is the fiber product $F \times_G G \simeq F$. ♠

DEFINITION 1.14. Let $\mathbf{P}$ be a property of arrows in $\mathcal{C}$. We say that $\mathbf{P}$ is *stable* if whenever

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a cartesian diagram and f has $\mathbf{P}$, then f' also has $\mathbf{P}$.

Examples of stable properties of continuous maps are being an embedding, being injective, being surjective, being a local homeomorphism, being open, being a covering map. Being closed, on the other hand, is not a stable property.

Any stable property of arrows in $\mathcal{C}$ can be extended to a property of representable morphisms. If $\mathbf{P}$ is a stable property of arrows in $\mathcal{C}$, we say that a representable morphism $F \to G$ has $\mathbf{P}$ if whenever $Y \to G$ is a morphism, with Y an object of $\mathcal{C}$, then the projection $F \times_G Y \to Y$ has $\mathbf{P}$. This makes sense, because $F \times_G Y$ is representable.

Consider a functor $F\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$; there is morphism $\delta_F\colon F \to F \times F$, the *diagonal of F*, defined by sending each object U of $\mathcal{C}$ into the diagonal function $FU \to FU \times FU = (F \times F)U$.

PROPOSITION 1.15. *Let $F\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ be a functor. Then the following three conditions are equivalent.*

(i) *The diagonal $\delta_F\colon F \to F \times F$ is representable.*
(ii) *If $X \to F$ and $Y \to F$ are morphisms, where X and Y are objects of $\mathcal{C}$, then the fiber product $X \times_F Y$ is representable.*
(iii) *All morphisms from representable functors into F are representable.*

PROOF. Parts (ii) and (iii) are equivalent by the definition of a representable morphism.

Assume that the diagonal $\delta_F\colon F \to F \times F$ is representable, and that $X \to F$ and $Y \to F$ are morphisms from objects of $\mathcal{C}$. It is a standard fact that there is a cartesian square

$$\begin{tikzcd} X \times_F Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ F \arrow[r, "\delta_F"] & F \times F \end{tikzcd},$$

which shows that the fiber product $X \times_F Y$ is representable. Hence (ii) holds.

Conversely, assume that (ii) holds, and that there is given a morphism $X \to F \times F$, where X is an object of $\mathcal{C}$. There is another cartesian diagram

$$\begin{tikzcd} F \times_{F\times F} X \arrow[r] \arrow[d] & X \arrow[d, "\delta_X"] \\ X \times_F X \arrow[r] \arrow[d] & X \times X \arrow[d] \\ F \arrow[r, "\delta_F"] & F \times F \end{tikzcd}$$

showing that $F \times_{F\times F} X$ is representable, as required. ♠

1.3. Sheaves in Grothendieck topologies

1.3.1. Grothendieck topologies. Now we need the notion of a sheaf on the category (Top). Consider a functor $F\colon (\mathrm{Top})^{\mathrm{opp}} \to (\mathrm{Set})$; for each topological space X we can consider the restriction F_X to the subcategory of (Top) whose objects are open subspaces of X, and whose arrows are the inclusion maps; this is a presheaf on X. We say that F is a *sheaf* on (Top) if F_X is a sheaf on X for all X.

For later use we are going to need the more general notion of sheaf in a Grothendieck topology; in this section we review this theory.

In a Grothendieck topology the "open sets" of a space are *maps* into this space; instead of intersections we have to look at fiber products, while unions play no role. The axioms do not describe the "open sets", but the coverings of a space.

DEFINITION 1.16. Let $\mathcal{C}$ be a category with fiber products. A *Grothendieck topology* on $\mathcal{C}$ is the assignement to each object U of $\mathcal{C}$ of a collection of sets of arrows $\{V_i \to U\}$, called *coverings of* U, so that the following conditions are satisfied.

(i) If $V \to U$ is an isomorphism, then the set $\{V \to U\}$ is a covering.
(ii) If $\{V_i \to U\}$ is a covering and $U' \to U$ is any arrow, then the collection of projections $\{V_i \times_U U' \to U'\}$ is a covering.
(iii) If $\{V_i \to U\}$ is a covering, and for each index i we have a covering $\{W_{ij} \to V_i\}$ (here j varies on a set depending on i), the collection of compositions $\{W_{ij} \to V_i \to U\}$ is a covering of U.

A category with a Grothendieck topology is called a *site*.

Notice that from (ii) and (iii) it follows that if $\{V_i \to U\}$ and $\{W_j \to U\}$ are two coverings of the same object, then $\{V_i \times_U W_j \to U\}$ is also a covering.

REMARK 1.17. In fact what we have defined here is what is called a *pretopology* in [**SGA1**]; a pretopology defines a topology, and very different pretopologies can define the same topology. The point is that the sheaf theory only depends on the topology, and not on the pretopology. So, for example, if two pretopologies on the same category satisfy the conditions of Proposition 1.25 below, the two induced topologies are the same, so the conclusion follows immediately.

Despite its unquestionable technical advantages, I do not find the notion of topology, as defined in [**SGA1**], very intuitive, so I prefer to avoid its use (just a question of habit, undoubtedly).

Here are some examples.

EXAMPLE 1.18 (The site of a topological space). Let X be a fixed topological space; call X_{cl} the category in which the objects are the open subsets of X, and the arrows are given by inclusions. Then we get a Grothendieck topology on X_{cl} by associating with each open subset $U \subseteq X$ the set of open coverings of U.

In this case if $V_1 \to U$ and $V_2 \to U$ are arrows, the fiber product $V_1 \times_U V_2$ is the intersection $V_1 \cap V_2$.

EXAMPLE 1.19 (The global classical topology). Here $\mathcal{C}$ is the category (Top) of topological spaces. If U is a topological space, then a covering of U will be a collection of open embeddings $V_i \to U$ whose images cover U.

Notice here we must interpret "open embedding" as meaning an open continuous injective map $V \to U$; if by an open embedding we mean the inclusion of an open subspace, then condition (i) of Definition 1.16 is not satisfied.

EXAMPLE 1.20 (The global étale topology for topological spaces). Here $\mathcal{C}$ is the category (Top) of topological spaces. If U is a topological space, then a covering of U will be a collection of local homeomorphisms $V_i \to U$ whose images cover U.

Here are the basic examples in algebraic geometry. Of course, a scheme is endowed with the Zariski topology, so it yields a site, according to Example 1.18. Of course, if this where the only significant example, this formalism would be useless.

EXAMPLE 1.21 (The small étale site of a scheme). Consider a scheme X. We can form a category $X_{\text{ét}}$, the full subcategory of the category (Sch/X) whose objects are morphism $U \to X$ that are locally of finite presentation and étale.

A covering $U_i \to U$ is a collection of morphisms of X-schemes whose images cover U. Recall that if U and each of the U_i is locally of finite presentation and étale over X, then each of the morphisms $U_i \to U$ is locally of finite presentation and étale, hence it has an open image.

EXAMPLE 1.22. Here are three topologies that one can put on the category (Sch/S) of schemes over a fixed scheme S. Several more have been used in different contexts.

The first is the *global Zariski topology.* Here a covering $\{U_i \to U\}$ is a collection of open embeddings covering U.

Then there is the *global étale topology.* A covering $\{U_i \to U\}$ is a collection of étale maps of finite presentation whose images cover U.

Finally, there is the *fppf topology*, where a covering $\{U_i \to U\}$ is a collection of flat maps locally of finite presentation whose images cover U.

1.3.2. Sheaves. If X is a topological space, a presheaf of sets on X is a functor ${X_{\mathrm{cl}}}^{\mathrm{opp}} \to (\mathrm{Set})$, where X_{cl} is the category of open subsets of X, as in Example 1.18. The condition that F be a sheaf can easily be generalized to any site, provided that we substitute intersections, that do not make sense, with fiber products.

DEFINITION 1.23. Let $\mathcal{C}$ be a site, $F\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ a functor.

(i) F is *separated* if, given a covering $\{U_i \to U\}$ and two sections a and b in FU whose pullbacks to each FU_i coincide, it follows that $a = b$.

(ii) F is a *sheaf* if the following condition is satisfied. Suppose that we are given a covering $\{U_i \to U\}$ in $\mathcal{C}$, and a set of sections $a_i \in FU_i$. Call $\mathrm{pr}_1\colon U_i \times_U U_j \to U_i$ and $\mathrm{pr}_2\colon U_i \times_U U_j \to U_j$ the first and second projection respectively, and assume that $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j \in F(U_i \times_U U_j)$ for all i and j. Then there is a unique section $a \in FU$ whose pullback to FU_i is a_i for all i.

If F and G are sheaves on a site $\mathcal{C}$, a *morphism of sheaves* $F \to G$ is simply a natural transformation of functors.

Of course one can also define sheaves of groups, rings, and so on, as usual: a functor from $\mathcal{C}^{\mathrm{opp}}$ to the category of groups, or rings, is a sheaf if its composition with the forgetful functor to the category of sets is a sheaf.

The reader might find our definition of sheaf rather pedantic, and wonder why we did not simply say “assume that the pullbacks of a_i and a_j to

$F(U_i \times_U U_j)$ coincide". The reason is the following: when $i = j$, in the classical case of a topological space we have $U_i \times_U U_i = U_i \cap U_i = U_i$, so the two possible pullbacks from $U_i \times_U U_i \to U_i$ coincide; but if the map $U_i \to U$ is not injective, then the two projections $U_i \times_U U_i \to U_i$ will be different. So, for example, in the classical case coverings with one subset are not interesting, and the sheaf condition is automatically verified for them, while in the general case this is very far from being true.

A sheaf on a site is clearly separated.

Sometimes two different topologies on the same category define the same sheaves.

DEFINITION 1.24. Let $\mathcal{C}$ be a category, $\{U_i \to U\}$ a set of arrows. A *refinement* $\{V_j \to U\}$ is a set of arrows such that for each index j there is some index i such that $V_j \to U$ factors through U_i.

PROPOSITION 1.25. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two Grothendieck topologies on the same category $\mathcal{C}$. Suppose that every covering in $\mathcal{T}$ is also in $\mathcal{T}'$, and that every covering in $\mathcal{T}'$ has a refinement in $\mathcal{T}$. Then a functor $\mathcal{C} \to$ (Set) is a sheaf in the topology $\mathcal{T}$ if and only if it is a sheaf on the topology $\mathcal{T}'$.*

In particular, the sheaves on (Top) *in the classical, the étale topology and the local fibration topology are the same.*

PROOF. Since $\mathcal{T}'$ contains all the coverings of $\mathcal{T}$, clearly any functor $F\colon \mathcal{C}^{\mathrm{opp}} \to$ (Set) that is a sheaf in the topology $\mathcal{T}'$ is also a sheaf in $\mathcal{T}$. On the other hand, assume that $F\colon \mathcal{C}^{\mathrm{opp}} \to$ (Set) is a sheaf in the topology $\mathcal{T}$, and take a covering $\{U_i \to U\}$ of an object U in the topology $\mathcal{T}'$. There is a refinement $\{V_j \to U\}$ of $\{U_i \to U\}$ in $\mathcal{T}$; for each index j choose a factorization $V_j \to U_{\iota_j} \to U$. If two section of FU coincide when pulled back to each FU_i they also coincide when pulled back to each FV_j, and therefore they coincide; hence the functor F is separated in the topology $\mathcal{T}'$.

Now, assume that we are given a collection of sections $\{a_i\} \in \prod_i FU_i$, such that the pullbacks $\mathrm{pr}_1^* a_i$ and $\mathrm{pr}_2^* a_{i'}$ to $F(U_i \times_U U_{i'})$ coincide for all indices i and i'. For each j call b_j the pullback of a_{ι_j} to V_j through the arrow $V_j \to U_{\iota_j}$. I claim that for every pair of indices j and j' the pullbacks of b_j and $b_{j'}$ to $V_j \times_U V_{j'}$ coincide. In fact, the composition of $\mathrm{pr}_1\colon V_j \times_U V_{j'} \to V_j$ with the arrow $V_j \to U_{\iota_j}$ factors through $\mathrm{pr}_1\colon U_{\iota_j} \times U_{\iota_{j'}} \to U_{\iota_j}$; and analogously for the second projection. Since the pullbacks $\mathrm{pr}_1^* a_{\iota_j}$ and $\mathrm{pr}_2^* a_{\iota_{j'}}$ to $F(U_i \times_U U_{i'})$ coincide, the thesis follows.

Since F is a sheaf in $\mathcal{T}$, there will exist some a in FU whose pullback to FV_j is b_j for all j. Now we need to show that the pullback of a to FU_i is a_i for all i. For each j and i there is a commutative diagram

$$\begin{array}{ccccc}
V_j \times_U U_i & \longrightarrow & U_{\iota_j} \times_U U_i & \longrightarrow & U_i \;; \\
\downarrow & & \downarrow & & \downarrow \\
V_j & \longrightarrow & U_{\iota_j} & \longrightarrow & U
\end{array}$$

since the pullbacks of a_{ι_j} and a_i to $F(U_{\iota_j} \times_U U_i)$ coincide, this shows that the pullbacks of $a \in FU$ and $a_i \in FU_i$ to $F(V_j \times_U U_i)$ are the same. But $\{V_j \times_U U_i \to U_i\}$ is a covering of U_i in the topology $\mathcal{T}'$, and since F is separated in the topology $\mathcal{T}'$ we conclude that in fact the pullback of a to FU_i equals a_i. ♠

DEFINITION 1.26. A topology $\mathcal{T}$ on a category $\mathcal{C}$ is called *saturated* if, whenever $\{U_i \to U\}$ is a set of arrows, $\{V_{ij} \to U_i\}$ is a covering of U_i for each i, and the set $\{V_{ij} \to U\}$ of compositions is a covering, then $\{U_i \to U\}$ is a covering.

If $\mathcal{T}$ is a topology of $\mathcal{C}$, the *saturation* of $\mathcal{T}$ is the set $\widetilde{\mathcal{T}}$ of all sets of arrows $\{U_i \to U\}$ with the property that there exists a covering $\{V_{ij} \to U_i\}$ for each i such that the set $\{V_{ij} \to U\}$ of compositions is a covering.

PROPOSITION 1.27. *The saturation $\widetilde{\mathcal{T}}$ of a topology $\mathcal{T}$ is a saturated topology. Furthermore, a functor $\mathcal{C}^{\mathrm{opp}} \to$ (Set) is a sheaf under $\widetilde{\mathcal{T}}$ if and only if it is a sheaf under $\mathcal{T}$.*

PROOF. **TO BE ADDED** ♠

EXAMPLE 1.28. The global étale topology on (Top) is a saturated topology. It is the saturation of the classical topology; hence the global étale topology and the classical topology have the same sheaves.

In the category (Sch/S) the étale topology and the fppf topology are both saturated. On the other hand, in the category of schemes an étale morphism is not Zariski-locally an open embedding, hence the global étale topology is not the saturation of the global Zariski topology.

There is also a statement saying that sometimes different sites have equivalent categories of sheaves.

PROPOSITION 1.29. *Let $\mathcal{C}$ be a full subcategory of a category $\mathcal{C}'$, closed under taking fiber products. Let $\mathcal{T}$ and $\mathcal{T}'$ be Grothendieck topologies on $\mathcal{C}$ and $\mathcal{C}'$ respectively. Assume that the following conditions hold.*

(i) *A covering in $\mathcal{T}$ is also a covering in $\mathcal{T}'$.*
(ii) *An object U of $\mathcal{C}$ has a covering $\{U_i \to U\}$ in which each U_i is in $\mathcal{C}$.*
(iii) *If U is an object of $\mathcal{C}$, any covering of U in $\mathcal{T}'$ has a refinement in $\mathcal{T}$.*

Then the restriction to $\mathcal{C}$ of a sheaf $\mathcal{C}'^{\mathrm{opp}} \to$ (Set) is a sheaf on $\mathcal{C}$. Furthermore, restriction induces an equivalence of the categories of sheaves on $\mathcal{C}'$ and on $\mathcal{C}$.

In particular, for a topological space X the categories of sheaves on X_{cl} and $X_{\text{ét}}$ are equivalent.

PROOF. Obviously condition (i) implies that the restriction of a sheaf is a sheaf. The restriction of a natural transformation of functors $\mathcal{C}'^{\mathrm{opp}} \to$ (Set) yields a natural transformation of functors $\mathcal{C}^{\mathrm{opp}} \to$ (Set), and this gives a functor from the category of sheaves on $\mathcal{C}'$ to the category of sheaves on $\mathcal{C}$.

Also, it is easy to check that the restriction of $\mathcal{T}'$ to $\mathcal{C}$ is a topology on $\mathcal{C}$, because $\mathcal{C}$ is closed under taking fiber products; conditions (i) and (iii), together with Proposition 1.25, imply that the sheaves on $\mathcal{C}$ relative to the restriction of $\mathcal{T}'$ and to $\mathcal{T}$ are the same. So we may assume that a covering $\{U_i \to U\}$ in $\mathcal{T}'$ in which both U and the U_i are in $\mathcal{C}$ is in $\mathcal{T}$. In this case we do not need to refer to the topology explicitly, and we will talk about a covering referring to a covering in $\mathcal{T}$ or in $\mathcal{T}'$.

First of all, let us check that this restriction functor is faithful. In fact, let F' and G' be sheaves on $\mathcal{C}'$, $\alpha', \beta' \colon F' \to G'$ natural transformations which coincide when restricted to $\mathcal{C}$. For any object U of $\mathcal{C}'$ there is a covering $\{U_i \to U\}$ in which the U_i are in $\mathcal{C}$; this implies that for any $x \in F'U$ the restrictions of $\alpha'_U x$ and $\beta'_U x$ to each U_i coincide. Since G' is a sheaf, this show that $\alpha = \beta$, and hence that the functor is faithful.

To show that is full, take a natural transformation $\alpha \colon F \to G$ between the restrictions of F' and G' to $\mathcal{C}$. For any object U in $\mathcal{C}'$ we have a covering $\{U_i \to U\}$ with U_i in $\mathcal{C}$; furthermore for each pair of indices i and j we fix a covering $\{U_{ijk} \to U_i \times_U U_j\}$ where each U_{ijk} is in $\mathcal{C}$ (here k varies over a set depending on i and j). Take an element a of $F'U$, and call $a_i \in F'U_i = FU_i$ the pullback of a to U_i. Set $b_i = \alpha_{U_i} a \in GU_i$; the restrictions of each b_i and b_j to each U_{ijk} coincide for all i and j, therefore they coincide in $U_i \times_U U_j$, because G' is a sheaf. It follows that there is a unique element $b \in G'U$ whose pullback to each $G'U_i$ is b_i. We define a function $\alpha'_U \colon F'U \to G'U$ by setting $\alpha'_U a = b$.

We have to check that α' is well defined, and that it defines a natural transformation. Take another covering $\{V_s \to U\}$ with the V_s in $\mathcal{C}$, and for each pair of indices a covering $\{V_{isk} \to U_i \times_U V_s\}$ with the V_{isk} in $\mathcal{C}$. Consider the element $c \in G'U$ obtained as above from the covering $\{V_a \to U\}$; it is easy to see that the pullbacks of b and c to V_{isk} coincide for each triple i, a and s, and this implies the thesis, because $\{V_{isk} \to U\}$ is a covering. It is easy to show that α' is a natural transformation, and this shows that the functor is full.

We only have left to prove that every sheaf on $\mathcal{C}$ is the restriction of a sheaf on $\mathcal{C}'$. Let $F \colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ be a sheaf. For each object U of $\mathcal{C}'$ choose a covering $\{U_i \to U\}$ with U_i in $\mathcal{C}$, and for each pair of indices i and j a covering $\{U_{ijk} \to U_i \times_U U_j\}$ where each U_{ijk} is in $\mathcal{C}$. For each triple of indices i, j and k there are arrows $\phi_{ijk} \colon U_{ijk} \to U_i$ and $\psi_{ijk} \colon U_{ijk} \to U_j$, obtained by composing the given arrow $U_{ijk} \to U_i \times_U U_j$ with the two projections $\mathrm{pr}_1 \colon U_i \times_U U_j \to U_i$ and $\mathrm{pr}_2 \colon U_i \times_U U_j \to U_2$.

We define a set $F'U$ as the equalizer of the two functions $\prod_{ij} FU_i \times FU_j \to \prod_{ijk} FU_{ijk}$ induced by ϕ_{ijk} and ψ_{ijk}: in other words, $F'U$ is the subset of $\prod_{ij} FU_i \times FU_j \to \prod_{ijk} FU_{ijk}$ consisting of elements $\big((a_i, b_j)\big)$ such that the pullbacks $F\phi_{ijk} a_i$ and $F\psi_{ijk} b_j$ in FU_{ijk} coincide. Of course when U is an object of $\mathcal{C}$, the restriction map $FU \to \prod_i FU_i$ induces a bijective correspondence between FU and $F'U$, by definition of sheaf. **TO BE ADDED**

♠

PROPOSITION 1.30. *A representable functor* (Top)$^{\mathrm{opp}} \to$ (Set) *is a sheaf in the classical topology.*

The proof is straightforward. It is similarly easy to show that a representable functor in the category (Sch/S) over a base scheme S is a sheaf in the Zariski topology. On the other hand the following is not straightforward at all.

PROPOSITION 1.31 (Grothendieck). *A representable functor in* (Sch/S) *is a sheaf in the fppf topology.*

For the proof, see [**SGA1**, ???]. From this it follows that a representable functor is also a sheaf in the étale topology, because every fppf covering is also an étale covering.

DEFINITION 1.32. A topology $\mathcal{T}$ on a category $\mathcal{C}$ is called *subcanonical* if every representable functor in $\mathcal{C}$ is a sheaf with respect to $\mathcal{T}$.
A *subcanonical site* is a category endowed with a subcanonical topology.

There are examples of sites that are not subcanonical, but I have never had dealings with any of them.
The name "subcanonical" comes from the fact that on a category $\mathcal{C}$ there is a topology, known as the *canonical topology*, which is the finest topology in which every representable functor is a sheaf. We will not be needing this fact.

1.3.3. The sheafification of a functor. The usual construction of the sheafification of a presheaf of sets on a topological space carries over to this more general context.

DEFINITION 1.33. Let $\mathcal{C}$ be a site, $F\colon \mathcal{C}^{\mathrm{opp}} \to$ (Set) a functor. A *sheafification* of F is a sheaf $F^{\mathrm{a}}\colon \mathcal{C}^{\mathrm{opp}} \to$ (Set), together with a natural transformation $F \to F^{\mathrm{a}}$, such that:

(i) given an object U of $\mathcal{C}$ and two objects ξ and η of $F(U)$ whose images ξ^{a} and η^{a} in $F^{\mathrm{a}}(U)$ are isomorphic, there exists a covering $\{\sigma_i\colon U_i \to U\}$ such that $\sigma_i^*\xi = \sigma_i^*\eta$, and
(ii) for each object U of $\mathcal{C}$ and each $\overline{\xi} \in F^{\mathrm{a}}(U)$, there exists a covering $\{\sigma_i\colon U_i \to U\}$ and elements $\xi_i \in F(U_i)$ such that $\xi_i^{\mathrm{a}} = \sigma_i^*\xi$.

THEOREM 1.34. *Let $\mathcal{C}$ be a site, $F\colon \mathcal{C}^{\mathrm{opp}} \to$ (Set) a functor.*

(i) *If $F^{\mathrm{a}}\colon \mathcal{C}^{\mathrm{opp}} \to$ (Set) is a sheafification of F, any morphism from F to a sheaf factors uniquely through F^{a}.*
(ii) *There exists a sheafification $F \to F^{\mathrm{a}}$, which is unique up to a canonical isomorphism.*

(iii) *The natural transformation $F \to F^{\mathrm{a}}$ is injective if and only if F is separated.*

SKETCH OF PROOF. For part s(i), let $\phi\colon F \to G$ be a natural transformation from F to a sheaf $G\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$.

Let us prove the first part. For each object U of $\mathcal{C}$, we define an equivalence relation $\sim$ on FU as follows. Given two sections a and b in FU, we write $a \sim b$ if there is a covering $U_i \to U$ such that the pullbacks of a and b to each U_i coincide. We check easily that this is an equivalence relation, and we define $F^{\mathrm{s}}U = FU/\sim$. We also verify that if $V \to U$ is an arrow in $\mathcal{C}$, the pullback $FU \to FV$ is compatible with the equivalence relations, yielding a pullback $F^{\mathrm{s}}U \to F^{\mathrm{s}}V$. This defines the functor F^{s} with the surjective morphism $F \to F^{\mathrm{s}}$. It is straightforward to verify that F^{s} is separated, and that every natural transformation from F to a separated functor factors uniquely through F^{s}.

To construct F^{a}, we take for each object U of $\mathcal{C}$ the set of pairs $(\{U_i \to U\}, \{a_i\})$, where $\{U_i \to U\}$ is a covering, and $\{a_i\}$ is a set of sections with $a_i \in F^{\mathrm{s}}U_i$ such that the pullback of a_i and a_j to $F^{\mathrm{s}}(U_i \times_U U_j)$, along the first and second projection respectively, coincide. On this set we impose an equivalence relation, by declaring $(\{U_i \to U\}, \{a_i\})$ to be equivalent to $(\{V_j \to U\}, \{b_j\})$ when the restrictions of a_i and b_j to $F^{\mathrm{s}}(U_i \times_U V_j)$, along the first and second projection respectively, coincide. To verify the transitivity of this relation we need to use the fact that the functor F^{s} is separated.

For each U we call $F^{\mathrm{a}}U$ the set of equivalence classes. If $V \to U$ is an arrow, we define a function $F^{\mathrm{a}}U \to F^{\mathrm{a}}V$ by associating with the class of a pair $(\{U_i \to U\}, \{a_i\})$ in $F^{\mathrm{a}}U$ the class of the pair $(\{U_i \times_U V\}, p_i^* a_i)$, where $p_i\colon U_i \times_U V \to U_i$ is the projection. Once we have checked that this is well defined, we obtain a functor $F^{\mathrm{a}}\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$. There is also a natural transformation $F^{\mathrm{s}} \to F^{\mathrm{a}}$, obtained by sending an element $a \in F^{\mathrm{s}}U$ into $(\{U = U\}, a)$. Then one verifies that F^{a} is a sheaf, and that the composition of the natural transformations $F \to F^{\mathrm{s}}$ and $F^{\mathrm{s}} \to F^{\mathrm{a}}$ has the desired universal property.

The unicity up to a canonical isomorphism follows immediately from part (i). Part (iii) follows easily from the definition. ♠

1.4. Equivalence relations

1.4.1. Equivalence relations as categories. The notion of category generalizes the notion of set: a set can be thought of as a category in which every arrow is an identity. Furthermore funtors between sets are simply functions.

It is also possible to characterize the categories that are equivalent to a set: these are the equivalence relations.

Suppose that $R \subseteq X \times X$ is an equivalence relation on a set X. We can produce a category (X, R) in which X is the set of objects, R is the set of arrows, and the source and target maps $R \to X$ are given by the first and second projection. Then given x and y in X, there is precisely one arrow (x, y) if x and y are in the same equivalence class, while there is none if they are not. Then transitivity assures us that we can compose arrows, while reflexivity tell us that over each object $x \in X$ there is a unique arrow (x, x), which is the identity. Finally symmetry tells us that any arrow (x, y) has an inverse (y, x). So, (X, R) is groupoid such that from a given object to another there is at most one arrow.

Conversely, given a groupoid such that from a given object to another there is at most one arrow, if we call X the set of objects and R the set of arrows, the source and target maps induce an injective map $R \to X \times X$, that gives an equivalence relation on X.

So an equivalence relation can be thought of as a groupoid such that from a given object to another there is at most one arrow. Equivalently, an equivalence relation is a groupoid in which the only arrow from an object to itself is the identity.

PROPOSITION 1.35. *A category is equivalent to a set if and only if it is an equivalence relation.*

PROOF. If a category is equivalent to a set, it is immediate to see that it is an equivalence relation. If (X, R) is an equivalence relation and X/R is the set of isomorphism classes of objects, that is, the set of equivalence classes, one checks immediately that the function $X \to X/R$ gives a functor that is fully faithful and essentially surjective, so it is an equivalence. ♠

1.4.2. Equivalence relations in a category. Recall that an arrow $X \to Y$ in a category $\mathcal{C}$ is *categorically injective*, or simply *injective*, when the induced function $X(U) \to Y(U)$ is injective for all objects U of $\mathcal{C}$. In other words, $X \to Y$ is injective if X is a subfunctor of Y. If $\mathcal{C}$ is the category of topological spaces, then a continuous function is injective in the categorical sense if and only if it is actually injective (to check that categorical injectivity implies injectivity use maps from a point).

Let $\mathcal{C}$ be a category, that we assume, as always, to have products and fiber products.

DEFINITION 1.36. An *equivalence relation in $\mathcal{C}$* consists of two objects X and R, together with an injective arrow $R \to X \times X$, such that for any object U of $\mathcal{C}$ the injective function $R(U) \to X(U) \times X(U)$ makes $R(U)$ into an equivalence relation on $X(U)$.

Given an equivalence relation (X, R), the *source* and *target* arrows $\mathrm{s}\colon R \to X$ and $\mathrm{t}\colon R \to X$ are respectively the first and second projection.

Here is another way of expressing this. Suppose that $\rho\colon R \to X \times X$ is an injective arrow; we will denote by $\mathrm{s}\colon R \to X$ and $\mathrm{t}\colon R \to X$ the first and second projection. For any object U of $\mathcal{C}$ we get a relation $R(U) \subseteq X(U) \times X(U)$ on the set $X(U)$. This relation is reflexive if and only if the diagonal function $X(U) \to X(U) \times X(U)$ lifts uniquely to a function $X(U) \to R(U)$; and if this happens this function for each U defines a natural transformation of functors $X \to R$, hence, by Yoneda's lemma, an arrow $\mathrm{e}\colon X \to R$ in $\mathcal{C}$, with the property that its composition with the given morphism $\rho\colon R \to X \times X$ is the diagonal $X \to X \times X$.

Similarly, symmetry can be expressed by saying that the involution $X \times X \simeq X \times X$ that switches the two factors lifts to an involution $\mathrm{i} : R \to R$.

Transitivity can also be expressed in this style, but this is a little trickier. Consider the fiber product $R \times_X R$, where the first factor is considered as an object over X via the arrow $\mathrm{s}\colon R \to X$, and the second factor via $\mathrm{t}\colon R \to X$. For any object U of $\mathcal{C}$, the set $(R \times_X R)(U)$ is the set of pairs of elements (u, v) of $R(U)$ such that the target of u equals the source of v; there is a natural function $(R \times_X R)(U) \to (X \times X)(U)$ to the pair $\big(\mathrm{s}(v), \mathrm{t}(u)\big)$. This defines a natural transformation, hence an arrow, $R \times_X R \to X \times X$. The equivalence relation is transitive if and only if this arrow $R \times_X R \to X \times X$ lifts to an arrow $\mathrm{m}\colon R \times_X R \to R$.

Note that the definition of m may look a little strange: for example, if X is a set and $R \subseteq X \times X$, we are defining $\mathrm{m}\big((y, z), (x, y)\big) = (x, z)$, while it might seem more natural to switch the roles of s and t and send $\big((x, y), (y, z)\big)$ to (x, z). The reason for our choice is that if we want to interpret an equivalence relation as a category, we think of the pair (x, y) as an arrow from x to y, and m gives the composition of arrows with the standard convention.

So we have the following alternate definition.

DEFINITION 1.37. An *equivalence relation* (X, R) in $\mathcal{C}$ is an injective arrow $\rho\colon R \to X \times X$ such that:

(i) The diagonal arrow $X \to X \times X$ lifts to an arrow $\mathrm{e}\colon X \to R$,
(ii) the composition of $\rho\colon R \to X \times X$ with the involution $X \times X \simeq X \times X$ that switches the two factors lifts to an arrow $R \to R$, and
(iii) The arrow $R \times_X R \to X \times X$ corresponding to the natural transformation sending each pair $(u, v) \in (R \times_X R)(U)$ to the pair $(\mathrm{s}(v), \mathrm{t}(u)) \in (X \times X)(U)$ lifts to an arrow $\mathrm{m}\colon R \times_X R \to R$.

There is an obvious notion of morphism of equivalence relations, that makes equivalence relations into a category.

DEFINITION 1.38. A morphism $f\colon (X, R) \to (X', R')$ of equivalence relations is an arrow $f\colon X \simeq X'$ such that the composition of $R \to X \times X$ with $f \times f\colon X \times X \to X' \times X'$ lifts to an arrow $R \to R'$.

Examples of equivalence relations are obtained from arrows in $\mathcal{C}$; an arrow $X \to Y$ in $\mathcal{C}$ yields an equivalence relation $X \times_Y X \hookrightarrow X \times X$. This gives a functor from the category of arrows in $\mathcal{C}$ to the category of equivalence relations in $\mathcal{C}$.

More generally, one can consider a representable morphism to a functor. The following will be very useful.

EXAMPLE 1.39. Let X be an object of $\mathcal{C}$, $F\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ a functor, $X \to F$ a representable morphism. Then fiber product $R = X \times_F X$ is representable, and with its natural injective arrow $R \to X \times X$ defines an equivalence relation on X.

CHAPTER 2

Fibered categories

2.1. Fibered categories

2.1.1. Definition and first properties. In this section we will fix a category $\mathcal{C}$ with products and fiber products; the topology will play no role. We will study categories over $\mathcal{C}$, that is, categories $\mathcal{F}$ equipped with a functor $\mathrm{p}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{C}$.

We will draw several commutative diagrams involving objects of $\mathcal{C}$ and $\mathcal{F}$; an arrow going from an object ξ of $\mathcal{F}$ to an object U of $\mathcal{C}$ will be of type "$\xi \mapsto U$", and will mean that $\mathrm{p}_{\mathcal{F}}\xi = U$. Furthermore the commutativity of the diagram

$$\begin{tikzcd}
\xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
U \arrow[r, "f"] & V
\end{tikzcd}$$

will mean that $\mathrm{p}_{\mathcal{F}}\phi = f$.

DEFINITION 2.1. Let $\mathcal{F}$ be a category over $\mathcal{C}$. An arrow $\phi\colon \xi \to \eta$ of $\mathcal{F}$ is *cartesian* if for any arrow $\psi\colon \zeta \to \eta$ in $\mathcal{F}$ and any arrow $h\colon \mathrm{p}_{\mathcal{F}}\zeta \to \mathrm{p}_{\mathcal{F}}\xi$ with $\mathrm{p}_{\mathcal{F}}\phi \circ h = \mathrm{p}_{\mathcal{F}}\psi$, there exists a unique arrow $\theta\colon \zeta \to \xi$ with $\mathrm{p}_{\mathcal{F}}\theta = h$ and $\theta \circ \phi = \zeta$, as in the commutative diagram

$$\begin{tikzcd}
\zeta \arrow[rrd, bend left, "\psi"] \arrow[rd, dashed, "\theta"'] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\
\mathrm{p}_{\mathcal{F}}\zeta \arrow[rd, "h"'] \arrow[rrd, bend left] & & \\
& \mathrm{p}_{\mathcal{F}}\xi \arrow[r] & \mathrm{p}_{\mathcal{F}}\eta
\end{tikzcd}$$

If $\xi \to \eta$ is a cartesian arrow of $\mathcal{F}$ mapping to an arrow $U \to V$ of $\mathcal{C}$, we also say that ξ is *a pullback of* η *to* U.

REMARK 2.2. The definition of cartesian arrow we give is more restrictive than the definition in [**SGA1**]; however, the resulting notions of fibered category coincide.

Notice that given two pullbacks $\phi\colon \xi \to \eta$ and $\widetilde{\phi}\colon \widetilde{\xi} \to \eta$ of η to U, the unique arrow $\theta\colon \widetilde{\xi} \to \xi$ that fits into the diagram

$$
\begin{tikzcd}
\widetilde{\xi} \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\widetilde{\phi}"] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\
U \arrow[dr, equal] \arrow[drr, bend left] & & \\
& U \arrow[r] & V
\end{tikzcd}
$$

is an isomorphism. In other words, a pullback is unique, up to a unique isomorphism.

The following facts are easy to prove, and are left to the reader.

PROPOSITION 2.3.

(i) *If $\mathcal{F}$ is a category over $\mathcal{C}$, the composition of cartesian arrows in $\mathcal{F}$ is cartesian.*

(ii) *A cartesian arrow of $\mathcal{F}$ whose image in $\mathcal{C}$ is an isomorphism is also an isomorphism.*

(iii) *If $\xi \to \eta$ and $\eta \to \zeta$ are arrows in $\mathcal{F}$ and $\eta \to \zeta$ is cartesian, then $\xi \to \eta$ is cartesian if and only if the composition $\xi \to \zeta$ is cartesian.*

(iv) *Let $\mathrm{p}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{C}$ and $\mathrm{p}_{\mathcal{G}}\colon \mathcal{G} \to \mathcal{C}$ be categories over $\mathcal{C}$. If $F\colon \mathcal{F} \to \mathcal{G}$ is a functor with $\mathrm{p}_{\mathcal{G}} \circ F = \mathrm{p}_{\mathcal{F}}$, $\xi \to \eta$ is an arrow in $\mathcal{F}$ that is cartesian over its image $F\xi \to F\eta$ in $\mathcal{F}$, and $F\xi \to F\eta$ is cartesian over its image $\mathrm{p}_{\mathcal{G}}\xi \to \mathrm{p}_{\mathcal{G}}\eta$ in $\mathcal{C}$, then $\xi \to \eta$ is cartesian over $\mathrm{p}_{\mathcal{G}}\xi \to \mathrm{p}_{\mathcal{G}}\eta$.*

DEFINITION 2.4. A *fibered category over $\mathcal{C}$* is a category $\mathcal{F}$ over $\mathcal{C}$, such that given an arrow $f\colon U \to V$ in $\mathcal{C}$ and an object η of $\mathcal{F}$ mapping to V, there is a cartesian arrow $\phi\colon \xi \to \eta$ with $\mathrm{p}_{\mathcal{F}}\phi = f$.

If $\mathcal{F}$ and $\mathcal{G}$ are fibered categories over $\mathcal{C}$, then a *morphism of fibered categories* $F\colon \mathcal{F} \to \mathcal{G}$ is a functor such that:

(i) F is *base-preserving*, that is, $\mathrm{p}_{\mathcal{G}} \circ F = \mathrm{p}_{\mathcal{F}}$;

(ii) F sends cartesian arrows to cartesian arrows.

In other words, in a fibered category $\mathcal{F} \to \mathcal{C}$ we can pull back objects of $\mathcal{F}$ along any arrow of $\mathcal{C}$.

Notice that in the definition above the equality $\mathrm{p}_{\mathcal{G}} \circ F = \mathrm{p}_{\mathcal{F}}$ must be interpreted as an actual equality. In other words, the existence of an isomorphism of functors between $\mathrm{p}_{\mathcal{G}} \circ F$ and $\mathrm{p}_{\mathcal{F}}$ is not enough.

PROPOSITION 2.5. *Let $\mathrm{p}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{C}$ and $\mathrm{p}_{\mathcal{G}}\colon \mathcal{G} \to \mathcal{C}$ be categories over $\mathcal{C}$, $F\colon \mathcal{F} \to \mathcal{G}$ a functor with $\mathrm{p}_{\mathcal{G}} \circ F = \mathrm{p}_{\mathcal{F}}$. Assume that $\mathcal{G}$ is fibered over $\mathcal{C}$.*

(i) *If $\mathcal{F}$ is fibered over $\mathcal{G}$, then it also fibered over $\mathcal{C}$.*

(ii) *If F is an equivalence of categories, then $\mathcal{F}$ is fibered over $\mathcal{C}$.*

PROOF. Part (i) follows from Proposition 2.3 (iv). Part (ii) follows from the easy fact that if F is an equivalence then $\mathcal{F}$ is fibered over $\mathcal{G}$. ♠

2.1.2. Fibered categories as lax 2-functors.

DEFINITION 2.6. Let $\mathcal{F}$ be a fibered category over $\mathcal{C}$. Given an object U of $\mathcal{C}$, the *fiber* $\mathcal{F}(U)$ of $\mathcal{F}$ over U is the subcategory of $\mathcal{F}$ whose objects are the objects ξ of $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}\xi = U$, and whose arrows are arrows ϕ in $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}\phi = \mathrm{id}_U$.

By definition, if $F\colon \mathcal{F} \to \mathcal{G}$ is a morphism of fibered categories over $\mathcal{C}$ and U is an object of $\mathcal{C}$, the functor F sends $\mathcal{F}(U)$ to $\mathcal{G}(U)$, so we have a restriction functor $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$.

Notice that formally we could give the same definition of a fiber for any functor $\mathrm{p}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{C}$, without assuming that $\mathcal{F}$ is fibered over $\mathcal{C}$. However, we would end up with a useless notion. For example, it may very well happen that we have two objects U and V of $\mathcal{C}$ which are isomorphic, but such that $\mathcal{F}(U)$ is empty while $\mathcal{F}(V)$ is not. This kind of pathology does not arise for fibered categories, and here is why.

Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$, and $f\colon U \to V$ an arrow in $\mathcal{C}$. For each object η over V, we choose a pullback $\phi_\eta\colon f^*\eta \to \eta$ of η to U. We also define a functor $f^*\colon \mathcal{F}(V) \to \mathcal{F}(U)$ by sending each object η of $\mathcal{F}(V)$ to $f^*\eta$, and each arrow $\beta\colon \eta \to \eta'$ of $\mathcal{F}(U)$ to the unique arrow $f^*\beta\colon f^*\eta \to f^*\eta'$ in $\mathcal{F}(V)$ making the diagram

$$\begin{tikzcd} f^*\eta \arrow[r] \arrow[d, dashed, "f^*\beta"] & \eta \arrow[d, "\beta"] \\ f^*\eta' \arrow[r] & \eta' \end{tikzcd}$$

commute.

In this way we associate with each object U of $\mathcal{C}$ a category $\mathcal{F}(U)$, and to each arrow $f\colon U \to V$ a functor $f^*\colon \mathcal{F}(V) \to \mathcal{F}(U)$. It is very tempting to believe that in this way we have defined a functor from $\mathcal{C}$ to the category of categories; however, this is not quite correct. First of all, pullbacks $\mathrm{id}_U^*\colon \mathcal{F}(U) \to \mathcal{F}(U)$ are not necessarily identities. Of course we could just choose all pullbacks along identites to be identities on the fiber categories: this would certainly work, but it is not very natural, as there are often natural defined pullbacks where this does not happen (in Example 2.7 and many others). What happens in general is that, when U is an object of $\mathcal{C}$ and ξ an object of $\mathcal{F}(U)$, we have the pullback $\epsilon_U(\xi)\colon \mathrm{id}_U^*\xi \to \xi$ is an isomorphism, because of Proposition 2.3 (ii), and this defines an isomorphism of functors $\epsilon_U\colon \mathrm{id}_U^* \simeq \mathrm{id}_{\mathcal{F}(U)}$.

A more serious problem is the following. Suppose that we have two arrows $f\colon U \to V$ and $g\colon V \to W$ in $\mathcal{C}$, and an object ζ of $\mathcal{F}$ over W. Then $f^*g^*\zeta$ is a pullback of ζ to U; however, pullbacks are not unique, so there is no reason why $f^*g^*\zeta$ should coincide with $(gf)^*\zeta$. However, there is a canonical isomorphism $\alpha_{f,g}(\zeta)\colon f^*g^*\zeta \simeq (gf)^*\zeta$ in $\mathcal{F}(U)$, because both are pullbacks, and this gives an isomorphism of functors $\alpha_{f,g}\colon f^*g^* \simeq (gf)^*\colon \mathcal{F}(W) \to \mathcal{F}(U)$.

So, a fibered category almost gives a functor from $\mathcal{C}$ to the category of categories, but not quite. The point is that the category of categories is not just a category, but what is known as a 2-category; that is, its arrows are functors, but two functors between the same two categories in turn form a category, the arrows being natural transformations of functors. Thus there are 1-arrows (functors) between objects (categories), but there are also 2-arrows (natural transformations) between 1-arrows.

Analogously, a morphism of fibered category $F\colon \mathcal{F} \to \mathcal{G}$ can be thought of as a "natural tranformation" from the "functor" $\mathcal{F}$ to the "functor" $\mathcal{G}$. In fact, the restriction $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a functor from the category $\mathcal{F}(U)$ to the category $\mathcal{G}(U)$. However, given an arrow $f\colon U \to V$ in $\mathcal{C}$, the diagram of functors

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "F_V"] \arrow[d, "f^*"] & \mathcal{G}(V) \arrow[d, "f^*"] \\ \mathcal{F}(U) \arrow[r, "F_U"] & \mathcal{G}(U) \end{tikzcd}$$

does not commute. In fact, since F carries cartesian arrows to cartesian arrows, for each object η of $\mathcal{F}(V)$, $F(f^*\xi) = F_U(f^*\xi)$ is a pullback of η to U; but there is no reason why it should coincide exactly with $f^*(F\eta)$. However, since both $F(f^*\eta)$ and $f^*(F\eta)$ are pullbacks, there is a canonical isomorphism $u_\eta\colon F(f^*\eta) \simeq f^*(F\eta)$ in $\mathcal{G}(U)$. When we assign the isomorphism u_η to each object η we define an isomorphism of functors $u\colon F_U \circ f^* \to f^* \circ F_V$. This means that in fact the square above is a commutative square, in the sense of Definition 2.31.

This point of view will not be used in this notes: it is interesting, and has many advantages, but it is technically rather involved; I will discuss it in a later version. Only the following construction will be useful in the rest of the notes: we are going to need the fibered category associated with an ordinary functor $\mathcal{C}^{\mathrm{opp}} \to (\mathrm{Cat})$.

2.1.3. The fibered category associated with a functor to the category of categories. Let $\Phi\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Cat})$ be a contravariant functor from the category $\mathcal{C}$ into the category of categories, considered as a 1-category. This means that with each object U of $\mathcal{C}$ we associate a category ΦU, and for each arrow $f\colon U \to V$ gives a functor $\Phi f\colon \Phi V \to \Phi U$, in such a way that $\Phi\mathrm{id}_U\colon \Phi U \to \Phi U$ is the identity, and $\Phi(g \circ f) = \Phi f \circ \Phi g$ everytime we have two composable arrows f and g in $\mathcal{C}$.

To this Φ we can associate a fibered category $\mathcal{F} \to \mathcal{C}$, such that for any object U in $\mathcal{C}$ the fiber $\mathcal{F}(U)$ is canonically equivalent to the category ΦU. An object of $\mathcal{F}$ is a pair (ξ, U) where U is an object of $\mathcal{C}$ and ξ is an object of $\mathcal{F}(U)$. An arrow $(a, f)\colon (\xi, U) \to (\eta, V)$ in $\mathcal{F}$ consists of an arrow $f\colon U \to V$ in $\mathcal{C}$, together with an arrow $\xi \to \Phi f(\eta)$.

The composition is defined as follows: if $(a,f)\colon (\xi,U)\to(\eta,V)$ and $(b,g)\colon(\eta,V)\to(\zeta,W)$ are two arrows, then

$$(b,g)\circ(a,f) = (\Phi b\circ f, g\circ f)\colon (\xi,U)\to(\zeta,W).$$

There is an obvious functor $\mathcal{F}\to\mathcal{C}$ that sends an object (ξ,U) into U and an arrow (a,f) into f; I claim that this functor makes $\mathcal{F}$ into a fibered category over $\mathcal{C}$. In fact, given an arrow $f\colon U\to V$ in $\mathcal{C}$ and an object (η,V) in $\mathcal{F}(V)$, then $\bigl(\Phi f(\eta),U\bigr)$ is an object of $\mathcal{F}(U)$, and it is easy to check that the pair $\bigl(f,\mathrm{id}_{\Phi f(\eta)}\bigr)$ gives a cartesian arrow $\bigl(\Phi f(\eta),U\bigr)\to(\eta,V)$.

The fiber of $\mathcal{F}$ is canonically equivalent to the category ΦU: the equivalence $\mathcal{F}(U)\to\Phi U$ is obtained at the level of objects by sending (ξ,U) to ξ, and at the level of arrows by sending (a,id_U) to a.

2.1.4. Examples of fibered categories.

EXAMPLE 2.7. Let $\mathcal{F}$ be the category of arrows in $\mathcal{C}$; the objects are the arrows in $\mathcal{C}$, while an arrow from $f\colon S\to U$ to $g\colon T\to V$ is a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, "f"] & T \arrow[d, "g"] \\ U \arrow[r] & V \end{tikzcd}.$$

The functor $\mathrm{p}_{\mathcal{F}}\colon\mathcal{F}\to\mathcal{C}$ sends each arrow $S\to U$ to its codomain U, and each commutative diagram to its bottom row.

I claim that $\mathcal{F}$ is a fibered category over $\mathcal{C}$. In fact, it easy to check that the cartesian diagrams are precisely the cartesian squares, so the statement follows from the fact that $\mathcal{C}$ has fibered products.

EXAMPLE 2.8. As a variant of the example above, consider a stable property (Definition 1.14) $\mathbf{P}$ of arrows in $\mathcal{C}$. We can consider the category whose objects are arrows in $\mathcal{C}$ having the property $\mathbf{P}$, where the arrows are given by commutative squares as above. This a category fibered over $\mathcal{C}$.

EXAMPLE 2.9. Let G a topological group. The *classifying stack* of G is the fibered category $\mathcal{B}G\to(\mathrm{Top})$ over the category of topological spaces, whose objects are principal G bundles $P\to S$, and whose arrows (ϕ,f) from $P\to U$ to $Q\to V$ are commutative diagrams

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d] & Q \arrow[d] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

where the function ϕ is G-equivariant. The functor $\mathcal{B}G\to(\mathrm{Top})$ sends a principal bundle $P\to U$ into the topological space U, and an arrow (ϕ,f) into f.

Contrary to the usual convention, in a principal G-bundle $P\to S$ we will write the action of G on the left.

It is important to notice that any such diagram is cartesian; so $\mathcal{B}G \to$ (Top) has the property that each of its arrows is cartesian.

2.2. Categories fibered in groupoids

DEFINITION 2.10. A *category fibered in groupoids over* $\mathcal{C}$ is a category $\mathcal{F}$ fibered over $\mathcal{C}$, such that the category $\mathcal{F}(U)$ is a groupoid for any object U of $\mathcal{C}$.

In the literature one often finds a different definition of a category fibered in groupoids.

PROPOSITION 2.11. *Let $\mathcal{F}$ be a category over $\mathcal{C}$. Then $\mathcal{F}$ is fibered in groupoids over $\mathcal{C}$ if and only if the following two conditions hold.*

(i) *Every arrow in $\mathcal{F}$ is cartesian.*
(ii) *Given an object η of $\mathcal{F}$ and an arrow $f\colon U \to p_{\mathcal{F}}\eta$ of $\mathcal{C}$, there exists an arrow $\phi\colon \xi \to \eta$ of $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}\phi = f$.*

PROOF. Suppose that these two conditions hold. Then it is immediate to see that $\mathcal{F}$ is fibered over $\mathcal{C}$. Also, if $\phi\colon \xi \to \eta$ is an arrow of $\mathcal{F}(U)$ for some object U of $\mathcal{C}$, then we see from condition 2.11 (i) that there exists an arrow $\psi\colon \eta \to \xi$ with $\mathrm{p}_{\mathcal{F}}\psi = \mathrm{id}_U$ and $\phi\psi = \mathrm{id}_\eta$; that is, every arrow in $\mathcal{F}(U)$ has a right inverse. But this right inverse ψ also must also have a right inverse, and then the right inverse of ψ must be ϕ. This proves that every arrow in $\mathcal{F}(U)$ is invertible.

Conversely, assume that $\mathcal{F}$ is fibered over $\mathcal{C}$, and each $\mathcal{F}(U)$ is a groupoid. Condition (ii) is trivially verified. To check condition (i), let $\phi\colon \xi \to \eta$ be an arrow in $\mathcal{C}$ mapping to $f\colon U \to V$ in $\mathcal{C}$. Choose a pullback $\phi'\colon \xi' \to \eta$ of η to U; by definition there will be an arrow $\alpha\colon \xi \to \xi'$ in $\mathcal{F}(U)$ such that $\phi'\alpha = \phi$. Since $\mathcal{F}(U)$ is a a groupoid, α will be an isomorphism, and this implies that ϕ is cartesian. ♠

COROLLARY 2.12. *Any base preserving functor from a fibered category to a category fibered in groupoids is a morphism.*

PROOF. This is clear, since every arrow in a category fibered in groupoids is cartesian. ♠

Of the examples of Section 2.1, 2.7 and 2.8 are not in general fibered in groupoids, while the classifying stack of a topological group introduced in 2.9 is always fibered in groupoids.

Give a fibered category $\mathcal{F} \to \mathcal{C}$, the subcategory $\mathcal{F}_{\mathrm{cart}}$ whose objects are the same as the objects of $\mathcal{F}$, but the arrows are the cartesian arrows, is fibered in groupoids. Any morphism $\mathcal{G} \to \mathcal{F}$ of fibered categories, where $\mathcal{G}$ is fibered in groupoids, factors uniquely through $\mathcal{F}_{\mathrm{cart}}$.

2.3. Functors and categories fibered in sets

As we remarked at the beginning of Section 1.4, categories are generalizations of sets: sets are categories in which all the arrows are identities. Similarly, fibered categories are generalizations of functors.

DEFINITION 2.13. A *category fibered in sets over* $\mathcal{C}$ is a category $\mathcal{F}$ fibered over $\mathcal{C}$, such that for any object U of $\mathcal{C}$ the category $\mathcal{F}(U)$ is a set.

Here is an useful characterization of categories fibered in sets.

PROPOSITION 2.14. *Let $\mathcal{F}$ be a category over $\mathcal{C}$. Then $\mathcal{F}$ is fibered in sets if and only if for any object η of $\mathcal{F}$ and any arrow $f\colon U \to \mathrm{p}_{\mathcal{F}}\eta$ of $\mathcal{C}$, there is a unique arrow $\phi\colon \xi \to \eta$ of $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}\phi = f$.*

PROOF. Suppose that $\mathcal{F}$ is fibered in sets. Given η and $f\colon U \to \mathrm{p}_{\mathcal{F}}\eta$ as above, pick a cartesian arrow $\xi \to \eta$ over f. If $\xi' \to \eta$ is any other arrow over f, by definition there exists an arrow $\xi' \to \xi$ in $\mathcal{F}(U)$ making the diagram

$$\begin{tikzcd}
\xi' \arrow[rr, dashed] \arrow[rd] & & \xi \arrow[ld] \\
& \eta &
\end{tikzcd}$$

commutative. Since $\mathcal{F}(U)$ is a set, it follows that this arrow $\xi' \to \xi$ is the identity, so the two arrows $\xi \to \eta$ and $\xi' \to \eta$ coincide.

Conversely, assume that the condition holds. Given a diagram

$$\begin{tikzcd}
\zeta \arrow[rrd, bend left, "\psi"] \arrow[rd, dashed, "\theta"'] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\
\mathrm{p}_{\mathcal{F}}\zeta \arrow[rrd, bend left] \arrow[rd, "h"'] & & \\
& \mathrm{p}_{\mathcal{F}}\xi \arrow[r] & \mathrm{p}_{\mathcal{F}}\eta
\end{tikzcd}$$

the condition implies that the only arrow $\theta\colon \zeta \to \xi$ over h makes the diagram commutative; so the category $\mathcal{F}$ is fibered.

It is obvious that the condition implies that $\mathcal{F}(U)$ is a set for all U. ♠

So, for categories fibered in sets the pullback of an object of $\mathcal{F}$ along an arrow of $\mathcal{C}$ is strictly unique. It follows from this that when $\mathcal{F}$ is fibered in sets over $\mathcal{C}$ and $f\colon U \to V$ is an arrow in $\mathcal{C}$, the pullback map $f^*\colon \mathcal{F}(V) \to \mathcal{F}(U)$ is uniquely defined, and the composition rule $f^*g^* = (gf)^*$ holds. Also for any object U of $\mathcal{C}$ we have that $\mathrm{id}_U^*\colon \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity. This means that we have defined a functor $\Phi_{\mathcal{F}}\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ by sending each object U of $\mathcal{C}$ to $\mathcal{F}(U)$, and each arrow $f\colon U \to V$ of $\mathcal{C}$ to the function $f^*\colon \mathcal{F}(V) \to \mathcal{F}(U)$.

Furthermore, if $F\colon \mathcal{F} \to \mathcal{G}$ is a morphism of categories fibered in sets, because of the condition that $\mathrm{p}_{\mathcal{G}} \circ F = \mathrm{p}_{\mathcal{F}}$, then every arrow in $\mathcal{F}(U)$, for some object U of $\mathcal{C}$, will be send to $\mathcal{F}(U)$ itself. So we get a function

$F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$. It is immediate to check that this gives a natural tranformation $\phi_F\colon \Phi_{\mathcal{F}} \to \Phi_{\mathcal{G}}$.

There is a category of categories fibered in sets over $\mathcal{C}$, where the arrows are morphisms of fibered categories; the construction above gives a functor from this category to the category of functors $\mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$.

PROPOSITION 2.15. *This is an equivalence of the category of categories fibered in sets over $\mathcal{C}$ and the category of functors $\mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$.*

PROOF. The inverse functor is obtained by the construction of 2.1.3. If $\Phi\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ is a functor, we construct a category fibered in sets $\mathcal{F}_\Phi$ as follows. The objects of $\mathcal{F}_\Phi$ will be pairs (U, ξ), where U is an object of $\mathcal{C}$, and $\xi \in \Phi U$. An arrow from (U,ξ) to (V,η) is an an arrow $f\colon U \to V$ of $\mathcal{C}$ with the property that $\Phi f \eta = \xi$. It follows from Proposition 2.14 that $\mathcal{F}_\Phi$ is fibered in sets over $\mathcal{C}$.

To each natural transformation of functors $\phi\colon \Phi \to \Phi'$ we associate a morphism $F_\phi\colon \mathcal{F}_\Phi \to \mathcal{F}_{\Phi'}$. An object (U,ξ) of $\mathcal{F}_\Phi$ will be sent to $(U, \phi_U\xi)$. If $f\colon (U,\xi) \to (V,\eta)$ is an arrow in $\mathcal{F}_\Phi$, then f is simply an arrow $f\colon U \to V$ in $\mathcal{C}$, with the property that $\Phi f(\eta) = \xi$. This implies that $\Phi'(f)\phi_V(\eta) = \phi_U\Phi(f)(\eta) = \phi_V\xi$, so the same f will yield an arrow $f\colon (U, \phi_U\xi) \to (V, \phi_V\eta)$.

We leave it the reader to check that this defines a functor from the category of functors to the category of categories fibered in sets. ♠

So, any functor $\mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ will give an example of a fibered category over $\mathcal{C}$.

In particular, given an object X of $\mathcal{C}$, we have the representable functor $\mathrm{h}_X\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$, defined on objects by the rule $\mathrm{h}_X U = \mathrm{Hom}_{\mathcal{C}}(U, X)$. The category in sets over $\mathcal{C}$ associated with this functor is the category $(\mathcal{C}/X)$, whose objects are arrows $U \to X$, and whose arrows are commutative diagrams

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd} .$$

So the situation is the following. From Yoneda's lemma we see that the category $\mathcal{C}$ is embedded into the category of functors $\mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$, while the category of functors is embedded into the category of fibered categories.

From now we will identify a functor $F\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ with the corresponding category fibered in sets over $\mathcal{C}$, and we will (inconsistently) call a category fibered in sets simply "a functor".

2.3.1. Categories fibered over an object.

PROPOSITION 2.16. *Let $\mathcal{G}$ be a category fibered in sets over $\mathcal{C}$, $\mathcal{F}$ another category, $F\colon \mathcal{F} \to \mathcal{G}$ a functor. Then $\mathcal{F}$ is fibered over $\mathcal{G}$ if and only if it is fibered over $\mathcal{C}$ via the composition $\mathrm{p}_{\mathcal{G}} \circ F\colon \mathcal{F} \to \mathcal{C}$.*

Furthermore, $\mathcal{F}$ is fibered in groupoids over $\mathcal{G}$ if and only if it fibered in groupoids over $\mathcal{C}$, and is fibered in sets over $\mathcal{G}$ if and only if it fibered in sets over $\mathcal{C}$.

PROOF. sOne sees immediately that an arrow of $\mathcal{G}$ is cartesian over its image in $\mathcal{F}$ if and only if it is cartesian over its image in $\mathcal{C}$, and the first statement follows from this.

Furthermore, one sees that the fiber of $\mathcal{F}$ over an object U of $\mathcal{C}$ is the disjoint union, as a category, of the fibers of $\mathcal{F}$ over all the objects of $\mathcal{G}$ over U; if these fiber are groupoids, or sets, so is their disjoint union. ♠

This is going to be used as follows. Suppose that S is an object of $\mathcal{C}$, and consider the category fibered in sets $(\mathcal{C}/S) \to \mathcal{C}$, corresponding to the representable functor $\mathrm{h}_S \colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$. By Proposition 2.16, a fibered category $\mathcal{F} \to (\mathcal{C}/S)$ is the same as a fibered category $\mathcal{F} \to \mathcal{C}$, together with a morphism $\mathcal{F} \to \mathcal{C}$.

It is interesting to describe this process for functors. Given a functor $F \colon (\mathcal{C}/S)^{\mathrm{opp}} \to (\mathrm{Set})$, this corresponds to a category fibered in sets $F \to (\mathcal{C}/S)$; this can be composed with the forgetful functor $(\mathcal{C}/S) \to \mathcal{C}$ to get a category fibered in sets $F \to \mathcal{C}$, which in turn corresponds to a functor $F' \colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$. What is this functor? One minute's thought will convince you that it can be described as follows: $F'(U)$ is the disjont union of the $F(U \xrightarrow{u} S)$ for all the arrows $u \colon U \to S$ in $\mathcal{C}$. The action of F' on arrows is the obvious one.

2.4. Equivalences of fibered categories

2.4.1. Natural transformations of functors. The fact that fibered categories are categories, and not functors, has strong implications, and does cause difficulties. As usual, the main problem is that functors between categories can be isomorphic without being equal; in other words, functors between two fixed categories form a category, the arrows being given by natural transformations.

DEFINITION 2.17. Let $\mathcal{F}$ and $\mathcal{G}$ be two categories fibered over $\mathcal{C}$, F, $G \colon \mathcal{F} \to \mathcal{G}$ two morphisms. A *base-preserving natural transformation* $\alpha \colon F \to G$ is a natural transformation such that for any object ξ of $\mathcal{F}$, the arrow $\alpha_\xi \colon F\xi \to G\xi$ is in $\mathcal{G}(U)$, where $U \overset{\mathrm{def}}{=} \mathrm{p}_{\mathcal{F}}\xi = \mathrm{p}_{\mathcal{G}}(F\xi) = \mathrm{p}_{\mathcal{G}}(G\xi)$.

An *isomorphism of F with G* is a base-preserving natural transformation $F \to G$ which is an isomorphism of functors.

It is immediate to check that the inverse of a base-preserving isomorphism is also base-preserving.

There is a category whose objects are the morphism from a $\mathcal{F}$ to $\mathcal{G}$, and the arrows are base-preserving natural transformations; we denote it by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$.

2.4.2. Equivalences.

DEFINITION 2.18. Let $\mathcal{F}$ and $\mathcal{G}$ be two fibered categories over $\mathcal{C}$. An *equivalence*, or *isomorphism*, of $\mathcal{F}$ with $\mathcal{G}$ is a morphism $F \colon \mathcal{F} \to \mathcal{G}$, such that there exists another morphism $G\colon \mathcal{G} \to \mathcal{F}$, together with isomorphisms of $G \circ F$ with $\mathrm{id}_{\mathcal{F}}$ and of $F \circ G$ with $\mathrm{id}_{\mathcal{G}}$.

We call G simply an *inverse* to F.

PROPOSITION 2.19. *Suppose that $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{G}$ and $\mathcal{G}'$ are categories fibered over $\mathcal{C}$. Suppose that $F\colon \mathcal{F}' \to \mathcal{F}$ and $G\colon \mathcal{G} \to \mathcal{G}'$ are equivalences. Then there an equivalence of categories*

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}', \mathcal{G}')$$

that sends each $\Phi\colon \mathcal{F} \to \mathcal{G}$ into the composition

$$G \circ \Phi \circ F\colon \mathcal{F}' \to \mathcal{G}'.$$

PROOF. **TO BE ADDED** ♠

The following is the basic criterion for checking whether a morphism of fibered categories is an equivalence.

PROPOSITION 2.20. *Let $F\colon \mathcal{F} \to \mathcal{G}$ be a morphism of fibered categories. Then F is an equivalence if and only if the restriction $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories for any object U of $\mathcal{C}$.*

PROOF. Suppose that $G\colon \mathcal{G} \to \mathcal{F}$ is an inverse to F; the two isomorphisms $F \circ G \simeq \mathrm{id}_{\mathcal{G}}$ and $G \circ F \simeq \mathrm{id}_{\mathcal{F}}$ restrict to isomorphisms $F_U \circ G_U \simeq \mathrm{id}_{\mathcal{G}(U)}$ and $G_U \circ F_U \simeq \mathrm{id}_{\mathcal{F}(U)}$, so G_U is an inverse to F_U.

Conversely, we assume that $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories for any object U of $\mathcal{C}$, and construct an inverse $G\colon \mathcal{G} \to \mathcal{F}$. Here is the main fact that we are going to need.

LEMMA 2.21. *Let $F\colon \mathcal{F} \to \mathcal{G}$ a morphism of fibered categories such that every restriction $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is fully faithful. Then the functor F is fully faithful.*

PROOF. We need to show that, given two objects ξ' and η' of $\mathcal{F}$ and an arrow $\phi\colon F\xi' \to F\eta'$ in $\mathcal{G}$, there is a unique arrow $\phi'\colon \xi' \to \eta'$ in $\mathcal{F}$ with $F\phi' = \phi$. Set $\xi = F\xi'$ and $\eta = F\eta'$. Let $\eta'_1 \to \eta'$ be a pullback of η' to U, $\eta_1 = F\eta'_1$. Then the image $\eta_1 \to \eta$ of $\eta'_1 \to \eta'$ is cartesian, so every morphism $\xi \to \eta$ factors uniquely as $\xi \to \eta_1 \to \eta$, where the arrow $\xi \to \xi_1$ is in $\mathcal{G}(U)$. Analogously all arrows $\xi' \to \eta'$ factor uniquely through through η_1; since every arrow $\xi \to \eta_1$ in $\mathcal{G}(U)$ lifts uniquely to an arrow $\xi' \to \eta'_1$ in $\mathcal{F}(U)$, we have proved the Lemma. ♠

For any object ξ of $\mathcal{G}$ pick an object $G\xi$ of $\mathcal{F}(U)$, where $U = \mathrm{p}_{\mathcal{G}}\xi$, together with an isomorphism $\alpha_\xi\colon \xi \simeq F(G\xi)$ in $\mathcal{G}(U)$; these $G\xi$ and α_ξ exist because $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories.

Now, if $\phi\colon \xi \to \eta$ is an arrow in $\mathcal{G}$, by the Lemma there is a unique arrow $G\phi\colon G\xi \to G\eta$ such that $F(G\phi) = \alpha_\eta \circ \phi \circ \alpha_\xi^{-1}$, that is, such that the diagram

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, "\alpha_\xi"] & \eta \arrow[d, "\alpha_\eta"] \\ F(G\xi) \arrow[r, "F(G\phi)"] & F(G\eta) \end{tikzcd}$$

commutes.

These operations define a functor $G \colon \mathcal{G} \to \mathcal{F}$. It is immediate to check that by sending each object ξ to the isomorphism $\alpha_\xi \colon \xi \simeq F(G\xi)$ we define an isomorphism of functors $\mathrm{id}_{\mathcal{F}} \simeq F \circ G \colon \mathcal{G} \to \mathcal{G}$.

We only have left to check that $G \circ F \colon \mathcal{F} \to \mathcal{F}$ is isomorphic to the identity $\mathrm{id}_{\mathcal{F}}$.

Fix an object ξ' of $\mathcal{F}$ over an object U of $\mathcal{C}$; we have a canonical isomorphism $\alpha_{F\xi'} \colon F\xi' \simeq F(G(F\xi'))$ in $\mathcal{G}(U)$. Since F_U is fully faithful there is a unique isomorphism $\beta_{\xi'} \colon \xi' \simeq G(F\xi')$ in $\mathcal{F}(U)$ such that $F\beta_{\xi'} = \alpha_{F\xi'}$; one checks easily that this defines an isomorphism of functors $\beta \colon G \circ F \simeq \mathrm{id}_{\mathcal{G}}$. ♠

2.4.3. Quasifunctors. A category can be equivalent to a set without being one: the categories equivalent to sets are precisely the equivalence relations. There is an analogous result for fibered categories.

DEFINITION 2.22. A category $\mathcal{F}$ over $\mathcal{C}$ is a *quasifunctor*, or that it is *fibered in equivalence relations* if it is fibered, and each fiber $\mathcal{F}(U)$ is an equivalence relation.

We have the following characterization of quasifunctors.

PROPOSITION 2.23. *A category $\mathcal{F}$ over $\mathcal{C}$ is a quasifunctor if and only if the following two conditions hold.*

(i) *Given an object η of $\mathcal{F}$ and an arrow $f \colon U \to p_{\mathcal{F}}\eta$ of $\mathcal{C}$, there exists an arrow $\phi \colon \xi \to \eta$ of $\mathcal{F}$ with $p_{\mathcal{F}}\phi = f$.*
(ii) *Given two objects ξ and η of $\mathcal{F}$ and an arrow $f \colon p_{\mathcal{F}}\xi \to p_{\mathcal{F}}\eta$ of $\mathcal{C}$, there exists at most one arrow $\xi \to \eta$ over f.*

The easy proof is left to the reader.

PROPOSITION 2.24. *A fibered category over $\mathcal{C}$ is a quasifunctor if and only if it is equivalent to a functor.*

PROOF. This is an application of Proposition 2.20.

Suppose that a fibered category $\mathcal{F}$ is equivalent to a functor Φ; then every category $\mathcal{F}(U)$ is equivalent to the set ΦU, so $\mathcal{F}$ is fibered in equivalence relations over $\mathcal{C}$ by Proposition 1.35.

Conversely, assume that $\mathcal{F}$ is fibered in equivalence relations. In particular it is fibered in groupoid, so every arrow in $\mathcal{F}$ is cartesian, by Proposition 2.11. For each object U of $\mathcal{C}$ choose a set of representatives ΦU for the isomorphism classes of objects in $\mathcal{F}(U)$. For each arrow $f \colon U \to V$ and each object η of ΦU, choose a pullback $f^*\eta \to \eta$ of η to U; we may assume that

$f^*\eta$ is in ΦU. Any two pullbacks are isomorphic in $\mathcal{F}(U)$, so in fact $f^*\eta$ is unique (the arrow $f^*\eta \to \eta$ might not be, though). We define a functor $\Phi\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ by sending each object U to ΦU, and associating with each arrow $f\colon U \to V$ the function $f^*\colon \Phi V \to \Phi U$. If we think of Φ as a category fibered in sets, then by construction Φ is a subcategory of $\mathcal{F}$; the embedding $\Phi \subseteq \mathcal{F}$ induces an equivalence of each sets ΦU with the equivalence relation $\mathcal{F}(U)$, so by Proposition 2.20 it is an equivalence. ♠

Here are a few useful facts.

PROPOSITION 2.25.
(i) *If $\mathcal{G}$ is groupoid, then so is* $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})$.
(ii) *If $\mathcal{G}$ is a quasifunctor, then* $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ *is an equivalence relation.*
(iii) *If $\mathcal{G}$ is a functor, then* $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ *is a set.*

We leave the easy proofs to the reader.

In 2-categorical terms, part (iii) says that the 2-category of categories fibered in sets is in fact just a 1-category, while part (ii) says that the 2-category of quasisets is equivalent to a 1-category.

2.5. Objects as fibered categories and the 2-Yoneda lemma

2.5.1. Representable fibered categories. In 1.1 we have seen how we can embed a category $\mathcal{C}$ into the functor category $\mathrm{Func}\big(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Set})\big)$, while in 2.3 we have seen how to embed the category $\mathrm{Func}\big(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Set})\big)$ into the 2-category of fibered categories over $\mathcal{C}$. By composing these embeddings we have embedded $\mathcal{C}$ into the 2-category of fibered categories: an object X of $\mathcal{C}$ is sent to the fibered category $(\mathcal{C}/X) \to \mathcal{C}$. Furthermore, an arrow $f\colon X \to Y$ goes to the morphism of fibered categories $(\mathcal{C}/f)\colon (\mathcal{C}/X) \to (\mathcal{C}/Y)$ that sends an object $U \to X$ of $(\mathcal{C}/X)$ to the composition $U \to X \xrightarrow{f} Y$. The functor $(\mathcal{C}/f)$ sends an arrow

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

of $(\mathcal{C}/X)$ to the commutative diagram obtained by composing both sides with $f\colon X \to Y$.

This is the 2-categorical version of the weak Yoneda lemma.

THE WEAK 2-YONEDA LEMMA. *The function that sends each arrow $f\colon X \to Y$ to the functor $(\mathcal{C}/f)\colon (\mathcal{C}/X) \to (\mathcal{C}/Y)$ is a bijection.*

DEFINITION 2.26. A fibered category over $\mathcal{C}$ is *representable* if it is equivalent to a category of the form $(\mathcal{C}/X)$.

So a representable category is necessarily a quasifunctor, by Proposition 2.24. However, we should be careful: if $\mathcal{F}$ and $\mathcal{G}$ are fibered categories, equivalent to $(\mathcal{C}/X)$ and $(\mathcal{C}/Y)$ for two objects X and Y of $\mathcal{C}$, then

$$\mathrm{Hom}(X,Y) = \mathrm{Hom}\big((\mathcal{C}/X),(\mathcal{C}/Y)\big),$$

and according to Proposition 2.19 we have an equivalence of categories

$$\operatorname{Hom}\big((\mathcal{C}/X),(\mathcal{C}/Y)\big) \simeq \operatorname{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G});$$

but $\operatorname{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ need not be a set, it could very well be an equivalence relation.

2.5.2. The 2-categorical Yoneda lemma. As in the case of functors, we have a stronger version of the 2-categorical Yoneda lemma. Suppose that $\mathcal{F}$ is a category fibered over $\mathcal{C}$, and that X is an object of $\mathcal{C}$. Suppose that we are given a morphism $F\colon (\mathcal{C}/X) \to \mathcal{F}$; to this we can associate an object $F(\mathrm{id}_X) \in \mathcal{F}(X)$. Also, to each base-preserving natural transformation $\alpha\colon F \to G$ of functors $F, G\colon (\mathcal{C}/X) \to \mathcal{F}$ we associate the arrow $\alpha_{\mathrm{id}_X}\colon F(\mathrm{id}_X) \to G(\mathrm{id}_X)$. This defines a functor

$$\operatorname{Hom}_{\mathcal{C}}\big((\mathcal{C}/X),\mathcal{F}\big) \longrightarrow \mathcal{F}(X).$$

Conversely, given an object $\xi \in \mathcal{F}(X)$ we get a functor $F_\xi\colon (\mathcal{C}/X) \to \mathcal{F}$ as follows. Given an object $\phi\colon U \to X$ of $(\mathcal{C}/X)$, we define $F_\xi(\phi) = \phi^*\xi \in \mathcal{F}(U)$; to an arrow

$$\begin{tikzcd}
U \arrow[rr, "f"] \arrow[rd, "\phi"'] & & V \arrow[ld, "\psi"] \\
& X &
\end{tikzcd}$$

in $(\mathcal{C}/X)$ we associate the only arrow $\theta\colon \phi^*\xi \to \psi^*\xi$ in $\mathcal{F}(U)$ making the diagram

$$\begin{tikzcd}
\phi^*\xi \arrow[rrd, bend left] \arrow[rd, dashed, "\theta"] \arrow[dd] & & \\
& \psi^*\xi s \arrow[r] \arrow[dd] & \xi \arrow[dd] \\
U \arrow[rd, "h"'] \arrow[rrd, bend left, "\phi"] & & \\
& V \arrow[r, "\psi"'] & X
\end{tikzcd}$$

commutative. We leave it to the reader to check that F_ξ is indeed a functor.

2-Yoneda Lemma. *The two functors above define an equivalence of categories*

$$\operatorname{Hom}_{\mathcal{C}}\big((\mathcal{C}/X),\mathcal{F}\big) \simeq \mathcal{F}(X).$$

Proof. To check that the composition

$$\mathcal{F}(X) \longrightarrow \operatorname{Hom}_{\mathcal{C}}\big((\mathcal{C}/X),\mathcal{F}\big) \longrightarrow \mathcal{F}(X)$$

is isomorphic to the identity, notice that for any object $\xi \in \mathcal{F}(X)$, the composition applied to ξ yields $F_\xi(\xi) = \mathrm{id}_X^*\xi$, which is canonically isomorphic to ξ. It is easy to check that this defines an isomorphism of functors.

For the composition

$$\operatorname{Hom}_{\mathcal{C}}\big((\mathcal{C}/X),\mathcal{F}\big) \longrightarrow \mathcal{F}(X) \longrightarrow \operatorname{Hom}_{\mathcal{C}}\big((\mathcal{C}/X),\mathcal{F}\big)$$

take a morphism $F\colon (\mathcal{C}/X) \to \mathcal{F}$ and set $\xi = F(\mathrm{id}_X)$. We need to produce a base-preserving isomorphism of functors of F with F_ξ. The identity id_X is a terminal object in the category $(\mathcal{C}/X)$, hence for any object $\phi\colon U \to X$ there is a unique arrow $\phi\colon \mathrm{id}_X$, which is clearly cartesian. Hence it will remain cartesian after applying F, because F is a functor: this means that $F(\phi)$ is a pullback of $\xi = F(\mathrm{id}_X)$ along $\phi\colon U \to X$, so there is a canonical isomorphism $F_\xi(\phi) = \phi^*\xi \simeq F(\phi)$ in $\mathcal{F}(U)$. It is easy to check that this defines a base-preserving isomorphism of funtors, and this ends the proof. ♠

We have identified an object X with the functor $\mathrm{h}_X\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ it represents, and we have identified the functor h_X with the corresponding category $(\mathcal{C}/X)$: so, to be consistent, we have to identify X and $(\mathcal{C}/X)$. So, we will write X for $(\mathcal{C}/X)$.

As for functors, the strong form of the 2-Yoneda lemma can be used to reformulate the condition of representability. A morphism $(\mathcal{C}/X) \to \mathcal{F}$ corresponds to an object $\xi \in \mathcal{F}(X)$, which in turn defines the functor $F'\colon (\mathcal{C}/X) \to \mathcal{F}$ described above; this is isomorphic to the original functor F. Then F' is an equivalence if and only if for each object U of $\mathcal{C}$ the restriction

$$F'_U\colon \mathrm{Hom}_{\mathcal{C}}(U, X) = (\mathcal{C}/X)(U) \longrightarrow \mathcal{F}(U)$$

that sends each $f\colon U \to X$ to the pullback $f^*\xi \in \mathcal{F}(U)$, is an equivalence of categories. Since $\mathrm{Hom}_{\mathcal{C}}(U, X)$ is a set, this is equivalent to saying that $\mathcal{F}(U)$ is a groupoid, and each object of $\mathcal{F}(U)$ is isomorphic to the image of a unique element of $\mathrm{Hom}_{\mathcal{C}}(U, X)$ via a unique isomorphism. Since the isomorphisms $\rho \simeq f^*\xi$ in U correspond to cartesian arrows $\rho \to \xi$, and in a groupoid all arrows are cartesian, this means that $\mathcal{F}$ is fibered in groupoids, and for each $\rho \in \mathcal{F}(U)$ there exists a unique arrow $\rho \to \xi$. We have proved the following.

PROPOSITION 2.27. *A category fibered in groupoids $\mathcal{F}$ in $\mathcal{C}$ is representable if and only if $\mathcal{F}$ is fibered in groupoids, and there is an object X of $\mathcal{C}$ and an object ξ of $\mathcal{F}(U)$, such that for any object ρ of $\mathcal{F}$ there exists a unique arrow $\rho \to \xi$ in $\mathcal{F}$.*

2.6. Fiber products of fibered categories

In this section all categories will be fibered over a fixed category $\mathcal{C}$, and all functors will be morphisms of fibered categories.

The notion of fiber product of fibered categories is a little subtle. There is an obvious definition, which is as follows.

DEFINITION 2.28. Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ be fibered categories over $\mathcal{C}$, $F_1\colon \mathcal{F}_1 \to \mathcal{G}$ and $F_2\colon \mathcal{F}_2 \to \mathcal{G}$ morphisms. The *naive fiber product* $\mathcal{F}_1 \overset{\text{naive}}{\times}_{\mathcal{G}} \mathcal{F}_2$ is the subcategory of the product $\mathcal{F}_1 \times \mathcal{F}_2$ consisting of those pairs of objects (ξ_1, ξ_2) with $F_1\xi_1 = F_2\xi_2$, and those arrows $(\phi_1, \phi_2)\colon (\xi_1, \xi_2) \to (\eta_1, \eta_2)$ with

$$F_1\phi_1 = F_2\phi_2 \in \mathrm{Hom}_{\mathcal{G}}\bigl(F_1\xi_1, F_1\eta_1\bigr) = \mathrm{Hom}_{\mathcal{G}}(F_2\xi_2, F_2\eta_2).$$

This fiber product may seem like the right one. First of all, there is a functor $\mathcal{F}_1 \overset{\text{naive}}{\times}_{\mathcal{G}} \mathcal{F}_2 \to \mathcal{C}$, sending each object (ξ_1, ξ_2) into $\mathrm{p}_{\mathcal{F}_1}\xi_1 = \mathrm{p}_{\mathcal{F}_2}\xi_2$; this makes $\mathcal{F}_1 \overset{\text{naive}}{\times}_{\mathcal{G}} \mathcal{F}_2$ into a fibered category over $\mathcal{C}$, in which the cartesian arrows are the pair (ϕ_1, ϕ_2) in which each ϕ_i is cartesian in $\mathcal{F}_i$. Also, the naive fiber product has the following universal property. There are two morphisms, the projections $\mathrm{pr}_1 \colon \mathcal{F}_1 \overset{\text{naive}}{\times}_{\mathcal{G}} \mathcal{F}_2 \to \mathcal{F}_1$ and $\mathrm{pr}_2 \colon \mathcal{F}_1 \overset{\text{naive}}{\times}_{\mathcal{G}} \mathcal{F}_2 \to \mathcal{F}_2$, such that, if $\mathcal{X}$ is a category, $H_1 \colon \mathcal{X} \to \mathcal{F}_1$ and $H_2 \colon \mathcal{X} \to \mathcal{F}_2$ are morphisms with $F \circ H = G \circ L$, there a unique morphism $(H_1, H_2) \colon \mathcal{X} \to \mathcal{F}_1 \overset{\text{naive}}{\times}_{\mathcal{G}} \mathcal{F}_2$ with $\mathrm{pr}_1(H_1, H_2) = H_1$ and $\mathrm{pr}_2 \circ (H_1, H_2) = H_2$.

However, the naive fiber product behaves very badly with respect to natural transformations. For example, if we take the fiber product $\mathcal{G} \overset{\text{naive}}{\times}_{\mathcal{G}} \mathcal{G}$, where both functors $\mathcal{G} \to \mathcal{G}$ are the indentity, we get a category that is canonically isomorphic to $\mathcal{G}$, as we should; but if one functor is the identity, while the other one is a functor $F \colon \mathcal{G} \to \mathcal{G}$ that is only *isomorphic* to the identity, but is such that the image $F\xi$ of each object ξ of $\mathcal{G}$, despite being isomorphic to ξ, is always *different* from ξ, we get that the fiber product $\mathcal{G} \overset{\text{naive}}{\times}_{\mathcal{G}} \mathcal{G}$ is the empty category. This should not be surprising: after all, two functors are rarely literally the same, they are usually only canonically isomorphic; this means that the *equality* of objects that intervenes in the notion of naive fiber product is often inappropriate.

Our first attempt to fix the definition might be to take as objects of the "correct" fiber product $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$ the pairs (ξ_1, ξ_2) lying over some object U of $\mathcal{C}$, such that $F_1\xi_1$ is isomorphic to $F_2\xi_2$ in $\mathcal{C}(U)$. However, then we do not know how to define arrows: if $F_1\xi_1$ is isomorphic to $F_2\xi_2$ and $F_1\eta_1$ is isomorphic to $F_2\eta_2$, this does not give us the bijective correspondence between $\mathrm{Hom}_{\mathcal{G}}(F_1\xi_1, F_1\eta_1)$ and $\mathrm{Hom}_{\mathcal{G}}(F_1\xi_1, F_2\eta_2)$ that we need to compare arrows $\xi_1 \to \eta_1$ with arrows $\xi_2 \to \eta_2$. To get this bijective correspondence we need to have a fixed isomorphism between $F_1\xi_1$ and $F_2\xi_2$; and this suggests the following definition.

Definition 2.29. Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ be categories fibered over a fixed category $\mathcal{C}$, $F_1 \colon \mathcal{F}_1 \to \mathcal{G}$ and $F_2 \colon \mathcal{F}_2 \to \mathcal{G}$ morphisms. The *fiber product* $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$ is the category whose objects are triples (ξ_1, ξ_2, u), where ξ_1 and ξ_2 are objects of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, mapping to the same object U of $\mathcal{C}$, u is an isomorphism between $F_1\xi_1$ and $F_2\xi_2$ in $\mathcal{G}(U)$. An arrow

$$(\phi_1, \phi_2) \colon (\xi_1, \xi_2, u) \to (\eta_1, \eta_2, v)$$

is a pair of arrows $\phi_1 \colon \xi_1 \to \eta_1$ in $\mathcal{F}_1(U)$ and $\phi_2 \colon \xi_2 \to \eta_2$ in $\mathcal{F}_2(V)$, where $U = \mathrm{p}_{\mathcal{F}_1}\xi_1 = \mathrm{p}_{\mathcal{F}_2}\xi_2$ and $V = \mathrm{p}_{\mathcal{F}_1}\eta_1 = \mathrm{p}_{\mathcal{F}_2}\eta_2$, such that the diagram

$$\begin{tikzcd}
F_1\xi_1 \arrow[r, "F_1\phi_1"] \arrow[d, "u"] & F_1\eta_1 \arrow[d, "v"] \\
F_2\xi_2 \arrow[r, "F_2\phi_2"] & F_2\eta_2
\end{tikzcd}$$

commutes.

Notice if we apply $\mathrm{p}_{\mathcal{G}}$ to the commutative diagram above we get the equality of arrows

$$\mathrm{p}_{\mathcal{F}_1}\phi_1 = \mathrm{p}_{\mathcal{F}_2}\phi_2 \colon U \to V$$

in $\mathcal{C}$.

There is an obvious functor

$$\mathrm{p}_{\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2} \colon \mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2 \to \mathcal{C}$$

sending an object (ξ_1, ξ_2, u) into $\mathrm{p}_{\mathcal{F}_1}\xi_1 = \mathrm{p}_{\mathcal{F}_2}\xi_2$.

Here we collect various properties of fiber products that we are going to need.

PROPOSITION 2.30. *In the situation of Definition 2.29, we have the following.*

(i) *The category $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$ is fibered over $\mathcal{C}$.*
(ii) *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are fibered in groupoids over $\mathcal{C}$, so is $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$.*
(iii) *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are functors over $\mathcal{C}$, then $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$ is also a functor, canonically isomorphic to the fiber product of functors defined in 1.2.1.*

We also have two projections $\mathrm{pr}_1 \colon \mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2 \to \mathcal{F}_1$ and $\mathrm{pr}_2 \colon \mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2 \to \mathcal{F}_2$, defined by sending an object (ξ_1, ξ_2, u) to ξ_1 and ξ_2 respectively, and an arrow (ϕ_1, ϕ_2) to ϕ_1 and ϕ_2 respectively. The compositions $F_1 \circ \mathrm{pr}_1 \colon \mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2 \to \mathcal{G}$ and $F_2 \circ \mathrm{pr}_2 \colon \mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2 \to \mathcal{G}$ are not equal; however, there is a tautological isomorphism of functors between them. In fact, given an object (ξ_1, ξ_2, u) of $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$, we have $F_1 \circ \mathrm{pr}_1(\xi_1, \xi_2, u) = F_1\xi_1$ and $F_2 \circ \mathrm{pr}_2(\xi_1, \xi_2, u) = F_2\xi_2$, and u is an isomorphism of $F_1\xi_1$ with $F_2\xi_2$ in $\mathcal{G}$.

This requires a change in the definition of the universal property for fiber products, and even of the notion of commutative diagram.

DEFINITION 2.31. A *commutative square* of categories

$$\begin{tikzcd} \mathcal{H} \arrow[r, "H_2"] \arrow[d, "H_1"] & \mathcal{F}_2 \arrow[d, "F_2"] \\ \mathcal{F}_1 \arrow[r, "F_1"] & \mathcal{G} \end{tikzcd}$$

consists of four categories and four functors as in the diagram, together with an isomorphism $\alpha \colon F_1 \circ H_1 \simeq F_2 \circ H_2$ of functors $\mathcal{H} \to \mathcal{G}$.

In the definition above, the isomorphism is part of the data that define the commutative square.

Given a square as above, we get a functor $(H_1, H_2, \alpha) \colon \mathcal{H} \to \mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$ by sending each object ξ of $\mathcal{H}$ to the triple $(H_1\xi, H_2\xi, \alpha_\xi)$, and each arrow ϕ of $\mathcal{H}$ to the arrow $(H_1\phi, H_2\phi)$. Conversely, given a functor $H \colon \mathcal{H} \to \mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$ we get a commutative diagram as above by defining $H_i = \mathrm{pr}_i \circ H \colon \mathcal{H} \to \mathcal{F}_i$, while the isomorphism $H_1\xi \simeq H_2\xi$, giving the isomorphism of functors $H_1 \simeq H_2$, is the third component of the triple $H\xi$.

2.7. The diagonal of a fibered category and its functors of arrows

2.7.1. The functors of arrows of a fibered category. Suppose that $\mathcal{F} \to \mathcal{C}$ is a fibered category; if U is an object in $\mathcal{C}$ and ξ, η are objects of $\mathcal{F}(U)$, we denote by $\mathrm{Hom}_U(\xi, \eta)$ the set of arrow from ξ to η in $\mathcal{F}(U)$.

Let ξ and η be two objects of $\mathcal{F}$ over the same object S of $\mathcal{C}$. Let $u_1 \colon U_1 \to S$ and $u_2 \colon U_2 \to S$ be arrows in $\mathcal{C}$; these are objects of the category $(\mathcal{C}/S)$. Suppose that $\xi_i \to \xi$ and $\eta_i \to \eta$ are pullbacks along $u_i \colon U_i \to S$ for $i = 1, 2$. For each arrow $f \colon U_1 \to U_2$ in $(\mathcal{C}/S)$, by definition of pullback there are two arrows, each unique, $\alpha_f \colon \xi_1 \to \xi_2$ and $\beta_f \colon \eta_1 \to \eta_2$, such that and the two diagrams

$$\begin{tikzcd} \xi_1 \arrow[rr, "\alpha_f"] \arrow[dr] & & \xi_2 \arrow[dl] \\ & \xi & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \eta_1 \arrow[rr, "\beta_f"] \arrow[dr] & & \eta_2 \arrow[dl] \\ & \xi & \end{tikzcd}$$

commute. By Proposition 2.3 (iii) the arrows α_f and β_f are cartesian; we define a pullback function

$$f^* \colon \mathrm{Hom}_{U_2}(\xi_2, \eta_2) \longrightarrow \mathrm{Hom}_{U_1}(\xi_1, \eta_1)$$

in which $f^*\phi$ is defined as the only arrow $f^*\phi \colon \xi_1 \to \eta_1$ in $\mathcal{F}(U_1)$ making the diagram

$$\begin{tikzcd} \xi_1 \arrow[r, "f^*\phi"] \arrow[d, "\alpha_f"] & \eta_1 \arrow[d, "\beta_f"] \\ \xi_2 \arrow[r, "\phi"] & \eta_2 \end{tikzcd}$$

commute. If we are given a third arrow $g \colon U_2 \to U_3$ in $(\mathcal{C}/S)$ with pullbacks $\xi_3 \to \xi$ and $\eta_3 \to \eta$, we have arrows $\alpha_g \colon \xi_2 \to \xi_3$ and $\beta_g \colon \eta_2 \to \eta_3$; it is immediate to check that

$$\alpha_{gf} = \alpha_g \circ \alpha_f \colon \xi_1 \to \xi_3 \quad \text{and} \quad \beta_{gf} = \beta_g \circ \beta_f \colon \eta_1 \to \eta_3$$

and this implies that

$$(gf)^* = f^* g^* \colon \mathrm{Hom}_{U_3}(\xi_3, \eta_3) \longrightarrow \mathrm{Hom}_{U_1}(\xi_1, \eta_1).$$

Now, assume that for each arrow $f \colon U \to V$ and each object η of $\mathcal{F}(V)$, we have chosen a pullback $f^*\eta \to \eta$ along f. In many cases there is a naturally given pullback; but in any case the axiom of choice insures that we can do this. Then we can define a functor

$$\underline{\mathrm{Hom}}_S(\xi, \eta) \colon (\mathcal{C}/S)^{\mathrm{opp}} \to (\mathrm{Set})$$

by sending each object $u \colon U \to S$ into the set $\mathrm{Hom}_U(u^*\xi, u^*\eta)$ of arrows in the category $\mathcal{F}(U)$. An arrow $f \colon U_1 \to U_2$ from $u_1 \colon U_1 \to S$ to $u_2 \colon U_2 \to S$ yields a function

$$f^* \colon \mathrm{Hom}_{U_2}(u_2^*\xi, u_2^*\eta) \longrightarrow \mathrm{Hom}_{U_1}(u_1^*\xi, u_1^*\eta);$$

and this defines the effect of $\underline{\mathrm{Hom}}_S(\xi, \eta)$ on arrows.

It is easy to check that the functor $\underline{\mathrm{Hom}}_S(\xi,\eta)$ is independent of the choice of pullbacks $f^*\xi$, in the sense that different choices of pullbacks give canonically isomorphic functors. Suppose that we have chosen for each $f\colon U \to V$ and each object ζ in $\mathcal{F}(V)$ another pullback $f^\vee\zeta \to \zeta$: then there is a canonical isomorphism $u^*\eta \simeq u^\vee\eta$ in $\mathcal{F}(U)$ for each arrow $u\colon U \to S$, and this gives a bijective correspondence

$$\mathrm{Hom}_U(u^*\xi, u^*\eta) \simeq \mathrm{Hom}_S(u^\vee\xi, u^\vee\eta),$$

yielding an isomorphism of the functors of arrows defined by the two pullbacks.

In fact, $\underline{\mathrm{Hom}}_S(\xi,\eta)$ can be more naturally defined as a quasifunctor $\mathcal{H}om_S(\xi,\eta) \to (\mathcal{C}/S)$; this does not require any choice of pullbacks.

From this point of view, the objects of $\mathcal{H}om_S(\xi,\eta)$ over some object $u\colon U \to S$ of $(\mathcal{C}/S)$ are triples

$$(\xi_1 \to \xi, \eta_1 \to \eta, \phi),$$

where $\xi_i \to \xi$ and $\eta_i \to \eta$ are cartesian arrows of $\mathcal{F}$ over $u\colon U \to S$, and $\phi\colon \xi_1 \to \eta_1$ is an arrow in $\mathcal{F}(U)$. An arrow from $(\xi_1 \to \xi, \eta_1 \to \eta, \phi_1)$ over $u_1\colon U_1 \to S$ and $(\xi_2 \to \xi, \eta_2 \to \eta, \phi_2)$ over $u_2\colon U_2 \to S$ is an arrow $f\colon U_1 \to U_2$ in $(\mathcal{C}/S)$ such that $f^*\phi_2 = \phi_1$.

From Proposition 2.23 we see that $\mathcal{H}om_S(\xi,\eta)$ is a quasifunctor over $\mathcal{C}$, and therefore, by Proposition 2.24, it is equivalent to a functor: of course this is the functor $\underline{\mathrm{Hom}}_S(\xi,\eta)$ obtained by the previous construction.

This can be proved as follows: the objects of $\underline{\mathrm{Hom}}_S(\xi,\eta)$, thought of as a category fibered in sets over $(\mathcal{C}/S)$ are pairs $(\phi, u\colon U \to S)$, where $u\colon U \to S$ is an object of $(\mathcal{C}/S)$ and $\phi\colon u^*\xi \to u^*\eta$ is an arrow in $\mathcal{F}(U)$; this also gives an object $(u^*\xi \to \xi, u^*\eta \to \eta, \phi)$ of $\mathcal{H}om_S(\xi,\eta)$ over U. The arrows between objects of $\underline{\mathrm{Hom}}_S(\xi,\eta)$ to another are precisely the arrows between the corresponding objects of $\mathcal{H}om_S(\xi,\eta)$, so we have an embedding of $\underline{\mathrm{Hom}}_S(\xi,\eta)$ into $\mathcal{H}om_S(\xi,\eta)$. But every object of $\mathcal{H}om_S(\xi,\eta)$ is isomorphic to an object of $\underline{\mathrm{Hom}}_S(\xi,\eta)$, hence the two fibered categories are equivalent.

2.7.2. The diagonal of a fibered category. These functors of arrows are linked with the diagonal functor $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ as follows.

Let S be an object of $\mathcal{C}$, ξ and η two objects of $\mathcal{F}(S)$. By the 2-Yoneda lemma, these two objects correspond to a morphism $S \to \mathcal{F} \times \mathcal{F}$, and we can consider the fiber product $\mathcal{F} \times_{\mathcal{F}\times\mathcal{F}} S$.

PROPOSITION 2.32. *The fiber product $\mathcal{F} \times_{\mathcal{F}\times\mathcal{F}} S$ is equivalent to the functor $\underline{\mathrm{Hom}}_S(\xi,\eta)$.*

PROOF. **TO BE ADDED** ♠

PROPOSITION 2.33. *For a fibered category $\mathcal{F} \to \mathcal{C}$, the following conditions are equivalent.*

(i) *For any object S of $\mathcal{C}$ and any two objects ξ and η of $\mathcal{F}(S)$, the functor*

$$\underline{\mathrm{Hom}}_S(\xi,\eta)\colon (\mathcal{C}/S)^{\mathrm{opp}} \to (\mathrm{Set})$$

is representable.

(ii) *The diagonal* $\delta \colon \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ *is representable.*

(iii) *For any two object* X *and* Y *of* $\mathcal{C}$, *and any two morphisms* $X \to \mathcal{F}$ *and* $Y \to \mathcal{F}$, *the fiber product* $X \times_{\mathcal{F}} Y$ *is representable.*

PROOF. The equivalence of (i) and (ii) follows from Proposition 2.32. From Proposition 2.34 below we see that (ii) implies (iii). ♠

PROPOSITION 2.34. *Let* $\mathcal{F} \to \mathcal{C}$ *be a fibered category,* X *and* Y *objects of* $\mathcal{C}$, $X \to \mathcal{F}$ *and* $Y \to \mathcal{F}$ *two arrows corresponding to objects* $\xi \in \mathcal{F}(X)$ *and* $\eta \in \mathcal{F}(Y)$. *Then the fiber product* $X \times_{\mathcal{F}} Y \to \mathcal{C}$ *is equivalent to the functor* $\underline{\mathrm{Hom}}_{X \times Y}(\mathrm{pr}_1^* \xi, \mathrm{pr}_2^* \eta) \colon (\mathcal{C}/X \times Y)^{\mathrm{opp}} \to (\mathrm{Set})$, *thought of as a fibered category over* $\mathcal{C}$.

The way to consider $\underline{\mathrm{Hom}}_{X \times Y}(\mathrm{pr}_1^* \xi, \mathrm{pr}_2^* \eta)$ as a fibered category over (Top) is to compose with the forgetful functor $(\mathrm{Top}/S) \to (\mathrm{Top})$, as explained in 2.3.1.

PROOF. **TO BE ADDED** ♠

CHAPTER 3

Stacks

3.1. Descent of objects of fibered categories

Descent theory has a somewhat formidable and totally undeserved reputation among algebraic geometers. In fact, it simply says that under certain conditions morphisms between objects can be glued together in some Grothendieck topology, while objects can be constructed locally and then glued together via isomorphisms that satisfy a cocycle condition.

3.1.1. Glueing continuous maps and topological spaces. The following is the archaetypical example of descent. Take (Cont) to be the category of continuous maps (that is, the category of arrows in (Top), as in Example 2.7); this category is fibered on (Top) via the functor $\mathrm{p}_{(\mathrm{Cont})}\colon (\mathrm{Cont}) \to (\mathrm{Top})$ sending each continuous map to its codomain. Now, suppose that $f\colon X \to U$ and $g\colon Y \to U$ are two objects of (Cont) mapping to the same object U in (Top); we want to construct a continuous map $\phi\colon X \to Y$ over U, that is, an arrow in $(\mathrm{Cont})(U) = (\mathrm{Top}/U)$. Suppose that we are given an open covering $\{U_i\}$ of U, and continuous maps $\phi_i\colon f^{-1}U_i \to g^{-1}U_i$ over U_i; assume furthermore that the restriction of ϕ_i and ϕ_j to $f^{-1}(U_i \cap U_j) \to g^{-1}(U_i \cap U_j)$ coincide. Then there is a unique continuous map $\phi\colon X \to Y$ over U whose restriction to each $f^{-1}U_i$ coincides with f_i.

This can be written as follows. The category (Cont) is fibered over (Top), and if $f\colon V \to U$ is a continuos map, $X \to U$ an object of $(\mathrm{Cont})(U) = (\mathrm{Top}/U)$, then a pullback of $X \to U$ to V is given by the projection $V \times_U X \to V$. The functor $f^*\colon (\mathrm{Cont})(U) \to (\mathrm{Cont})(V)$ sends each object $X \to U$ to $V \times_U X \to V$, and each arrow in (Top/U), given by continuous function $\phi\colon X \to Y$ over U, to the continuous function $f^*\phi = \mathrm{id}_V \times_U f\colon V \times_U X \to V \times_U Y$.

Suppose that we are given two topological spaces X and Y with continuous maps $X \to S$ and $Y \to S$. Consider the functor

$$\underline{\mathrm{Hom}}_S(X, Y)\colon (\mathrm{Top}/S) \to (\mathrm{Set})$$

from the category of topological spaces over S, defined in Section 2.7. This sends each arrow $U \to S$ to the set of continuous maps $\mathrm{Hom}_U(U \times_S X, U \times_S Y)$ over U. The actions on arrows is obtained as follows: Given a continuous function $f\colon V \to U$, we send each continuous function $\phi\colon U \times_S X \to U \times_S Y$ to the function

$$f^*\phi = \mathrm{id}_V \times \phi\colon V \times_S X = V \times_U (U \times_S X) \to V \times_U (U \times_S Y) = V \times_S Y$$

Then the fact that continuous functions can be constructed locally and then glued together can be expressed by saying that the functor

$$\underline{\mathrm{Hom}}_S(X,Y)\colon (\mathrm{Top}/S)^{\mathrm{opp}} \to (\mathrm{Set})$$

is a sheaf in the classical topology of (Top).

But there is more: not only we can construct continuous functions locally: we can also do this for spaces, altough this is more complicated.

PROPOSITION 3.1. *Suppose that we are given a topological space U with an open covering $\{U_i\}$; for each triple of indices i, j and k set $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. Assume that for each i we have a continuous map $u_i\colon X_i \to U_i$, and that for each pair of indices i and j we have a homeomorphism $\phi_{ij}\colon u_j^{-1}U_{ij} \simeq u_i^{-1}U_{ij}$ over U_{ij}, satifying the cocycle condition*

$$\phi_{ik} = \phi_{ij} \circ \phi_{jk}\colon u_k^{-1}U_{ijk} \to u_j^{-1}U_{ijk} \to u_i^{-1}U_{ijk}.$$

Then there exists a continuous map $u\colon X \to U$, together with isomorphisms $\phi_i\colon u^{-1}U_i \simeq X_i$, such that $\phi_{ij} = \phi_i \circ \phi_j^{-1}\colon u_j^{-1}U_{ij} \to u^{-1}U_{ij} \to u_iU_{ij}$ for all i and j.

PROOF. Consider the disjoint union U' of the U_i; the fiber product $U' \times_U U'$ is the disjoint union of the U_{ij}. The disjoint union X' of the X_i, maps to U'; consider the subset $R \subseteq X' \times X'$ consising of pairs $(x_i, x_j) \in X_i \times X_j \subseteq X' \times X'$ such that $x_i = \phi_{ij}x_j$. I claim that R is an equivalence relation in X'. Notice that the cocycle condition $\phi_{ii} = \phi_{ii} \circ \phi_{ii}$ implies that ϕ_{ii} is the identity on X_i, and this show that the equivalence relation is reflexive. The fact that $\phi_{ii} = \phi_{ij} \circ \phi_{ji}$, and therefore $\phi_{ji} = \phi_{ij}^{-1}$, prove that it is symmetric; and transitivity follows directly from the general cocycle condition. We define X to be the quotient X'/R.

If two points of X' are equivalent, then their images in U coincide; so there is an induced continuous map $u\colon X \to U$. The restriction to $X_i \subseteq X'$ of the projection $X' \to X$ gives a continuous map $\phi_i\colon X_i \to u^{-1}U_i$, that's easily checked to be a homeomorphism. One also sees that $\phi_{ij} = \phi_i \circ \phi_j^{-1}$, and this completes the proof. ♠

The facts that we can glue continuous maps and topological spaces say that (Cont) is a *stack* over (Top).

3.1.2. Descent in arbitrary sites and stacks. Let $\mathcal{C}$ be a site. We have seen that a fibered category over $\mathcal{C}$ should be thought of as a contravariant functor from $\mathcal{C}$ to the category of categories, that is, presheaves of categories over $\mathcal{C}$. A stack is, morally, a sheaf of categories over $\mathcal{C}$.

Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$. From now on we will always assume that for each object ξ of $\mathcal{F}$ and each arrow $f\colon U \to \mathrm{p}_{\mathcal{F}}\xi$ of $(\mathcal{C}/X)$, we have chosen a cartesian arrow $f^*\xi \to \xi$. This can be avoided, and everything can be written in terms of cartesian arrows; in this way one would have treatment which is more elegant and more precise, but, in my opinion, less transparent.

Given a covering $\{\sigma\colon U_i \to U\}$, set $U_{ij} = U_i \times_U U_j$ and $U_{ijk} = U_i \times_U U_j \times_U U_k$ for each triple of indices i, j and k.

DEFINITION 3.2. Let $\mathcal{U} = \{\sigma_i\colon U_i \to U\}$ be a covering in $\mathcal{C}$. An *object with descent data* $(\{\xi_i\}, \{\phi_{ij}\})$ on $\mathcal{U}$, is a collection of objects $\xi_i \in \mathcal{F}(U_i)$, together with isomorphisms $\phi_{ij}\colon \mathrm{pr}_2^* \xi_j \simeq \mathrm{pr}_1^* \xi_i$ in $\mathcal{F}(U_i \times_U U_j)$, such that the following cocycle condition is satisfied.

For any triple of indices i, j and k, we have the equality

$$\mathrm{pr}_{13}^* \phi_{ik} = \mathrm{pr}_{12}^* \phi_{ij} \circ \mathrm{pr}_{23}^* \phi_{jk}\colon\ \mathrm{pr}_3^* \xi_k \longrightarrow \mathrm{pr}_1^* \xi_i$$

where the pr_{ab} and pr_a are projections on the a^{th} and b^{th} factor, or the a^{th} factor respectively.

In understanding the definition above it may be useful to contemplate the cube

$$\begin{tikzcd}
& U_{ijk} \arrow[rr, "\mathrm{pr}_{23}"] \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dd, "\mathrm{pr}_{13}"] & & U_{jk} \arrow[dl] \arrow[dd] \\
U_{ij} \arrow[rr] \arrow[dd] & & U_j \arrow[dd] & \\
& U_{ik} \arrow[rr] \arrow[dl] & & U_k \arrow[dl] \\
U_i \arrow[rr] & & U &
\end{tikzcd}$$

in which all arrows are given by projections, and every face is cartesian.

We will denote the category of objects with descent data on the fibered category $\mathcal{F}$ by $\mathcal{F}(\mathcal{U}) = \mathcal{F}(\{U_i \to U\})$.

For each object ξ of $\mathcal{F}(U)$ we can construct an object with descent data on a covering $\{\sigma_i\colon U_i \to U\}$ as follows. The objects are the pullbacks $\sigma_i^* \xi$; the isomorphisms $\phi_{ij}\colon \mathrm{pr}_2^* \sigma_j^* \xi \simeq \mathrm{pr}_1^* \sigma_i^* \xi$ are the isomorphisms that come from the fact that both $\mathrm{pr}_2^* \sigma_j^* \xi$ and $\mathrm{pr}_1^* \sigma_i^* \xi$ are pullbacks of ξ to U_{ij}. If we identify $\mathrm{pr}_2^* \sigma_j^* \xi$ with $\mathrm{pr}_1^* \sigma_i^* \xi$, as is commonly done, then the ϕ_{ij} are identities.

Given an arrow $\alpha\colon \xi \to \eta$ in $\mathcal{F}(U)$, we get arrows $\sigma_i^*\colon \sigma_i^* \xi \to \sigma_i^* \eta$, yielding an arrow from the object with descent associated with ξ to the associated with η. This gives a functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$.

DEFINITION 3.3. Let $\mathcal{F} \to \mathcal{C}$ be a fibered category on a site $\mathcal{C}$.

(i) $\mathcal{F}$ is a *prestack* over $\mathcal{C}$ if for each covering $\{U_i \to U\}$ in $\mathcal{C}$, the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is fully faithful.

(ii) $\mathcal{F}$ is a *stack* over $\mathcal{C}$ if for each covering $\{U_i \to U\}$ in $\mathcal{C}$, the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is an equivalence of categories.

This condition can be restated.

The category $(\mathcal{C}/S)$ inherits a Grothendieck topology from the given Grothendieck topology on $\mathcal{C}$; simply, a covering of an object $U \to S$ of $(\mathcal{C}/S)$ is a collection of arrows

$$\begin{tikzcd} U_i \arrow[rr, "f_i"] \arrow[dr] & & U \arrow[dl] \\ & S & \end{tikzcd}$$

such that the collection $\{f_i \colon U_i \to U\}$ is a covering in $\mathcal{C}$. In other words, the coverings of $U \to S$ are simply the coverings of U.

Finally, we have the following definition.

DEFINITION 3.4. An object with descent data $(\{\xi_i\}, \{\phi_{ij}\})$ in $\mathcal{F}(\{U_i \to U\})$ is *effective* if it is isomorphic to the image of an object of $\mathcal{F}(U)$.

PROPOSITION 3.5. *Let $\mathcal{F}$ be a fibered category over a site $\mathcal{C}$.*

(i) *$\mathcal{F}$ is a prestack if and only if for any object S of $\mathcal{C}$ and any two objects ξ and η in $\mathcal{F}(S)$, the functor $\underline{\mathrm{Hom}}_S(\xi, \eta) \colon (\mathcal{C}/S) \to (\mathrm{Set})$ is a sheaf.*

(ii) *$\mathcal{F}$ is a stack if and only if it is a prestack, and all objects with descent data in $\mathcal{F}$ are effective.*

EXAMPLE 3.6. If G is a topological group, the fibered category $\mathcal{B}G \to (\mathrm{Top})$ is a stack.

PROPOSITION 3.7. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fibered categories over a site $\mathcal{C}$.*

(i) *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent and $\mathcal{F}_1$ is a stack, then $\mathcal{F}_2$ is also a stack.*

(ii) *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are stacks over $\mathcal{C}$, then the fiber product $\mathcal{F}_1 \times_{\mathcal{C}} \mathcal{F}_2$ is also a stack.*

From a stack we get a stack in groupoids.

PROPOSITION 3.8. *If $\mathcal{F} \to \mathcal{C}$, the associated stack in groupoids $\mathcal{F}_{\mathrm{cart}} \to \mathcal{C}$ is a stack.*

PROOF. **TO BE ADDED** ♠

Stacks are the correct generalization of sheaves. This may not be obvious now, but at least we should prove the following statement.

PROPOSITION 3.9. *Let $\mathcal{C}$ be a site, $F \colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ a functor; we can also consider it as a category fibered in sets $F \to \mathcal{C}$.*

(i) *F is a prestack if and only if it is a separated functor.*

(ii) *F is stack if and only if it is a sheaf.*

PROOF. Consider a covering $\{U_i \to U\}$. The fiber of the category $F \to \mathcal{C}$ over U is precisely the set $F(U)$, while the category $F(\{U_i \to U\})$ is the set of elements $(\xi_i) \in \prod_i F(U_i)$ such that the pullbacks of ξ_i and ξ_j to $F(U_i \times_U U_j)$, via the first and second projections $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_i$, coincide. The functor $F(U) \to F(\{U_i \to U\})$ is the function that sends each element $\xi \in F(U)$ to the collection of restrictions $(\xi\mid_{U_i})$.

Now, to say that a function, thought of as a functor between categories, is fully faithful is equivalent to saying that it is injective; while to say that it is an equivalence it is like saying that it is a bijection. From this both statements follow. ♠

3.2. The stackification of a fibered category

In this section we describe a procedure, analogous to the sheafification of a functor described in Section 1.3.

DEFINITION 3.10. Let $\mathcal{F} \to \mathcal{C}$ be a fibered category over a site $\mathcal{C}$. A *stackification* of $\mathcal{F}$ is a stack $\mathcal{F}^{\mathrm{a}} \to \mathcal{C}$ with a morphism $\mathcal{F} \to \mathcal{F}^{\mathrm{a}}$ such that:

(i) given an object U of $\mathcal{C}$ and two arrows $\phi, \psi \colon \xi \to \eta$ in $\mathcal{F}(U)$, if the two images $\phi^{\mathrm{a}} \colon \xi^{\mathrm{a}} \to \eta^{\mathrm{a}}$ and $\psi^{\mathrm{a}} \colon \xi^{\mathrm{a}} \to \eta^{\mathrm{a}}$ in $\mathcal{F}^{\mathrm{a}}$ are equal, then there is a covering $\{\sigma_i \colon U_i \to U\}$ such that $\sigma_i^*\phi = \sigma_i^*\psi \colon \sigma_i^*\xi \to \sigma_i^*\eta$ for all U_i;

(ii) given two objects ξ and η of $\mathcal{F}(U)$ and an arrow $\overline{\phi} \colon \xi^{\mathrm{a}} \to \eta^{\mathrm{a}}$ in $\mathcal{F}^{\mathrm{a}}(U)$, there exists a covering $\{\sigma_i \colon U_i \to U\}$ and arrows $\phi_i \colon \sigma_i^*\xi \to \sigma_i^*\eta$ such that the diagram **TO BE ADDED**

(iii) Given an object U of $\mathcal{C}$ and an object $\overline{\xi}$ of $\mathcal{F}^{\mathrm{a}}(U)$, there exists a covering $\{\sigma_i \colon U_i \to U\}$ and objects ξ_i of $\mathcal{F}(U_i)$ such that $\sigma_i^*\xi \in \mathcal{F}^{\mathrm{a}}(U_i)$ is isomorphic to ξ_i^{a}.

REMARK 3.11. Conditions (i) and (ii) are equivalent to saying that the functor

$$\underline{\mathrm{Hom}}_S(\xi^{\mathrm{a}}, \eta^{\mathrm{a}}) \colon (\mathcal{C}/S)^{\mathrm{opp}} \to (\mathrm{Set})$$

is the sheafification of the functor

$$\underline{\mathrm{Hom}}_S(\xi, \eta) \colon (\mathcal{C}/S)^{\mathrm{opp}} \to (\mathrm{Set})$$

for any object S of $\mathcal{C}$ and any two objects ξ and η of $\mathcal{F}(S)$. Hence, if we find a stack $\mathcal{F}^{\mathrm{a}} \to \mathcal{C}$ and a fully faithful morphism $\mathcal{F} \to \mathcal{F}^{\mathrm{a}}$ satisfying (iii), where $\mathcal{F}^{\mathrm{a}}$ is a stack, then $\mathcal{F}$ is a prestack, and $\mathcal{F}^{\mathrm{a}}$ is a stackification of $\mathcal{F}$.

THEOREM 3.12. *Let $\mathcal{F} \to \mathcal{C}$ be a fibered category over a site $\mathcal{C}$.*

(i) *There exists a stackification $\mathcal{F} \to \mathcal{F}^{\mathrm{a}}$ of $\mathcal{F}$.*

(ii) *If $P \colon \mathcal{F} \to \mathcal{F}^{\mathrm{a}}$ is a stackification, then for each stack $\mathcal{G} \to \mathcal{C}$ and each morphism of fibered categories $F \colon \mathcal{F} \to \mathcal{G}$, there is a morphism $F^{\mathrm{a}} \colon \mathcal{G} \to \mathcal{C}$ together with an isomorphism $u \colon F^{\mathrm{a}} \circ P \simeq F$ of base-preserving transformations. Furthermore, if $G \colon \mathcal{F}^{\mathrm{a}} \to \mathcal{G}$ is any other morphism, and $v \colon G \circ P \simeq F$ is an isomorphism of base-preserving functors, then there is a unique isomorphism $w \colon G \simeq F^{\mathrm{a}}$ such that for any object ξ of $\mathcal{F}$ we have $v_\xi = u_\xi \circ w_{P\xi} \colon GP\xi \to F^{\mathrm{a}}P\xi \to F\xi$.*

(iii) *The stackfication of $\mathcal{F}$ is unique up to an equivalence; the equivalence is unique up a unique isomorphism.*

(iv) *The stackifications of two equivalent fibered categories are equivalent.*

(v) *The functor $\mathcal{F} \to \mathcal{F}^{\mathrm{a}}$ is fully faithful if and only if $\mathcal{F}$ is a prestack.*

(vi) *If $\mathcal{F}$ is fibered in groupoids, then the stackification $\mathcal{F}^{\mathrm{a}}$ is also fibered in groupoids.*

(vii) *If $\mathcal{F}$ is a functor, then $\mathcal{F}^{\mathrm{a}}$ is naturally equivalent to the sheafification of $\mathcal{F}$ as a functor.*

(viii) *$\mathcal{F}$ is fibered in groupoids if and only if $\mathcal{F}^{\mathrm{a}}$ is fibered in groupoids.*

PROOF. First of all we are going to define a prestack $\mathcal{F}^{\mathrm{s}}$ with a morphism $\mathcal{F} \to \mathcal{F}^{\mathrm{s}}$ which is universal for morphisms into prestacks.

The category $\mathcal{F}^{\mathrm{s}}$ will have the same objects as $\mathcal{F}$. The arrows are defined as follows.

Suppose that we have **TO BE ADDED** ♠

3.2.1. The stack of sheaves. Let $\mathcal{C}$ be a site, and call $\mathcal{T}$ its topology. For each object X of $\mathcal{C}$ there is an induced topology $\mathcal{T}_X$ on the category $(\mathcal{C}/X)$, in which a set of arrows

$$\left\{\begin{tikzcd}[column sep=small] U_i \arrow[rr] \arrow[dr] & & U \arrow[dl] \\ & X & \end{tikzcd}\right\}$$

is a covering if and only if the set $\{U_i \to U\}$ is a covering in $\mathcal{C}$. We will refer to a sheaf in the site $(\mathcal{C}/X)$ as a *sheaf on* X, and denote the category of sheaves on X by $\operatorname{Sh} X$.

If $f : X \to Y$ is an arrow in $\mathcal{C}$, there is a corresponding restriction functor $f^*\colon \operatorname{Sh} Y \to \operatorname{Sh} X$, defined as follows.

If G is a sheaf on Y and $U \to X$ is an object of $(\mathcal{C}/X)$, we define $f^*G(U \to Y) = G(U \to Y)$, where $U \to Y$ is the composition of $U \to X$ with f.

If $U \to X$ and $V \to X$ are objects of $(\mathcal{C}/X)$ and $\phi\colon U \to V$ is an arrow in $(\mathcal{C}/X)$, then ϕ is also an arrow from $U \to Y$ to $V \to Y$, hence it induces a function $\phi^*\colon F^*(V \to X) = F(U \to Y) \to F(V \to Y) = f^*F(V \to X)$. This gives f^*F the structure of a functor $(\mathcal{C}/X)^{\mathrm{opp}} \to (\mathrm{Set})$. One checks immediately that f^*F is a sheaf on $(\mathcal{C}/X)$.

If $\phi\colon F \to G$ is a natural transformation of sheaves on $(\mathcal{C}/Y)$, there is an induces natural transformation $f^*\phi\colon f^*F \to f^*G$ of sheaves on $(\mathcal{C}/X)$, defined in the obvious way. This defines a functor $f^*\colon (\mathcal{C}/Y) \to (\mathcal{C}/X)$.

It is immediate to check that, if $f : X \to Y$ and $g\colon Y \to Z$ are arrows in $\mathcal{C}$, we have an *equality* of functors $(gf)^* = f^*g^*\colon (\mathcal{C}/Z) \to (\mathcal{C}/X)$. This means that we have defined a functor from $\mathcal{C}$ to the category of categories, sending an object X into the category of categories. According to the result of 2.1.3, this yields a category $(\mathrm{Sh}/\mathcal{C}) \to \mathcal{C}$, whose fiber over X is $\operatorname{Sh} X$.

3.3. Groupoids in a category

A groupoid is a set of objects X, a set of arrows R, together with the following data:

(i) source and target maps $\mathrm{s}, \mathrm{t}\colon R \to X$, sending each arrow to its domain and codomain, respectively;

(ii) an identity map $\mathrm{e}\colon X \to R$, sending each object into the corresponding identity arrow;

(iii) a composition map $\mathrm{m}\colon R \times_X R \to R$, where in $R \times_X R$ the first factor is considered as a set over X via the source map, the second one via the target map, that sends a pair of arrows (g, f) with $\mathrm{s}g = \mathrm{t}f$ to the composition gf, and finally

(iv) the inverse map $\mathrm{i}\colon R \to R$, that sends an arrow to its inverse.

Equivalence relations are groupoids; a groupoid is an equivalence relation if and only if the function

$$(\mathrm{s}, \mathrm{t})\colon R \to X \times X$$

is injective.

Now we can define a groupoid in a category like we have defined an equivalence relation. As usual, we will assume that our category $\mathcal{C}$ has products and fiber products.

DEFINITION 3.13. A groupoid $R \rightrightarrows X$ in a category $\mathcal{C}$ is pair of objects R and X, together with arrows $\mathrm{s}, \mathrm{t}\colon R \to X$, $\mathrm{e}\colon X \to R$, $\mathrm{i}R \to R$ and $\mathrm{m}\colon R \times_X R \to R$, where in $R \times_X R$ the first factor is considered as a set over X via s and the second one via t, such that for any object U of $\mathcal{C}$ the sets $X(U)$ and $R(U)$, with the maps induced by s, t, e, m and i, is a groupoid.

In fact, we could observe that the functions $\mathrm{e}_U\colon X(U) \to R(U)$ and $\mathrm{i}_U\colon R(U) \to R(U)$ are determined by the other functions s_U, t_U and m_U, so, by Yoneda's lemma, e and i are determined by s, t and m.

The conditions that define a gropoid can be also expressed in diagramatic terms.

PROPOSITION 3.14. *Let R and X be objects of a category $\mathcal{C}$, with arrows $\mathrm{s}, \mathrm{t}\colon R \to X$, $\mathrm{e}\colon X \to R$, $\mathrm{i}\colon R \to R$ and $\mathrm{m}\colon R \times_X R \to R$, where in $R \times_X R$ the first factor is considered as a set over X via s and the second one via t. Then these data define a groupoid in $\mathcal{C}$ if and only if all of the following diagrams commute.*

(i) *The source and target of the identity on an object are the object itself:*

$$\begin{tikzcd} X \arrow[r, "\mathrm{e}"] \arrow[rr, bend right, "\mathrm{id}_X"'] & R \arrow[r, "\mathrm{s}"] & X \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "\mathrm{e}"] \arrow[rr, bend right, "\mathrm{id}_X"'] & R \arrow[r, "\mathrm{t}"] & X \end{tikzcd}$$

The commutativity of these two diagrams implies that the two arrows $\mathrm{id}_R \times (\mathrm{s} \circ \mathrm{e})\colon R \to R \times R$ and $(\mathrm{t} \circ \mathrm{e}) \times \mathrm{id}_R\colon R \to R \times R$ factor through $R \times_X R$.

(ii) *The identity is a right and left identity:*

$$\begin{tikzcd} R \arrow[r, "\mathrm{id}_R\times(\mathrm{s}\circ\mathrm{e})"] \arrow[dr, equal, "\mathrm{id}_R"'] & R\times_X R \arrow[d, "\mathrm{m}"] \\ & R \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[r, "(\mathrm{t}\circ\mathrm{e})\times\mathrm{id}_R"] \arrow[dr, equal, "\mathrm{id}_R"'] & R\times_X R \arrow[d, "\mathrm{m}"] \\ & R \end{tikzcd}$$

(iii) *Multiplication is associative:*

$$\begin{tikzcd} R\times_X R\times_X R \arrow[r, "\mathrm{m}\times\mathrm{id}_R"] \arrow[d, "\mathrm{id}_R\times\mathrm{m}"] & R\times_X R \arrow[d, "\mathrm{m}"] \\ R\times_X R \arrow[r, "\mathrm{m}"] & R \end{tikzcd}$$

(iv) *The source and target of the inverse of an arrow are the target and source of the arrow:*

$$\begin{tikzcd} R \arrow[rr, "\mathrm{i}"] \arrow[dr, "\mathrm{s}"'] & & R \arrow[dl, "\mathrm{t}"] \\ & X & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[rr, "\mathrm{i}"] \arrow[dr, "\mathrm{t}"'] & & R \arrow[dl, "\mathrm{s}"] \\ & X & \end{tikzcd}$$

The commutativity of these two diagrams imply that the two arrows $\mathrm{id}_R \times \mathrm{i}\colon R \to R \times R$ *and* $\mathrm{i} \times \mathrm{id}_R\colon R \to R \times R$ *factor through* $R \times_X R$.

(v) *The inverse of an arrow is a right and left inverse:*

$$\begin{tikzcd} R \arrow[r, "\mathrm{id}_R\times\mathrm{i}"] \arrow[d, "\mathrm{t}"] & R\times_X R \arrow[d, "\mathrm{m}"] \\ X \arrow[r, "\mathrm{e}"] & R \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[r, "\mathrm{i}\times\mathrm{id}_R"] \arrow[d, "\mathrm{s}"] & R\times_X R \arrow[d, "\mathrm{m}"] \\ X \arrow[r, "\mathrm{e}"] & R \end{tikzcd}$$

PROOF. $R \rightrightarrows X$ is a groupoid in $\mathcal{C}$ if and only if each of the diagrams above becomes commutative when all the arrows are evaluated at all the objects of $\mathcal{C}$. The thesis follows by Yoneda's lemma. ♠

An equivalence relation in a category gives a groupoid in a category, because of the discussion in Section 1.4. A groupoid is an equivalence relation if and only if the arrow $(\mathrm{s},\mathrm{t})\colon R \to X \times X$ is injective.

Furthermore, if X is a final object in the category $\mathcal{C}$, then a groupoid $R \rightrightarrows X$ is a groups object (see 1.1.4). Thus group objects are groupoids: the following shows, that more generally, actions give rise to groupoids.

Other very important examples are obtained from group actions.

EXAMPLE 3.15. If a group object G acts on a set X, we get a groupoid $G \times X \rightrightarrows X$, in which $\mathrm{t}\colon G \times X \to X$ is the second projection, $\mathrm{s}\colon G \times X \to X$ is the action. The other arrows are best described as natural transformations of functors.

(i) $\mathrm{m}\colon (G \times X) \times_X (G \times X) \to G \times X$ sends a pair

$$\big((h,y),(g,x)\big) \in \big((G \times X) \times_X (G \times X)\big)(U)$$

with $gx = y$ into

$$(hg, x) \in (G \times X)(U),$$

(ii) $\mathrm{e}\colon X \to G \times X$ sends $x \in X(U)$ into $(1,x) \in (G \times X)(U)$, and
(iii) $\mathrm{i}\colon G{\times}X \to G{\times}X$ sends $(g,x) \in (G{\times}X)(U)$ into $(\mathrm{i}g, gx) \in (G{\times}X)(U)$.

Suppose that (X,R) is an equivalence relation in the category of sets. Then the equivalence relation, thought of as a category, is equivalent to the set X/R (see the proof of Proposition 1.35). This is not true of groupoids in general; we can still consider the quotient X/R, which will be the set of isomorphism classes in the category (X,R), but this is equivalent to the groupoid precisely when this is an equivalence relation.

We have seen how an equivalence relation (X,R) in $\mathcal{C}$ determines a functor $[X|R]\colon \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Set})$ that sends an object U of $\mathcal{C}$ into the quotient $X(U)/R(U)$; we could also have defined a quasifunctor whose fiber over U is the equivalence category $\big(X(U), R(U)\big)$. This would have been equivalent to the functor above.

When we are dealing with a groupoid $R \rightrightarrows X$ in $\mathcal{C}$, we can still consider a functor that sends each object U of $\mathcal{C}$ into the set $X(U)/R(U)$ of isomorphism classes in the groupoid $R(U) \rightrightarrows X(U)$, but this will often carry very little information. For example, if G is a topological group, and we let G act on a point pt, the quotient $\mathrm{pt}(U)/G(U)$ has only one element, so the functor tells us nothing about G.

The right thing to do is to construct a fibered category $[X|R]$ whose fiber over U is equivalent to the groupoid $R(U) \rightrightarrows X(U)$, using the construction of 2.1.3. The point is that if $f\colon U \to V$ is an arrow in $\mathcal{C}$, we get a functor from $R(V) \rightrightarrows X(V)$ to $R(U) \rightrightarrows X(U)$ by sending each object $\eta\colon V \to X$ of $X(V)$ into its pullback $f^*\eta = \eta \circ f\colon U \to X$, and each arrow $a\colon V \to R$ of $R(V)$ into $f^*a = a \circ f \in R(U)$. So we get a functor from $\mathcal{C}$ to the category of categories. According to 2.1.3, we can construct the desired category by taking the objects of $[X|R]$ to be pairs (ξ, U), where U is an object of $\mathcal{C}$ and $\xi\colon U \to X$ is an element of $X(U)$, while an arrow $(a,f)\colon (\xi,U) \to (\eta,V)$ consists of an arrow $f\colon U \to V$ in $\mathcal{C}$, while $a\colon U \to R$ is an element of $R(U)$ with source ξ and target $f^*\eta$.

The composition is defined as follows: given $\xi \in X(U)$, $\eta \in X(V)$ and $\zeta \in X(W)$, arrows $(f,a)\colon \xi \to \eta$ and $(g,b)\colon \eta \to \zeta$, the element $b \in R(V)$ has source η and target $g^*\zeta$, therefore f^*b will have source $f^*\eta$ and target $f^*g^*\zeta = (gf)^*\zeta$. Consider the function $\mathrm{m}_U\colon R(U) \times_{X(U)} R(U) \to R(U)$ that sends two arrows in $R(U)$ into their composition; it can be applied to the pair (f^*b, a) to yield an element of $R(U)$ whose target is $(gf)^*\zeta$. We define

$$(g,b) \circ (f,a) = \big(gf, \mathrm{m}_U(f^*b, a)\big).$$

Then, according to 2.1.3, the category $[X|R]$ is fibered in groupoids over $\mathcal{C}$, and its fiber over an object U of $\mathcal{C}$ is canonically equivalent to the groupoid $R(U) \rightrightarrows X(U)$.

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