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Lecture Notes on Relative GW-Invariants

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LECTURE NOTES ON RELATIVE GW-INVARIANTS

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1. Introduction

1.1. **The GW-invariants.** Let W be a smooth projective variety. It is known that we can construct the GW-invariants of W via the (virtual) intersection theory on the moduli of stable morphisms to W.

Let us review the key ingredient of such construction. We fix integers g, k and algebraic class $d \in H_2(X)$. We form the moduli space

$$\mathfrak{M}_{g,k}(X,d) = \{f : X \to W \mid X = (X, p_1, \cdots, p_k),\ g(X) = g,\ \deg f = d\}/\sim .$$

Here we of course impose that X be a connected nodal curve and f be stable. We know $\mathfrak{M}_{g,k}(X,d)$ is a proper, separated DM-stack, of expected dimension

$$\exp.\dim \mathfrak{M}_{g,k}(X,d) = (3 - \dim W)(g-1) + k + d \cdot c_1(W).$$

We let

$$\mathrm{ev} : \mathfrak{M}_{g,k}(X,d) \longrightarrow X^k$$

be the evaluation map. Then the GW-invariants are

$$\Phi^X_{g,k,d}(\beta) = \int_{[\mathfrak{M}_{g,k}(X,d)]^{\mathrm{vir}}} \mathrm{ev}^* \beta, \quad \beta \in H^*(X^n).$$

When $\mathfrak{M}_{g,k}(X,d)$ has pure dimension and is equal to the expected dimension, $[\mathfrak{M}_{g,k}(X,d)]^{\mathrm{vir}}$ is the fundamental class of $\mathfrak{M}_{g,k}(X,d)$. In general, we need to use the notion of virtual cycles $[\mathfrak{M}_{g,k}(X,d)]^{\mathrm{vir}}$, constructed by Li-Tian and Behrend-Fantechi [16, 1, 2].

1.2. **An easy example.** An easy case is when $W = \mathbf{P}^2$, $g = 0$ and d is a multiple of a line in $\mathbf{P}^2$. Then

$$\mathfrak{M}_{0,k}(\mathbf{P}^2, d)$$

is the space of all degree d stable $f : X \to \mathbf{P}^2$ with $g(X) = 0$. It is easy to see that $\mathfrak{M}_{0,k}(\mathbf{P}^2, d)$ is smooth, since the first order deformation of the f fits into the exact sequence

$$0 \to H^0(T_X) \to H^0(f^*T\mathbf{P}^2) \to Def^1(f) \to Def^1(X) \to H^1(f^*T\mathbf{P}^2) = 0.$$

By Riemann-Roch theorem, $Def^1(f)$ is independent of f and is equal to the expected dimension above. This is the first indication that $\mathfrak{M}_{0,d}(\mathbf{P}^2, d)$

is smooth. (To prove the smoothness of the moduli, we need to prove the vanishing of the obstruction classes.)

We pick k so that $\dim \mathfrak{M}_{0,k}(\mathbf{P}^2, d) = 2k$. We then pick k general points $q_1, \cdots, q_k \in \mathbf{P}^2$, look at the subset of the moduli space $\mathfrak{M}_{0,k}(\mathbf{P}^2, d)$:

$$\{f \in \mathfrak{M}_{0,k}(\mathbf{P}^2, d) \mid f(p_i) = q_i, \ \forall i\}$$

their degrees are the GW-invariants of $\mathbf{P}^2$.

For instance, when $k = 3d - 1$, $\dim \mathfrak{M}_{0,k}(\mathbf{P}^2, d) = 2k$ and their GW-invariants are

$$N_1 = 1, \ N_2 = 1, \ N_3 = 12, \ N_4 = 87304, \cdots .$$

In general there is an explicit formula expressing N_d.

1.3. **The game of degeneration.** In algebraic geometry, we like to play the game of degeneration. One direction in GW-invariants is to degenerate the domain of the curves. Say, restricting to a nodal curves. This is the basis of the associativity law of the GW-invariants. we will not talk about this in our lecture.

What we like to do it to degenerate the target space. Simply speaking, we let C be a connected smooth curve, $0 \in C$ be a fixed closed point and let $W \to C$ be a family of projective schemes so that the fibers W_t of W over $t \neq 0 \in C$ are smooth and the central fiber W_0 is the union of two smooth varieties Y_1 and Y_2 intersecting transversally along a smooth irreducible divisor. We denote by $D_i \subset Y_i$ the divisor $Y_1 \cap Y_2 \subset Y_i$. We will say W_0 is the gluing of (D_1, Y_1) and (Y_2, D_2) along $D_1 = D_2$. We call $Y_i^{\mathrm{rel}} = (Y_i, D_i)$ the relative pair.

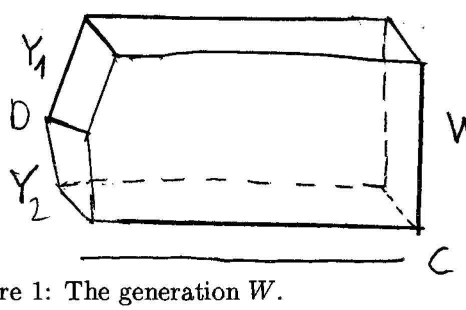

Figure 1: The generation W.

As for $t \neq 0$, since W_t are members of a connected family of smooth varieties, the Gromov-Witten invariants of W_t are all equivalent.

Main Question: *How to relate the Gromov-Witten invariants of W_t with that of W_0, and then with that of the pairs (Y_i, D_i).*

The goal of this lecture series is to show how to construct the relative Gromov-Witten invariants of the pairs (Y_i, D_i), and how to relate the Gromov-Witten invariants of W_t with the relative Gromov-Witten invariants of the pairs (Y_1, D_1) and (Y_2, D_2). The hoped relation will be of the form

$$GW(W_t) = GW(Y_1^{\mathrm{rel}}) * GW(Y_2^{\mathrm{rel}}).$$

Here we use $GW(W_t)$ to denote the full GW-invariants of W_t and use $GW(Y_i^{\mathrm{rel}})$ to denote the full relative GW-invariants of (Y_i, D_i). The operator $*$ is an involution type product linear in both arguments.

1.4. **Continuation of the easy example.** Let us continue with the example $\mathfrak{M}_{0,k}(\mathbf{P}^2, d)$. We like to play the game of degenerating the $\mathbf{P}^2$ to a reducible surface with two smooth component. One example is as follows: We blow up a line $L \times 0 \subset \mathbf{P}^2 \times \mathbf{A}^1$. We let the resulting 3-fold be $W \to \mathbf{A}^1$. Clearly, $W_t \cong \mathbf{P}^2$ while W_0 is the union of $\mathbf{P}^2$ with the Hirzebruch surface $\mathbf{H}$, intersecting along the line L. Now let's see how to trace the moduli space

$$\mathfrak{M}_{0,5}(W_t, 2).$$

If this space is too big for the moment to visualize, let us pick five sections $z_i(t)$ of $W \to \mathbf{A}^1$, and look at the subspace

$$\Theta_t \triangleq \{f \in \mathfrak{M}_{0,5}(W_t, 2) \mid f(p_i) \in z_i(t),\ i = 1, \cdots, 5\}.$$

If we choose z_i in general positions, for almost all closed $t \neq 0 \in \mathbf{A}^1$ we have $\Theta_t = pt$. Thus we expect $\Theta_0 = pt$ as well.

Since $W_0 = \mathbf{P}^2 \cup \mathbf{H}$, we can choose z_i so that

Case 1. All $z_i(0) \in \mathbf{P}^2$: In this case, one checks that Θ_0 consists of a single map

$$f : (X_1, a_1, a_2) \cup (X_2, b_1, b_2) \longrightarrow \mathbf{P}^2 \cup \mathbf{H},$$

where $X_1 = \mathbf{P}^1$, $f(X_1) \subset \mathbf{P}^2$ is a quadric; $X_2 = \mathbf{P}^1 \cup \mathbf{P}^1$ is a disjoint union and $f(X_2) \subset \mathbf{H}$ are fibers of $\mathbf{H}$ and $a_i = b_i$ and $f(a_i) = f(b_i)$.

Case 2. $z_1(0)$ and $z_2(0) \in \mathbf{P}^2$ and the others are in $\mathbf{H}$: In this case, one checks that Θ_0 consists of a single map

$$f : (X_1, a) \cup (X_2, b) \longrightarrow \mathbf{P}^2 \cup \mathbf{H},$$

where $X_1 = \mathbf{P}^1$ and $f(X_1)$ is a line, $X_2 = \mathbf{P}^1$ and $f(X_2) \subset \mathbf{H}$ is in the divisor class $L + 2F$ where F is the fiber $\mathbf{H}$, $a = b$ and $f(a) = f(b)$.

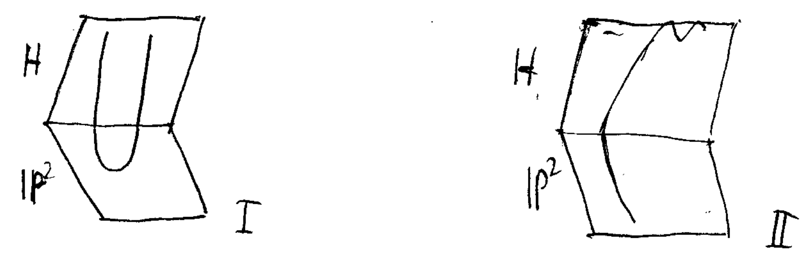

Figure 2: The specializations of families of stable maps.

The general philosophy is that when W_t specialize (degenerate) to $W_0 = Y_1 \cup Y_2$, the stable maps $f_t : X_t \longrightarrow W_t$ will degenerate to $f_0 : X_0 \longrightarrow W_0$.

Since $W_0 = Y_1 \cup Y_2$, then f_0 can be split to a pair of maps

$$f_1 : X_1 \to Y_1 \quad \text{and} \quad f_2 : X_2 \to Y_2$$

so that they satisfy the incidence relations: 1. X_0 is the gluing of X_1 and X_2 and 2. f_0 is the gluing of f_1 and f_2.

The counting of $f : X \to W_t$ should be equivalent to the counting of pairs (f_1, f_2) satisfying the above incidence relations.

1.5. **The need of tangential condition.** This suggests that we should consider the relative stable maps: Let $D \subset Y$ be a smooth divisor in a smooth projective variety. We should consider

$$f : (X, p_1, \cdots, p_n, q_1, \cdots, q_m) \longrightarrow Y$$

such that $f(q_1), \cdots, f(q_m)$ are all lie in D. (Here p_i are the usual marked points.)

A moment of thought suggests that we need to specify the order of contacts of f at q_i. In other words, we should insist on

$$f^{-1}(D) = \sum \mu_i q_i$$

as Cartier divisor, for a set of pre-chosen integers $\mu_1, \cdots, \mu_m$. In other words, q_i is where $f(X)$ intersects D, and we specify the order of tangent of $f(X)$ with D at q_i to be μ_i.

On reason for doing this is that it is nature to impose such contact order condition. The other reason is due to the requirement of the composition formula we hope to develop.

We illustrate this by a simple example: A (local model of a) family of smooth curves degenerates to a nodal curve with images in a family of varieties is as follows:

$$\begin{array}{ccc} \cup X_t = \mathbf{A}^2 & \xrightarrow[f]{} & W = \mathbf{A}^2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathbf{A}^1 & \longrightarrow & \mathbf{A}^1 \end{array} \qquad \begin{array}{ccc} (z_1, z_2) & \xrightarrow[w_i = z_i^{\mu}]{} & (w-1, w_2) \\ \downarrow{\scriptstyle s = z_1 z_2} & & \downarrow{\scriptstyle t = w_1 w_2} \\ s & \xrightarrow{t = s^{\mu}} & t \end{array}$$

Then the contact order of $f|_{X_1}$ and $f|_{X_2}$ are both e. The upshot is that they have to be *identical.*

It can be proved that this is a general rule: *If we have a flat C-scheme S, a family of curves $\mathcal{X}/S$ and a C-morphism $F : \mathcal{X} \to W$ so that $F^{-1}(D)$ does not contain irreducible components of fibers of $\mathcal{X}/S$. Then the upper and the lower contact orders of F at each $q \in F^{-1}(D)$ are identical.*

Hence in order to establish a workable decomposition formula, we need to keep track of the contact orders of the maps when they intersect D.

1.6. **The intuitive definition of relative stable maps.** We fix the topological type of the domains of the relative stable maps we are interesting at. It consists of the genus g, the number of marked points k, the number and the weights of the distinguished marked points $\mu_1, \cdots, \mu_r$. (This is the case where the domain is connected. The general case will be specified later.) We then fix the degree $d \in H_2(Y)$ and look at the stable maps:

Definition 1. *A regular relative stable map $f : X \to Y^{rel}$ of the given topological type consists of the data (f, X, q_i, p_j) as follows:*
1. X is a connected nodal curve of arithmetic genus g;
2. $q_i \in X$, $i = 1, \cdots, r$ and $p_j \in X$, $j = 1, \cdots, k$, are disjoint points away from the singular locus of X;.
3. $f : X \to Y$ is a morphism so that as Cartier divisors $f^{-1}(D) = \sum_{i=1}^{r} \mu_i q_i$ and $\deg(f(X)) = d$;
4. The morphism f considered as a morphism whose domain is X with all marked points p_i, $q_j \in X$ is a stable morphism.

It is standard to show that the moduli of all regular relative stable morphisms is a DM-stack, is separated.

1.7. **It does not work.** Unfortunately, this construction of the moduli of regular relative stable morphisms will not produce us a relative Gromov-Witten invariant. The reason is that it is usually not proper. Here is an example: We let (Y, D) be the pair of a line in $\mathbf{P}^2$. We pick $d = 2$, $r = 1$ and $\mu_1 = 2$. Then regular relative stable maps of the type given are quadric in $\mathbf{P}^2$ tangent to L. It is easy to see that regular relative stable morphisms can specialize to an $f : X \to \mathbf{P}^2$ with $X = \mathbf{P}^1 \cup \mathbf{P}^1$ and $f(\mathbf{P}^1) = L$.

One solution is to take the closure of the relative stable maps just defined, and hope for the best. However, there is a serious draw back in this approach. Recall that in general, GW-invariants are defined using the virtual moduli cycles, which are defined based on a good understanding of the obstruction theory of the moduli spaces. As one can imagine, it is almost impossible to track the obstruction theory by simply taking the closure in some ambient spaces.

What we need is an intrinsic way of compactifying the moduli space of regular relative stable so that we can still keep track of its obstruction theory. In algebraic geometry, the best way to achieve this is to expand the moduli problem and show that the new moduli space is the desired compactification.

2. Relative stable morphisms

2.1. **The way out.** As a warming up, let us look at a simple case where degenerate relative stable morphisms arise. Let $f_t : (\mathbf{P}^1, 0) \to (\mathbf{P}^2, L)$, $t \in \mathbb{C} - 0$, be a family of relative stable morphisms from $\mathbf{P}^1$ to $\mathbf{P}^2$ of degree 2 with $f_t^{-1}(L) = 2[0]$. We assume f_t specialize to $f_0 : \mathbf{P}^1 \to L \subset \mathbf{P}^2$. Of course f_0 is not a regular relative map.

We change out view point a bit. We think of f_t as a map $\mathbf{P}^1 \times \mathbf{A}^1$ to $\mathbf{P}^2 \times \mathbf{A}^1$, via $F(z,t) = (f(z), t)$. To resolve the degeneracy, we blow up $L \times 0 \subset \mathbf{P}^2 \times \mathbf{A}^1$, with $W[1]$ the resulting family. We denote $W[1]_0 = \Delta \cup \mathbf{P}^1$, where Δ is a ruled surface and denote $D[1]$ the proper transform of $L \times \mathbf{A}^1$. Then the maps f_t specialize to $\bar{f}_0 : \mathbf{P}^1 \to \Delta \subset W[1]_0$ with $\bar{f}_0^{-1}(D[1]) = 2[0]$.

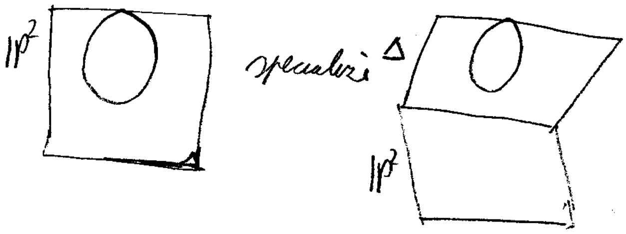

Figure 3: After replacing $\mathbf{P}^2$ by $\mathbf{P}^2 \cup \Delta$, f_t specialize to a new kind of relative stable map.

The up shot is that after we change the target, from $\mathbf{P}^2$ to $\mathbf{P}^2 \cup \Delta$, we obtain a map with the same relative condition.

In general, a "good" specialization of a family of relative stable maps to $(\mathbf{P}^1, L)$ can be a relative stable morphism to $\mathbf{P}^2 \cup \Delta \cdots \cup \Delta$, the gluing of $\mathbf{P}^2$ with several copies of Δ, relative to a divisor in the last copy of Δ.

2.2. **The extended relative target spaces.** Using blowing up to resolve degenerate specialization is not new in algebraic geometry. What is new is that we need to put all maps to different target spaces together to form a moduli space of which we can still work out its obstruction theory. We now show how to do such moduli problem.

We consider a pair of a smooth divisor in a smooth projective variety $D \subset Y$.

We let $Y[1]$ be the blowing up of $Y \times \mathbf{A}^1$ along $D \times 0$, and let $D[1]$ be the proper transform of $D \times \mathbf{A}^1$.

We let $Y[2]$ be the blowing up of $Y[1] \times \mathbf{A}^1$ along $D[1] \times 0$, and let $D[2]$ be the proper transform of $D[1] \times \mathbf{A}^1$.

We construct pairs $(D[n], Y[n])$ accordingly.

We let $G[n]$ be the direct product of n copies of $\mathbb{C}^*$. Let $G[1]$ acts on $\mathbf{A}^1$ via $t^\sigma = \sigma t$. Then the induced $G[1]$ action on $Y \times \mathbf{A}^1$ lifts to an action on $Y[1]$. In general, the induced product action $G[n]$ on $\mathbf{A}^n$ lifts to an action on $Y[n]$.

Figure 4: The new spaces $Y[1]$ and $Y[2]$, with arrows indicating the group action.

Note that $Y[n]_0$ is the union of Y with n copies of Δ. The Δ is a ruled variety over D, with two sections D_- and D_+. There is a $\mathbb{C}^*$ action on Δ, preserving each fibers and fixing the two sections $D_\pm$. The $G[n]$ action on $Y[n]$ leave invariant the $Y[n]_0$. Its action on $Y[n]_0$ is induced by the n $\mathbb{C}^*$-action on n Δ's.

Figure 5: The $Y[n]_0$.

The up shot is that $Y[n]$ is a family that combines all members $Y[k]_0$ for $k \le n$. Following Grothendieck's view point, this is how $Y[k]_0$ deforms to other $Y[k']_0$ with $k' \le k$.

For instance, if we let $\mathbf{A}^1_l$ be the l-th coordinate axis of $\mathbf{A}^n$, then the union of all fibers of $Y[n]$ over $\mathbf{A}^n_l$ is a smooth of $Y[n]_0$ along its l-th singular divisor. On the other hand, if we let L be the line $t_1 = t_2$ while all other $t_i = 0$. Then the union of all fibers of $Y[n]$ over L is a smoothing of the first two singular divisors of $Y[n]_0$. Unlike the first case, this time the total space of the smoothing is not smooth.

Figure 6: The fibers over $\mathbf{A}^1_1$ and L, which are smoothing of $Y[n]_0$. The blur part means that the total space is not smooth.

What we will use in constructing the relative stable morphisms is the quotient $Y[n]/G[n]$, as an Artin stack. Since there are inclusions

$$Y[n-1]/G[n-1] \subset Y[n]/G[n],$$

we can talk about the limit stack

$$\mathfrak{Y} = \lim Y[n]/G[n].$$

2.3. **The domain of relative stable maps.** For simplicity, we fix an ample line bundle on Y. For any map $f: X \to Y$ we define its degree to be the degree of f^*H. For $f: X \to Y[n]_0$, we define its degree to be the degree of the pull back of the ample line bundle on Y via $X \to Y[n]_0 \to Y$.

In this paper, by a graph Γ we mean a finite collection of vertices, edges, legs and roots. Here an edge is as usual a line segment with both ends attached to vertices of Γ. A leg or a root is a line segment with only one end attached to a vertex of Γ. We will denote by $V(\Gamma)$ the set of vertices of Γ.

Definition 2. *An admissible weighted graph Γ is a graph without edges coupled with the following additional data:*
1. An ordered collection of legs, an ordered collection of weighted roots and two weight functions $g: \mathrm{V}(\Gamma) \to \mathbb{Z}_{\geq 0}$ and $b: V(\Gamma) \to \mathbb{Z}_{\geq 0}$.
3. The graph is relatively connected in the sense that either $\mathrm{V}(\Gamma)$ consists of a single element or each vertex in $\mathrm{V}(\Gamma)$ has at least one root attached to it.

Figure 7: These are examples of domain and their associated graphs.

2.4. **Relative morphisms to extended targets.**

Definition 3. *Let Γ be an admissible graph with r roots, k legs and l vertices $v_1, \cdots, v_l$. A relative morphism to $(Y[n]_0, D[n]_0)$ of type Γ is a quadruple (f, X, q_i, p_j) as follows:*
1. X is a disjoint union of $X_1, \cdots, X_l$ such that each X_i is a pre-stable curve of arithmetic genus $g(v_i)$.
2. $q_i \in X$, $i = 1, \cdots, r$ and $p_j \in X$, $j = 1, \cdots, k$, are distinct points away from the singular locus of X so that $q_i \in X_j$ (resp. $p_i \in X_j$) if the i-th root (resp. i-th leg) is attached to the j-th vertex of Γ.
3. $f: X \to Y[n]_0$ is a morphism so that as Cartier divisors $f^{-1}(D[n]_0) = \sum_{i=1}^r \mu_i q_i$, and that $\deg(f(X)) = b(v)$.

4. Finally, each morphism $f|_{X_i}$, considered as a morphism whose domain is X_i with all marked points in X_i, is a stable morphism to $Y[n]_0$.

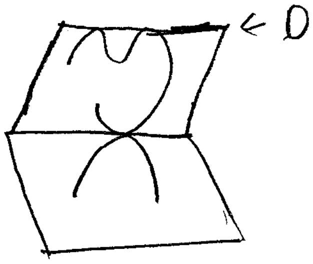

Figure 8: Examples of relative morphisms.

2.5. **The stability condition.** Now we come to the notion of relative stable morphisms.

Let $f: X \to Y[n]_0$ be a relative morphism. We let $D_1, \cdots, D_n \subset Y[n]_0$ be the n singular divisor. For f to be stable, f first of all should satisfy the so called admissible condition:

Definition 4. *We say f is admissible if the following holds:*
1. $f^{-1}(D_i) = \{z_{i,1}, \cdots, z_{i,j_i}\}$ is a discrete set;
2. Each $z_{i,j}$ must be a node of X, and is the intersection of two irreducible components, say A_- and A_+, of X. Let $e_\pm$ be the contact order of $f|_{A_\pm}$ with D_i at $z_{i,j}$, then $e_- = e_+$.

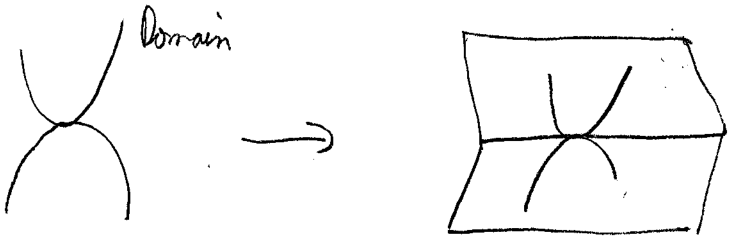

Figure 9: The upper and lower contact orders

This condition is necessary because if f is a specialization of relative maps to (Y, D), then this identity automatically holds on all $f^{-1}(D_i)$.

Now we come to the definition of relative stable maps.

We define the automorphism group of f as

$$\mathfrak{Aut}(f) = \{(h, \sigma) \mid \sigma \in G[n],\ h: X \xrightarrow{\cong} X,\ \sigma \circ h = h\}.$$

Definition 5. *We say f is a relative stable morphism if f is admissible and if $\mathfrak{Aut}(f)$ is finite.*

2.6. **Examples.** We let $Y = \mathbf{P}^1 \times C$, C a curve. $Y[1]_0 = Y \cup \Delta$. We let $X = X_1 \cup X_2$ with $X_i \cong \mathbf{P}^1$ and $0 \in X_1$ is glued to $\infty \in X_2$. We let $f: X \to Y[1]_0$ always send X_1 to Δ and the second X_2 to Y. We fix a $\xi \in C$.
Example 1. $f|_{X_1}(z) = (z^2, \xi)$ and $f|_{X_2}(z) = (z^2, \xi)$. This is not stable since $\mathfrak{Aut}(f) = \mathbb{C}^*$.
Example 2. $f|_{X_1}(z) = (z^2/(z+1), \xi)$ and $f|_{X_2}(z) = (z^2, \xi)$ is stable.

Note that in the first case f has one distinguished marked point with order 2; In the second case f has two distinguished marked points with order 1.
Example 3. $f|_{X_1}(z) = (z^2/(z+1), \xi)$ and $f|_{X_2}(z) = (z, \xi)$ is not admissible.

2.7. **The equivalence relation.** We say two $f: X \to Y[n]_0$ and $f': X' \to Y[n]_0$ are equivalent if there is an isomorphism h and an $\sigma \in G[n]$ so that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y[n]_0 \\ \downarrow h & & \downarrow \sigma \\ X' & \xrightarrow{f'} & Y[n]_0. \end{array}$$

2.8. **The existence theorem.** We list the main theorem concerning the relative stable morphisms. For the proof please consult [15].

Theorem 6. *The moduli functor of relative stable morphisms to (Y, D) of fixed topological type Γ is represented by a proper, separated DM stack. We denote this moduli space by $\mathfrak{M}(Y^{rel}, \Gamma)$.*

2.9. **Relative GW-invariants.** We define the relative Gromov-Witten invariants of Y^{rel} as follows:

We let

$$\mathrm{ev} : \mathfrak{M}(Y^{\mathrm{rel}}, \Gamma) \longrightarrow Y^n, \quad f \mapsto (f(p_1), \cdots, f(p_n)) \in Y^n$$

and

$$\mathbf{q} : \mathfrak{M}(Y^{\mathrm{rel}}, \Gamma) \longrightarrow D^r, \quad f \mapsto (f(q_1), \cdots, f(q_r)) \in D^r$$

be the evaluation maps. The relative GW-invariants is

$$\Psi_\Gamma^{Z^{\mathrm{rel}}} : H^*(Z \times k) \longrightarrow H_*(D^r)$$

defined by

$$\Psi_\Gamma^{Z^{\mathrm{rel}}}(\alpha) = \mathbf{q}_*\big(\mathrm{ev}^*(\alpha)\, [\mathfrak{M}(Y^{\mathrm{rel}}, \Gamma)]^{\mathrm{vir}}\big) \in H_*(D^r).$$

Here k (resp. r) is the number of ordinary (resp. distinguished marked points of the domain curves).

2.10. **Example.** One example is the classical Hurwize number.

3. Degeneration formula of GW-invariants

We will sketch how to establish the degeneration formula of the GW-invariants.

3.1. **Degeneration of targets.** We let C be a smooth irreducible curve, $0 \in C$ a closed point and $\pi: W \to C$ a flat and projective family of schemes satisfying the following condition: The morphism π is smooth away from the central fiber $W_0 = W \times_C 0$ and the central fiber W_0 is reducible with normal crossing singularity and has two smooth irreducible components Y_1 and Y_2 intersecting along a smooth divisor $D \subset W_0$. When we view D as a divisor in Y_i, we will denote it by $D_i \subset Y_i$.

Figure 10. A degeneration of targets.

3.2. **Degeneration of moduli spaces.** We fix g, k two integers and let b be a homology class. We consider the moduli space (stack) $\mathfrak{M}_{g,k}(W_t, b)$. The union

$$\mathfrak{M}_{g,k}(W_{C^\circ}, b) = \cup_{t \in C^\circ} \mathfrak{M}_{g,k}(W_t, b), \quad C^\circ = C - 0,$$

is a proper family over C°. Inserting the central fiber $\mathfrak{M}_{g,k}(W_0, b)$ gives one extension (which we will call a degeneration of moduli spaces). However, the natural obstruction theory of this new family is no longer perfect near degenerate stable morphisms. Here we say a stable morphism $f: X \to W_0$ is degenerate if some irreducible components of X are mapped entirely to the singular locus of W_0.

The solution we propose is to work out a new degeneration of $\mathfrak{M}_{g,k}(W_{C^\circ}, b)$ that will allow us to apply the machinery of virtual moduli cycles. The idea is parallel to the construction of relative stable morphisms.

3.3. **Extended target degenerations.** We now construct the class of expanded degenerations of W. We let Δ be the projective bundle over D:

$$\Delta = \mathbf{P}(\mathbf{1}_D \oplus N_{D_2/Y_2}),$$

where $\mathbf{1}_D$ is the trivial holomorphic line bundle on D and N_{D_2/Y_2} is the normal bundle of D_2 in Y_2. Δ has two distinguished divisors

$$D_- = \mathbf{P}(\mathbf{1}_D \oplus 0) \quad \text{and} \quad D_+ = \mathbf{P}(0 \oplus N_{D_2/Y_2}).$$

For convenience, we call D_- the left distinguished and D_+ the right distinguished divisors. Using this identification, we can *glue* an ordered chain of n Δ's by identifying the right distinguished divisor (i.e. D_+) in the k-th Δ with the left distinguished divisor (i.e. D_-) of the $(k+1)$-th Δ, for $k = 1, \cdots, n-1$. We denote the resulting scheme by $\Delta[n]$. It is connected, has normal crossing singularity and has n-irreducible component all isomorphic to Δ. We then *glue* Y_1 to $\Delta[n]$ by identifying D_1 in Y_1 with the left distinguished divisor D_- in the first Δ of $\Delta[n]$, and then *glue* Y_2 to this scheme by identifying the right distinguished divisor D_+ of the last Δ in $\Delta[n]$ with D_2 in Y_2. We denote the resulting scheme by $W[n]_0$. Note that $W[n]_0$ has $(n+2)$-irreducible components.

Figure 11. The picture of $W[n]_0$.

We need to construct a stack $\mathfrak{W}$ representing all extended degenerations which include $W[n]_0$ as their special fibers.

For simplicity, we assume $C = \mathbf{A}^1$. We let $G[n]$ acts on $\mathbf{A}^{n+1}$ via

$$\mathbf{t}^\sigma = (\sigma_1 t_1, \sigma_1^{-1}\sigma_2 t_2, \cdots, \sigma_{n-1}^{-1}\sigma_n t_n, \sigma_n^{-1} t_{n+1}). \tag{3.1}$$

Note that if we let

$$\mathbf{p} : \mathbf{A}^{n+1} \to \mathbf{A}^1, \quad \mathbf{p}(\mathbf{t}) = t_1 \times \cdots \times t_{n+1},$$

then $\mathbf{p}$ is $G[n]$-equivariant with the trivial $G[n]$-action on $\mathbf{A}^1$.

The standard model $W[n]$ will be constructed as a desingularization of

$$W \times_{\mathbf{A}^1} \mathbf{A}^{n+1}.$$

We construct $W[n]$ by induction on n. For $n = 0$, $W[0] = W$. For $n = 1$, $W_1 = W \times_{\mathbf{A}^1} \mathbf{A}^2$ is smooth except along the locus $D \times_{\mathbf{A}^1} 0$, which is a smooth codimension 3 subscheme in W_1.[1] After blowing up the subscheme $D \times_{\mathbf{A}^1} 0$ in $W \times_{\mathbf{A}^1} \mathbf{A}^2$ we obtain the scheme $\tilde{W}_1$ whose exceptional divisor is isomorphic to a $\mathbf{P}^1 \times \mathbf{P}^1$-bundle over D. We then contract one factor of this $\mathbf{P}^1 \times \mathbf{P}^1$-bundle to obtain a smooth scheme $W[1]$ over $\mathbf{A}^2$. Note that either way, the fiber of $W[1]$ over $0 \in \mathbf{A}^2$ is the $W[1]_0$ introduced before.

[1]The formal completions (the germs) of its normal slices in W_1 is isomorphic to the formal completion of

$$X = \{z_1 z_2 = t_1 t_2\} \subset \mathbf{A}^4 \tag{3.2}$$

along its origin. Here we use (z_1, z_2, t_1, t_2) to denote the coordinate of $\mathbf{A}^4$.

We choose the construction so that the fibers of $W[1]$ over the first and the second coordinate line of $\mathbf{A}^2$ is a smoothing of the first and second singular divisor D_1 and D_2 of $W[1]_0$.

Figure 12: The fibers over the two coordinate lines are smoothing of two singular divisors.

We now construct $W[2]$. We let W_2 be the fiber product $W[1] \times_{\mathbf{A}^2} \mathbf{A}^3$ with $\mathbf{A}^3 \to \mathbf{A}^2$ the morphism $(t_1, t_2, t_3) \mapsto (t_1, t_2 t_3)$. We let $W[2] \to \mathbf{A}^3$ be a small resolution of W_2. We choose the small resolution so that the fiber of $W[2]$ over $0 \in \mathbf{A}^3$ is $W[2]_0$, that the fibers over the i-th coordinate line is a smoothing of the i-th singular divisor $D_i \subset W[2]_0$.

We construct $W[n] \to \mathbf{A}^{n+1}$ inductively following the same line.

3.4. **The group action.** The $G[n]$ action on $\mathbf{A}^{n+1}$ lifts uniquely to a $G[n]$ action on $W[n] \to \mathbf{A}^{n+1}$. Its action on $W[n]_0$ is the product of the $\mathbb{C}^*$ action on Δ leaving $D_\pm$ fixed.

Figure 13: The arrows indicate the $\mathbb{C}^* \times \mathbb{C}^*$ action on $W[2]$.

We define

$$\mathfrak{W} = \lim W[n]/G[n].$$

It is the stack of the extended degenerations of $W/\mathbf{A}^1$.

3.5. **Degeneration of moduli spaces.** We need to construct the new central fiber of the family of the moduli spaces.

Definition 7. *A stable map to $\mathfrak{W}_0$ is an ordinary stable map $f : X \to W[n]_0$ for some n such that it is admissible at each point in $f^{-1}(D_i)$, $i = 1, \cdots, n$, and such that $\mathfrak{Aut}(f)$ is finite.*

The notion of admissible is similar to that of relative stable maps.

We define $\mathfrak{Aut}(f)$ be the set of all pairs (h, σ), where $h : X \cong X$ and $\sigma \in G[n]$, so that $\sigma \circ h = h$.

Two stable morphisms $f : X \to W[n]_0$ and $f' : X' \to W[n]_0$ are equivalent if there is an isomorphism h and an $\sigma \in G[n]$ so that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & W[n]_0 \\ \downarrow h & & \downarrow \sigma \\ X' & \xrightarrow{f'} & W[n]_0. \end{array}$$

We have the following existence theorem.

Theorem 8. *Given g, k and degree d, the moduli functor of all equivalent classes of stable morphisms to $\mathfrak{W}_0$ of genus g, k marked points and degree d is represented by a proper, separated DM stack $\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$.*

We will use $\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$ as the new central fiber of the family of the moduli space we seek to complete. We let

$$\mathfrak{M}_{g,k}(\mathfrak{W}, d) = \mathfrak{M}_{g,k}(\mathfrak{W}_0, d) \cup \cup_{t \in \mathbb{C}^*} \mathfrak{M}_{g,k}(W_t, d).$$

It is proved in [15] that

Theorem 9. *$\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ is a separated, $\mathbf{A}^1$-proper DM stack. It also admits a perfect obstruction theory, thus the virtual moduli cycle $[\mathfrak{M}_{g,k}(\mathfrak{W}, d)]^{\mathrm{vir}}$.*

To prove the theorem, we first define the corresponding moduli functor. For this we need to make sense of family of stable morphisms to $\mathfrak{W}$, which is rather straight forward. Then we prove that the functor is represented by a stack, prove that this stack is separated, is $\mathbf{A}^1$-proper and is indeed a DM stack. For details please consult [15].

4. The degeneration formula

In this lecture, we will demonstrate how to establish the decomposition formula of the GW invariants associated to the family $W/\mathbf{A}^1$.

4.1. Family of GW-invariants.

Using the perfect obstruction theory of $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$, $\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$ we obtain their virtual cycles $[\mathfrak{M}_{g,k}(\mathfrak{W}, d)]^{\mathrm{vir}}$ and $[\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)]^{\mathrm{vir}}$ and their respective GW-invariants.

The GW-invariants of W_t are homomorphisms

$$\Psi_{\Gamma}^{W_t} : H^*(W_t)^{\times k} \longrightarrow \mathbb{Q}$$

defined by

$$\Psi_{g,k,d}^{W_t}(\alpha) = \left[\mathrm{ev}^*(\alpha)\, [\mathfrak{M}_{g,k}(\mathfrak{W}_t, d)]^{\mathrm{vir}}\right]_0.$$

The Gromov-Witten invariants of $\mathfrak{W}$ is the homomorphism

$$\Psi^{W/\mathbf{A}^1}_{g,k,d}: H^0_{\mathbf{A}^1}(R^*\pi_*\mathbb{Q}_W)^{\times k} \longrightarrow H_2^{\mathrm{BM}}(\mathbf{A}^1) \cong \mathbb{Q}$$

defined by

$$\Psi^{W/\mathbf{A}^1}_{g,k,d}(\alpha) = \mathbf{q}_{*1}\big(\mathrm{ev}^*(\alpha)\,[\mathfrak{M}_{g,k}(\mathfrak{W},d)]^{\mathrm{vir}}\big).$$

Here $\mathbb{Q}_W$ is the sheaf of locally constant functions on W, $\pi: W \to \mathbf{A}^1$ is the tautological projection and H_2^{BM} is the Borel-Moore homology of the open complex curve $\mathbf{A}^1$.

4.2. **The invariance of the GW-invariants.** The invariance of the GW-invariants can be stated as the commutativity of the following diagram.

Let $\xi \in \mathbf{A}^1$ be any closed point and let $H^0_{\mathbf{A}^1}(R^*\pi_*\mathbb{Q}_W) \to H^*(W_\xi)$ be induced by $W_\xi \to W$. For $d \in H^0_{\mathbf{A}^1}(R^*\pi_*\mathbb{Q}_W)$ we denote by $d(\xi)$ its image in $H^*(W_\xi)$. We let $H_2^{\mathrm{BM}}(\mathbf{A}^1) \to \mathbb{Q}$ be the Gysin homomorphism defined by intersecting with the divisor $\xi \in \mathbf{A}^1$. Then we have the commutative diagram

$$\begin{array}{ccc} H^0_{\mathbf{A}^1}(R^*\pi_*\mathbb{Q}_W)^{\times k} & \xrightarrow{\Psi^{W/\mathbf{A}^1}_{g,k,d}} & H_2^{\mathrm{BM}}(\mathbf{A}^1) \cong \mathbb{Q} \\ \downarrow & & \downarrow \\ H^*(W_\xi)^{\times k} & \xrightarrow{\Psi^{W_\xi}_{g,k,d}} & \mathbb{Q}. \end{array} \qquad (4.1) \quad [3.22]$$

The proof is quite involved, and will be omitted.

4.3. **From $\mathfrak{W}_0$ to the relative stable morphisms.** We consider a stable $f \in \mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$ represented by

$$f: X \longrightarrow W[n]_0$$

for some n. We let $D_l \subset W[n]_0$ be its l-th singular divisor. (The D_i is from $i = 0, \cdots, n$.) Then $W[n]_0$ is the gluing of $Y_1[l]_0$ and $Y_2[n-l]_0$ along their distinguished divisors.

By the stability condition on f, The nodes $f^{-1}(D_l) = \{q_i\}$ splits X into two parts $X = (X_1, q_i') \sqcup (X_2, q_i')$, where $X_1 = f^{-1}(Y_1[l]_0)$ and $X_2 = f^{-1}(Y_2[n-l]_0)$. The union is by gluing X_1 and X_2 via $q_i = q_i'$ for $i = 1, \cdots, r$.

It is direct to check that both $h_i = f|_{X_i}: X_i \to Y_i[\cdot]_0$ are stable relative morphisms.

Proposition 10. *Any stable map $f \in \mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$ is a gluing of a pair of relative stable morphisms to $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$. If the target of f is $W[n]_0$, then there are exactly $n+1$ different ways of decomposition.*

Figure 14: This is an example of decomposing stable morphism to $W[2]_0$ into two pairs of relative stable maps.

Let $f : X \to W[n]_0$ be a stable morphism to $\mathfrak{W}_0$. For each $l \in [0, n]$ we can decompose f into a pair of relative stable morphisms:

$$h_1^l : X_1^l \longrightarrow Y_1[l]_0 \quad \text{and} \quad h_2^l : X_2^l \longrightarrow Y_2[n-l]_0.$$

Let Γ_i^l be the associated graph of h_i^l.

Lemma 11. *The $n+1$ pairs of graphs (Γ_1^k, Γ_2^k) are all distinct.*

This is true because of the stability condition imposed on f.

Now let $\gamma = (\Gamma_1, \Gamma_2)$ a pair that appear in such a decomposition. We consider any pair of relative stable morphisms

$$(h_1, h_2) \in \mathfrak{M}(\mathfrak{Y}_1^{\text{rel}}, \Gamma_1) \times \mathfrak{M}(\mathfrak{Y}_2^{\text{rel}}, \Gamma_2).$$

Because γ is derived from a splitting of some $f \in \mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$, X_i has distinguished marked points q_i and q_i', $i = 1, \cdots, r$, in their domains with same orders $\mu_1, \cdots, \mu_r$. In case $h_1(q_i) = h_2(q_i') \in D$, then we can glue h_1 and h_2 along $q_i = q_i'$, to obtain a stable morphism

$$h_1 \sqcup h_2 : X_1 \sqcup X_2 \longrightarrow Y_1[l_1]_0 \sqcup Y_2[l_2]_0 = W[l_1 + l_2]_0.$$

We let

$$\mathfrak{M}(\mathfrak{Y}^{\text{rel}}, \Gamma_i) \longrightarrow D^r$$

be the evaluation of the distinguished marked points. Then the above gluing construction defines a morphisms

$$\Phi_\gamma : \mathfrak{M}(\mathfrak{Y}_1^{\text{rel}}, \Gamma_1) \times_{D^r} \mathfrak{M}(\mathfrak{Y}_2^{\text{rel}}, \Gamma_2) \longrightarrow \mathfrak{M}_{g,k}(\mathfrak{W}_0, d).$$

It is easy to show that the above morphism is a local immersion.

So as sets we have the union

$$\mathfrak{M}_{g,k}(\mathfrak{W}_0, d) = \cup_\gamma \Phi_\gamma\big(\mathfrak{M}(\mathfrak{Y}_1^{\text{rel}}, \Gamma_1) \times_{D^{r_\gamma}} \mathfrak{M}(\mathfrak{Y}_2^{\text{rel}}, \Gamma_2)\big).$$

Because the decomposition are not unique, the above is not a disjoint union.

4.4. **The union as Cartier divisors.** We know that a deformation of $f \in \mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$ in $f \in \mathfrak{M}_{g,k}(\mathfrak{W}, d)$ involves possibly the smooth of the singular divisor of the target. Now in case f is in the image of the above map, the target of f is $W[n]_0$ and the above map is associated to the decomposition along the l-th singular divisor $D_l \subset W[n]_0$, then deformation of f in the image of the above map can possibly smooth all singular divisor $D_{l'}$ except D_l.

In other words, if we let $\mathbf{H}_l \subset \mathbf{A}^{n+1}$ be the coordinate hyperplane defined by $t_l = 0$, then all deformations of f will have target inside

$$W[n] \times_{\mathbf{A}^{n+1}} \mathbf{H}_l.$$

Here is the key to our next step:

Lemma 12. *There is a $G[n]$-linearized pair of a line bundle with a section (L_l, s_l) on $\mathbf{A}^{n+1}$ so that $s_l^{-1}(0) = \mathbf{H}_l$.*

This can be constructed by hands.

The benefit of this is that if we have an S-family of stable morphisms in $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$, say represented by an $S \to \mathbf{A}^{n+1}$ and

$$F : \mathcal{X} \longrightarrow \mathcal{W}_S = W[n] \times_{\mathbf{A}^{n+1}} S,$$

then we can define $S_l = s_l^{-1}(0) \times_{\mathbf{A}^{n+1}} S$ as a closed subscheme of S.

Now we cover $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ by S_α with F_α be the tautological families over S_α. For each k we obtain closed subscheme $S_{d,l}$. But, for a fixed k all $S_{d,l}$ *do not* define a closed substack of $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$.

Here is the reason: Suppose we decomposition $f : X \to W[n]_0$ to a pair (h_1, h_2) along $D_l \subset W[n]_0$ with $\gamma = (\gamma_1, \Gamma_2)$ its pair of graphs. Then $f \in \mathrm{Im}(\Phi_\gamma)$. When we deform f inside $\mathrm{Im}(\Phi_\gamma)$, say f_t, they all decompose into $(h_{t,1}, h_{t,2})$ along D_{l_t} with graph γ_t. From the construction, γ_t is constant but k_t may not be constant in t.

Because the pairs (L_l, s_l) are $G[n]$-equivariant, a moment of thought proves to us

Lemma 13. *For each γ appear in the morphism Φ_γ, there is pair of a line bundle and a section $(\mathbf{L}_\gamma, \mathbf{s}_\gamma)$ over $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ so that as sets*

$$\mathrm{Im}(\Phi_\gamma) = \mathbf{s}_\gamma^{-1}(0).$$

4.5. **The multiplicities are different.** It turns out that

$$\mathrm{Im}(\Phi_\gamma) \subset \mathbf{s}_\gamma^{-1}(0)$$

as stacks, they are equal as sets but usually they are different as stacks.

Here is an easy way to see this phenomenon. We consider an example. We let W be the blowing up of $\mathbf{P}^1 \times \mathbf{A}^1$ at $(0,0)$, as family over $\mathbf{A}^1 = \mathbf{A}^1$. Then W_0 is a union of two $\mathbf{P}^1$, intersecting at one point while $W_t \cong \mathbf{P}^1$. We let $f_s : \mathcal{X}_s \longrightarrow W$, $s \in S = \mathbf{A}^1$, be a family of stable maps so that they have the following local property: We let $\mathbf{A}^2 \subset W$ be a chart so that $W \to \mathbf{A}^1$ is given by $t = (w_1, w_2)$; We can find a chart of $\mathbf{A}^2 \subset \mathcal{X}$ so that $\mathcal{X} \to S$ is via $s = z_1 z_2$ and the map f_s locally is of the form

$$f_s(z_1, z_2) = (z_1^{\mu}, z_2^{\mu}).$$

Then the induced $S \to \mathbf{A}^1$ is of the form $t = s^{\mu}$. In this case, the subscheme defined by $t = 0$ (which is $s_l = 0$ in the previous subsection) is $s^{\mu} = 0$. However, the subscheme associated to the image substack $\mathrm{Im}(\Phi_\gamma)$ is defined by $s = 0$. They differ by a multiple of μ, which is exactly the order associated to the node where the decomposition of the domain is taking place.

Lemma 14. *Both* $\mathrm{Im}(\phi_\gamma)$ *and* $\mathbf{s}_\gamma^{-1}(0)$ *are (virtual) Cartier divisors of the moduli space* $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$. *Further, as divisors they satisfy*

$$m(\gamma)\,\mathrm{Im}(\phi_\gamma) = \mathbf{s}_\gamma^{-1}(0).$$

Here $m(\gamma)$ is the product of all the orders of the distinguished marked points of $\gamma = (\Gamma_1, \Gamma_2)$. Also, by virtually we mean if all relevant subsets are of the expected dimension, then they are actual divisors. Otherwise this is understood in terms of their respective obstruction theory.

4.6. **One more identity.** The bridge between $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ and $\mathfrak{M}(\mathfrak{Y}_i^{\mathrm{rel}}, \Gamma_i)$ is via the morphisms Φ_γ and the following

$$\otimes_{\gamma \text{ admissible}}(\mathbf{L}_\gamma, \mathbf{s}_\gamma) \cong (\mathbf{L}_0, \mathbf{s}_0),$$

where $\mathbf{L}_0$ is the trivial line bundle on $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ and $\mathbf{s}_0$ is the pull back of $t \in \Gamma(\mathcal{O}_{\mathbf{A}^1})$.

The proof relies on the identity $\otimes_{k=0}^{n}(L_l, s_l) \cong \pi^*(\mathbf{1}_{\mathbf{A}^1}, t)$, where $\pi : \mathbf{A}^{n+1} \to \mathbf{A}^1$ is the projection.

4.7. **Statement of the decomposition formula.** We have identities

$$c_1(\mathbf{L}_0, \mathbf{s}_0)[\mathfrak{M}_{g,k}(\mathfrak{W}, d)]^{\mathrm{vir}} = [\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)]^{\mathrm{vir}} \in A_*\mathfrak{M}_{g,k}(\mathfrak{W}_0, d),$$

$$[\mathbf{s}_\gamma^{-1}(0)]^{\mathrm{vir}} = c_1(\mathbf{L}_\gamma, \mathbf{s}_\gamma)[\mathfrak{M}_{g,k}(\mathfrak{W}, d)]^{\mathrm{vir}} \in A_*\mathfrak{M}_{g,k}(\mathfrak{W}_0, d),$$

and

$$\mathbf{m}(\gamma)[\mathrm{Im}(\Phi_\gamma)]^{\mathrm{vir}} = [\mathbf{s}_\gamma^{-1}(0)]^{\mathrm{vir}} \in A_*\mathfrak{M}_{g,k}(\mathfrak{W}_0, d).$$

Combined together, we have

$$[\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)]^{\mathrm{vir}} = \sum_{\gamma} m(\gamma)[\mathrm{Im}(\Phi_\gamma)]^{\mathrm{vir}}.$$

We now state how virtual moduli cycles $[\mathrm{Im}(\Phi_\gamma)]^{\mathrm{vir}}$ is related to $[\mathfrak{M}(\mathfrak{Y}_i^{\mathrm{rel}}, \Gamma_i)]^{\mathrm{vir}}$. Using the natural evaluation morphism $\mathbf{q}_i : \mathfrak{M}(\mathfrak{Y}_i^{\mathrm{rel}}, \Gamma_i) \to D^r$ we form the

Cartesian diagram

(4.2)

[3.51]

$$\begin{array}{ccc} \mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1) \times_{D^r} \mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2) & \longrightarrow & \mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1) \times \mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2) \\ \downarrow & & \downarrow \\ D^r & \stackrel{\Delta}{\longrightarrow} & D^r \times D^r \end{array}$$

Here the arrow Δ is the diagonal morphism.

We have the identity

$$\frac{1}{|\operatorname{Eq}(\gamma)|} \Phi_{\gamma *} \Delta^! \big([\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1)]^{\mathrm{vir}} \times [\mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2)]^{\mathrm{vir}} \big) = [\operatorname{Im}(\Phi_\gamma)]^{\mathrm{vir}}.$$

The main degeneration formula of the Gromov-Witten invariants of $W/\mathbf{A}^1$ follows immediately from these identities.

Theorem 15. *Let the notation be as before. Then*

$$[\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)]^{\mathrm{vir}} = \sum_\gamma \frac{\mathbf{m}(\gamma)}{|\operatorname{Eq}(\gamma)|} \Phi_{\gamma *} \Delta^! \big([\mathfrak{M}(\mathfrak{Y}_1^{rel}, \Gamma_1)]^{\mathrm{vir}} \times [\mathfrak{M}(\mathfrak{Y}_2^{rel}, \Gamma_2)]^{\mathrm{vir}} \big).$$

Here are a few words on $\operatorname{Eq}(\gamma)$. When we decompose $f : X \to W[n]_0$, we assign an order to the distinguished marked points. This assignment is artificial, and has to quotient out to get the right answer. Also, the automorphism of the decomposed curves will introduce extra quotient. The term $\operatorname{Eq}(\gamma)$ is a combination of all these.

Finally, we state the numerical corollary of this theorem. Let $\jmath_i : Y_i \to W$ be the inclusion and let

$$\jmath_i^* : H^0_{\mathbf{A}^1}(R^*\pi_* \mathbb{Q}_W)^{\times k_i} \to H^*(Y_i, \mathbb{Q})^{\times k_i}$$

be the induced pull back homomorphism. Now let $\gamma = (\Gamma_1, \Gamma_2, I) \in \Omega$ be any admissible triple.

Corollary 16. *Let $W/\mathbf{A}^1$ be the family and let $\Gamma = (g, b, k)$ be as before. Then for any closed point $\xi \neq 0 \in \mathbf{A}^1$, $d \in H^0_{\mathbf{A}^1}(R^*\pi_* \mathbb{Q}_W)^{\times k}$ and $\beta \in H^*(\mathfrak{M}_{g,k})$ as before,*

$$\Phi_\Gamma^{W_\xi}(\alpha(\xi)) = \sum_\gamma \frac{\mathbf{m}(\gamma)}{|\operatorname{Eq}(\gamma)|} \big[\Psi_{\Gamma_1}^{Y_1^{rel}}(\jmath_1^* d) \bullet \Psi_{\Gamma_2}^{Y_2^{rel}}(\jmath_2^* \alpha) \big]_0.$$

Here $\bullet$ is the intersection of the homology groups

$$H_*(D^r) \times H_*(D^r) \stackrel{\cap}{\longrightarrow} H_*(D^r)$$

and $[\gamma]_0$ is the degree of the degree 0 part of the homology class $\gamma \in H_*(D^r)$.

5. The Obstruction theory

There is a symplectic version of relative GW-invariants developed by [5, 11, 17]. The benefit of an algebraic version of the relative GW-invariants is because we can apply virtual localization to in many applications. For this a good understanding of the deformation of the moduli of relative GW-invariants is in order. In this part, we give a brief description of such obstruction. For details see [15, Appendix].

5.1. **The first order deformations.** Let $Y^{\mathrm{rel}} = (Y, D)$ be as before and let $f : X \to Y[n]_0$ be a relative stable morphism. We like to describe the tangent space $T^1 = T_f \mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ and the obstruction space T^2 to deformation of f.

Obviously, the first order deformation of f is the first order deformation of $f : X \to Y[n]$, observing the admissible condition and the contact order requirement at the distinguished marked points, modulo the $G[n]$ action.

For simplicity, we consider the case $n = 1$. The general case is similar.

We let $q_1, \cdots, q_r$ be the distinguished marked points of X, with contact order $\mu_1, \cdots, \mu_r$. We let $E \subset X$ be the divisor of all marked points of f.

We let $\xi_1, \cdots, \xi_l$ be the nodes of X in $f^{-1}(D_1)$.

We have two exact sequences relating T^1 and T^2 to some known cohomology groups:

$$0 \longrightarrow \mathrm{Ext}^0_X(\Omega_X(E), \mathcal{O}_X) \longrightarrow A^0 \longrightarrow T^1 \oplus \mathbb{C} \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^1_X(\Omega_X(E), \mathcal{O}_X) \longrightarrow A^1 \longrightarrow T^2 \longrightarrow 0$$

and

$$0 \longrightarrow H^0(f^*\Omega^\vee_{Y[1]^\dagger/\mathbf{A}^{1\dagger}}) \longrightarrow A^0 \xrightarrow{b_0} \mathbb{C}^{\oplus r} \longrightarrow$$
$$\xrightarrow{\delta} H^1(f^*\Omega^\vee_{Y[1]^\dagger/\mathbf{A}^{1\dagger}}) \longrightarrow A^1 \xrightarrow{b_1} \mathbb{C}^{\oplus r}/\mathbb{C} \longrightarrow 0.$$

Here $\Omega_{Y[1]^\dagger/\mathbf{A}^{1\dagger}}$ is the sheaf of log-differentials of the pair of log-schemes (cf. [12, 13]).

Here is a heuristic explanation of the these two exact sequences without going into the technical details.

First, the term $T^1 \oplus \mathbb{C}$ is the space of the first order deformations of $f : X \to Y[1]$, as admissible map. Since $f \in \mathfrak{M}_{g,k}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ is the equivalence class of f via the $\mathbb{C}^*$ action, the tangent space $T_f \mathfrak{M}_{g,k}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma) = (T^1 \oplus \mathbb{C})/\mathbb{C} = T^1$.

If $f_s : X_s \to Y[1]$ is a deformation of f, then X_s induces a deformation of the domain curve X. This explain the arrow $T^1 \oplus \mathbb{C} \to \mathrm{Ext}^1_X(\Omega_X(E), \mathcal{O}_X)$.

Now assume X_s is a constant family. Then the deformation is a deformation of maps only $f_s : X \to Y[1]$. We now describe the arrow b_0. Let q be a node in $f^{-1}(D_1)$. We pick an (analytic) chart of $q \in X$, via (z_1, z_2) subject to $z_1 z_2 = 0$. We then pick a chart of $f(q) \in Y[1]$, say $(w - 1, w_2, \cdots)$ with

$Y[1] \to \mathbf{A}^1$ defined by $t = w_1 w_2$. We assume the map f is given by $w_i = z_i^{\mu}$. The deformation f_s will be of the form $w_i = z_i^{\mu}(1 + c_i(z, s))$ so that

$$(1 + c_1(z, s))(1 + c_2(z, s)) = 1 + c(s).$$

The arrow b_0 will send $\dot{f}_0$ to $\dot{c}(0) \in \mathbb{C}$. Clearly, if $b_0(\dot{f}_0) = 0$, then the first order deformation $\dot{f}_0$ is given by a (local) section of the log differential $f^*\Omega^{\vee}_{Y[1]^{\dagger}/\mathbf{A}^{1\dagger}}$. This explains that the kernel of b_0 is $H^0(f^*\Omega^{\vee}_{Y[1]^{\dagger}/\mathbf{A}^{1\dagger}})$.

But then we need to cancel those in A^0 that can arise from vector fields of X. Hence the kernel of $T^1 \oplus \mathbb{C} \to \mathrm{Ext}^1_X(\Omega_X(E), \mathcal{O}_X)$ is the cokernel of $\mathrm{Ext}^0_X(\Omega_X(E), \mathcal{O}_X) \to A^0$.

5.2. **An example of obstruction bundle.** For amusement, let us determine the obstruction bundle of an example of the moduli of relative stable morphisms.

Let $Y_{\mathrm{rel}} = (Y, D)$ be a pair of smooth variety and a smooth divisor. We let Γ be the graph consisting of one vertex and one leg. We assign the weights of the vertex to be $g = 1$ and $d = 0$. Thus $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ is the moduli of relative stable morphisms to Y from 1-pointed genus 1 curves to Y of degree 0. Since $d = 0$, all $f : X \to Y$ in $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ are constant maps. Hence $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ is isomorphic to $\mathfrak{M}_{1,1} \times Y$. We now show that its obstruction sheaf is

$$\mathcal{O}b = \pi_2^* \Omega_Y(\log D)^{\vee},$$

where $\pi_2 : \mathfrak{M}_{1,1} \times Y \to Y$ is the second projection.

Let $f_0 \in \mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ be a relative stable morphism. Since $d = 0$, we can always represent f_0 by a morphism $f_0 : X \to Y[1]^o$, where $Y[1]^o = Y[1] - D[1] \cup Y[1]_{0,\mathrm{sing}}$ with $Y[1]_{0,\mathrm{sing}}$ is the singular locus of $Y[1]_0$. Then $Y[1]^o/\mathbb{C}^* \equiv Y$. The obstruction to deforming f_0 as morphism to $Y[1]^o$ is $H^1(f_0^* T_{Y[1]^o/\mathbf{A}^1}) \equiv T_{Y[1]^o/\mathbf{A}^1}|_{f_0(X)}$, where $Y[1]^o \to \mathbf{A}^1$ is the tautological projection and $T_{Y[1]^o/\mathbf{A}^1}$ is the relative tangent bundle. We let $f : \mathcal{X} \to Y[1]^o$ be the family over $\mathfrak{M}_{1,1} \times Y[1]^o$ so that $\mathcal{X}$ is the pull back of the universal family over $\mathfrak{M}_{1,1}$ while the morphism f is the composite of the projection $\mathcal{X} \to \mathfrak{M}_{1,1} \times Y[1]^o$ with the second projection $\mathfrak{M}_{1,1} \times Y[1]^o \to Y[1]^o$. Clearly, f is the universal family of $\mathfrak{M}(Y[1]^{\mathrm{rel}}, \Gamma)$. The obstruction bundle to the moduli space $\mathfrak{M}(Y[1]^{\mathrm{rel}}, \Gamma)$ over $\mathfrak{M}_{1,1} \times Y[1]^o$ is $p_2^* T_{Y[1]^o/\mathbf{A}^1}$, where p_2 is the second projection of $\mathfrak{M}_{1,1} \times Y[1]^o$. The $\mathbb{C}^*$-action lifts canonically to $p_2^* T_{Y[1]^o/\mathbf{A}^1}$ and the obstruction sheaf of $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ is the descent of $p_2^* T_{Y[1]^o/\mathbf{A}^1}$. It is direct to check that under the quotient map $\mathfrak{M}_{1,1} \times Y[1]^o/\mathbb{C}^* \cong \mathfrak{M}_{1,1} \times Y$, equivariant part $(p_2^* T_{Y[1]^o/\mathbf{A}^1})^{\mathbb{C}^*}$ is canonically isomorphic to $\pi_2^* \Omega_Y(\log D)^{\vee}$. This proves the identity.

6. Applications

There are several known applications, worked out by various people. I will list the references here.

The first application is the enumeration of curves in $\mathbf{P}^2$. This was worked out by Caporaso-Harris [4]. Ionel-Parker showed that their result can be proved using degeneration of GW-invariants [11].

The second application is to derive some vanishing results on $\mathfrak{M}_{g,k}$ by using the degeneration formula of GW-invariants. This was first worked on by A.Li-Zhao-Zheng [14] and Ionel [10]

Another one is the study of Hurwize number. As explained, these number are exactly the relative GW-invariants of curves. There are some work in this direction [14, 9].

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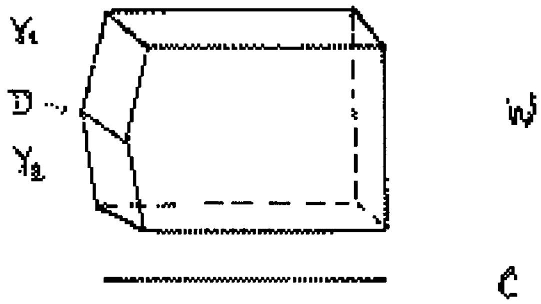

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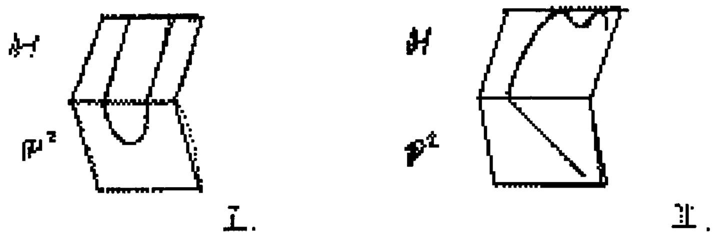

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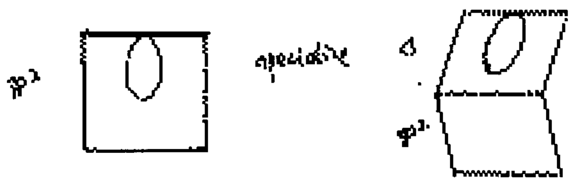

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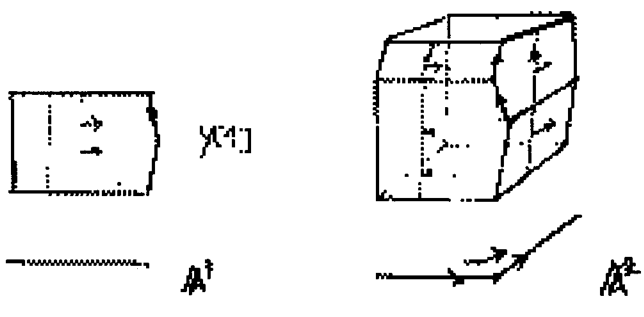

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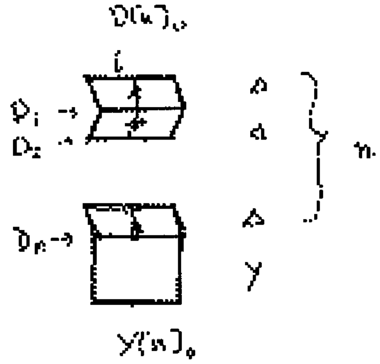

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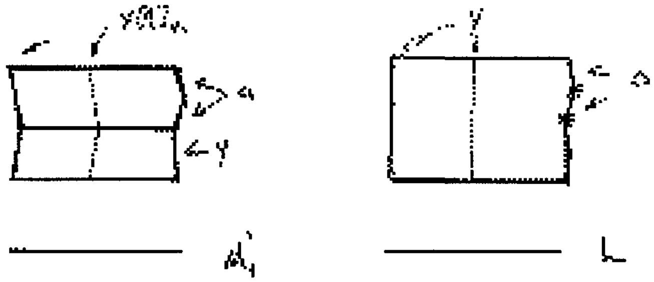

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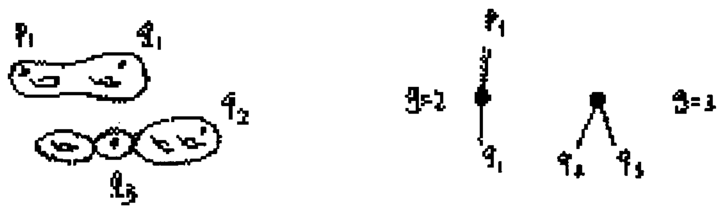

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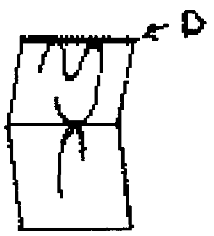

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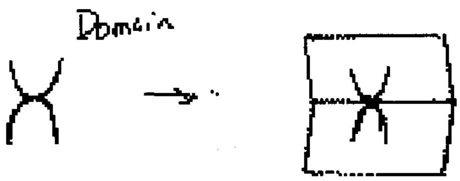

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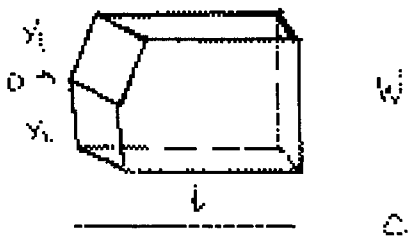

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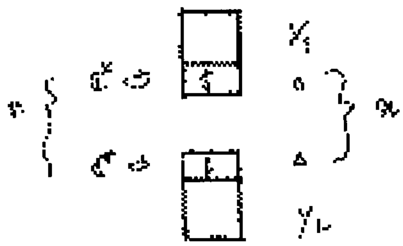

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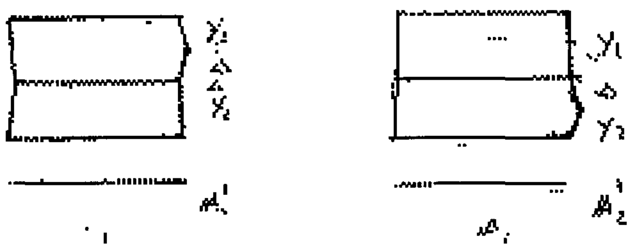

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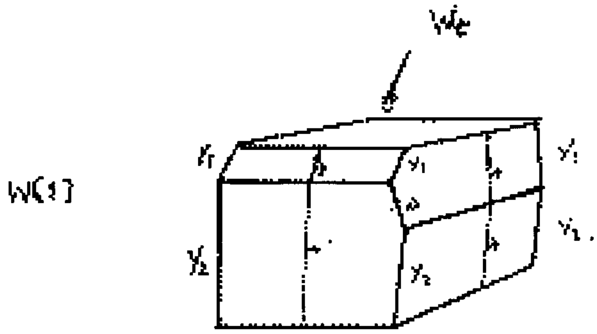

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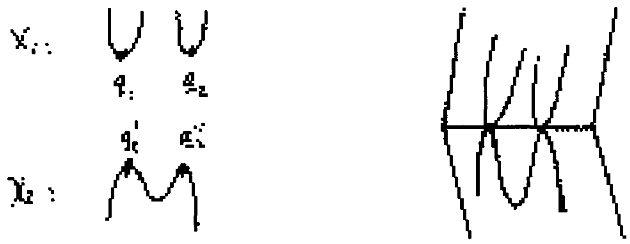

Figure 14