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# Lectures on moduli spaces of hyperkähler manifolds and mirror symmetry

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# Lectures on moduli spaces of hyperkähler manifolds and mirror symmetry

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In these notes we will try to explain certain aspects of the theory of moduli spaces of compact hyperkähler manifolds. After recalling the main definitions and facts concerning the complex and metric structure of these manifolds in Section 1 we will soon turn to the global aspects of their moduli spaces. In Sections 2 and 3 we introduce these moduli spaces as well as the corresponding period domains. The geometric moduli spaces are studied via maps into the period domains. This will be explained in Section 4. Some of the main results about compact hyperkähler manifolds can be translated into global aspects of these maps.

Compared to other texts (e.g. [1]) on moduli spaces of K3 surfaces we will try to develop the theory as far as possible for compact hyperkähler manifolds of arbitrary dimension. The second main difference is that we also treat the less classical moduli spaces of certain CFTs. This will be done from a purely mathematical point of view by considering hyperkähler manifolds which are enriched by a B-field, i.e. an additional real cohomology class of degree two. This will lead to new features starting in Section 5, where we let act a certain discrete group on the various moduli spaces. This section follows papers by Aspinwall, Morrison, and others. Using this action mirror symmetry of K3 surfaces will be explained in Section 6. The advantage of this slightly technical approach is that various versions of mirror symmetry for (e.g. lattice polarized or elliptic) K3 surfaces can be explained by the same group action. Of course, explaining mirror symmetry in these terms is only possible for K3 surfaces or hyperkähler manifolds. Mirror symmetry for general Calabi–Yau manifolds will usually change the topology.

The text contains little or none original material. The main goal was to explain global phenomena of moduli spaces of K3 surfaces, or more generally of compact hyperkähler manifolds, and to give a concise introduction into the main constructions used in establishing mirror symmetry for K3 surfaces.

We encourage the reader to consult the survey [1] and the original articles [3, 4].

## 1 Basics

In this section we collect the basic definitions and facts concerning irreducible holomorphic symplectic manifolds and compact hyperkähler manifolds. Most of the material will be presented without proofs and we shall refer to other sources for more details (e.g. [6, 19]).

**Definition 1.1** *An* irreducible holomorphic symplectic manifold *(*IHS*, for short) is a simply connected compact Kähler manifold $X$, such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic two-form $\sigma$.*

Since an IHS is in particular a compact Kähler manifold, Hodge decomposition holds. In degree two it yields

$$\begin{aligned} H^2(X,\mathbb{C}) &= H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X) \\ &= \mathbb{C}\sigma \oplus H^{1,1}(X) \oplus \mathbb{C}\bar{\sigma}. \end{aligned}$$

The existence of an everywhere non-degenerate two-form $\sigma \in H^0(X, \Omega_X^2)$ implies that the manifold has even complex dimension $\dim_{\mathbb{C}}(X) = 2n$. Moreover, $\sigma$ induces an alternating homomorphism $\sigma : \mathcal{T}_X \to \Omega_X$. Since the two-form is everywhere non-degenerate, this homomorphism is bijective. Thus, the tangent bundle and the cotangent bundle of an IHS are isomorphic. Moreover, the canonical bundle $K_X = \Omega_X^{2n}$ is trivialized by the $(2n, 0)$-form $\sigma^n$. Thus, an IHS has trivial canonical bundle and, therefore, vanishing first Chern class $c_1(X)$.

In dimension two IHS are also called K3 surfaces (K3=**K**ähler, **K**odaira, **K**ummer). More precisely, by definition a K3 surface is a compact complex surface with trivial canonical bundle $K_X$ and such that $H^1(X, \mathcal{O}_X) = 0$. It is a deep fact that any such surface is also Kähler [29]. Moreover, $H^1(X, \mathcal{O}_X) = 0$ does indeed imply that such a surface is simply-connected.

Here are the basic examples.

**Examples 1.2** **i)** Any smooth quartic hypersurface $X \subset \mathbb{P}^3$ is a K3 surface, e.g. the Fermat quartic $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$.

**ii)** Let $T = \mathbb{C}^2/\Gamma$ be a compact two-dimensional complex torus. The involution $x \mapsto -x$ has 16 fixed points and, thus, the quotient $T/\pm$ is singular in precisely 16 points. Blowing-up those yields a *Kummer* surface $X \to T/\pm$, which is a K3 surface containing 16 smooth irreducible rational curves.

**iii)** An elliptic K3 surface is a K3 surface $X$ together with a surjective morphism $\pi : X \to \mathbb{P}^1$. The general fibre of $\pi$ is a smooth elliptic curve.

It is much harder to construct higher dimensional examples of IHS and all known examples are constructed by means of K3 surfaces or two-dimensional complex tori. The list of known examples has been discussed in length in the lectures of Lehn (see also [19]).

So far we have discussed IHS purely from the complex geometric point of view. However, the most important feature of this type of manifolds is the existence of a very special metric.

**Definition 1.3** *A compact oriented Riemannian manifold* $(M, g)$ *of dimension* $4n$ *is called* hyperkähler *(*HK, *for short) if the holonomy group of* $g$ *equals* $\mathrm{Sp}(n)$. *In this case* $g$ *is called a hyperkähler metric.*

**Remark 1.4** If $g$ is a hyperkähler metric, then there exist three complex structures $I$, $J$, and $K$ on $M$, such that $g$ is Kähler with respect to all three of them and such that $K = I \circ J = -J \circ I$. Thus, $I$ is orthogonal with respect to $g$ and the Kähler form $\omega_I := g(I(\ ), \ )$ is closed (similarly for $J$ and $K$). Often, this is taken as a definition of a hyperkähler metric. Note that our condition is stronger, as we not only want the holonomy be contained in $\mathrm{Sp}(n)$, but be equal to it.

**Proposition 1.5** *Let* $(M, g)$ *be a HK. Then for any* $(a, b, c) \in \mathbb{R}^3$ *with* $a^2 + b^2 + c^2 = 1$ *the complex manifold* $(M, aI + bJ + cK)$ *is an IHS.*

Thus, for any HK $(M, g)$ there exists a two-sphere $S^2 \subset \mathbb{R}^3$ of complex structures compatible with the Riemannian metric $g$.

**Remark 1.6** Let $(M, g)$ be a HK. The associated Kähler forms $\omega_I$, $\omega_J$, $\omega_K$ span a three-dimensional subspace $H^2_+(M, g) \subset H^2(M, \mathbb{R})$. In fact, this space will always be considered as a three-dimensional space endowed with the natural orientation. If $X = (M, I)$, then $H^2_+(M, g) = (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}} \oplus \mathbb{R}\omega_I$, where the orientation is given by the base $(\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), \omega_I)$. In order to see this, one verifies that the holomorphic two-form $\sigma$ on $X = (M, I)$ can be given as $\sigma = \omega_J + i\omega_K$ (cf. [19]).

**Definition 1.7** *Let* $X$ *be an IHS. The* Kähler cone $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ *is the open convex cone of all Kähler classes on* $X$*, i.e. classes that can be represented by some Kähler form.*

The most important single result on IHS is the following consequence of the celebrated theorem of Calabi–Yau:

**Theorem 1.8** *Let* $X$ *be an IHS. Then for any* $\alpha \in \mathcal{K}_X$ *there exists a unique hyperkähler metric* $g$ *on* $M$*, such that* $\alpha = [\omega_I]$ *for* $\omega_I = g(I(\ ), \ )$.

Thus, on any IHS $X$ the Kähler cone $\mathcal{K}_X$ parametrizes all possible hyperkähler metrics $g$ compatible with the given complex structure. Below we will explain how the Kähler cone $\mathcal{K}_X$ can be described as a subset of $H^{1,1}(X)$.

**Remark 1.9** Thus, an IHS $X$ together with a Kähler class $\alpha \in \mathcal{K}_X$ is the same thing as a HK $(M, g)$ together with a compatible complex structure $I$. As a short hand, we write $(X, \alpha) = (M, g, I)$ in this case.

**Definition 1.10** *The BB(Beauville–Bogomolov)-form of an IHS $X$ is the quadratic form on $H^2(X,\mathbb{R})$ given by*

$$q_X(\alpha) = (n/2)\int_X \alpha^2(\sigma\bar\sigma)^{n-1} + (1-n)(\int_X \alpha\sigma^{n-1}\bar\sigma^n)(\int_X \alpha\sigma^n\bar\sigma^{n-1}),$$

*where $\sigma \in H^{2,0}(X)$ is chosen such that $\int_X(\sigma\bar\sigma) = 1$*

For any Kähler class $[\omega]$ we obtain a $q_X$-orthogonal decomposition $H^2(X,\mathbb{R}) = (H^{2,0}(X)\oplus H^{0,2}(X))_{\mathbb{R}} \oplus \mathbb{R}\omega \oplus H^{1,1}(X)_\omega$. Here, $H^{1,1}(X)_\omega$ is the space of $\omega$-primitive real $(1,1)$-classes. Note that we get a different decomposition for every Kähler class $[\omega] \in \mathcal{K}_X$, but that the quadratic form $q_X$ does not depend on the chosen Kähler class.

The following proposition collects the main facts about the BB-form $q_X$.

**Proposition 1.11** *i) For any Kähler class $[\omega] \in \mathcal{K}_X$ on an IHS $X$ the BB-form $q_X$ is positive definite on $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}} \oplus \mathbb{R}\omega$ and negative definite on $H^{1,1}(X)_\omega$.*
*ii) There exists a positive real scalar $\lambda_1$ such that $q_X(\alpha)^n = \lambda_1 \cdot \int_X \alpha^{2n}$ for all $\alpha \in H^2(X)$.*
*iii) There exists a positive real scalar $\lambda_2$ such that $\lambda_2 \cdot q_X$ is a primitive integral form on $H^2(X,\mathbb{Z})$.*
*iv) There exists a positive real scalar $\lambda_3$ such that $q_X(\alpha) = \lambda_3 \cdot \int_X \alpha^2\sqrt{\mathrm{td}(X)}$ for all $\alpha \in H^2(X)$.*

After eliminating the denominator of $\sqrt{\mathrm{td}(X)}$ by multiplying with a universal coefficient $c_n$ that only depends on $n$ we obtain an integral quadratic form $c_n \cdot \int \alpha^2\sqrt{\mathrm{td}(X)}$. In general this form need not be primitive, but this will be of no importance for us. Moreover, since any IHS has vanishing odd Chern classes, $\sqrt{\mathrm{td}(X)} = \sqrt{\hat{A}(X)}$. (Everythings that matters here is that $\sqrt{\mathrm{td}(X)}$ is purely topological in this case.) Therefore, in these lectures we will use the following modified version of the BB-form.

**Definition 1.12** *The BB-form $q_X$ of an $2n$-dimensional IHS $X$ is given by*

$$q_X(\alpha) = c_n \cdot \int_X \alpha^2\sqrt{\hat{A}(X)}.$$

With this definition we see that $q_X$ only depends on the underlying manifold $M$, i.e. for two different hyperkähler metrics $g$ and $g'$ and two compatible complex structures $I$ resp. $I'$ the BB-forms with the above definition of $X = (M, I)$ and $X' = (M, I')$ coincide.

Note for $n = 1$ we have $c_1 = 1$ and thus $q_X$ is nothing but the intersection pairing $\alpha^2$ of the four-manifold underlying a K3 surface. The quadratic form in this case is even, unimodular and indefinite and can thus be explicitely determined:

**Proposition 1.13** *The intersection form $(H^2(X,\mathbb{Z}),\cup)$ of a K3 surface $X$ is isomorphic to the K3 lattice $2(-E_8) \oplus 3U$, where $U$ is the standard hyperbolic plane $\left(\mathbb{Z}^2, \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right)$.*

**Definition 1.14** *The BB-volume of a HK* $(M,g)$ *is*

$$q(M,g) := q_X([\omega_I]),$$

*where* $X = (M,I)$ *is the IHS associated to one of the compatible complex structures* $I$ *and* $\omega_I$ *is the induced Kähler form.*

Note that the BB-volume does not depend on the chosen complex structure. Analogously one can define the volume of an IHS endowed with a Kähler class $\alpha$ as $q_X(\alpha)$. For a K3 surface one has $q(M,g) = \int \omega_I^2$, which is the usual volume up to the scalar factor 1/2. In higher dimension the usual volume is of degree $2n$ and the BB-volume is quadratic. Of course, due to Proposition 1.11 one knows that up to a scalar factor $q(M,g)^n$ equals the standard volume, but this factor might depend on the topology of $M$.

What makes the theory of K3 surfaces and higher-dimensional HK so pleasant is that they can be studied by means of their period.

**Definition 1.15** *Let* $X$ *be an IHS. The* period *of* $X$ *is the lattice* $(H^2(X,\mathbb{Z}), q_X)$ *endowed with the weight-two Hodge structure* $H^2(X,\mathbb{Z}) \otimes \mathbb{C} = H^2(X,\mathbb{C}) = \mathbb{C}\sigma \oplus H^{1,1}(X,\mathbb{C}) \oplus \mathbb{C}\bar{\sigma}$.

Since $H^{1,1}(X,\mathbb{C})$ is orthogonal with respect to $q_X$ and $\mathbb{C}\bar{\sigma}$ is the complex conjugate of $\mathbb{C}\sigma$, the period of the IHS $X$ is in fact given by the lattice $(H^2(X,\mathbb{Z}), q_X)$ and the line $\mathbb{C}\sigma \subset H^2(X,\mathbb{C})$.

The theory of K3 surfaces is crowned by the so called Global Torelli Theorem (due to Pjateckii-Sapiro, Shafarevich, Burns, Rapoport, Looijenga, Peters, Friedman):

**Theorem 1.16** *Let* $X$ *and* $X'$ *be two K3 surfaces and let* $\varphi : H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ *be an isomorphism of their periods such that* $\varphi(\mathcal{K}_X) \cap \mathcal{K}_{X'} \neq \emptyset$. *Then there exists a unique isomorphism* $f : X' \cong X$ *such that* $f^* = \varphi$.

Moreover, an arbitrary isomorphism between the periods of two K3 surfaces is in general not induced by an isomorphism of the K3 surfaces, but the K3 surfaces are nevertheless isomorphic.

The uniqueness assertion in the Global Torelli Theorem is roughly proven as follows (cf. [25]): If $f$ is an automorphism of finite order with $f^* = \mathrm{id}$ then the holomorphic two-form $\sigma$ is invariant under $f$ and the action at the fixed points is locally of the form $(u,v) \mapsto (\xi \cdot u, \xi^{-1} \cdot v)$. Using Lefschetz fixed point formular and again $f^* = \mathrm{id}$ one finds that there are 24 fixed points. Thus the minimal resolution $\tilde{X}$ of the quotient $X/\langle f \rangle$ contains 24 pairwise disjoint curves. Moreover, one verifies that $\tilde{X}$ is again a K3 surface. The last two statements together yield a contradiction for $|\langle f \rangle| \neq 1$.

For the time being there exists not even a convincing conjectural version of the Global Torelli theorem in higher dimensions. E.g. if $f : X \cong X$ is an automorphism of a K3 surface

$X$ such that $f^* = \text{id}$, then $f = \text{id}$. This does not hold in higher dimensions [7]. Even worse, due to a recent counterexample of Namikawa [28] one knows that higher dimensional IHS $X$ and $X'$ might have isomorphic periods without even being birational.

Often, a certain type of K3 surfaces is distinguished by the form of the period. We explain this in the three examples presented earlier. In fact, the proofs of these descriptions are all quite involved.

**Example 1.17** **i)** Let $X$ be a K3 surface such that $\text{Pic}(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ is generated by a class $\alpha$ with $\alpha^2 = 4$. Then $X$ is isomorphic to a quartic hypersurface in $\mathbb{P}^3$ and $\alpha$ corresponds to $\mathcal{O}(1)$ (cf. [1, Exp. VI]).

**ii)** Let $X$ be a K3 surface such that $\text{Pic}(X)$ contains 16 disjoint smooth irreducible rational curves $C_1, \ldots, C_{16} \subset X$ such that $\sum [C_i] \in H^2(X,\mathbb{Z})$ is two-divisible. This description of Kummer surfaces is not entirely in terms of the period. Later we will rather use the following description of an even more special type of K3 surfaces: Let $X$ be a K3 surface such that the lattice $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{Z}}$ is of rank two and any vector $x$ in this lattice satisfies $x^2 \equiv 0$ mod 4. Then $X$ is a Kummer surface. It turns out that K3 surfaces with this type of period are exactly the exceptional Kummer surfaces, i.e. Kummer surfaces with $\text{rk}(\text{Pic}(X)) = 20$ (cf. [1, Exp.VIII]).

**iii)** Let $X$ be a K3 surface such that there exists a class $\alpha \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ with $\alpha^2 = 0$. Then $X$ is an elliptic K3 surface. Clearly, if $X \to \mathbb{P}^1$ is an elliptic K3 surface then the class of the fibre defines such a class. But note that conversely not every class $\alpha$ with $\alpha^2 = 0$ is automatically a fibre class of some elliptic fibration, but by applying certain reflections it can be be made into one (cf. [8]).

In order to get a better feeling for the set of all possible hyperkähler structures on an IHS $X$ we shall discuss the Kähler cone in some more detail.

**Definition 1.18** *The* positive cone $\mathcal{C}_X$ *of an IHS X is the connected component of* $\{\alpha \mid q_X(\alpha) > 0\} \subset H^{1,1}(X,\mathbb{R})$ *that contains the Kähler cone* $\mathcal{K}_X$.

(Here we use the fact that $q_X(\alpha) > 0$ for any Kähler class $\alpha$.) Thus, $\mathcal{C}_X \cup (-\mathcal{C}_X)$ can be entirely read off the period of $X$. This is no longer possible for the Kähler cone, but one can at least try to find a minimal set of further geometric information that determines $\mathcal{K}_X$ as an open subcone of $\mathcal{C}_X$.

**Proposition 1.19** *The Kähler cone* $\mathcal{K}_X \subset \mathcal{C}_X$ *is the open subset of all* $\alpha \in \mathcal{C}_X$ *such that* $\int_C \alpha > 0$ *for all rational curves* $C \subset X$. *If X is a K3 surface it suffices to test smooth rational curves (cf. [1, 5, 19]).*

Since any smooth irreducible rational curve $C$ in a K3 surface $X$ defines a $(-2)$-class $[C] \in H^{1,1}(X,\mathbb{Z})$, one can use this result to show that for any class $\alpha \in \mathcal{C}_X$ there exists a finite number of smooth rational curves $C_1, \dots, C_k \subset X$ such that $s_{C_1} \dots s_{C_k}(\alpha) \in \overline{\mathcal{K}}_X$, where $s_C$ is the reflection in the hyperplane $[C]^\perp$. Of course, these reflection $s_C$ are contained in the discrete orthogonal group $\mathrm{O}(\Gamma)$ of the lattice $\Gamma = (H^2(X,\mathbb{Z}), \cup)$.

## 2 Moduli spaces

Ultimately, we will be interested in moduli spaces of irreducible holomorphic symplectic manifolds (IHS), hyperkähler manifolds (HK), etc. In this section we will introduce moduli spaces of such manifolds endowed with an additional marking. A marking in general refers to an isomorphism of the second cohomology with a fixed lattice. The choice of such an isomorphism gives rise to the action of a discrete group and the quotients by this group will eventually yield the true moduli spaces. For this section we fix a lattice $\Gamma$ of signature $(3, b-3)$ and an integer $n$.

### 2.1 Moduli spaces of marked IHS

**Definition 2.1** *A marked IHS is a pair $(X,\varphi)$ consisting of an IHS of complex dimension $2n$ and a lattice isomorphism $\varphi : (H^2(X,\mathbb{Z}), q_X) \cong \Gamma$. We say that two marked IHS $(X,\varphi)$ and $(X',\varphi')$ are equivalent, $(X,\varphi) \sim (X',\varphi')$, if there exists an isomorphism $f : X \cong X'$ of complex manifolds such that $\varphi' = \varphi \circ f^*$.*

**Definition 2.2** *The moduli space of marked IHS is the space*

$$\mathcal{T}_\Gamma^{\mathrm{cpl}} := \{(X,\varphi) = \text{marked IHS}\}/\!\sim .$$

A priori, $\mathcal{T}_\Gamma^{\mathrm{cpl}}$ is just a set, but, as we will see later, it can be endowed with the structure of a topological space locally isomorphic to a complex manifold of dimension $b-2$.

Let $X$ be an IHS and $\varphi$ a marking of $X$. If $\mathcal{X} \to \mathrm{Def}(X)$ is the universal deformation of $X = \mathcal{X}_0$, then $\mathrm{Def}(X)$ is a smooth germ of dimension $h^1(X,\mathcal{T}_X)$. We may represent $\mathrm{Def}(X)$ by a small disc in $\mathbb{C}^{h^1(X,\mathcal{T}_X)}$. The marking $\varphi$ induces in a canonical way a marking $\varphi_t$ of the fibre $\mathcal{X}_t$ for any $t \in \mathrm{Def}(X)$. Using the Local Torelli Theorem (cf. Section 4) we see that the induced map $\mathrm{Def}(X) \to \mathcal{T}_\Gamma^{\mathrm{cpl}}$ is injective, i.e. any two fibres of the family $\mathcal{X} \to \mathrm{Def}(X)$ define non-equivalent marked IHS. The various $\mathrm{Def}(X) \subset \mathcal{T}_\Gamma^{\mathrm{cpl}}$ for all possible choices of $X$ and markings $\varphi$ cover the moduli space $\mathcal{T}_\Gamma^{\mathrm{cpl}}$. Since the universal deformation $\mathcal{X} \to \mathrm{Def}(X)$ of $X = \mathcal{X}_0$ is, at the same time, also the universal deformation of all its fibres $\mathcal{X}_t$, one can define a natural topology on $\mathcal{T}_\Gamma^{\mathrm{cpl}}$ by gluing the complex manifolds $\mathrm{Def}(X)$. Thus, locally $\mathcal{T}_\Gamma^{\mathrm{cpl}}$ is a smooth complex manifold of dimension $h^1(X,\mathcal{T}_X) = b-2$. However, $\mathcal{T}_\Gamma^{\mathrm{cpl}}$ is not a complex manifold, as it does not need to be Hausdorff. In fact, not a single example is known, where $\mathcal{T}_\Gamma^{\mathrm{cpl}}$ would be Hausdorff and conjecturally this never happens.

A family $(\mathcal{X}, \varphi) \to S$ of marked IHS is a family $\mathcal{X} \to S$ of IHS of dimension $2n$ and a family of markings $\varphi_t$ of the fibres $\mathcal{X}_t$ locally constant with respect to $t$.

**Lemma 2.3** *If $(\mathcal{X}, \varphi) \to S$ is a family of marked IHS, then there exists a canonical holomorphic map $\eta : S \to \mathcal{T}_\Gamma^{\mathrm{cpl}}$, such that $\eta(t) = [(\mathcal{X}_t, \varphi_t)]$.*

*Proof.* This follows directly from the universality of $\mathcal{X} \to \mathrm{Def}(X)$. □

**Remark 2.4** In order to construct a universal family over $\mathcal{T}_\Gamma^{\mathrm{cpl}}$ one would need to glue universal families $\mathcal{X} \to \mathrm{Def}(X)$, $\mathcal{Y} \to \mathrm{Def}(Y)$, where $(X, \varphi)$ and $(Y, \psi)$ are marked IHS, over the intersection $\mathrm{Def}(X) \cap \mathrm{Def}(Y) \subset \mathcal{T}_\Gamma^{\mathrm{cpl}}$. This is only possible if for $t \in \mathrm{Def}(X) \cap \mathrm{Def}(Y)$ there exists a unique isomorphism $f : \mathcal{X}_t \cong \mathcal{Y}_t$ with $\psi_t = \varphi_t \circ f^*$. For K3 surfaces the uniqueness can be ensured due to the strong version of the Global Torelli Theorem (see Thm. 1.16), but in higher dimensions this fails. Thus, $\mathcal{T}_\Gamma^{\mathrm{cpl}}$ is, in general, only a coarse moduli space.

## 2.2 Moduli spaces of marked HK

**Definition 2.5** *A marked HK is a triple $(M, g, \varphi)$, where $(M, g)$ is a compact HK of dimension $4n$ in the sense of Prop. 1.3 and $\varphi$ is an isomorphism $(H^2(M, \mathbb{Z}), q) \cong \Gamma$. Two triples $(M, g, \varphi)$, $(M', g', \varphi')$ are equivalent, $(M, g, \varphi) \sim (M', g', \varphi')$, if there exists an isometry $f : (M, g) \cong (M', g')$ with $\varphi' = \varphi \circ f^*$.*

**Definition 2.6** *The moduli space of marked HK is the space*

$$\mathcal{T}_\Gamma^{\mathrm{met}} := \{(M, g, \varphi) = \text{marked HK}\}/\sim .$$

A slightly different approach towards $\mathcal{T}_\Gamma^{\mathrm{met}}$ will be explained in Section 2.5. There, the manifold $M$ is fixed and only the metric $g$ is allowed to vary.

## 2.3 Moduli spaces of marked complex HK or Kähler IHS

Recall (cf. Remark 1.9) that there is a bijection between HK with a compatible complex structure and IHS with a chosen Kähler class. Thus, the two moduli spaces are naturally equivalent.

**Definition 2.7** *A* marked complex HK *is a tuple $(M, g, I, \varphi)$, where $(M, g, \varphi)$ is a marked HK and $I$ is a compatible complex structure on $(M, g)$. A* marked Kähler IHS *is a triple $(X, \alpha, \varphi)$, where $(X, \varphi)$ is a marked IHS and $\alpha \in \mathcal{K}_X$ is a Kähler class. Two marked complex HK $(M, g, I, \varphi)$, $(M', g', I', \varphi')$ are equivalent if there exists an isometry $f : (M, g) \cong (M', g')$ with $I = f^*I'$ and $\varphi' = \varphi \circ f^*$. Analogously, one defines the equivalence of marked Kähler IHS.*

Note that the equivalence relation is compatible with the natural bijection $\{(M, g, I, \varphi)\} \leftrightarrow \{(X, \alpha, \varphi)\}$.

**Definition 2.8** *The moduli space of complex HK or, equivalently, of Kähler IHS is the space*

$$\begin{aligned} \mathcal{T}_\Gamma &:= \{(M, g, I, \varphi) = \text{marked complex HK}\}/\sim \\ &= \{(X, \alpha, \varphi) = \text{marked Kähler IHS}\}/\sim . \end{aligned}$$

Obviously, there are two forgetful maps

$$\begin{array}{ccc} \mathcal{T}_\Gamma & \xrightarrow{m} & \mathcal{T}_\Gamma^{\mathrm{met}} \\ \downarrow{\scriptstyle c} & & \\ \mathcal{T}_\Gamma^{\mathrm{cpl}} & & \end{array}$$

**Proposition 2.9** *The set $\mathcal{T}_\Gamma$ has the structure of a real manifold of dimension $3(b-2)$. The fibre $c^{-1}(X, \varphi) = \mathcal{K}_X$ is a real manifold of dimension $b-2$. The fibre $m^{-1}(M, g, \varphi)$ is naturally isomorphic to the complex manifold $\mathbb{P}^1$. The induced map $c : \mathbb{P}^1 = m^{-1}(M, g, \varphi) \to \mathcal{T}_\Gamma^{\mathrm{cpl}}$ is a holomorphic embedding. The map $m : c^{-1}(X, \varphi) \to \mathcal{T}_\Gamma^{\mathrm{met}}$ is a real embedding.*

The line $\mathbb{P}^1 \subset \mathcal{T}_\Gamma^{\mathrm{cpl}}$ is also called 'twistor line'. Having of a global deformation like this, is one of the key tools in studying moduli spaces of IHS.

## 2.4 CFT moduli spaces of HK

From a geometric point of view the following moduli space is an almost trivial extension of $\mathcal{T}_\Gamma$. However, it will become of central interest in later sections, when we will let act the full modular group on it. This group action will relate very different HK and thus gives rise to mirror symmetry phenomena.

**Definition 2.10** *A marked complex HK with a B-field is a tuple $(M, g, I, B, \varphi)$, where $(M, g, I, \varphi)$ is a marked complex HK and $B \in H^2(M, \mathbb{R})$. Two such tuples $(M, g, I, B, \varphi)$, $(M', g', I', B', \varphi')$ are equivalent if there exists an isometry $f : (M, g) \cong (M', g')$ with $I = f^*I'$, $\varphi' = \varphi \circ f^*$, and $f^*(B') = B$.*

**Definition 2.11** *The $(2,2)$-CFT moduli space of HK is the space*

$$\mathcal{T}_\Gamma^{(2,2)} := \{(M, g, I, B, \varphi) = \text{marked complex HK with B-field}\}/\sim .$$

Clearly, the moduli space $\mathcal{T}_\Gamma^{(2,2)}$ is naturally isomorphic to $\mathcal{T}_\Gamma \times \Gamma \otimes \mathbb{R}$ by mapping $(M, g, I, B, \varphi)$ to $((M, g, I, \varphi), \varphi_{\mathbb{R}}(B))$. In particular, $\mathcal{T}_\Gamma^{(2,2)}$ is a real manifold of dimension $4b - 6$.

Analogously, one defines the $(4,4)$-CFT moduli space

$$\mathcal{T}_\Gamma^{(4,4)} := \{(M, g, B, \varphi) = \text{marked HK with B-field}\}/\!\sim .$$

In partiular, there is a natural forgetful map $\mathcal{T}_\Gamma^{(2,2)} \to \mathcal{T}_\Gamma^{(4,4)}$ which is surjective with fibre $S^2$.

## 2.5 Moduli spaces without markings

All previous moduli spaces parametrize various geometric objects with an additional marking of the second cohomology. Of course, what we are really interested in are the true moduli spaces $\mathcal{M}_\Gamma^{\mathrm{cpl}}$, $\mathcal{M}_\Gamma^{\mathrm{met}}$, $\mathcal{M}_\Gamma$, $\mathcal{M}_\Gamma^{(2,2)}$, and $\mathcal{M}^{(4,4)}$. E.g. $\mathcal{M}_\Gamma^{\mathrm{cpl}}$ is the moduli space of IHS $X$ of dimension $2n$ such that $(H^2(X,\mathbb{Z}), q_X)$ is isomorphic to $\Gamma$, but without actually fixing the isomorphism. Analogously for the other spaces. In other words one has:

$$\mathcal{M}_\Gamma^{\mathrm{cpl}} = \mathrm{O}(\Gamma) \setminus \mathcal{T}_\Gamma^{\mathrm{cpl}}, \qquad \mathcal{M}_\Gamma^{\mathrm{met}} = \mathrm{O}(\Gamma) \setminus \mathcal{T}_\Gamma^{\mathrm{met}}, \quad \mathcal{M}_\Gamma = \mathrm{O}(\Gamma) \setminus \mathcal{T}_\Gamma,$$
$$\mathcal{M}_\Gamma^{(2,2)} = \mathrm{O}(\Gamma) \setminus \mathcal{T}_\Gamma^{(2,2)}, \qquad \mathcal{M}_\Gamma^{(4,4)} = \mathrm{O}(\Gamma) \setminus \mathcal{T}_\Gamma^{(4,4)}.$$

The Teichmüller spaces $\mathcal{T}_\Gamma^*$ are in general better behaved. E.g. the moduli spaces are usually singular at points that correspond to manifolds with a bigger automorphism group than expected. This usually leads to orbifold singularities. However, sometimes the passage from the Teichmüller space to the moduli space is really ill-behaved. E.g. the action of $\mathrm{O}(\Gamma)$ on $\mathcal{T}_\Gamma^{\mathrm{cpl}}$ is not properly discontinuously. Thus, $\mathcal{T}_\Gamma$ which already is not Hausdorff, becomes even worse when dividing out by $\mathrm{O}(\Gamma)$ (cf. the discussion in Section 5).

There is yet another approach to these moduli spaces where one actually fixes the underlying manifold and constructs the moduli space as a quotient of the space of hyperkähler metrics by the diffeomorphism group. We will briefly discuss this.

Let $M$ be a compact oriented differentiable manifold of real dimension $4n$ and let $q_M$ be the quadratic form on $H^2(M,\mathbb{Z})$ given by $q_M(\alpha) = c_n \cdot \int_M \alpha^2 \sqrt{\hat{A}(M)}$. We wrire $\Gamma = (H^2(M,\mathbb{Z}), q_M)$ and call this identification $\varphi_0$.

By $\mathrm{Diff}(M)$ we denote the group of orientation-preserving diffeomorphisms of $M$. In fact, at least for $b_2 \neq 6$, the group $\mathrm{Diff}(M)$ is the full diffeomorphism group of $M$, as any orientation-reversing diffeomorphism $f$ would induce an isomorphism $(H^2(M,\mathbb{Z}), q_M) \cong (H^2(M,\mathbb{Z}), -q_M)$ which is impossible for $b_2(M) \neq 6$. The set of all hyperkähler metrics $g$ on $M$ is denoted by $\mathrm{Met}^{\mathrm{HK}}(M)$. Clearly, $\mathrm{Diff}(M)$ acts naturally on $\mathrm{Met}^{\mathrm{HK}}(M)$ by $(f,g) \mapsto f^*g$.

**Definition 2.12** *The group* $\mathrm{Diff}_o(M) \subset \mathrm{Diff}(M)$ *is the connected component of* $\mathrm{Diff}(M)$ *containing the identity* $\mathrm{id}_M \in \mathrm{Diff}(M)$. *The group* $\mathrm{Diff}_*(M) \subset \mathrm{Diff}(M))$ *is the kernel of the natural representation* $\mathrm{Diff}(M) \to \mathrm{O}(H^2(M,\mathbb{Z}), q_M)$.

Mapping $g \in \mathrm{Met}^{\mathrm{HK}}(M)$ to $(M, g, \varphi) \in \mathcal{T}_\Gamma^{\mathrm{met}}$ induces a commutative diagramm

$$\begin{array}{ccc} \mathrm{Met}^{\mathrm{HK}}(M)/\mathrm{Diff}_*(M) & \xrightarrow{\eta} & \mathcal{T}_\Gamma^{\mathrm{met}} \\ \downarrow & & \downarrow \\ \mathrm{Met}^{\mathrm{HK}}(M)/\mathrm{Diff}(M) & \xrightarrow{\bar{\eta}} & \mathcal{M}_\Gamma^{\mathrm{met}} \end{array}$$

Note that $\eta$ is well-defined. Indeed, if $f \in \mathrm{Diff}_*(M)$, then $(M, g, \varphi_0) \sim (M, f^*g, \varphi_0 \circ f^*) = (M, f^*g, \varphi_0)$.

**Remark 2.13** It seems, nothing is known about the quotient of the natural inclusion $\mathrm{Diff}_o(M) \subset \mathrm{Diff}_*(M)$, not even for K3 surfaces, i.e. $n = 1$.

Clearly, the image of $\eta$ (and $\bar{\eta}$) contains only those HK $(M', g', \varphi) \in \mathcal{T}_\Gamma^{\mathrm{met}}$ whose underlying real manifold $M'$ is diffeomorphic to $M$. Let $\mathcal{T}_\Gamma^{\mathrm{met}}(M)$ and $\mathcal{M}_\Gamma^{\mathrm{met}}(M)$ denote the union of all those connected components.

**i)** In general, $\eta : \mathrm{Met}^{\mathrm{HK}}(M)/\mathrm{Diff}_*(M) \to \mathcal{T}_\Gamma^{\mathrm{met}}(M)$ is injective, but not surjective.
Only the surjectivity needs a proof. If $(M, g, \varphi) \in \mathrm{Im}(\eta)$ and $\psi \in \mathrm{O}(H^2(M, \mathbb{Z}), q_M)$, then $(M, g, \varphi_0 \circ \psi) \in \mathrm{Im}(\eta)$ if and only if there exists $f \in \mathrm{Diff}(M)$ with $f^* = \psi$ but $\mathrm{Diff}(M) \to \mathrm{O}(H^2(M, \mathbb{Z}), q_M)$ is not necessarily surjective. E.g. for K3 surfaces the image does not contain $-\mathrm{id}$ and, more precsiely, $\mathrm{O}(H^2(M, \mathbb{Z}), \cup)/\mathrm{Diff}(M) \cong \mathbb{Z}/2\mathbb{Z}$ (cf. 4.3). However in this case the situation is rather simple, as $\mathcal{T}_\Gamma^{\mathrm{met}}$ consists of two components and $\mathrm{Met}^{\mathrm{HK}}(M)/\mathrm{Diff}_*(M)$ is one of them. For higher dimensional HK nothing is known about the image of $\mathrm{Diff}(M) \to \mathrm{O}(H^2(M, \mathbb{Z}), q_M)$.

**ii)** The map $\bar{\eta} : \mathrm{Met}^{\mathrm{HK}}(M)/\mathrm{Diff}(M) \to \mathcal{M}_\Gamma^{\mathrm{met}}(M)$ is bijective.
Indeed, if $(M, g, \varphi) \in \mathcal{T}_\Gamma^{\mathrm{met}}(M)$, then $[(M, g, \varphi_0)] = [(M, g, (\varphi_0\varphi^{-1})\varphi)] = (\varphi_0\varphi^{-1})[(M, g, \varphi)] = [(M, g, \varphi)] \in \mathcal{M}_\Gamma^{\mathrm{met}}(M)$. Thus, $\bar{\eta}$ is surjective. If $\bar{\eta}(M, g) = \bar{\eta}(M, g')$, then there exists $\psi \in \mathrm{O}(\Gamma)$ such that $(M, g, \varphi_0) \sim (M, g', \psi \circ \varphi_0)$ and hence there exists $f \in \mathrm{Diff}(M)$ with $f^*g = g'$ (note that for $b_2(M) = 6$ one would have to argue that $f$ can be chosen orientation-preserving) and $\varphi_0 = \psi \circ \varphi_0 \circ f^*$. Thus, $[(M, g)] = [(M, g')]$ in $\mathrm{Met}^{\mathrm{HK}}(M)/\mathrm{Diff}(M)$ and hence $\bar{\eta}$ is injective.

One last word concerning the stabilizer of the action of $\mathrm{Diff}(M)$. Clearly, the stabilizer of a hyperkähler metric $g$ is the isometry group $\mathrm{Isom}(M, g)$ of $(M, g)$. This group is compact (cf. [9]). Hence the stabilizer of $g \in \mathrm{Met}^{\mathrm{HK}}(M)$ is a compact group. Moreover, $\mathrm{Isom}(M, g) \cap \mathrm{Diff}_*(M)$ is finite. Indeed, if we fix in addition a compatible complex structure $I$ on $(M, g)$ then $\mathrm{Aut}(M, I)$ is discrete, as $H^0((M, I), \mathcal{T}) = 0$, and hence $\mathrm{Aut}(M, I) \cap \mathrm{Isom}(M, g)$ is finite. Since any $f \in \mathrm{Isom}(M, g) \cap \mathrm{Diff}_*(M)$ acts trivially on cohomology, it must preserve $I$. Hence, the action of $\mathrm{Diff}_*(M)$ on $\mathrm{Met}^{\mathrm{HK}}(M)$ has finite stabilizer.

# 3 Period domains

The moduli spaces that have been introduced in the last section will be studied by means of various period maps. In this section we define and discuss the spaces in which these maps take their values, the period domains.

Let $\Gamma$ be a lattice of signature $(m, n)$. The standard example for $\Gamma$ is the K3 lattice $2(-E_8) \oplus 3U$, where $U$ denotes the hyperbolic plane $\left(\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$. However, $\Gamma$ might in general be non-unimodular. This will be of no importance in this section, as only the real vector space $\Gamma_{\mathbb{R}} := \Gamma \otimes \mathbb{R}$ is going te be used. In fact, usually we will work with an arbitrary vector space $V$, but $\Gamma$ will nevertheless occur in the notation. I hope this will not lead to any confusion.

## 3.1 Positive definite (oriented) subspaces

Let $V$ be a real vector space that is endowed with a bilinear form $\langle\ ,\ \rangle$ of signature $(m, n)$, e.g. $V = \Gamma_{\mathbb{R}}$. We will also write $x^2$ for $\langle x, x\rangle$. Fix $k \leq m$ and consider the space of all $k$-dimensional subspaces $W \subset V$ such that $\langle\ ,\ \rangle$ restricted to $W$ is positive definite. We will denote this space by $\mathrm{Gr}_k^{\mathrm{p}}(V)$. Clearly, $\mathrm{Gr}_k^{\mathrm{p}}(V)$ is an open non-empty subset of the Grassmannian $\mathrm{Gr}_k(V)$.

In order to describe $\mathrm{Gr}_k^{\mathrm{p}}(V)$ as a homogeneous space we consider the natural action of $\mathrm{O}(V)$ on $\mathrm{Gr}_k(V)$ given by: $(\varphi, W) \mapsto \varphi(W)$. The stabilizer of a point $W_0 \in \mathrm{Gr}_k^{\mathrm{p}}(V)$ is $\mathrm{O}(W_0) \times \mathrm{O}(W_0^{\perp})$. Since the action is transitive, one obtains the following description

$$\boxed{\mathrm{Gr}_k^{\mathrm{p}}(V) \cong \mathrm{O}(V)/\mathrm{O}(W_0) \times \mathrm{O}(W_0^{\perp}) \cong \mathrm{O}(m,n)/\mathrm{O}(k) \times \mathrm{O}(m-k, n)}$$

The second isomorphism depends on the choice of a basis of the spaces $W_0$ and $W_0^{\perp}$.

Next consider the space $\mathrm{Gr}_k^{\mathrm{po}}(V)$ of all oriented positive definite subspaces $W \subset V$ of dimension $k$. Clearly, the natural map $\mathrm{Gr}_k^{\mathrm{po}}(V) \to \mathrm{Gr}_k^{\mathrm{p}}(V)$ is a $2:1$ cover. Again, $\mathrm{O}(V)$ acts transitively on $\mathrm{Gr}_k^{\mathrm{po}}(V)$ and the stabilizer of an oriented positive definite subspace $W_0$ is $\mathrm{SO}(W_0) \times \mathrm{O}(W_0^{\perp})$. Thus,

$$\boxed{\mathrm{Gr}_2^{\mathrm{po}}(V) \cong \mathrm{O}(V)/\mathrm{SO}(W_0) \times \mathrm{O}(W_0^{\perp}) \cong \mathrm{O}(m,n)/\mathrm{SO}(k) \times \mathrm{O}(m-k, n)}$$

## 3.2 Planes and complex lines

For $k = 2$ the space $\mathrm{Gr}_2^{\mathrm{po}}(V)$ allows an alternative description. It turns out that there is a natural bijection between this space and the space

$$Q_\Gamma := \{x \mid x^2 = 0,\ (x + \bar{x})^2 > 0\} \subset \mathbb{P}(\Gamma_{\mathbb{C}}),$$

where we use the $\mathbb{C}$-linear extension of $\langle\ ,\ \rangle$. Note that the second condition in the definition of $Q_\Gamma$ is well posed, i.e. independent of the representative $x \in \Gamma_{\mathbb{C}}$ of the line $x \in \mathbb{P}(\Gamma_{\mathbb{C}})$, as long as the first condition $x^2 = 0$ is satisfied. Clearly, $Q_\Gamma$ is an open subset of a non-singular quadric hypersurface in $\mathbb{P}(\Gamma_\mathbb{C})$.

To any $x \in Q_\Gamma$ one associates the plane $W_x := \Gamma_\mathbb{R} \cap (x\mathbb{C} \oplus \bar{x}\mathbb{C}) \subset \Gamma_\mathbb{R}$ endowed with the orientation given by $(\mathrm{Re}(x), \mathrm{Im}(x))$. Since $x\mathbb{C} \oplus \bar{x}\mathbb{C}$ is invariant under conjugation, this space is indeed a real plane. Moreover, $W_{\lambda x} = \Gamma_\mathbb{R} \cap (\lambda x\mathbb{C} \oplus \bar{\lambda}\bar{x}\mathbb{C}) = W_x$ and $(\mathrm{Re}(\lambda x), \mathrm{Im}(\lambda x)) = (\mathrm{Re}(x), \mathrm{Im}(x))\left(\begin{smallmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{smallmatrix}\right)$, where the matrix has positive determinant. Hence, the oriented plane $W_x$ is well-defined, i.e. it only depends on $x \in \mathbb{P}(\Gamma_\mathbb{C})$. It is positive, since $(\lambda x + \bar{\lambda}\bar{x})^2 = \lambda\bar{\lambda}(x + \bar{x})^2 > 0$ for $\lambda \neq 0$.

Conversely, if $W \in \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_\mathbb{R})$, then choose a positively oriented orthonormal basis $w_1, w_2 \in W$ and set $x := w_1 + iw_2$. Then $W = W_x$ and $x^2 = 0$, $(x + \bar{x})^2 = (2w_1)^2 > 0$. Moreover, $x \in \mathbb{P}(\Gamma_\mathbb{C})$ does not depend on the choice of the basis and any $x \in Q_\Gamma$ can be written in this form.

Thus, one has a bijection

$$\boxed{Q_\Gamma \cong \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_\mathbb{R})}$$

### 3.3 Planes and three-spaces

For our purpose the spaces $\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_\mathbb{R})$, $\mathrm{Gr}_3^{\mathrm{po}}(\Gamma_\mathbb{R})$, and $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_\mathbb{R} \oplus U_\mathbb{R})$ are the most interesting ones. In the next two sections we will study how they are related to each other. To this end let us first introduce the space

$$\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R}) := \{(P, \omega) \mid P \in \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_\mathbb{R}) \ , \ \omega \in P^\perp \subset \Gamma_\mathbb{R} \ , \ \omega^2 > 0\}.$$

Clearly, this space projects naturally to $\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_\mathbb{R})$ by $(P, \omega) \mapsto P$. The fibre over the point $P$ is the quadratic cone $\{\omega \mid \omega^2 > 0\} \subset P^\perp \subset \Gamma_\mathbb{R}$. If $\Gamma$ has signature $(3, b-3)$, this cone consists of exactly two connected components, which can be identified with each other by $\omega \mapsto -\omega$. Thus, the fibre of $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R}) \to \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_\mathbb{R})$ over $P$ in this case is the disjoint union of two copies of a connected cone, which will be called $\mathcal{C}_P$.

In fact, $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R}) \to \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_\mathbb{R})$ is a trivial cover, i.e. $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R})$ splits into two components. This can either be deduced from the fact that $\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_\mathbb{R}) \cong \mathrm{O}(3, b-3)/(\mathrm{SO}(2) \times \mathrm{O}(1, b-3))$ is simply connected (cf. Sect. 3.6) or from the following argument: If we fix a positive oriented three-space $F \in \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_\mathbb{R})$, then the orthogonal projection $P \oplus \mathbb{R}\omega \to F$ for any $\omega \in \pm\mathcal{C}_X$ must be an isomorphism, since $F^\perp$ is negative definite. Thus, we can distinguish one of the two connected components of $\pm\mathcal{C}_X$ by requiring that $P \oplus \mathbb{R}\omega \cong F$ is compatible with the orientations on both spaces.

Mapping $(P, \omega)$ to the oriented positive definite three-space $F(P, \omega) := P \oplus \omega\mathbb{R}$ and the scalar $\omega^2 \in \mathbb{R}_{>0}$ defines a map $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R}) \to \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_\mathbb{R}) \times \mathbb{R}_{>0}$. The map is surjective and the fibre over a point $(F, \lambda)$ can be identified with the set of all $\omega \in F$ with $\omega^2 = \lambda$ which is a two-dimensional sphere.

Thus, one has the following diagram

$$\begin{array}{ccl}
\mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) & \xrightarrow{S^2} & \mathrm{Gr}^{\mathrm{po}}_3(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \cong \Big(\mathrm{O}(m,n)/\mathrm{SO}(3) \times \mathrm{O}(m-3,n)\Big) \times \mathbb{R}_{>0} \\
\big\downarrow {\scriptstyle \pm c_P} & & \\
\mathrm{Gr}^{\mathrm{po}}_2(\Gamma_{\mathbb{R}}) & \xrightarrow{\cong} & \mathrm{O}(m,n)/\mathrm{SO}(2) \times \mathrm{O}(m-2,n)
\end{array}$$

Note that the two natural compositions $S^2 \subset \mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) \to \mathrm{Gr}^{\mathrm{po}}_2(\Gamma_{\mathbb{R}})$ and $\mathcal{C}_P \subset \mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) \to \mathrm{Gr}^{\mathrm{p}}_3(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0}$ are both injective.

### 3.4 Three- and four-spaces

From now on we will assume that $\Gamma$ has signature $(3, b-3)$. Furthermore, let us fix a standard basis $(w, w^*)$ of $U$, i.e. $w^2 = w^{*2} = 0$ and $\langle w, w^* \rangle = 1$. We will see that the space of four-spaces in $\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}$ relates naturally to the space of three-spaces in $\Gamma_{\mathbb{R}}$. Explicitely, we will show

$$\boxed{\begin{array}{rcl}
\mathrm{Gr}^{\mathrm{po}}_3(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \times \Gamma_{\mathbb{R}} & \cong & \mathrm{Gr}^{\mathrm{po}}_4(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \\
 & \cong & \mathrm{O}(4, b-2)/\mathrm{SO}(4) \times \mathrm{O}(b-2)
\end{array}}$$

The second isomorphism follows from Sect. 3.1. The first one is given as follows.

$$\phi : (F, \alpha, B) \mapsto \Pi := F' \oplus B'\mathbb{R},$$

where $F' := \{f - \langle f, B \rangle w | f \in F\}$ and $B' := B + \frac{1}{2}(\alpha - B^2)w + w^*$. Clearly, $\langle f - \langle f, B \rangle w, B' \rangle = \langle f - \langle f, B \rangle w, B + w^* \rangle = 0$ and thus the decomposition is orthogonal. Furthermore, $(f - \langle f, B \rangle w)^2 = f^2 > 0$ for $0 \neq f \in F$ and $B'^2 = B^2 + \alpha - B^2 = \alpha > 0$. Hence, $\Pi$ is a positive four-space. Its orientation is induced by the orientation of $F \cong F'$ and the decomposition $\Pi = F' \oplus B'\mathbb{R}$.

In order to see that $\phi$ is bijective we study the inverse map $\psi : \Pi \mapsto (F, B'^2, B)$, where $F$, $B'$, and $B$ are defined as follows: One first introduces $F' := \Pi \cap w^{\perp}$. This space is of dimension three, since otherwise $\Pi \subset w^{\perp} = \Gamma_{\mathbb{R}} \oplus w\mathbb{R}$ and the latter space does not contain any positive four-space. Again by the positivity of $\Pi$ one finds $w \notin F' \subset \Pi$. Hence, $F := \pi(F') \subset \Gamma_{\mathbb{R}}$ is a positive three-space, where $\pi : \Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}} \to \Gamma_{\mathbb{R}}$ is the natural projection. Furthermore, there exists a $B' \in \Pi$ such that $\Pi = F' \oplus B'\mathbb{R}$ is an orthogonal splitting. As before $B'$ cannot be contained in $w^{\perp}$. Thus, one can rescale $B'$ such that $\langle B', w \rangle = 1$. This determines $B'$ uniquely. Since $B' \in \Pi$, one has $B'^2 > 0$. The B-field is by definition $B := \pi(B')$. One easily verifies that $\psi$ and $\phi$ are indeed inverse to each other.

### 3.5 Pairs of planes

The last space we will discuss in this series of period domains is the space of orthogonal oriented positive definite planes in $\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}$, i.e.

$$\mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) = \{(H_1, H_2) \mid H_i \in \mathrm{Gr}^{\mathrm{po}}_2(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}), H_1 \perp H_2\}.$$

Using the same techniques as before this space can also be described as an homogeneous space as follows

$$\begin{aligned}\mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) &\cong \mathrm{O}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})/\mathrm{SO}(H_1) \times \mathrm{SO}(H_2) \times \mathrm{O}((H_1 \oplus H_2)^{\perp}) \\ &\cong \mathrm{O}(4, b-2)/\mathrm{SO}(2) \times \mathrm{SO}(2) \times \mathrm{O}(b-2);\end{aligned}$$

for some chosen point $(H_1, H_2) \in \mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$, respectively basis of the spaces $H_1$, $H_2$, and $(H_1 \oplus H_2)^{\perp}$.

We will be interested in the natural projection

$$\pi : \mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \twoheadrightarrow \mathrm{Gr}^{\mathrm{po}}_4(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}), \ (H_1, H_2) \mapsto \Pi := H_1 \oplus H_2$$

and in the injection

$$\gamma : \mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \hookrightarrow \mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$$

which is compatible with $\mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) \to \mathrm{Gr}^{\mathrm{po}}_3(\Gamma_{\mathbb{R}})$.

Let us first study the projection. Using the above description of both spaces as homogeneous spaces this map corresponds to dividing by $\mathrm{SO}(4)/(\mathrm{SO}(2) \times \mathrm{SO}(2))$. The fibre of $\pi$ over $\Pi \in \mathrm{Gr}^{\mathrm{po}}_4(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ is canonically isomorphic to $\mathrm{Gr}^{\mathrm{po}}_2(\Pi)$ by $(H_1, H_2) \mapsto H_1$. The inverse image of $H \in \mathrm{Gr}^{\mathrm{po}}_2(\Pi)$ is $(H, H^{\perp})$, where $H^{\perp}$ gets its orientation from $\Pi$ and the decomposition $\Pi = H \oplus H^{\perp}$.

Thus one obtains the following description of the fibre

$$\pi^{-1}(\Pi) \cong \mathrm{Gr}^{\mathrm{po}}_2(\Pi) \cong S^2 \times S^2.$$

The second isomorphism is derived as in Sect. 3.2 from

$$\mathrm{Gr}^{\mathrm{po}}_2(\Pi) = \{x \in \mathbb{P}(\Pi_{\mathbb{C}}) \mid x^2 = 0\} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Note that $(x + \bar{x})^2 > 0$ is automatically satisfied, for $\langle\ ,\ \rangle$ on $\Pi$ is positive definite by assumption.

Let us now turn to the injection $\gamma$, which is defined as follows. We set $\gamma((P, \omega), B) = (H_1, H_2)$ with

$$H_1 := \{x - \langle x, B\rangle w \mid x \in P\}$$

and

$$H_2 := \left(\frac{1}{2}(\alpha - B^2)w + w^* + B\right)\mathbb{R} \oplus (\omega - \langle \omega, B\rangle w)\,\mathbb{R},$$

where as before $(w, w^*)$ is the standard basis of $U$ and $\alpha = \omega^2$.

The isomorphism $P \cong H_1$, $x \mapsto x - \langle x, B\rangle w$ endowes $H_1$ with an orientation. A natural orientation of $H_2$ is given by definition. Observe that $H_1$ only depends on $P$ and $B$, whereas $H_2$ on $\omega$ and $B$. One easily verifies that the map $\gamma$ is injective and that it commutes with the projections to

$$\mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \times \mathbb{R}_{>0} \times \Gamma_{\mathbb{R}} \cong \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}).$$

Recall that the fibre of $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \to \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ is $S^2$, whereas the fibre of $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \to \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ is $S^2 \times S^2$. It can be checked that the embedding $\gamma$ does not identify the fibre $S^2$ with the diagonal.

**Remark 3.1** Note that the projection $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \to \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \to \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})$ does not extend, at least not canonically, to a map $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \to \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})$. Geometrically this will be interpreted by the fact that not any point in the $(2,2)$-CFT moduli space of K3 surfaces canonically defines a complex structure.

We summarize the discussion of this paragraph in the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) & \xrightarrow{S^2 \times S^2} & \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \\
\uparrow & & \| \\
\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} & \xrightarrow{S^2} & \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \times \Gamma_{\mathbb{R}} \\
\downarrow{\scriptstyle \Gamma_{\mathbb{R}}} & & \downarrow{\scriptstyle \Gamma_{\mathbb{R}}} \\
\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) & \xrightarrow{S^2} & \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \\
\downarrow{\scriptstyle \pm c_P} & & \\
\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}}) & &
\end{array}$$

## 3.6 Topology of period domains

Let us study some basic aspects of the topology of the period domains that are of interest for us. Let $\Gamma$ be a lattice of signature $(m, n)$.

$$\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \cong \mathrm{O}(3,n)/\mathrm{SO}(2) \times \mathrm{O}(1,n), \quad \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \cong \mathrm{O}(4,n+1)/\mathrm{SO}(4) \times \mathrm{O}(n+1)$$

$$\mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \cong \mathrm{O}(3,n)/\mathrm{SO}(3) \times \mathrm{O}(n), \quad \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \cong \mathrm{O}(4,n+1)/\mathrm{SO}(2) \times \mathrm{SO}(2) \times \mathrm{O}(n+1)$$

For simplicity we will suppose that $n > 0$.

**Lemma 3.2** *The group $\mathrm{O}(m,n)$ with $m, n > 0$ has exactly four connected components.*

*Proof.* Write $\mathrm{O} := \mathrm{O}(m,n)$. Then there are the following disjoint unions $\mathrm{O} = \mathrm{O}^+ \cup \mathrm{O}^-$, $\mathrm{O} = \mathrm{O}_+ \cup \mathrm{O}_-$, and $\mathrm{O} = \mathrm{O}^+_+ \cup \mathrm{O}^+_- \cup \mathrm{O}^-_+ \cup \mathrm{O}^-_-$. Here, $\mathrm{O}^\pm_\pm$ are defined as follows: Write $\mathbb{R}^{m+n} = W_0 \oplus W_0^\perp$ with $W_0 \subset \mathbb{R}^{m+n}$ a positive subspace, which is endowed with an orientation. Then let $\mathrm{O}^+$ and $\mathrm{O}^-$ (respectively, $\mathrm{O}_+$ and $\mathrm{O}_-$) be the subsets of all linear maps $A \in \mathrm{O}$ such that the orthogonal projection $AW_0 \to W_0$ (respectively, $AW_0^\perp \to W_o^\perp$) is orientation preserving resp. orientation reversing. By definition $\mathrm{O}^+_+ = \mathrm{O}^+ \cap \mathrm{O}_+$, etc. For any $A_0 \in \mathrm{O}^\pm_\pm$ the map $\mathrm{O}^+_+ \to \mathrm{O}^\pm_\pm$, $A \mapsto AA_0$ defines a homeomorphism. Thus, it suffices to show that $\mathrm{O}^+_+$ is connected. □

Note that $\mathrm{O}^+_+(m,n)$ is the connected component of the identity. It will thus also be denoted $\mathrm{O}_o(m,n)$.

**Corollary 1** *The space* $\mathrm{Gr}^{\mathrm{po}}_2(\Gamma_\mathbb{R})$ *is connected, whereas the spaces* $\mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_\mathbb{R})$, $\mathrm{Gr}^{\mathrm{po}}_3(\Gamma_\mathbb{R})$, $\mathrm{Gr}^{\mathrm{po}}_4(\Gamma_\mathbb{R} \oplus U_\mathbb{R}$, *and* $\mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_\mathbb{R} \oplus U_\mathbb{R})$ *consist of two connected components.*

*Proof.* Use the obvious fact that the inclusion $\mathrm{SO}(2) \times \mathrm{O}(1,n) \subset \mathrm{O}(3,n)$ respects the decomposition into connected components, i.e. $\mathrm{O}^\pm_\pm(1,n) \subset \mathrm{O}^\pm_\pm(3,n)$. Thus, $\pi_0(\mathrm{SO}(2) \times \mathrm{O}(1,n)) \cong \pi_0(\mathrm{O}(3,n)) \cong \mathbb{Z}/4\mathbb{Z}$. Similarly for $\mathrm{Gr}^{\mathrm{po}}_3(\Gamma_\mathbb{R})$. Here $\pi_0(\mathrm{O}(3,n)) = \mathbb{Z}/4\mathbb{Z}$, but $\pi_0(\mathrm{SO}(3) \times \mathrm{O}(n)) = \mathbb{Z}/2\mathbb{Z}$, i.e. the components $\mathrm{O}^-_\pm$ do not intersect the image of the inclusion. Hence, $\pi_0(\mathrm{Gr}^{\mathrm{po}}_3(\Gamma_\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$. The remaining assertions are proven analogously. □

We are also interested in the fundamental groups of these spaces. In order to compute those, we recall the following classical facts.

**Proposition 3.3** *One has* $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$, $\pi_1(\mathrm{SO}(k)) = \mathbb{Z}/2\mathbb{Z}$ *for* $k > 2$, *and* $\pi_1(\mathrm{O}_o(m,n)) \cong \pi_1(\mathrm{SO}(m)) \times \pi_1(\mathrm{SO}(n))$.

*Proof.* The first assertion follows from $\mathrm{SO}(2) \cong S^2$. The universal cover of $\mathrm{SO}(k)$ for $k \geq 3$ is the two-to-one cover $\mathrm{Spin}(k) \to \mathrm{SO}(k)$. The isomorphism in the last assertion is induced by the natural inclusion $\mathrm{SO}(m) \times \mathrm{SO}(n) \hookrightarrow \mathrm{O}_o(m,n)$. □

**Corollary 2** *All period domains* $\mathrm{Gr}^{\mathrm{po}}_2(\Gamma_\mathbb{R})$, $\mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_\mathbb{R})$, $\mathrm{Gr}^{\mathrm{po}}_3(\Gamma_\mathbb{R})$, $\mathrm{Gr}^{\mathrm{po}}_4(\Gamma_\mathbb{R} \oplus U_\mathbb{R})$, *and* $\mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_\mathbb{R} \oplus U_\mathbb{R})$ *are simply-connected, i.e. every connected component is simply connected.*

*Proof.* Since

$$\mathrm{Gr}^{\mathrm{po}}_2(\Gamma_\mathbb{R}) = \mathrm{O}(3,n)/\mathrm{SO}(2) \times \mathrm{O}(1,n) \cong \mathrm{O}_o(3,n)/\mathrm{SO}(2) \times \mathrm{O}_o(1,n),$$

we may use the exact sequence

$$\pi_1(\mathrm{SO}(2) \times \mathrm{O}_o(1,n)) \xrightarrow{a} \pi_1(\mathrm{O}_o(3,n)) \to \pi_1(\mathrm{Gr}^{\mathrm{po}}_2(\Gamma_\mathbb{R})) \to \pi_0(\quad) \cong \pi_0(\quad).$$

The map $a$ is compatible with the natural isomorphisms $\pi_1(\mathrm{SO}(2)\times \mathrm{O}_o(1,n))\cong \pi_1(\mathrm{SO}(2))\times \pi_1(\mathrm{O}_o(1,n))\cong \pi_1(\mathrm{SO}(2))\times \pi_1(\mathrm{SO}(1))\times \pi_1(\mathrm{SO}(n))$, $\pi_1(\mathrm{O}_o(3,n))\cong \pi_1(\mathrm{SO}(3))\times \pi_1(\mathrm{SO}(n))$ and the natural maps $\mathbb{Z}\cong \pi_1(\mathrm{SO}(2))\times\pi_1(\mathrm{SO}(1))\to \pi_1(\mathrm{SO}(3))\cong \mathbb{Z}/2\mathbb{Z}$. Thus, $a$ is surjective and hence $\pi_1(\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}}))=0$. The other assertions are proven analogously. □

**Remark 3.4** Eventually, we list the dimensions of our period spaces, which can easily be computed starting from $\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})\cong Q_\Gamma$. We have $\dim \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})=2(n+1)$, $\dim \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}})=3n+1$, $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}})=3n-1$, and $\dim \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}})=3n+3$.

### 3.7 Density results

Here we shall be interested in those points $P\in Q_\Gamma$ whose orthogonal complement $P^\perp\subset\Gamma_{\mathbb{R}}$ contains integral elements $\alpha\in\Gamma$ of given length. For simplicity we shall assume that $\Gamma$ is the K3 lattice $2(-E_8)\oplus 3U$, but all we will use is that $\Gamma$ is even of index $(3,b-3)$ and that any primitive isotropic element of $\Gamma$ can be complemented to a sublattice of $\Gamma$ which is isomorphic to the hyperbolic plane. First note the following easy fact.

**Lemma 3.5** *If $0\neq\alpha\in\Gamma_{\mathbb{R}}$ then $\alpha^\perp\cap Q_\Gamma$ is not empty.*

*Proof.* Indeed, $\alpha^\perp\subset\Gamma_{\mathbb{R}}$ is a hyperplane containing at least two linearly independent positive vectors $x,y$. Thus, $P:=\langle x,y\rangle\in\alpha^\perp\cap Q_\Gamma$. □

The quadric in $\mathbb{P}(\Gamma_{\mathbb{C}})$ defined by the quadratic form $\langle\ ,\ \rangle$ on $\Gamma$ will be denoted $Z$, its real points form the set $Z_{\mathbb{R}}=\mathbb{P}(\Gamma_{\mathbb{R}})\cap Z$.

**Proposition 3.6** *Let $0\neq\alpha\in\Gamma$. Then the set*

$$\bigcup_{g\in\mathrm{O}(\Gamma)} g(\alpha^\perp\cap Q_\Gamma)=\bigcup_{g\in\mathrm{O}(\Gamma)} g(\alpha)^\perp\cap Q_\Gamma$$

*is dense in $Q_\Gamma$.*

*Proof.* We start out with the following observation: Let $\Gamma=\Gamma'\oplus U$ be an orthogonal decomposition and let $(v,v^*)$ be a standard basis of $U$. For $B\in\Gamma'$ with $B^2\neq 0$ we define $\varphi_B\in\mathrm{O}(\Gamma)$ by $\varphi_B(v)=v$, $\varphi_B(v^*)=B+v^*-B^2/2\cdot v$, and $\varphi_B(x)=x-\langle B,x\rangle v$ for $x\in\Gamma'$. It is easy to see that indeed with this definition $\varphi_B\in\mathrm{O}(\Gamma)$. (We shall study a similarly defined automorphism $\varphi_B\in\mathrm{O}(\Gamma\oplus U)$ in Sect. 5).

This automorphism has the remarkable property that for any $y\in\Gamma_{\mathbb{R}}$ one has

$$\lim_{k\to\infty}\varphi_N^k[y]=[v]\in\mathbb{P}(\Gamma_{\mathbb{R}}).$$

In particular, we find that in the closure of the orbit $\mathrm{O}:=\mathrm{O}(\Gamma)\cdot[\alpha]\subset\mathbb{P}(\Gamma_{\mathbb{R}})$ there exists an isotropic vector, i.e. $\overline{\mathrm{O}}\cap Z_{\mathbb{R}}\neq\emptyset$.

In order to prove the assertion of the proposition we have to show that for any $P \in Q_\Gamma$ there exists an automorphism $g \in \mathrm{O}(\Gamma)$ such that $g(\alpha)$ is arbitrarily close to $P^\perp$. Indeed, in this case we find a codimension two subspace $W \subset \Gamma_\mathbb{R}$ close to $P^\perp$ containing $g(\alpha)$ and, therefore, $W^\perp \in Q_\Gamma$ is close to $P$ and orthogonal to $g(\alpha)$.

Since $P^\perp$ contains some isotropic vector, it suffices to show that any vector $[y] \in Z_\mathbb{R} \subset \mathbb{P}(\Gamma_\mathbb{R})$ is contained in $\overline{\mathrm{O}}$. As explained before, $\overline{\mathrm{O}} \cap Z_\mathbb{R} \neq \emptyset$. On the other hand, $\overline{\mathrm{O}} \cap Z_\mathbb{R}$ is closed and $\mathrm{O}(\Gamma)$-invariant. Thus, it suffices to show that any $\mathrm{O}(\Gamma)$-orbit $\mathrm{O}_y := \mathrm{O}(\Gamma)\cdot[y] \subset Z_\mathbb{R}$ is dense. This is proved in two steps.

i) The closure $\overline{\mathrm{O}}_y$ contains the subset $\{[x] \in Z \mid x \in \Gamma\}$. Indeed, for any $x \in \Gamma$ primitive with $x^2 = 0$ one finds an orthogonal decomposition $\Gamma = \Gamma' \oplus U$ with $x = v$, where $(v, v^*)$ is a standard basis of the hyperbolic plane $U$. If we choose $B \in \Gamma'$ with $B^2 \neq 0$, then $\lim_{k\to\infty} \varphi_B^k[y] = [v] = [x]$, as we have seen before. Hence, $[x] \in \overline{\mathrm{O}}_y$.

ii) The set $\{[x] \in Z \mid x \in \Gamma\}$ is dense in $Z$. Indeed, if we write $\Gamma = \Gamma' \oplus U$ as before, then the dense open subset $V \subset Z_\mathbb{R}$ of points of the form $[x' + \lambda v + v^*]$ with $\lambda \in \mathbb{R}$, $x' \in \Gamma'_\mathbb{R}$ is the affine quadric $\{(x', \lambda) \mid 2\lambda + x'^2 = 0\} \subset \Gamma_\mathbb{R} \times \mathbb{R}$ and thus is given as the graph of the rational polynomial $\Gamma'_\mathbb{R} \to \mathbb{R}$, $x' \mapsto -x'^2/2$. Therefore, the rational points are dense in $V$.

Combining both steps yields the assertion. □

**Corollary 3** *For any $m \in \mathbb{Z}$ the subset*

$$\{P \in Q_\Gamma \mid \text{there exists a primitive } \alpha \in \Gamma \cap P^\perp \text{ with } \alpha^2 = 2m\}$$

*is dense in $Q_\Gamma$.*

*Proof.* In order to apply the proposition we only have to ensure that there is a primitive element $0 \neq \alpha \in \Gamma$ with $\alpha^2 = 2m$. If $(w, w^*)$ is the standard base of a copy of the hyperbolic plane $U$ contained in $\Gamma$, we can choose $\alpha = w + mw^*$. □

In fact, if $\alpha_1, \alpha_2 \in \Gamma$ are primitive elements with $\alpha_1^2 = \alpha_2^2$ then there exists an automorphism $\varphi \in \mathrm{O}(\Gamma)$ with $\varphi(\alpha_1) = \alpha_2$ (cf. [25, Thm.2.4] or Remark 6.4). Thus, the assertion of the corollary is essentially equivalent to the proposition (see [1] page 111). Note that for general HKs we don't know which values of $2m$ can be realized.

As a further trivial consequence one sees that the set of those $P \in Q_\Gamma$ such that $P^\perp \cap \Gamma \neq 0$ is dense in $Q_\Gamma$. One can now go on and ask for those $P \in Q_\Gamma$ such that $P^\perp \cap \Gamma$ has higher rank. Those with maximal rank, i.e. $\mathrm{rk}(P^\perp \cap \Gamma) = \mathrm{rk}(\Gamma) - 2$, are called exceptional. An equivalent definition is

**Definition 3.7** *A period point $P \in Q_\Gamma$ is exceptional if $P \subset \Gamma_\mathbb{R}$ is defined over $\mathbb{Q}$, i.e. $P \in Q_\Gamma \cap \mathbb{P}(\Gamma_{\mathbb{Q}(i)})$.*

Clearly, $P$ is exceptional if there exist linearly independent elements $\alpha_1, \ldots, \alpha_{\mathrm{rk}(\Gamma)-2} \in \Gamma$ such that $P \subset \alpha_i^\perp$ for all $i$. Note that if $P \in Q_\Gamma$ is exceptional, the orthogonal complement $P^\perp$ always contains a lattice vector $x \in \Gamma$ with $x^2 > 0$ (use that $\Gamma$ has signature $(3, b-3)$).

Next we will prove that also the exceptional points are dense in $Q_\Gamma$. For K3 surfaces one can add further restrictions.

**Definition 3.8** *Let $\Gamma$ be the K3 lattice. A period point $P \in Q_\Gamma$ is called* exceptional Kummer *if $P \subset \Gamma_{\mathbb{R}}$ is defined over $\mathbb{Q}$ and for all $x \in P \cap \Gamma$ one has $x^2 \equiv 0 \mod 4$.*

**Proposition 3.9** *Let $\Gamma$ be the K3 lattice. Then the set of exceptional Kummer points $P \in Q_\Gamma$ is a dense subset of $Q_\Gamma$.*

*Proof.* We first prove the following statement. Let $L$ be an arbitrary lattice. Then the set

$$\{[x] \mid x \in L \text{ is primitive and } x^2 \equiv 0 \mod 4\} \subset \mathbb{P}(L_{\mathbb{R}})$$

is empty or dense. Indeed, if $[x]$ is contained in this set and $y \in L$ is arbitrary, then $[x + N \cdot y] \in \mathbb{P}(L_{\mathbb{R}})$ converges towards $[y]$ for $N \to \infty$. Moreover, $(x + N \cdot y)^2 \equiv x^2 \equiv 0 \mod 4$ if $N$ is even. If $y \in L$ is primitive and $y \neq x$ then there exist arbitrarily large even $N$ such that $x + N \cdot y$ is again primitive. Since the set of all $[y]$ with $y \in L$ primitive is dense in $\mathbb{P}(L_{\mathbb{R}})$, this proves the assertion.

Now let $P \in Q_\Gamma$ be spanned by orthogonal vectors $y_1, y_2 \in \Gamma_{\mathbb{R}}$. Then by what was explained before we can find $x_1 \in \Gamma$ primitive with $x_1^2 \equiv 0 \mod 4$ such that $[x_1]$ is arbitrarily close to $[y_1] \in \mathbb{P}(\Gamma_{\mathbb{R}})$. Furthermore, choose $x_2 \in x_1^\perp \subset \Gamma$ primitive and arbitrarily close to $y_2 \in y_1^\perp$ with $x_2^2 \equiv 0 \mod 4$ and set $P' := (\mathbb{Z}x_1 \oplus \mathbb{Z}x_2)_{\mathbb{R}}$. Such an element $x_2$ can be found, as $x_1^\perp \subset \Gamma$ contains a copy of the hyperbolic plane $U$ and thus an element whose square is divisible by four, e.g. $2v + v^*$, where $(v, v^*)$ is a standard basis of $U$. Then $P'$ is close to $P$ and $(ax_1 + bx_2)^2 = a^2x_1^2 + b^2x_2^2 \equiv 0 \mod 4$. □

We leave it to the reader to modify the above proof to obtain

**Corollary 4** *Let $\Gamma$ be an arbitrary lattice of signature $(3, b-3)$. Then the set of exceptional period points is dense in $Q_\Gamma$.* □

# 4 Period maps

The aim of this section is to compare the various moduli spaces introduced in Section 2 with the period domains of Section 3 via period maps $\mathcal{P}^{\mathrm{cpl}}$, $\mathcal{P}$, $\mathcal{P}^{\mathrm{met}}$, $\mathcal{P}^{(2,2)}$, and $\mathcal{P}^{(4,4)}$.

## 4.1 Definition of the period maps

The period maps we are about to define will fit into the following two commutative diagrams:

$$
\begin{array}{lccc}
\mathcal{P}^{\mathrm{cpl}}: & \mathcal{T}_\Gamma^{\mathrm{cpl}} & \longrightarrow & \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_\mathbb{R}) \cong Q_\Gamma \\
 & \uparrow & & \uparrow \\
\mathcal{P}: & \mathcal{T}_\Gamma & \longrightarrow & \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R}) \\
 & \downarrow{\scriptstyle s^2} & & \downarrow{\scriptstyle s^2} \\
\mathcal{P}^{\mathrm{met}}: & \mathcal{T}_\Gamma^{\mathrm{met}} & \longrightarrow & \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_\mathbb{R}) \times \mathbb{R}_{>0}
\end{array}
$$

and

$$
\begin{array}{lccccc}
\mathcal{P}^{(2,2)}: & \mathcal{T}_\Gamma^{(2,2)} & \longrightarrow & \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R}) \times \Gamma_\mathbb{R} & \hookrightarrow & \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_\mathbb{R} \oplus U_\mathbb{R}) \\
 & \downarrow{\scriptstyle s^2} & & \downarrow{\scriptstyle s^2} & \swarrow{\scriptstyle s^2 \times s^2} & \\
\mathcal{P}^{(4,4)}: & \mathcal{T}_\Gamma^{(4,4)} & \longrightarrow & \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_\mathbb{R} \oplus U_\mathbb{R}) & &
\end{array}
$$

The latter should be compatible with the two diagrams

$$
\begin{array}{ccc}
\mathcal{T}_\Gamma^{(2,2)} & \twoheadrightarrow & \mathcal{T}_\Gamma \\
\downarrow & & \downarrow \\
\mathcal{T}_\Gamma^{(4,4)} & \twoheadrightarrow & \mathcal{T}_\Gamma^{\mathrm{met}}
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R}) \times \Gamma_\mathbb{R} & \twoheadrightarrow & \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R}) \\
\downarrow & & \downarrow \\
\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_\mathbb{R} \oplus U_\mathbb{R}) & \twoheadrightarrow & \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_\mathbb{R})
\end{array}
$$

and the period maps $\mathcal{P}$ and $\mathcal{P}^{\mathrm{met}}$.

The definition of the maps $\mathcal{P}$, $\mathcal{P}^{\mathrm{met}}$, $\mathcal{P}^{\mathrm{cpl}}$, $\mathcal{P}^{(2,2)}$, and $\mathcal{P}^{(4,4)}$ is straightforward. Let $(X, \alpha, \varphi) = (M, g, I, \varphi) \in \mathcal{T}_\Gamma$ and $B \in H^2(X, \mathbb{R}) = H^2(M, \mathbb{R})$ a B-field. By $\sigma$ we denote a generator of $H^{2,0}(X)$.

Then we set:

$$
\begin{aligned}
\mathcal{P}^{\mathrm{cpl}}(X, \varphi) &= \varphi(\sigma) \in Q_\Gamma \subset \mathbb{P}(\Gamma_\mathbb{C}) \\
&= \varphi\langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma)\rangle \in \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_\mathbb{R}) \\
\mathcal{P}(X, \alpha, \varphi) &= \Big(\mathcal{P}^{\mathrm{cpl}}(X, \varphi), \varphi(\alpha)\Big) \in \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R}) \\
\mathcal{P}^{\mathrm{met}}(M, g, \varphi) &= \big(\varphi(H_+^2(M, g)), q(M, g)\big) \in \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_\mathbb{R}) \times \mathbb{R}_{>0} \\
\mathcal{P}^{(2,2)}(M, g, I, B, \varphi) &= (\mathcal{P}(M, g, I, \varphi), \varphi(B)) \in \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R}) \times \Gamma_\mathbb{R} \\
\mathcal{P}^{(4,4)}(M, g, B, \varphi) &= \big(\mathcal{P}^{\mathrm{met}}(M, g, \varphi), \varphi(B)\big) \in \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_\mathbb{R}) \times \mathbb{R}_{>0} \times \Gamma_\mathbb{R} \cong \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_\mathbb{R} \oplus U_\mathbb{R})
\end{aligned}
$$

We leave it to the reader to verify that all period maps are $\mathrm{O}(\Gamma)$-equivariant and that one indeed obtains the above commutative digrams.

## 4.2 Geometry and period maps

Without going too much into the details we collect in the following some important results about period maps. In particular, we will translate geometric results, like the Global Torelli Theorem into global properties of the period maps.

**Local Torelli.** *The map* $\mathcal{P}^{\rm cpl} : \mathcal{T}_\Gamma^{\rm cpl} \to Q_\Gamma$ *is holomorphic and locally (in* $\mathcal{T}_\Gamma^{\rm cpl}$*) an isomorphism* (cf. [6]).

Recall that $\mathcal{T}_\Gamma^{\rm cpl}$ has a natural complex structure, but that the underlying topological space is not Hausdorff. On the other hand, $Q_\Gamma$ is an open subset of a non-singular quadric in $\mathbb{P}(\Gamma_\mathbb{C})$ and, therefore, a nice complex manifold.

Of course, the Local Torelli Theorem in the above version immediately carries over to the other period maps. Thus, $\mathcal{P}$, $\mathcal{P}^{\rm met}$, $\mathcal{P}^{(2,2)}$, and $\mathcal{P}^{(4,4)}$ are all locally injective. Since the Teichmüller spaces $\mathcal{T}_\Gamma$, $\mathcal{T}_\Gamma^{\rm met}$ $\mathcal{T}_\Gamma^{(2,2)}$, and $\mathcal{T}^{(4,4)}$ are all Hausdorff, this shows that except $\mathcal{P}^{\rm cpl}$ all period maps define covering maps on their open images.

**Twistor lines.** *Under the period map* $\mathcal{P}^{\rm cpl}$ *the twistor line* $\mathbb{P}^1 = cm^{-1}(M,g,\varphi) \subset \mathcal{T}_\Gamma^{\rm cpl}$ *(cf. Proposition 2.9) is identified with a quadric in some linear subspace* $\mathbb{P}^2 \subset \mathbb{P}(\Gamma_\mathbb{C})$.

Indeed, the $\mathbb{P}^2$ is given as $\mathbb{P}(\varphi(H^2_+(M,g)_\mathbb{C})) \subset \mathbb{P}(\Gamma_\mathbb{C})$.

**Surjectivity of the period map.** *The map* $\mathcal{P}^{\rm cpl} : \mathcal{T}_\Gamma^{\rm cpl} \to Q_\Gamma$ *maps every connected component of* $\mathcal{T}_\Gamma^{\rm cpl}$ *onto* $Q_\Gamma$ (cf. [22]).

Analogous statements for the other period maps do not hold. In these cases the assertion has to be modified. To see this let us look at the fibres of $\mathcal{T}_\Gamma \to \mathcal{T}_\Gamma^{\rm cpl}$ over $(X,\varphi)$. By definition of $\mathcal{T}_\Gamma$ this is the Kähler cone $\mathcal{K}_X$ which, via the period map $\mathcal{P}$, is identified with an open subcone of the positive cone $\mathcal{C}_{\mathcal{P}^{\rm cpl}(X,\varphi)}$ which is one connected component of the fibre of $\mathrm{Gr}^{\rm po}_{2,1}(\Gamma_\mathbb{R}) \to Q_\Gamma$ over $\mathcal{P}^{\rm cpl}(X,\varphi)$. For a very general marked IHS $(X,\varphi) \in \mathcal{T}_\Gamma^{\rm cpl}$ the Kähler cone $\mathcal{K}_X$ is maximal, i.e. $\mathcal{K}_X = \mathcal{C}_X$. Thus, for those points $\mathcal{P}$ maps the fibre of $\mathcal{T}_\Gamma \to \mathcal{T}_\Gamma^{\rm cpl}$ bijectively onto one of the connected components $\mathcal{C}_P$ or $-\mathcal{C}_P$ of the fibre of $\mathrm{Gr}^{\rm po}_{2,1}(\Gamma_\mathbb{R}) \to Q_\Gamma$ over $P = \mathcal{P}^{\rm cpl}(X,\varphi)$. For special marked IHS $(X,\varphi)$, which usually (e.g. for K3 surfaces) nevertheless form a dense subset of $\mathcal{T}_\Gamma^{\rm cpl}$, the Kähler cone is strictly smaller.

**Density of the image.** *The image of every connected component of* $\mathcal{T}_\Gamma$ *under the period map* $\mathcal{P}$ *is dense in the connected component of the period domain* $\mathrm{Gr}^{\rm po}_{2,1}(\Gamma_\mathbb{R})$ *containing it. Analogous statements hold true for* $\mathcal{P}^{\rm met}$, $\mathcal{P}^{(2,2)}$, *and* $\mathcal{P}^{(4,4)}$.

Let us say a few words about how the density is proved and how the boundary $\mathrm{Gr}^{\rm po}_{2,1}(\Gamma_\mathbb{R}) \setminus \mathcal{P}(\mathcal{T}_\Gamma)$ can be interpreted.

Since $\mathcal{P}^{\rm cpl}$ is surjective, we may consider $(X,\varphi) \in \mathcal{T}^{\rm cpl}$ and study the fibre of $\mathrm{Gr}^{\rm po}_{2,1}(\Gamma_\mathbb{R}) \to \mathrm{Gr}^{\rm po}_2(\Gamma_\mathbb{R}) \cong Q_\Gamma$ over $\mathcal{P}^{\rm cpl}(X,\varphi)$, which is $\pm\varphi(\mathcal{C}_X)$. The $\pm$-sign distinguishes the two connected components of $\mathrm{Gr}^{\rm po}_{2,1}(\Gamma_\mathbb{R})$. The image of the fibre $\mathcal{T}_\Gamma \to \mathcal{T}_\Gamma^{\rm cpl}$ over $(X,\varphi)$ is the open subcone $\varphi(\mathcal{K}_X) \subset \varphi(\mathcal{C}_X)$. We will discuss its boundary and its complement: If $\alpha \in \mathcal{C}_X$ is general, then there exists $(X', \varphi') \in \mathcal{T}_\Gamma$ which cannot be separated from $(X, \varphi)$ such that $\mathcal{P}(X, \varphi) = \mathcal{P}(X', \varphi')$ and $\varphi(\alpha) \in \varphi'(\mathcal{K}_{X'})$ (see [19]). (Moreover, $X$ and $X'$ are birational.) Thus, the disjoint union $\pm \bigcup \varphi(\mathcal{K}_X)$ over all $(X, \varphi)$ in the same connected component and with the same period $\mathcal{P}(X, \varphi) \in Q_\Gamma$ is dense in $\varphi(\mathcal{C}_X)$.

For a point $\alpha \in \partial\varphi(\mathcal{K}_X)$ there always exists a rational curve $C \subset X$ with $\int_C \alpha = 0$ (see [10], i.e. under the degenerate Kähler structure $\alpha$ the volume of the rational curve $C$ shrinks to zero. Thus, points in the boundary of $\mathcal{P}(\mathcal{K}_X)$ should be thought of as singular IHS/HK which are obtained by contracting certain rational curves. Unfortunately, neither are we able to make this statement more precise nor do we know that any point $\alpha \in \varphi(\mathcal{C}_X)$ is actually contained in the closure of some $\varphi'(\mathcal{K}_{X'})$, where $(X', \varphi')$ is as above. However, for K3 surfaces the situation is much better understood (cf. [24]).

**Projective IHS.** *The set of projective marked IHS forms a countable dense union of hyperplane section of $Q_\Gamma$. If $\mathcal{M}_\Gamma^{\mathrm{proj}} \subset \mathcal{M}_\Gamma$ denotes the set of all Kähler IHS for which the underlying IHS is projective, then $\mathcal{M}_\Gamma^{\mathrm{proj}} \to \mathcal{M}_\Gamma^{\mathrm{met}}$ is surjective.*

In fact, due to a general projectivity criterion for surfaces and an analogous result for IHS (cf. [19]) one knows that an IHS $X$ is projective if and only if there exists an integral $(1,1)$-class $\alpha$ with $q(\alpha) > 0$. Thus, $(X, \varphi) \in \mathcal{T}_\Gamma^{\mathrm{cpl}}$ is projective if and only if $\mathcal{P}^{\mathrm{cpl}}(X, \varphi)$ is contained in a hyperplane orthogonal to some $\alpha \in \Gamma$ with $\alpha^2 > 0$. As we have seen before, the set of such periods is dense in the period domain $Q_\Gamma$. Since the fibre of $\mathcal{T}_\Gamma \to \mathcal{T}_\Gamma^{\mathrm{met}}$ is identified with a quadric curve $\mathbb{P}^1 \subset \mathbb{P}(\Gamma_\mathbb{C})$ under the projection $\mathcal{T}_\Gamma \to \mathcal{T}_\Gamma^{\mathrm{cpl}}$ and as such is intersected non-trivially by every such hyperplane, the fibre contains at least one Kähler $(X, \alpha, \varphi)$ with $X$ projective. In other words, for any hyperkähler metric $g$ on a manifold $M$ at least one of the complex structures $\lambda = aI + bJ + cK$ defines a projective IHS. In fact, the set of projective IHS is also dense among the $(M, \lambda)$.

**Finiteness.** *The induced period maps*

$$\overline{\mathcal{P}}^{\mathrm{cpl}}: \quad \mathcal{M}_\Gamma^{\mathrm{cpl}} \to \mathrm{O}(\Gamma) \setminus Q_\Gamma \cong \mathrm{O}(\Gamma) \setminus \mathrm{O}(3, b-3)/\mathrm{SO}(2) \times \mathrm{O}(1, b-3)$$

$$\overline{\mathcal{P}}: \quad \mathcal{M}_\Gamma \to \mathrm{O}(\Gamma) \setminus \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_\mathbb{R})$$

$$\overline{\mathcal{P}}^{\mathrm{met}}: \quad \mathcal{M}_\Gamma^{\mathrm{met}} \to \mathrm{O}(\Gamma) \setminus \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_\mathbb{R}) \times \mathbb{R}_{>0} \cong \mathrm{O}(\Gamma) \setminus \mathrm{O}(3, b-3)/\mathrm{SO}(3) \times \mathrm{O}(b-3) \times \mathbb{R}_{>0}$$

*are finite trivial covers of their images, i.e. every moduli space has only finitely many connected components and each connected component is mapped bijectively onto its image.*

Note that e.g. $\mathcal{T}_\Gamma^{\mathrm{cpl}}$ might *a priori* have infinitely many components. That this is no longer possible for the quotient $\mathcal{M}_\Gamma^{\mathrm{cpl}} = \mathrm{O}(\Gamma) \setminus \mathcal{T}_\Gamma^{\mathrm{cpl}}$ is a consequence of the finiteness result in [23, Thm. 4.3] which says that there are only finitely many different deformation types of IHS with the same BB–form $q_X$. Since $Q_\Gamma$ is simply connected and $\mathcal{P}^{\rm cpl}$ is surjective, the cover $\mathcal{P}^{\rm cpl}$ has to be trivial. In fact, in order to make this precise one first should construct the 'Hausdorff reduction' of $\mathcal{T}_\Gamma^{\rm cpl}$ by identifying all points that cannot be separated from each other. This Hausdorff space then is an honest étale cover of the simply connected space $Q_\Gamma$ and, therefore, consists of several copies of $Q_\Gamma$.

We leave it to the reader to deduce similar statements for the maps $\mathcal{P}^{(2,2)}$ and $\mathcal{P}^{(4,4)}$.

**Remark 4.1** This is essentially all that is known in the general case. For **K3 surfaces** however the above results can be strengthened considerably as follows. The Global Torelli for K3 surfaces shows that $\mathcal{T}_\Gamma^{\rm cpl}$ consists of two connected components which are identified with each other by $(X,\varphi) \mapsto (X,-\varphi)$ and which are not distinguished by $\mathcal{P}^{\rm cpl}$. The two components are separated by the map $\mathcal{P} : \mathcal{T}_\Gamma \to {\rm Gr}^{\rm po}_{2,1}(\Gamma_\mathbb{R})$, which is injective in the case of K3 surfaces. Analogously, $\mathcal{P}^{\rm met}$, $\mathcal{P}^{(2,2)}$, and $\mathcal{P}^{(4,4)}$ are all injective.

The density results of Sect. 3.7 together with the description of the periods of our list of examples of K3 surfaces in Section 1 and the above information about the period maps (i.e. the Global Torelli Theorem) yield:

**Proposition 4.2** *The following three sets are dense in the moduli space of marked K3 surfaces:* i) $\{(X,\varphi) \mid X \subset \mathbb{P}^3$ is a quartic hypersurface$\}$
ii) $\{(X,\varphi) \mid X$ is an elliptic K3 surface$\}$
iii) $\{(X,\varphi) \mid X$ is a(n exceptional) Kummer surface$\}$. □

## 4.3 The diffeomorphism group of a K3 surface

**Proposition 4.3** *Let $X$ be a K3 surface. The image of the natural map* $\rho : {\rm Diff}(X) \to {\rm O}(H^2(X,\mathbb{Z}),\cup)$ *is the subgroup* ${\rm O}^+(H^2(X,\mathbb{Z}),\cup)$.

Recall (cf. Section 3.6) that ${\rm O}^+$ is the group of all $A \in {\rm O}$ that preserve the orientation of positive three-space (but not of a negative 19-space). The proposition is due to Borcea [11], who showed the inclusion ${\rm O}^+ \subset {\rm Im}(\rho)$, and Donaldson [16], who showed equality. We only reproduce Borcea's argument here.

*Proof.* First note the following. If $(X_t,\varphi_t)$ is a connected path in $\mathcal{T}_\Gamma^{\rm cpl}$, then there exists a sequence of diffeomorphisms $f_t : X_0 \cong X_t$ such that $\varphi_0 \circ f^* = \varphi_t$.

Let now $\varphi$ be any marking of $X$ and consider $(X,\varphi) \in \mathcal{T}_\Gamma^{\rm cpl}$. By $\mathcal{T}_0$ we denote the connected component of $\mathcal{T}_\Gamma^{\rm cpl}$ that contains this point. Pick $A \in {\rm O}^+(H^2(X,\mathbb{Z}),\cup)$. Then $A$ acts on $\mathcal{T}_\Gamma^{\rm cpl}$ and $Q_\Gamma$ by $\varphi A \varphi^{-1}$ and the period map $\mathcal{P}^{\rm cpl} : \mathcal{T}_\Gamma^{\rm cpl} \to Q_\Gamma$ is equivariant. Since the restriction of the period map $\mathcal{P}^{\rm cpl}$ yields a surjective map $\mathcal{T}_0 \to Q_\Gamma$, there exists a marked K3 surface $(X',\varphi')$ with $\mathcal{P}^{\rm cpl}(X',\varphi') = A\mathcal{P}^{\rm cpl}(X,\varphi) = \mathcal{P}(X,A\varphi)$. If $X$ is a general K3 surface such that $\mathcal{K}_X \cong \mathcal{C}_X$, then $\pm\varphi'^{-1} \circ (\varphi A) : H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ is an isomorphism of periods mapping $\mathcal{K}_X$ to $\mathcal{K}_{X'}$. By the Global Torelli Theorem there exists a (unique) isomorphism $g : X' \cong X$

such that $g^* = \pm\varphi'^{-1} \circ (\varphi A)$. By the remark above we also find a diffeomorphism $f : X \cong X'$ such that $\varphi \circ f^* = \varphi'$. Hence, $\varphi \circ f^* g^* = \pm(\varphi A)$ and thus $(g \circ f)^* = \pm A$ is realized by a diffeomorphism of $X$. In fact, the sign must be "+", as $g^*$, $f^*$, and $A$ preserve the orientation of a positive three-space.

It remains to show that $-\mathrm{id}$ is not contained in the image and this was done by Donaldson using zero-dimensional moduli spaces of stable bundles on a double cover of the projective plane. □

**Remark 4.4** In the proof above we used the assumption that $n = 1$ twice: When we applied the Global Torelli Theorem and, of course, when using Donaldson invariants. The surjectivity which is also crucial holds true also for $n > 1$. Somehow, the use of the Global Torelli Theorem seems a little strong, as we have no need to know that $g^*$ is induced by a biholomorphic map, a diffeomorphism would be enough.

In [28] Namikawa constructs an example of two four-dimensional IHS $X$ and $X'$ together with an isomorphism of their periods which preserves the Kähler cone, but such that $X$ and $X'$ are not even birational. To be more precise, he let $X = \mathrm{K}_2(T)$ and $X' = \mathrm{K}_2(T^*)$ be generalized Kummer varieties associated to a complex torus $T$ and its dual $T^*$. As the moduli space of complex tori is connected, one can endow $X$ and $X'$ with markings $\varphi$ respectively $\varphi'$ such that $(X, \varphi)$ and $(X', \varphi')$ are contained in the same connected component $\mathcal{T}_0$ of $\mathcal{T}_\Gamma^{\mathrm{cpl}}$. His example shows that $\mathrm{O}^+(\Gamma)$ does not preserve $\mathcal{T}_0$, i.e. there exists $A \in \mathrm{O}^+$ such that $(X', A\varphi') \notin \mathcal{T}_0$ (with $\mathcal{P}(X', A\varphi') = \mathcal{P}(X, \varphi)$). Indeed, after identifying non-separated points in $\mathcal{T}_\Gamma^{\mathrm{cpl}}$ the period map $\mathcal{P}^{\mathrm{cpl}} : \mathcal{T}_\Gamma^{\mathrm{cpl}} \to Q_\Gamma$ is a covering and thus, since $Q_\Gamma$ is simply connected, every connected component $\mathcal{T}_0$ of $\mathcal{T}_\Gamma^{\mathrm{cpl}}$ is generically mapped one-to-one onto $Q_\Gamma$.

# 5 Discrete group actions

All spaces considered in Section 3 are quotients either of $\mathrm{O}(\Gamma_\mathbb{R})$ or $\mathrm{O}(\Gamma_\mathbb{R} \oplus U_\mathbb{R})$. So from a mathematical point of view it seems very natural to study the action of the discrete groups $\mathrm{O}(\Gamma)$ respectively $\mathrm{O}(\Gamma \oplus U)$ on these spaces. In fact, from a geometric point of view one has to divide out by the smaller group in order to obtain moduli spaces of unmarked (complex) HK or (kähler) IHS with or without B-fields.

But in [4] it is argued that dividing out $\mathcal{T}_\Gamma^{(4,4)}$ or $\mathcal{T}_\Gamma^{(2,2)}$ by $\mathrm{O}(\Gamma \oplus U)$ yields the true moduli space of CFTs on K3 surfaces. In order to recover the full symmetry of the situation they proceed as follows:

**i) Maximal discrete subgroups.** Find a discrete group $G$ that acts on a certain moduli space of relevant theories and show that it is maximal in the sense that any bigger group would no longer act properly discontinuously. (Recall that the quotient of a properly discontinuous group action is Hausdorff.)

**ii) Geometric symmetries.** Describe the part of $G$ (the geometric symmetries) that identifies geometrically identical theories and the part that is responsible for trivial identifications (e.g. integral shifts of the B-field).

**iii) Mirror symmetries.** Show that $G$ is generated by the symmetries in **ii)** and a few others that are responsible for mirror symmetry phenomena.

## 5.1 Maximal discrete subgroups

We first recall the following facts:

• Let $G$ be a topological group which is Hausdorff and locally compact. If $K \subset G$ is a compact subgroup then any other subgroup $H$ acts properly discontinuously from the left on the quotient space $G/K$ if and only if $H \subset G$ is a discrete subgroup. (For the elementary proof see e.g. [34, Lemma 3.1.1].)

• Let $L$ be a non-trivial definite even unimodular lattice and let $q \geq 3$. Then $\mathrm{O}(L \oplus U^{\oplus q}) \subset \mathrm{O}(L_{\mathbb{R}} \oplus U_{\mathbb{R}}^{\oplus q})$ is a maximal discrete subgroup (cf. [2]).

The second result in particular applies to the K3 surface lattice $\Gamma = 2(-E_8) \oplus 3U$ and yields that $\mathrm{O}(\Gamma) \subset \mathrm{O}(\Gamma_{\mathbb{R}})$ and $\mathrm{O}(\Gamma \oplus U) \subset \mathrm{O}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ are both maximal discrete subgroups.

The group $\mathrm{O}(\Gamma)$ acts on $\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})$ and $\mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}})$. As we have seen

$$\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \cong \mathrm{O}(3,19)/\mathrm{SO}(2) \times \mathrm{O}(1,19) \quad \text{and} \quad \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \cong \mathrm{O}(3,19)/\mathrm{SO}(3) \times \mathrm{O}(19).$$

In the second case we are in the above situation, i.e. the quotient is taken with respect to the compact subgroup $\mathrm{SO}(3) \times \mathrm{O}(19)$. Hence, $\mathrm{O}(\Gamma)$ acts properly discontinuously on $\mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}})$ and there is no bigger subgroup of $\mathrm{O}(\Gamma_{\mathbb{R}})$ than $\mathrm{O}(\Gamma)$ with the same property. However, the action of $\mathrm{O}(\Gamma)$ on $\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})$ is badly behaved, as the subgroup $\mathrm{SO}(2) \times \mathrm{O}(1,19)$ is not compact. In fact, in the proof of Proposition 3.6 we have already seen that the action of $\mathrm{O}(\Gamma)$ is properly discontinuous.

We are more interested in the action of $\mathrm{O}(\Gamma \oplus U)$ on $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \cong \mathrm{O}(4,20)/(\mathrm{SO}(4) \times \mathrm{O}(20))$. Again $\mathrm{O}(\Gamma \oplus U)$ is maximal discrete and thus there is no bigger properly discontinuous subgroup action on $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$, as $\mathrm{SO}(4) \times \mathrm{O}(20)$ is compact. Analogously, one finds that $\mathrm{O}(\Gamma \oplus U)$ is a maximal discrete subgroup of $\mathrm{O}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ acting properly discontinuously on $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$.

Presumably, all these arguments also apply to any HK manifold, but details need to be checked. (Recall that $(H^2(X,\mathbb{Z}), q_X))$ is not necessarily unimodular in higher dimensions.)

## 5.2 Geometric symmetries

We will try to identify "geometric" symmetries and integral shifts of the B-field inside $\mathrm{O}(\Gamma \oplus U)$. To this end we use the identification

$$\phi : \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \times \Gamma_{\mathbb{R}} \cong \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$$

described in Sect. 3.4. The natural inclusion $\mathrm{O}(\Gamma) \subset \mathrm{O}(\Gamma \oplus U)$ is compatible with this is isomorphism, i.e. if $\Pi = \phi(F, \alpha, B)$ and $\varphi \in \mathrm{O}(\Gamma) \subset \mathrm{O}(\Gamma \oplus U)$, then $\varphi(\Pi) = \phi(\varphi(F), \alpha, \varphi(B))$. This is a straightforward calculation which we leave to the reader. Clearly, $\mathrm{O}(\Gamma)$ acts naturally on all spaces $\mathcal{T}$, $\mathcal{T}^{\mathrm{met}}$, $\mathcal{T}^{\mathrm{cpl}}$, $\mathcal{T}^{(2,2)}$, and $\mathcal{T}^{(4,4)}$ and the period maps are equivariant. Thus, $\mathrm{O}(\Gamma)$ is the subgroup that identifies geometrically equivalent theories.

Next let $B_0 \in \Gamma$ and let $\varphi_{B_0} \in \mathrm{O}(\Gamma \oplus U)$ be the automorphism $w \mapsto w$, $w^* \mapsto B_0 + w^* - (B_0^2/2)w$, and $x \in \Gamma \mapsto x - \langle B_0, x\rangle w$. One easily verifies that this really defines an isometry. We claim that if $\Pi = \phi(F, \alpha, B)$, then $\varphi_{B_0}(\Pi) = \phi(F, \alpha, B + B_0)$.

In order to do this let us more generally consider an element $\varphi \in \mathrm{O}(\Gamma \oplus U)$ such that $\varphi(w) = w$. For $\Pi \in \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$, let $\tilde{\Pi} := \varphi(\Pi)$. Then $\tilde{F}' = \tilde{\Pi} \cap w^{\perp} = \varphi(\Pi) \cap \varphi(w)^{\perp} = \varphi(\Pi \cap w^{\perp}) = \varphi(F')$. Moreover, one has the two orthogonal splittings $\tilde{\Pi} = \tilde{F}' \oplus \tilde{B}'\mathbb{R}$ and $\tilde{\Pi} = \varphi(F') \oplus \varphi(B')\mathbb{R}$, where $\tilde{B}'$ is determined by $\langle \tilde{B}', w\rangle = 1$. Since $\langle \varphi(B'), w\rangle = \langle \varphi(B'), \varphi(w)\rangle = \langle B', w\rangle = 1$, one concludes $\tilde{B}' = \varphi(B')$. In particular, $\tilde{B}'^2 = B'^2$. The B-field $B$ is given by $B' = \alpha w + w^* + B$. Hence, $\tilde{B}' = \alpha w + \varphi(w^*) + \varphi(B)$ and thus the B-field determined by $\tilde{B}'$ is nothing but $\varphi(B)$.

All this applied to $\varphi = \varphi_0$ one finds that under the isomorphism $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) = \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \times \Gamma_{\mathbb{R}}$ the integral B-shift by $B_0$ that maps $(F, \alpha, B)$ to $(F, \alpha, B + B_0)$ corresponds to $\varphi_{B_0}$.

We leave it to the reader to verify that also the $\mathrm{O}(\Gamma \oplus U)$-action on $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ is well-behaved in the sense that $\mathrm{O}(\Gamma) \subset \mathrm{O}(\Gamma \oplus U)$ and the maps $\varphi_{B_0}$ for $B_0 \in \Gamma$ act on the subspace $\gamma(\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}) \subset \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ in the natural way.

## 5.3 Mirror symmetries

The next result (due to C. T. C. Wall, [33]) explains which additional group elements have to be added in order to pass from $\mathrm{O}(\Gamma)$ to $\mathrm{O}(\Gamma \oplus U)$.

**Proposition 5.1** *Let $\Gamma$ be a unimodular lattice of index $(m, n)$ with $m, n \geq 2$. Then $\mathrm{O}(\Gamma \oplus U)$ is generated by the following three subgroups:*

$$\mathrm{O}(\Gamma), \quad \mathrm{O}(U), \text{ and } \{\varphi_{B_0} \mid B_0 \in \Gamma\}.$$

Thus, the result applies to the K3 surface lattice $2(-E_8) \oplus 3U$, but presumably something similar can be said for the case of the lattice $2(-E_8) \oplus 3U \oplus n\mathbb{Z}$, which is realized by the Hilbert scheme of a K3 surface..

In [4] passing from $\mathrm{O}(\Gamma)$ to $\mathrm{O}(\Gamma \oplus U)$ is motivated on the base of physical insight. As usual in mathematical papers on mirror symmetry we will take this for granted and rather study the effects of these additional symmetries in geometrical terms. Thus, the rest of this paragraph is devoted to the study a few special elements of $G$ that are not contained in the subgroup generated by $\mathrm{O}(\Gamma)$ and $\{\varphi_{B_0} \mid B_0 \in \Gamma\}$. In particular, we will be interested in the their induced action on $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$.

So far we have argued that $O(\Gamma \oplus U)$ is a maximal discrete subgroup of $O(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ that acts on the two period spaces that interest us: $\mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ and $\mathrm{Gr}^{\mathrm{po}}_{4}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$. However, there seems to be a bigger group that naturally acts on the space $\mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ (which thus cannot be realized as a subgroup of $O(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$).

**Definition 5.2** *The group* $\tilde{O}(\Gamma \oplus U)$ *is the group acting on* $\mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ *which is generated by* $O(\Gamma \oplus U)$ *and the involution* $\iota : (H_1, H_2) \mapsto (H_2, H_1)$.

Note that one could actually go further and consider the maps $(H_1, H_2) \mapsto (\bar{H}_1, \bar{H}_2)$ or $(H_1, H_2) \mapsto (\bar{H}_1, H_2)$ , where $\bar{H}$ is the space $H$ with the opposite orientation. However, for the versions of mirror symmetry that will be discussed in these lectures $\iota$ will do.

## 5.4 $-\mathrm{id}_U$

Consider the automorphism $\psi_0 \in O(\Gamma \oplus U)$ that acts trivially on $\Gamma$ and as $-\mathrm{id}$ on $U$.

**Lemma 5.3** *The automorphism* $\psi_0$ *preserves the subspace* $\mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ *and acts on it by*

$$((P, \omega), B) \mapsto ((P, -\omega), -B).$$

*Proof.* If $(H_1, H_2) = \gamma((P, \omega), B)$, then by definition of $\psi_0$:

$$\psi_0(H_1) = \{x + \langle x, B\rangle w \mid x \in P\} = \{x - \langle x, (-B)\rangle w \mid x \in P\}$$

and

$$\begin{aligned} \psi_0(H_2) &= (\frac{1}{2}(\alpha - B^2)(-w) - w^* + B)\mathbb{R} \oplus (\omega + \langle \omega, B\rangle w)\mathbb{R} \\ &= -(\frac{1}{2}(\alpha - (-B)^2)w + w^* - B)\mathbb{R} \oplus (\omega - \langle \omega, (-B)\rangle w)\mathbb{R} \end{aligned}$$

Thus, the sign of $\omega$ has to be changed in order to get the correct orientation $\psi_0(H_2)$. □

## 5.5 $\mathbf{w} \leftrightarrow \mathbf{w}^*$

Consider the automorphism $\psi_1 \in O(\Gamma \oplus U)$ that acts trivially on $\Gamma$ and as $\psi_1(w) = w^*$, $\psi_1(w^*) = w$ on $U$.

**Lemma 5.4** *The automorphism* $\psi_1$ *preserves the subspace* $\{((P, \omega), B) \mid B \in (P, \omega)^{\perp}, \alpha \neq B^2\}$ *of* $\mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ *and acts on it by*

$$((P, \omega), B) \mapsto \frac{2}{\alpha - B^2}((P, \omega), B).$$

*Proof.* Indeed, by definition of $\psi_1$ one has $\psi_1(H_1) = H_1$ and

$$\begin{aligned}\psi_1(H_2) &= \left(\frac{1}{2}(\alpha - B^2)w^* + w + B\right)\mathbb{R} \oplus (\omega - \langle\omega, B\rangle w^*)\mathbb{R} \\ &= \left(w^* + \frac{2}{\alpha - B^2}w + \frac{2}{\alpha - B^2}B\right)\mathbb{R} \oplus \left(\frac{2}{\alpha - B^2}\omega\right)\mathbb{R}\end{aligned}$$

Then check that for $\tilde{\omega} := \frac{2}{\alpha - B^2}\omega$ and $\tilde{B} := \frac{2}{\alpha - B^2}B$ one indeed has $\frac{2}{\alpha - B^2} = \frac{1}{2}(\tilde{\omega}^2 - \tilde{B}^2)$. □

It is interesting to observe that on the yet smaller subspace $\{((P, \omega), 0)\}$ the automorphism $\psi_1$ acts by $(P, \omega) \to \frac{2}{\sqrt{\omega^2}}(P, \omega)$. In the geometric context this will be interpreted as invertion of the volume.

## 5.6 $\mathbf{U} \leftrightarrow \mathbf{U}'$

If the lattice can be written as $\Gamma = \Gamma' \oplus U'$, where $U'$ is a copy of the hyperbolic plane $U$, then by Wall's result Proposition 5.1 the group $G$ is generated by $\mathrm{O}(\Gamma)$, $\{\varphi_{B_0} \mid B_0 \in \Gamma\}$, the involution $\iota$, and $\xi \in \mathrm{O}(\Gamma \oplus U)$ which is the identity on $\Gamma'$ and switches $U$ and $U'$. Here we use an isomorphism $U \cong U'$ which we fix once and for all. We consider $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ as a subspace of $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ via the injection $\gamma$.

Neither $\iota$ nor $\xi$ leave the subspace $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ invariant. Indeed, if $((P, \omega), B)$ then $H_1 \subset \Gamma_{\mathbb{R}} \oplus \mathbb{R}w$ and $H_2 \not\subset \Gamma_{\mathbb{R}} \oplus \mathbb{R}w$ and therefore $(H_2, H_1) = \iota(H_1, H_2)$ cannot be contained in the image of $\gamma$. Similarly, for a general $(H_1, H_2)$ the pair of planes $(\xi(H_1), \xi(H_2))$ will not satisfy $\xi(H_1) \subset \Gamma_{\mathbb{R}} \oplus \mathbb{R}w$.

**Definition 5.5** $\tilde{\xi} := \iota \circ \xi \in \tilde{O}(\Gamma \oplus U)$.

By definition $\tilde{\xi}$ acts naturally on $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ and $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$. The action on the latter coincides with the action of $\xi$. We will show that $\tilde{\xi}$ can be used to identify certain subspaces of $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$, but the whole $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ will again not be invariant. Maybe it is worth emphasizing that $\tilde{\xi}$ is an involution. Indeed, $\iota$ commutes with the action of $\mathrm{O}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ and both transformations $\iota$ and $\xi$ are of order two.

Note that different decompositions of $\Gamma$ yield different $\xi$, which then relate different pairs of subspaces of $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$. The following easy lemma shows that we dispose of such a decomposition whenever we find a hyperbolic plane contained in $\Gamma$.

**Lemma 5.6** *If $U'$ is a hyperbolic plane contained in a lattice $\Gamma$, then $\Gamma = U'^{\perp} \oplus U'$.*

*Proof.* Choose a basis $(v, v^*)$ of $U'$ that corresponds to the basis $(w, w^*)$ of $U$ under the identification $U' \cong U$. Furthermore, let $\Gamma' := U'^{\perp}$ and let $V$ be the subspace of the $\mathbb{Q}$-vector space $\Gamma_{\mathbb{Q}}$ that is orthogonal to $U'_{\mathbb{Q}}$. Thus, $\Gamma_{\mathbb{Q}} = V \oplus U'_{\mathbb{Q}}$. Clearly, $\Gamma' \subset V$ and, conversely, for any $v \in V$ there exists $\lambda \in \mathbb{Q}^*$ with $\lambda v \in V \cap \Gamma \subset \Gamma'$. Hence, $V = \Gamma'_{\mathbb{Q}}$. Let $x \in \Gamma$ and write

$x = y + (\lambda v + \mu v^*)$ with $y \in V$ and $\lambda, \mu \in \mathbb{Q}$. Then $\langle x, v\rangle, \langle x, v^*\rangle \in \mathbb{Z}$ implies $\lambda, \mu \in \mathbb{Z}$ and, therefore, $y = x - (\lambda v + \mu v^*) \in \Gamma \cap V = \Gamma'$. □

For the rest of this section we fix the orthogonal splitting $\Gamma = \Gamma' \oplus U'$ together with an identification $U' = U$. By $\mathrm{pr} : \Gamma_{\mathbb{R}} \to \Gamma'_{\mathbb{R}}$we denote the orthogonal projection.

**Proposition 5.7** *Let* $((P, \omega), B) \in \mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ *such that* $\omega, B \in \Gamma'_{\mathbb{R}} \oplus \mathbb{R}v$. *Then the* $\tilde{\xi}$*-mirror image* $((P^{\vee}, \omega^{\vee}), B^{\vee}) := \iota(\xi((P, \omega), B))$ *is again contained in* $\mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$. *It is explicitly given as*

$$\begin{aligned} \sigma^{\vee} &:= \langle \mathrm{Re}(\sigma), v\rangle^{-1} \left(\mathrm{pr}(B + i\omega) - \frac{1}{2}(B + i\omega)^2 v + v^*\right) \\ B^{\vee} + i\omega^{\vee} &:= \langle \mathrm{Re}(\sigma), v\rangle^{-1} \left(\mathrm{pr}(\sigma) - \langle \sigma, B\rangle v\right) \end{aligned}$$

*Here, we have replaced* $P$ *by the corresponding line* $[\sigma] \in Q_{\Gamma} \subset \mathbb{P}(\Gamma_{\mathbb{C}})$. *Furthermore, we have chosen* $\sigma$ *such that* $\mathrm{Im}(\sigma)$ *is orthogonal to* $v$.

*Proof.* By definition the positive plane $P$ is contained in $\omega^{\perp}$. Since the intersection of $\omega^{\perp}$ with $\Gamma'_{\mathbb{R}} \oplus \mathbb{R}v$ and $\Gamma'_{\mathbb{R}} \oplus \mathbb{R}v^*$ have both only one positive direction, $P$ cannot be contained in either of them. Thus, we may choose $\sigma$ such that $v^{\perp} \cap P = \mathrm{Im}(\sigma)\mathbb{R}$ and $\langle \mathrm{Re}(\sigma), v\rangle \neq 0$. This justifies the above choices. Also note that $\omega^{\vee}$ and $B^{\vee}$ do not change when $\sigma$ is changed by a real scalar. The defining equations for $B^{\vee} + \omega^{\vee}$ and $\sigma^{\vee}$ are spelled out as follows

$$\begin{aligned} \sigma^{\vee} &:= \langle \mathrm{Re}(\sigma), v\rangle^{-1} \left(-\frac{1}{2}(B + i\omega + v^*)^2 v + B + i\omega + v^*\right) \\ \omega^{\vee} &:= \langle \mathrm{Re}(\sigma), v\rangle^{-1} \left(\mathrm{Im}(\sigma) - \langle \mathrm{Im}(\sigma), v^*\rangle v - \langle \mathrm{Im}(\sigma), B\rangle v\right) \\ B^{\vee} &:= \langle \mathrm{Re}(\sigma), v\rangle^{-1} \left(\mathrm{Re}(\sigma) - \langle \mathrm{Re}(\sigma), v\rangle v^* - \langle \mathrm{Re}(\sigma), v^*\rangle v - \langle \mathrm{Re}(\sigma), B\rangle v\right) \end{aligned}$$

Let us now compute $\xi(H_1, H_2)$. We denote $\gamma((\sigma^{\vee}, \omega^{\vee}), B^{\vee})$ by $(H_1^{\vee}, H_2^{\vee})$, where $\sigma^{\vee}, \omega^{\vee}$, and $B^{\vee}$ are as above.

The space $H_1^{\vee}$ is spanned by the real and imaginary part of $\sigma^{\vee} - \langle \sigma^{\vee}, B^{\vee}\rangle w$. A simple calculation yields

$$\langle \sigma^{\vee}, B^{\vee}\rangle = -\langle \mathrm{Re}(\sigma), v\rangle^{-1} \left(\langle B, v^*\rangle + i\langle \omega, v^*\rangle\right).$$

Thus, $H_1^{\vee}$ is spanned by

$$\begin{aligned} & B + v^* + \frac{1}{2}(\omega^2 - B^2 - 2\langle B, v^*\rangle)v + \langle B, v^*\rangle w \\ = \; & \frac{1}{2}(\omega^2 - B^2)v + v^* + (B - \langle B, v^*\rangle v) + \langle B, v^*\rangle w \\ = \; & \xi\left(\frac{1}{2}(\omega^2 - B^2)w + w^* + (B - \langle B, v^*\rangle v) + \langle B, v^*\rangle v\right) \\ = \; & \xi\left(\frac{1}{2}(\omega^2 - B^2)w + w^* + B\right) \end{aligned}$$

and

$$\begin{aligned} & \omega - \langle\omega, B + v^*\rangle v + \langle\omega, v^*\rangle w \\ = & (\omega - \langle\omega, v^*\rangle v) + \langle\omega, v^*\rangle w - \langle\omega, B\rangle v \\ = & \xi\left(\omega - \langle\omega, v^*\rangle v + \langle\omega, v^*\rangle v - \langle\omega, B\rangle w\right) \\ = & \xi\left(\omega - \langle\omega, B\rangle w\right). \end{aligned}$$

Hence, $H_1^{\mathrm{v}} = \xi(H_2)$. Similarly, one proves $H_2^{\mathrm{v}} = \xi(H_1)$. First one computes

$$\begin{aligned} \omega^{\mathrm{v}2} &= \langle\mathrm{Re}(\sigma), v\rangle^{-2}\mathrm{Im}(\sigma)^2, \\ B^{\mathrm{v}2} &= \langle\mathrm{Re}(\sigma), v\rangle^{-2}\left(\mathrm{Re}(\sigma)^2 - 2\langle\mathrm{Re}(\sigma), v\rangle\langle\mathrm{Re}(\sigma), v^*\rangle\right) \\ \langle\omega^{\mathrm{v}}, B^{\mathrm{v}}\rangle &= -\langle\mathrm{Re}(\sigma), v\rangle^{-1}\langle\mathrm{Im}(\sigma), v^*\rangle, \end{aligned}$$

where one uses $\langle\mathrm{Im}(\sigma), v\rangle = 0$. Since $\mathrm{Im}(\sigma)^2 = \mathrm{Re}(\sigma)^2$, this yields

$$\omega^{\mathrm{v}2} - B^{\mathrm{v}2} = 2\langle\mathrm{Re}(\sigma), v\rangle^{-1}\langle\mathrm{Re}(\sigma), v^*\rangle.$$

Hence, $H_2^{\mathrm{v}}$ is spanned by

$$\begin{aligned} & \frac{1}{2}(\omega^{\mathrm{v}2} - B^{\mathrm{v}2})w + w^* + B^{\mathrm{v}} \\ = & \langle\mathrm{Re}(\sigma), v\rangle^{-1}\langle\mathrm{Re}(\sigma), v^*\rangle w + w^* \\ & + \langle\mathrm{Re}(\sigma), v\rangle^{-1}\left(\mathrm{Re}(\sigma) - \langle\mathrm{Re}(\sigma), v\rangle v^*\langle\mathrm{Re}(\sigma), v^*\rangle v - \langle\mathrm{Re}(\sigma), B\rangle v\right) \end{aligned}$$

and

$$\begin{aligned} & \omega^{\mathrm{v}} - \langle\omega^{\mathrm{v}}, B^{\mathrm{v}}\rangle w \\ = & \langle\mathrm{Re}(\sigma), v\rangle^{-1}\left((\mathrm{Im}(\sigma) - \langle\mathrm{Im}(\sigma), v^*\rangle v) - \langle\mathrm{Im}(\sigma), B\rangle v + \langle\mathrm{Im}(\sigma), v^*\rangle w\right) \end{aligned}$$

Thus, $\xi(H_2^{\mathrm{v}})$ is generated by $\mathrm{Re}(\sigma) - \langle\mathrm{Re}(\sigma), B\rangle w$ and $\mathrm{Im}(\sigma) - \langle\mathrm{Im}(\sigma), B\rangle w$. Hence, $\xi(H_2^{\mathrm{v}}) = H_1$. □

**Examples 5.8** The proposition can be used to identify certain subspaces of $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ via the mirror symmetry $\iota \circ \xi$. We will present a few examples, which will be interpreted geometrically later on. As the B-field from a geometric point of view is not well-understood we will be especially interested in those points with vanishing B-field.

**i)** Fix an orthogonal decomposition $\Gamma'_{\mathbb{R}} = V \oplus V^{\mathrm{v}}$, such that both subspaces $V$ and $V^{\mathrm{v}}$ contain a positive line. The automorphism $\iota \circ \xi \in \tilde{O}(\Gamma \oplus U)$ induces a bijection between the two subspaces:

$$\{((P, \omega), B) \mid B, \omega \in V,\ P \subset V^{\mathrm{v}} \oplus U'_{\mathbb{R}}\} \text{ and } \{((P, \omega), B) \mid B, \omega \in V^{\mathrm{v}},\ P \subset V \oplus U'_{\mathbb{R}}\}.$$

Note that in this case the formulae for $(\sigma^\vee, \omega^\vee, B^\vee)$ simplify slightly to: $\sigma^\vee = \langle \mathrm{Re}(\sigma), v\rangle^{-1}(-\frac{1}{2}(B+i\omega)^2 v + v^* + B + i\omega)$, $\omega^\vee = \langle \mathrm{Re}(\sigma), v\rangle^{-1}(\mathrm{Im}(\sigma) - \langle \mathrm{Im}(\sigma), v^*\rangle v)$, and $B^\vee = \langle \mathrm{Re}(\sigma), v\rangle^{-1}(\mathrm{Re}(\sigma) - \langle \mathrm{Re}(\sigma), v\rangle v^* - \langle \mathrm{Re}(\sigma), v^*\rangle v)$. From here it is easy to verify that $\iota \circ \xi$ maps these two subspaces into each other. Note that $B^\vee + i\omega^\vee$ is up to the scalar factor $\langle \mathrm{Re}(\sigma), v\rangle^{-1}$ nothing but the projection of $\sigma \in V_{\mathbb{C}}^\vee \oplus U'_{\mathbb{C}}$ to $V_{\mathbb{C}}^\vee$.

**ii)** It might be interesting to see what happens in the previous example if we set the B-filed zero. Under the assumption of **i)** the symmetry $\iota \circ \xi$ induces a bijection between the following two subspaces

$$\begin{aligned} &\{((P,\omega), B = 0) \mid \omega \in V,\ \mathrm{Re}(\sigma) \in U'_{\mathbb{R}},\ \mathrm{Im}(\sigma) \in V^\vee\} \\ \text{and} \quad &\{((P,\omega), B = 0) \mid \omega \in V^\vee,\ \mathrm{Re}(\sigma) \in U'_{\mathbb{R}},\ \mathrm{Im}(\sigma) \in V\} \end{aligned}$$

Indeed, $\mathrm{Im}(\sigma^\vee) = \langle \mathrm{Re}(\sigma), v\rangle^{-1}(-\langle B, \omega\rangle v + \omega)$ and $\mathrm{Re}(\sigma^\vee) = \langle \mathrm{Re}(\sigma), v\rangle^{-1}(\frac{1}{2}(\omega^2 - B^2)v + v^* + B)$. Thus, if $B = 0$ one has $\mathrm{Im}(\sigma^\vee) = \langle \mathrm{Re}(\sigma), v\rangle^{-1}\omega$ and $\mathrm{Re}(\sigma^\vee) = \langle \mathrm{Re}(\sigma), v\rangle^{-1}(\frac{\omega^2}{2}v + v^*)$, and $B^\vee = 0$. Conversely, if $\mathrm{Re}(\sigma^\vee) \in U'_{\mathbb{R}}$ then $B = 0$. Moreover, $\mathrm{Im}(\sigma) \in V$ implies $\omega \in V$ and $\omega^\vee \in V^\vee$ implies $\mathrm{Im}(\sigma) \in V^\vee$. Eventually, $B^\vee$ yields $\mathrm{Re}(\sigma) \in U'_{\mathbb{R}}$.

**iii)** In this example we will not need any further decomposition of $\Gamma'_{\mathbb{R}}$. The automorphism $\iota \circ \xi \in \tilde{O}(\Gamma \oplus U)$ induces an involution on the subspace

$$\{((P,\omega), B) \mid \omega, B \in \Gamma'_{\mathbb{R}} \oplus \mathbb{R}v\} \subset \mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}).$$

This follows again easily from the explicit description of $(\sigma^\vee, \omega^\vee, B^\vee)$.

**iv)** Also in **iii)** one finds a smaller subset parametrizing only objects with trivial B-field that is respected by $\iota \circ \xi$. Indeed, the subpace

$$\{((P,\omega), 0) \mid \omega \in \Gamma'_{\mathbb{R}},\ P \cap U'_{\mathbb{R}} \neq 0\}$$

is mapped onto itself under $\iota \circ \xi$. □

**Remark 5.9** **i)** Note that we have not used any further normalization, e.g. $\omega^2 = \mathrm{Re}(\sigma)^2$ as in [20].
**ii)** If $((P,\omega), B)$ such that $B^\vee = 0$ and $\varphi \in \mathrm{O}(\Gamma')$, then also $\tilde{\xi}((\varphi(P), \varphi(\omega)), \varphi(B))$ has vanishing B-field. Geometrically this is used to argue that if the mirror $X^\vee$ of $X$ has vanishing B-field then the same holds for the mirror of $f^*X$ under any diffeomorphism $f$ of $X$ with $f^*|_{U'} = \mathrm{id}$. The assertion is an immediate consequence of the explicit description of $B^\vee$ given above.

Note that $\iota \circ \xi$ is by far the most interesting automorphism considered so far, as it really mixes the 'complex direction' $\sigma$ with the 'metric direction' $(\omega, B)$. However, at least for the case of the K3 lattice $\Gamma = 2(-E_8) \oplus 3U$ the automorphisms $\xi$ respectively $\{-\mathrm{id}_U, w \leftrightarrow w^*\}$ together with $\{\mathrm{O}(\Gamma), \varphi_{B_0 \in \Gamma}\}$ generate both the same group, namely $\mathrm{O}(\Gamma \oplus U)$. This is a consequence of Proposition 5.1, where one uses $\xi \mathrm{O}(U)\xi = \mathrm{O}(U')$ and thus $\mathrm{O}(U) \subset \langle \xi, \mathrm{O}(\Gamma)\rangle$. So in this sense, $\xi \in \mathrm{O}(\Gamma \oplus U)$ as an automorphism of $\mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ is not more or less interesting than those in 5.4 and 5.5, but for the latter ones the interesting things happen outside the 'geometric world' of $\mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$.

# 6 Geometric interpretation of mirror symmetry

## 6.1 Lattice polarized mirror symmetry

Let $\Gamma$ as before be the K3 lattice $2(-E_8) \oplus 3U$ and fix a sublattice $N \subset \Gamma$ of signature $(1, r)$.

**Definition 6.1** *An $N$-polarized marked K3 surface is a marked K3 surface $(X, \varphi)$ such that $N \subset \varphi(\mathrm{Pic}(X))$.*

Note that any $N$-polarized K3 surface is projective. If $\mathcal{T}^{\mathrm{cpl}}_{\Gamma}$ is the moduli space of marked K3 surfaces we denote by $\mathcal{T}^{\mathrm{cpl}}_{N\subset\Gamma}$ the subspace that consists of $N$-polarized marked K3 surfaces. Analogously, one defines

$$\mathcal{T}^{(2,2)}_{N\subset\Gamma} \subset \mathcal{T}^{(2,2)}_{\Gamma}$$

as the subset of all marked Kähler K3 surfaces with B-field $(X, \omega, B, \varphi)$ such that $N \subset \mathrm{Pic}(X)$ and $\omega, B \in N_{\mathbb{R}}$. Here and in the following, we omit the marking in the notation, i.e. the identification $H^2(X, \mathbb{Z}) \cong \Gamma$ via $\varphi$ will be understood.

The condition $N \subset \mathrm{Pic}(X)$ is in fact equivalent to $V := N_{\mathbb{R}} \subset \mathrm{Pic}(X)_{\mathbb{R}}$. The latter can furthermore be rephrased as $V \subset (H^{2,0}(X) \oplus H^{0,2}(X))^{\perp}$, i.e. $\sigma \in V^{\perp}_{\mathbb{C}}$.

By construction there exists a natural map

$$\mathcal{T}^{(2,2)}_{N\subset\Gamma} \to \mathcal{T}^{\mathrm{cpl}}_{N\subset\Gamma}.$$

The fibre over $(X, \varphi) \in \mathcal{T}^{\mathrm{cpl}}_{N\subset\Gamma}$ is isomorphic to $V_{\mathbb{R}} + i(\mathcal{K}_X \cap V_{\mathbb{R}})$ via $(\omega, B) \mapsto B + i\omega$.

Using the period map, the space $\mathcal{T}^{(2,2)}_{N\subset\Gamma}$ can be realized as a subspace of $\mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \subset \mathrm{Gr}^{\mathrm{po}}_{2,2}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$. Its closure $\overline{\mathcal{T}}^{(2,2)}_{N\subset\Gamma}$ consists of all points $((P, \omega), B) \in \mathrm{Gr}^{\mathrm{po}}_{2,1}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ such that $P \subset V^{\perp}$ and $\omega, B \in V$. Indeed, via the period map $\mathcal{T}^{(2,2)}_{N\subset\Gamma}$ is identified with an open subset of $\{((P, \omega), B) \mid B, \omega \in V, P \subset V^{\perp}\}$ and the latter is irreducible.

Let us now assume that the orthogonal complement $N^{\perp} \subset \Gamma$ contains a hyperbolic plane $U' \subset N^{\perp}$. Then $N^{\perp} = N^{\vee} \oplus U'$ by Lemma 5.6 for some sublattice $N^{\vee} \subset \Gamma$ of signature $(1, 18 - r)$. The real vector space $N^{\vee}_{\mathbb{R}}$ is denoted by $V^{\vee}$. As above one introduces $\mathcal{T}^{(2,2)}_{N^{\vee}\subset\Gamma}$ and $\mathcal{T}^{\mathrm{cpl}}_{N^{\vee}\subset\Gamma}$.

**Proposition 6.2** *The mirror symmetry map $\tilde{\xi}$ associated to the splitting $\Gamma = \Gamma' \oplus U'$ induces a bijection*

$$\overline{\mathcal{T}}^{(2,2)}_{N\subset\Gamma} \cong \overline{\mathcal{T}}^{(2,2)}_{N^{\vee}\subset\Gamma}.$$

*Proof.* By the description of $\overline{\mathcal{T}}^{(2,2)}_{N\subset\Gamma}$ as the set $\{((P,\omega),B) \mid B,\omega \in V, P \subset V^{\perp}\}$, it suffices to show that the mirror map identifies the two sets $\{((P,\omega),B) \mid B,\omega \in V, P \subset V^{\perp}\}$ and $\{((P,\omega),B) \mid B,\omega \in V^{\vee}, P \subset (V^{\vee})^{\perp}\}$, which has been observed already in the Examples in Section 5.6. $\square$

**Remark 6.3 i)** In general, we cannot expect to have a bijection $\mathcal{T}^{(2,2)}_{N\subset\Gamma} \cong \mathcal{T}^{(2,2)}_{N^{\vee}\subset\Gamma}$. Indeed, for a point in $\mathcal{T}^{(2,2)}_{N\subset\Gamma}$ that corresponds to a triple $((P,\omega),B)$ the image $((P^{\vee},\omega^{\vee}),B^{\vee}) = \tilde{\xi}((P,\omega),B)$ might admit a $(-2)$-class $c \in (P^{\vee})^{\perp} \cap \Gamma$ with $\langle c,\omega^{\vee}\rangle = 0$. In fact, these two conditions on the $(-2)$-class $c$ translate into the equations $\langle c + \langle c,v\rangle B,\omega\rangle = 0$ and $\langle c - \langle c,v\rangle v^*, \mathrm{Im}(\sigma)\rangle = 0$. To exclude this possibility one would need to derive from this fact that there exists a $(-2)$-class $c'$ with $\langle c',\sigma\rangle = 0$ and $\langle c',\omega\rangle = 0$ and this doesn't seem possible in general.

One should regard this phenomenon as a very fortunate fact. As points in the boundary are interpreted as singular K3 surfaces, it enables us to compare smooth K3 surfaces with singular ones. One should try to construct examples of (singular) Kummer surfaces in this context.

**ii)** Also note that if $\omega \in \Gamma$, i.e. $\omega$ corresponds to a line bundle, then $\omega^{\vee}$ does not necessarily have the same property.

**iii)** We also remark that the lattices $N$ and $N^{\vee}$ are rather unimportant in all this. Indeed, what really matters are the two decompositions $\Gamma = \Gamma \oplus U'$ and $\Gamma'_{\mathbb{R}} = V \oplus V^{\vee}$.

To conclude this section, we shall compare the above discussion with [15]. Let $N \subset \Gamma$ and $N^{\perp} = N^{\vee} \oplus U'$ be as before. Following [15] one defines

$$\Omega := N^{\vee}_{\mathbb{R}} \oplus i(N^{\vee}_{\mathbb{R}} \cap \mathcal{C}) \text{ and } D_N := Q_{\Gamma} \cap \mathbb{P}(N^{\perp}_{\mathbb{C}}).$$

Then by [15, Thm.4.2,Rem.4.5] the map

$$\alpha : \Omega \to D_N, z \mapsto [z - \frac{1}{2}z^2 \cdot v + v^*]$$

is an isomorphism. This map obviously coincides with $(B + i\omega) \mapsto [\sigma^{\vee}]$ as described in Prop. 5.7, since for $B,\omega \in N^{\vee}_{\mathbb{R}} \subset \Gamma'_{\mathbb{R}}$ one has $\mathrm{pr}(B+i\omega) = B+i\omega$. Thus, the map $\alpha$ coincides with the map given by the isomorphism $\overline{\mathcal{T}}^{(2,2)}_{N\subset\Gamma} \cong \overline{\mathcal{T}}^{(2,2)}_{N^{\vee}\subset\Gamma}$. To make this precise note that $\overline{\mathcal{T}}^{\mathrm{cpl}}_{N^{\vee}\subset\Gamma} \cong D_N$ via the period map and that $((P,\omega),B) \mapsto B + i\omega$ defines a surjection $\overline{\mathcal{T}}^{(2,2)}_{N\subset\Gamma} \twoheadrightarrow \Omega$. This yields a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{T}}^{(2,2)}_{N\subset\Gamma} & \xrightarrow{\cong} & \overline{\mathcal{T}}^{(2,2)}_{N^{\vee}\subset\Gamma} \\ \downarrow & & \downarrow \\ \Omega & \xrightarrow{\cong} & D_N \cong \mathcal{T}^{\mathrm{cpl}}_{N^{\vee}\subset\Gamma} \end{array}$$

which emphasizes the fact that the mirror isomorphism identifies Kähler deformations with complex deformations.

**Remark 6.4** We also mention the following result of Looijenga and Peters [25], which shows that lattices of small rank can always be realized. Let $\Gamma$ be the K3 lattice and $N$ any even lattice of rank at most three. Then there exists a primitive embedding $N \subset \Gamma$. If the rank is smaller than three then this primitive embedding is unique up to automorphisms of $\Gamma$, i.e. elements of $\mathrm{O}(\Gamma)$.

## 6.2 Mirror symmetry for elliptic K3 surfaces

Let $\pi : Y \to \mathbb{P}^1$ be an elliptic K3 surface with a section $\sigma_0 \subset Y$. The cohomology class $f$ of the fibre and $[\sigma_0]$ generate a sublattice $U' \subset H^2(Y, \mathbb{Z})$. It can be identified with the standard hyperbolic plane by choosing as a basis $v = f$ and $v^* = f + \sigma_0$. Thus, we obtain a decomposition $\Gamma := H^2(Y, \mathbb{Z}) = \Gamma' \oplus U'$.

Let us now study the action of $\tilde{\xi}$ on K3 surfaces that are related to $Y$. If we fix a HK-metric $g$ on $Y$, then we may write $Y = (M, J)$, where $J$ is one of the compatible complex structures $\{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$ associated with $g$. A holomorphic two-form on $Y$ can be given as $\sigma_J = \omega_K + i\omega_I$. The reason why the complex structure that defines $Y$ is denoted $J$ is that we will actually not describe the mirror of $Y$, but rather of $X := (M, I)$.

Clearly $X$ inherits the torsu fibration from $Y$ which gives rise to a differentiable map $\pi : X \to Y$.

**Lemma 6.5** *The torus fibration $\pi : X \to \mathbb{P}^1$ is a SLAG fibration.*

*Proof.* Indeed, since the holomorphc two-form $\sigma_J$ vanishes on any holomorphic curve in $Y$, the form $\omega_I = \mathrm{Im}(\sigma_J)$ vanishes in particular on every fibre of $X \to \mathbb{P}^1$, i.e. all fibres are Lagrangian. Moreover, since $\sigma_I = \omega_J + i\omega_K$ and $\omega_K = \frac{1}{2}(\sigma_J + \bar{\sigma}_J)$, we see that $\mathrm{Im}(\sigma_I)|_{\pi^{-1}(t)} = 0$ and $\mathrm{Re}(\sigma_I)|_{\pi^{-1}(t)} = \omega_J|_{\pi^{-1}(t)}$. Hence, the (smooth) fibres are special Lagrangians of phase 0. □

**Proposition 6.6** *The $\tilde{\xi}$-mirror of $(X, \omega_I)$ is the K3 surface $X^\vee$ given by the period*

$$\sigma^\vee = \frac{1}{\mathrm{vol}(f)} \left( \frac{\omega_I^2}{2} f + \sigma_0 + f + i\omega_I \right),$$

*which is endowed with the Kähler class*

$$\omega^\vee = \frac{1}{\mathrm{vol}(f)} \mathrm{Im}(\sigma_I),$$

*where $\mathrm{vol}(f) = \langle \omega_J, f \rangle$ is the volume of the fibre of the elliptic fibration $Y \to \mathbb{P}^1$.*

*Proof.* Using $B = 0$, $\langle \omega_I, v \rangle = 0$, and $\langle \mathrm{Re}(\sigma_I), f \rangle = 0$, this is a direct consequence of the general formula in Proposition 5.7. □

Of course, an explicit formula could also be given for the mirror of $X$ endowed with the Kähler form and an auxiliary B-field. We leave this to the reader.

**Remark 6.7** A priori, the mirror described by the proposition above could be singular, i.e. there could be a $(-2)$-class $c \in \Gamma$ such that $\langle c, \omega^\vee \rangle = \langle c, \sigma^\vee \rangle = 0$. For such a class we would have $\langle c, \omega_I \rangle = \langle c, \mathrm{Im}(\sigma_I) \rangle = 0$. Of course, if we also had $\langle c, \omega_I \rangle = 0$, then already $(X, \omega_I)$ would be singular.

The description of the mirror as above does not really give an idea of the kind of K3 surface we have to expect for the mirror $X^\vee$. However, there is a very special case where the mirror can be understood by hyperkähler rotation.

Let $\pi : Y \to \mathbb{P}^1$ be an elliptic K3 surface with fibre $f$ and section $\sigma_0$. Assume that $(\alpha+1)f + \sigma_0$ is a Kähler class for some $\alpha > 0$, e.g. $Y$ has Picard number two. Let $\pi : X \to \mathbb{P}^1$ be as before the SLAG fibration obtained from hyperkähler rotating $Y$ with the complex structure $J$ and Kähler class $\omega_J$ to $X$ with the complex structure $I$ and Kähler class $\omega_I$.

**Proposition 6.8** *The $\tilde{\xi}$-mirror of $(X, \omega_I)$ is the K3 surface given by the complex structure $-K = JI$ on the K3 surface $Y$.*

Thus, in this very special case mirror symmetry is indeed given by hyperkähler rotation. Stronger statements, i.e. for less special elliptic K3 surfaces, as made e.g. in [12], cannot be justified by the approach of these lectures.

*Proof.* The general formual shows that $\sigma^\vee = \mathrm{vol}(f)^{-1}(\omega_J + i\omega_I)$. On the other hand, $\sigma_{-K} = \omega_J + i\omega_I$. Hence, $X^\vee$ is given by the complex structure $-K$. □

There is one tiny subtlety. If we compute also the mirror Kähler form $\omega^\vee$, we obtain $-\omega_{-K}$, but this is of no importance as we can always apply the harmless global transformation $-\mathrm{id} \in \mathrm{O}(\Gamma)$.

**Remark 6.9** If we go back to the more general case, where $\omega_J$ on $Y$ might be arbitrary, then we still see that $\omega^\vee, \mathrm{Im}(\sigma^\vee) \in \langle v, v^* \rangle^\perp$, i.e. at least cohomologically the classes $f$ and $\sigma_0$ are still Lagrangian on the mirror, as in the more special case above where $X^\vee$ was given by $-K$. Also note that in any case, the fibre of the volume is reversed, for $\langle \mathrm{Re}(\sigma^\vee), v \rangle = \langle \mathrm{Re}(\sigma_I), v \rangle^{-1}$.

## 6.3 FM transforms

## 6.4 Kummer surfaces

In this section we plan to include a description of the toroidal theories insides the K3 surface moduli space. See [27].

## 6.5 Large complex structure limit

A discussion of large complex structures should come here cf. [20].

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