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Cohomology and Intersection theory of Algebraic stacks (I)

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1.1 Lie groupoids and de Rham cohomology

Differentiable stacks are stacks over the category of differentiable manifolds. They are the stacks associated to Lie groupoids. A groupoid $X_1 \rightrightarrows X_0$, is a *Lie groupoid* if both X_0 and X_1 are smooth C^∞-manifolds and source and target map are C^∞-submersions.

Two Lie groupoids $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ give rise to essentially the same stack, if and only if they are *Morita equivalent*, which means that there is a third Lie groupoid $Z_1 \rightrightarrows Z_0$, together with Morita morphisms $Z_\bullet \to X_\bullet$ and $Z_\bullet \to Y_\bullet$. A morphism of Lie groupoids $f : X_\bullet \to Y_\bullet$ is a *Morita morphism* if $f_0 : X_0 \to Y_0$ is a surjective submersion and the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{(s,t)} & X_0 \times X_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0 \times f_0} \\ Y_1 & \xrightarrow{(s,t)} & Y_0 \times Y_0 \end{array}$$

is cartesian, i.e., a pullback diagram of differentiable manifolds. We say that a Morita morphism $f : X_\bullet Y_\bullet$ *admits a section* if $X_0 \to Y_0$ admits a section.

Any section $s : X_0 \to Y_0$ of a Morita morphism $f : X_\bullet \to Y_\bullet$ induces uniquely a groupoid morphism $s : Y_\bullet \to X_\bullet$ with the properties

- $f \circ s = \mathrm{id}_{Y_\bullet}$,
- $s \circ f \cong \mathrm{id}_{X_\bullet}$, which means that there exits a 2-isomorphism $\theta : s \circ f \Rightarrow \mathrm{id}_{X_\bullet}$.

(Recall that a 2-isomorphism of morphisms of Lie groupoids $X_\bullet \to Y_\bullet$ is a differentiable map $\theta : X_0 \to Y_1$ satisfying the formal properties of a natural transformation between functors.)

The simplicial nerve of a Lie groupoid

Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. Then we can produce a simplicial manifold $X_\bullet$ as follows. For every $p \geq 0$ we let X_p be the manifold of composable sequences of elements of X_1 of length p. In other words,

$$X_p = \underbrace{X_1 \times_{X_0} X_1 \times_{X_0} \ldots \times_{X_0} X_1}_{p} .$$

Then we have $p+1$ C^∞-maps $\partial_i : X_p \to X_{p-1}$, for $i = 0, \ldots, p$, where ∂_i is given by 'leaving out the i-th object'. Thus ∂_0 leaves out the first arrow, ∂_p leaves out the last arrow, and $\partial_1, \ldots, \partial_{p-1}$ are given by composing two successive arrows. (There are also maps $X_{p-1} \to X_p$, given by inserting identity arrows, but they are less important for us.) Note that for the composition of maps $X_p \to X_{p-2}$ we have the relations

$$[\texttt{rel}]\ \partial_i \partial_j = \partial_{j-1} \partial_i , \quad \text{for all } 0 \leq i < j \leq p. \tag{1}$$

We summarize this data by the diagram of manifolds

$$[\texttt{sim.ma}]\ \cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} X_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} X_1 \rightrightarrows X_0 . \tag{2}$$

Čech cohomology

Let $X_1 \rightrightarrows X_0$ be a Lie groupoid and $X_\bullet$ the associated simplicial manifold. Letting Ω^q be the sheaf of q-forms, we get an induced cosimplicial set

$$[\texttt{cosim}] \quad \Omega^q(X_0) \rightrightarrows \Omega^q(X_1) \;\overrightarrow{\rightrightarrows}\; \Omega^q(X_2) \;\overrightarrow{\overrightarrow{\rightrightarrows}}\; \cdots \tag{3}$$

Since this is, in fact, a cosimplicial abelian group, we can associate a complex

$$\Omega^q(X_0) \xrightarrow{\partial} \Omega^q(X_1) \xrightarrow{\partial} \Omega^q(X_2) \xrightarrow{\partial} \cdots$$

Here $\partial : \Omega^q(X_k) \to \Omega^q(X_{k+1})$ is given by $\partial = \sum_{i=0}^{k+1}(-1)^i \partial_i^*$. We call this complex the *Čech complex* associated to the sheaf Ω^q and the simplicial manifold $X_\bullet$. Its cohomology groups $H^k(X_1 \rightrightarrows X_0, \Omega^q)$ are called *Čech cohomology groups* of the groupoid $X_1 \rightrightarrows X_0$ with values in the sheaf of q-forms Ω^q.

Remark 1.1 (naturality) Given a morphism of Lie groupoids $f : X_\bullet \to Y_\bullet$, we get an induced homomorphism of Čech complexes

$$f^* : \check{C}^*(Y_\bullet, \Omega^q) \to \check{C}^*(X_\bullet, \Omega^q).$$

It is given by the formula

$$f^*(\omega)(\phi_1 \dots \phi_p) = \omega\big(\mathrm{f}(\phi_1) \dots \dots f(\phi_p)\big),$$

for $\omega \in \Omega^q(Y_p)$. Here $\phi_1 \dots \phi_p$ abbreviates the element

$$x_0 \xrightarrow{\phi_1} x_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_p} x_p$$

in X_p. This follows directly from the presheaf property of Ω^q and the functoriality of f.

More interestingly, if we have a 2-isomorphism $\theta : f \Rightarrow g$ between the two morphisms $f, g : X_\bullet \to Y_\bullet$, then we get an induced homotopy $\theta^* : f^* \Rightarrow g^*$, between the two induced homomorphisms $f^*, g^* : \check{C}(Y_\bullet, \Omega^q) \to \check{C}(X_\bullet, \Omega^q)$. In fact, $\theta^* : \Omega^q(Y_{p+1}) \to \Omega^q(X_p)$ is defined by the formula

$$\theta^*(\omega)(\phi_1 \dots \phi_p) = \sum_{i=0}^{p}(-1)^i \omega\big(f(\phi_1) \dots f(\phi_i)\theta(x_i)g(\phi_{i+1}) \dots g(\phi_p)\big).$$

One checks (this is straightforward but tedious) that

$$\partial\theta^* + \theta^*\partial = g^* - f^*.$$

As a consequence of these naturality properties we deduce that

- groupoid morphisms induce homomorphisms on Čech cohomology groups,
- 2-isomorphic groupoid morphisms induce identical homomorphisms on Čech cohomology groups,
- a Morita morphism admitting a section induces *isomorphisms* on Čech cohomology groups.

Proposition 1.2 [banal] *If $X_1 \rightrightarrows X_0$ is the banal groupoid associated to a surjective submersion of manifolds $X_0 \to Y$, then the Čech cohomology groups $H^k(X_\bullet, \Omega^q)$ vanish, for all $k > 0$ and all $q \geq 0$. Moreover, $H^0(X_\bullet, \Omega^q) = \Gamma(Y, \Omega^q)$.*

PROOF. Recall that the banal groupoid associated to a submersion $X_0 \to Y$ is defined by setting $X_1 = X_0 \times_Y X_0$. Since $X_1 \to X_0 \times X_0$ is then an equivalence relation on X_0, we get a groupoid structure on $X_1 \rightrightarrows X_0$. Note that such a banal groupoid comes with a canonical Morita morphism $X_\bullet \to Y$, where Y is considered as a groupoid $Y \rightrightarrows Y$ in the trivial way.

For example, if $\{U_i\}$ is an open cover of Y, and $X_0 = \coprod U_i$, then $X_1 = \coprod U_{ij}$, where $U_{ij} = U_i \cap U_j$. In this case the proposition is a standard fact, which follows essentially from the existence of partitions of unity. See for example Proposition 8.5 of [1], where this result is called the generalized Mayer-Vietoris sequence.

Another case where the proof is easy, is the case of a surjective submersion with a section. This is because a section $s : Y \to X_0$ induces a section of the Morita morphism $X_\bullet \to Y$. Thus by naturality we have $H^k(X_\bullet, \Omega^q) = H^k(Y \rightrightarrows Y, \Omega^q)$, which vanishes for $k > 0$, and equals $\Gamma(Y, \Omega^q)$, for $k = 0$.

The general case now follows from these two special cases by a double fibration argument. Let $\{U_i\}$ be an open cover of Y over which $X_0 \to Y$ admits local sections and let $V = \coprod U_i$. We consider the banal groupoid $V_\bullet$ given by $V \to Y$.

The key is to introduce $W = X_0 \times_Y V$. Thus $W \to V$ is now a surjective submersion which admits a section. We define $W_{mn} = X_m \times_Y V_n$, for all $m, n \geq 0$.

$$\begin{array}{ccc} W_{mn} & \longrightarrow & V_n \\ \downarrow & & \downarrow \\ X_m & \longrightarrow & Y \end{array}$$

Then $W_{\bullet\bullet}$ is a bisimplicial manifold. This means that we have an array

$$\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots \\
 & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\cdots \Rrightarrow & W_{22} & \Rrightarrow & W_{12} & \rightrightarrows & W_{02} & \longrightarrow & V_2 \\
 & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\cdots \Rrightarrow & W_{21} & \Rrightarrow & W_{11} & \rightrightarrows & W_{01} & \longrightarrow & V_1 \\
 & \downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\cdots \Rrightarrow & W_{20} & \Rrightarrow & W_{10} & \rightrightarrows & W_{00} & \longrightarrow & V_0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \Rrightarrow & X_2 & \Rrightarrow & X_1 & \rightrightarrows & X_0 & \longrightarrow & Y
\end{array}$$

It is important to notice that $W_{\bullet n}$ is the simplicial nerve of the banal groupoid associated to $W_{0n} \to V_n$, and $W_{m\bullet}$ is the simplicial nerve of the banal groupoid associated to $W_{m0} \to X_m$. All $W_{0n} \to V_n$ are submersions admitting sections and all $W_{m0} \to X_m$ are submersions coming from open covers. Thus we already know the proposition for all of these submersions.

We apply Ω^q to this array to obtain a double complex $\Omega^p(W_{\bullet\bullet})$ mapping to the two complexes $\Omega^q(X_\bullet)$ and $\Omega^q(V_\bullet)$.

$$\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & \Omega^q(W_{22}) & \longrightarrow & \Omega^q(W_{12}) & \longrightarrow & \Omega^q(W_{02}) & \longrightarrow & \Omega^q(V_2) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & \Omega^q(W_{21}) & \longrightarrow & \Omega^q(W_{11}) & \longrightarrow & \Omega^q(W_{01}) & \longrightarrow & \Omega^q(V_1) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & \Omega^q(W_{20}) & \longrightarrow & \Omega^q(W_{10}) & \longrightarrow & \Omega^q(W_{00}) & \longrightarrow & \Omega^q(V_0) \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & \Omega^q(X_2) & \longrightarrow & \Omega^q(X_1) & \longrightarrow & \Omega^q(X_0) & &
\end{array}$$

Passing to cohomology we get a commutative diagram

$$\begin{array}{ccc} H^*(W_{\bullet\bullet}, \Omega^q) & \longrightarrow & H^*(V_\bullet, \Omega^q) \\ \downarrow & & \downarrow \cong \\ H^*(X_\bullet, \Omega^q) & \longrightarrow & H^*(Y \rightrightarrows Y, \Omega^q) \end{array}$$

and noticing that the two arrows originating at $H^*(W_{\bullet\bullet}, \Omega^q)$ are isomorphisms, which follows by calculating cohomology of the double complex in two different ways, we get the required result. □

Corollary 1.3 *Any Morita morphism of Lie groupoids $f : X_\bullet \to Y_\bullet$ induces isomorphisms on Čech cohomology groups $f^* : H^k(Y_\bullet, \Omega^q) \xrightarrow{\sim} H^k(X_\bullet, \Omega^q)$. Morita equivalent groupoids have canonically isomorphic Čech cohomology groups with values in Ω^q.*

PROOF. We first prove the latter statement. Let $\mathfrak{X}$ be the differentiable stack given by the first groupoid $X_\bullet$. Then there also exists a morphism $Y_\bullet \to \mathfrak{X}$, identifying $\mathfrak{X}$ as the stack given by $Y_\bullet$. Form the fibered product $Z_{00} = X_0 \times_{\mathfrak{X}} Y_0$. Then define a bisimplicial manifold $Z_{\bullet\bullet}$ as in the previous proof and apply the same kind of double fibration argument to produce isomorphisms $H^*(Z_{\bullet\bullet}, \Omega^q) \to H^*(X_\bullet)$ and $H^*(Z_{\bullet\bullet}, \Omega^q) \to H^*(Y_\bullet)$. □

Thus we can make the following definition.

Definition 1.4 Let $\mathfrak{X}$ be a differentiable stack. Then

$$H^k(\mathfrak{X}, \Omega^q) = H^k(X_\bullet, \Omega^q),$$

for any Lie groupoid $X_\bullet$ giving an atlas for $\mathfrak{X}$. In particular, this defines

$$\Gamma(\mathfrak{X}, \Omega^q) = H^0(\mathfrak{X}, \Omega^q).$$

Example 1.5 If G is a Lie group then $H^k(BG, \Omega^0)$ is the group cohomology of G calculated with differentiable cochains. Thus there are stacks for which these cohomology groups are non-trivial.

The Čech-de Rham complex

The exterior derivative $d : \Omega^q \to \Omega^{q+1}$ connects the various Čech complexes of a Lie groupoid with each other. We thus get a double complex

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\ \Omega^2(X_0) & \xrightarrow{\partial} & \Omega^2(X_1) & \xrightarrow{\partial} & \Omega^2(X_2) & \xrightarrow{\partial} & \cdots \\ \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\ \Omega^1(X_0) & \xrightarrow{\partial} & \Omega^1(X_1) & \xrightarrow{\partial} & \Omega^1(X_2) & \xrightarrow{\partial} & \cdots \\ \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\ \Omega^0(X_0) & \xrightarrow{\partial} & \Omega^0(X_1) & \xrightarrow{\partial} & \Omega^0(X_2) & \xrightarrow{\partial} & \cdots \end{array} \quad [\mathrm{dr}] \tag{4}$$

We make a total complex out of this by setting

$$C^n_{DR}(X_1 \rightrightarrows X_0) = \bigoplus_{p+q=n} \Omega^q(X_p)$$

and defining the total differential $\delta : C^n_{DR}(X_1 \rightrightarrows X_0) \to C^{n+1}_{DR}(X_1 \rightrightarrows X_0)$ by

$$\delta(\omega) = \partial(\omega) + (-1)^p d(\omega), \quad \text{for } \omega \in \Omega^q(X_p).$$

The sign change is introduced in order that $\delta^2 = 0$.

Definition 1.6 The complex $C^*_{DR}(X_1 \rightrightarrows X_0)$ is called the *de Rham complex* of the Lie groupoid $X_1 \rightrightarrows X_0$. Its cohomology groups

$$H^n_{DR}(X_1 \rightrightarrows X_0) = h^n\big(C^*_{DR}(X_1 \rightrightarrows X_0)\big)$$

are called the *de Rham* cohomology groups of $X_1 \rightrightarrows X_0$.

If $X_1 \rightrightarrows X_0$ is the étale banal groupoid associated to an open cover $X_0 \to Y$ of a manifold Y, then the de Rham complex of $X_1 \rightrightarrows X_0$ is just the usual Čech-de Rham complex as treated, for example, in Chapter II of [1].

Remark 1.7 One can use Proposition 1.2 and a double fibration argument to prove that de Rham cohomology is invariant under Morita equivalence and hence well-defined for differentiable stacks:

$$H^n_{DR}(\mathfrak{X}) = H^n_{DR}(X_1 \rightrightarrows X_0),$$

for any groupoid atlas $X_1 \rightrightarrows X_0$ of the stack $\mathfrak{X}$.

Moreover, there is a canonical isomorphism

$$H^n_{DR}(\mathfrak{X}) = H^n(\mathfrak{X}, \mathbb{R}),$$

where $\mathbb{R}$ is the big sheaf of locally constant $\mathbb{R}$-valued functions. This latter claim follows from the exact sequence of big sheaves (see Section 1.3)

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \dots$$

Example 1.8 Equivariant cohomology.

Multiplicative structure

We define a multiplication on the double complex (4) as follows. Let $\omega \in \Omega^q(X_p)$ and $\eta \in \Omega^{q'}(X_{p'})$. Then we set

$$[\text{cup}] \quad \omega \cup \eta = (-1)^{qp'} s^*\omega \wedge t^*\eta \quad \in \Omega^{q+q'}(X_{p+p'}) \,. \tag{5}$$

Here the map $s : X_{p+p'} \to X_p$ projects the element

$$[\texttt{elt}] \quad \circ \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_p} \circ \xrightarrow{\phi_{p+1}} \cdots \xrightarrow{\phi_{p+p'}} \circ \quad \in X_{p+p'} \tag{6}$$

to

$$\circ \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_p} \circ$$

and $t : X_{p+p'} \to X_{p'}$ projects the same element (6) to

$$\circ \xrightarrow{\phi_{p+1}} \cdots \xrightarrow{\phi_{p+p'}} \circ \,.$$

One checks that

$$\delta(\omega \cup \eta) = \delta(\omega) \cup \eta + (-1)^{p+q} \omega \cup \delta(\eta) \,,$$

and so we get an induced cup product

$$H^n_{DR}(X_\bullet) \otimes H^m_{DR}(X_\bullet) \longrightarrow H^{n+m}_{DR}(X_\bullet) \,.$$

Note that there is no reason why the cup product should be skew commutative on the level of cochains. On the other hand, on the level of cohomology this is the case.

Cohomology with compact supports

As with cohomology, cohomology with compact supports is defined via a double complex. As usual, let $X_1 \rightrightarrows X_0$ be a Lie groupoid. But now we have to also assume that $X_1 \rightrightarrows X_0$ is oriented. This means that both the manifolds X_1 and X_0 and the submersions s and t are oriented, in a compatible way. Moreover, assume that both X_0 and X_1 have constant dimension. Define two numbers r, n by the formulas

$$r = \dim X_1 - \dim X_0, \quad n = 2 \dim X_0 - \dim X_1 \,.$$

Note that n is the dimension of the stack defined by $X_1 \rightrightarrows X_0$ and r is the relative dimension of X_1 over X_0.

Let $\Omega_c^q(X_p)$ denote the space of differential forms on X_p which have compact support. Note that Ω_c^q is not a sheaf. We consider the double complex

$$
\text{[cs]}\qquad
\begin{array}{ccccccc}
\cdots \xrightarrow{\partial_!} & \Omega_c^{n+3r}(X_2) & \xrightarrow{\partial_!} & \Omega_c^{n+2r}(X_1) & \xrightarrow{\partial_!} & \Omega_c^{n+r}(X_0) \\
 & \uparrow d & & \uparrow d & & \uparrow d \\
\cdots \xrightarrow{\partial_!} & \Omega_c^{n+3r-1}(X_2) & \xrightarrow{\partial_!} & \Omega_c^{n+2r-1}(X_1) & \xrightarrow{\partial_!} & \Omega_c^{n+r-1}(X_0) \\
 & \uparrow d & & \uparrow d & & \uparrow d \\
\cdots \xrightarrow{\partial_!} & \Omega_c^{n+3r-2}(X_2) & \xrightarrow{\partial_!} & \Omega_c^{n+2r-2}(X_1) & \xrightarrow{\partial_!} & \Omega_c^{n+r-2}(X_0) \\
 & \uparrow d & & \uparrow d & & \uparrow d \\
 & \vdots & & \vdots & & \vdots
\end{array}
\qquad (7)
$$

Here d is usual exterior derivative. The boundary map $\partial_!$ is the alternating sum of the maps obtained from the various $\partial_i : X_p \to X_{p-1}$ be integration over the fiber $\partial_{i!} : \Omega_c^{q+r}(X_p) \to \Omega_c^q(X_{p-1})$.

To make a single complex out of (7), we define

$$C_c^\nu(X_1 \rightrightarrows X_0) = \bigoplus_{q-rp-p-r=\nu} \Omega_c^q(X_p)\,,$$

and set the total differential equal to

$$\delta(\omega) = \partial_!(\omega) + (-1)^p d(\omega), \quad \text{for } \omega \in \Omega_c^q(X_p)\,.$$

Thus the total degree of an element of $\Omega_c^q(X_p)$ is equal to $q - rp - p - r = q - (p+1)(r+1) + 1$.

We also introduce notation for the horizontal cohomology of (7). Namely, we denote the k-th homology of $\big(\Omega_c^{(*+1)r-q}(X_*), \partial_!\big)$ by $H_c^k(X_\bullet, \Omega^q)$. This defines $H_c^k(X_\bullet, \Omega^q)$ for $k \leq 0$ and $q \leq n$. We also denote $H_c^0(X_\bullet, \Omega^q)$ by $\Gamma_c(X_\bullet, \Omega)$.

Module structure

Now we shall turn (7) into a module over (4). Given $\omega \in \Omega^q(X_p)$ and $\gamma \in \Omega_c^{q'}(X_{p'})$, we set

$$\omega \cap \gamma = (-1)^{qp'} t_!(s^*\omega \wedge \gamma)\,,$$

where s and t have similar meanings as in (5). More precisely, they are defined according to the cartesian diagram

$$
\begin{array}{ccc}
X_{p'} & \xrightarrow{\;t\;} & X_{p'-p} \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \text{0-th projection}} \\
X_p & \xrightarrow[\text{p-th projection}]{} & X_0\,.
\end{array}
$$

Note that $\omega \cap \gamma \in \Omega_c^{q+q'-pr}(X_{p'-p})$ and hence we have

$$\deg(\omega \cap \gamma) = \deg \omega + \deg \gamma .$$

Of course, if $p' < p$, then it is understood that $\omega \cap \gamma = 0$. One checks the formula

$$\delta(\omega \cap \gamma) = \delta\omega \cap \gamma + (-1)^{\deg \omega} \omega \cap \delta\gamma ,$$

which implies that the cap product passes to cohomology, and we have that $H_c^*(X_\bullet)$ is a graded module over the graded ring $H^*(X_\bullet)$.

The integral

We can define and integral

$$\int_{X_\bullet} : H_c^n(X_\bullet) \longrightarrow \mathbb{R}$$

by noticing that the integral $\Omega_c^{n+r}(X_0) \to \mathbb{R}$ vanishes on coboundaries of the total complex $C *_c (X_\bullet)$.

Finally, we define a pairing

$$\begin{aligned} [\mathrm{PD}] \quad H^*(X_\bullet) \otimes H_c^*(X_\bullet) &\longrightarrow \mathbb{R} \\ \omega \otimes \gamma &\longmapsto \int_{X_\bullet} \omega \cap \gamma , \end{aligned} \tag{8}$$

For Poincaré duality, let us assume that X_1 and X_0 have compatible finite good covers.

Proposition 1.9 (Poincaré duality) *Under this assumption, the pairing (8) sets up a perfect pairing*

$$H^p(X_\bullet) \otimes H_c^{n-p}(X_\bullet) \longrightarrow \mathbb{R},$$

for all $p \geq 0$.

PROOF. Consider the homomorphism of complexes

$$\begin{aligned} C^*(X_\bullet) &\longrightarrow \left(C_c^*(X_\bullet)[n]\right)^\vee \\ \omega &\longmapsto \int_{X_0} \omega \cap (\cdot) . \end{aligned}$$

It suffices to prove that this is a quasi-isomorphism. But this we can check by considering the associated spectral sequences whose E_1-terms are given by $H^q(X_p)$ and $H_c^{n-q}(X_p)^\vee$, respectively. Thus conclude using usual Poincaré duality for manifolds (see for example [1, §5]). □

Deligne-Mumford stacks

From now on, we will assume that our Lie groupoids are étale, which means that $s : X_1 \to X_0$ and $t : X_1 \to X_0$ are étale. We will also assume that $X_1 \to X_0 \times X_0$ is proper and unramified, with finite fibers. This means that the associated differentiable stack is of Deligne-Mumford type.

Definition 1.10 A *partition of unity* for the groupoid $X_1 \rightrightarrows X_0$ is an $\mathbb{R}$-valued C^∞-function ρ on X_0 with the property that $s^*\rho$ has proper support with respect to $t : X_1 \to X_0$ and

$$t_! s^* \rho \equiv 1\,.$$

Partitions of unity may not exist, unless we pass to a Morita equivalent groupoid. This process works as follows.

For groupoid as above there always exists an open cover $\{U_i\}$ of X_0, with the property that the restricted groupoid $V_i \rightrightarrows U_i$ (which is the restriction of $X_1 \rightrightarrows X_0$ via $U_i \to X_0$) is a transformation groupoid associated to the action of a finite group G_i on U_i. Given such a cover, we let $U = \coprod U_i$ and $V \rightrightarrows U$ be the restriction of $X_1 \rightrightarrows X_0$ via $U \to X_0$. Thus we have a Morita morphism from $X_1 \rightrightarrows X_0$ to $V \rightrightarrows U$.

Now we consider the moduli space $\overline{X}$ of $X_\bullet$, which is also the moduli space of $V \rightrightarrows U$. The open cover $\{U_i\}$ of X_0 induces an open cover of $\overline{X}$. Choose a differentiable partition of unity for $\overline{X}$ subordinate to this cover. Pull back to U. This gives over each U_i a G_i-invariant differentiable function ρ_i. Define $\rho : U \to \mathbb{R}$ by setting $\rho \,|\, U_i = \frac{1}{\#G_i}\rho_i$. It is then straightforward to check that ρ is, indeed, a partition of unity for the groupoid $V \rightrightarrows U$.

Proposition 1.11 *Assume that the groupoid $X_1 \rightrightarrows X_0$ admits a partition of unity. Then for every q we have long exact sequences*

$$\cdots \xrightarrow{\partial_!} \Omega^q_c(X_1) \xrightarrow{\partial_!} \Omega^q_c(X_0) \longrightarrow \Gamma_c(X_\bullet, \Omega^q) \longrightarrow 0$$

and

$$0 \longrightarrow \Gamma(X_\bullet, \Omega^q) \longrightarrow \Omega^q(X_0) \xrightarrow{\partial} \Omega^q(X_1) \xrightarrow{\partial} \cdots$$

If we can find a partition of unity with compact support, then there is a long exact sequence

$$[\texttt{les}] \quad \cdots \xrightarrow{\partial_!} \Omega^q_c(X_1) \xrightarrow{\partial_!} \Omega^q_c(X_0) \longrightarrow \Omega^q(X_0) \xrightarrow{\partial} \Omega^q(X_1) \xrightarrow{\partial} \cdots \tag{9}$$

Here the central map $\Omega^q_c(X_0) \to \Omega^q(X_0)$ is given by $\omega \mapsto s_! t^* \omega = t_! s^* \omega$. So in this latter case, we have a canonical isomorphism

$$\Gamma_c(X_\bullet, \Omega^q) \xrightarrow{\sim} \Gamma(X_\bullet, \Omega^q)\,.$$

Proof. To prove (9), let $\rho : X_0 \to \mathbb{R}$ be a partition of unity for $X_\bullet$, such that ρ has compact support. We define a contraction operator

$$\begin{aligned} K : \Omega^q(X_p) &\longrightarrow \Omega^q(X_{p-1}) \\ \omega &\longmapsto \partial_{0!}\big((\pi_0^*\rho)\omega\big) . \end{aligned}$$

Here $\pi_0 : X_p \to X_0$ maps onto the zeroth object, $\partial_0 : X_p \to X_{p-1}$ leaves out the zeroth object. This definition is valid for $p > 0$. We also define

$$\begin{aligned} K : \Omega^q_c(X_p) &\longrightarrow \Omega^q_c(X_{p+1}) \\ \omega &\longmapsto \pi_0^*\rho\, \partial_0^*\omega . \end{aligned}$$

This definition is valid for $p \geq 0$. We finally define $K : \Omega^q(X_0) \to \Omega^q_c(X_0)$ as multiplication by ρ. This defines a contraction operator for the total complex (9), i.e. we have $K\delta + \delta K = \mathrm{id}$, where δ is the boundary operator of (9).

The only place where we used properness was when we used multiplication by ρ to define $\Omega^q(X_0) \to \Omega_c(X_0)$. For this, ρ needs to have compact support, which is only true if $\mathfrak{X}$ is proper. In this case, we may choose X_0 to come from a finite cover U_i.

The first two claims follow by just using part of K. □

Corollary 1.12 *For a differentiable Deligne-Mumford stack $\mathfrak{X}$ we have:*

- *the de Rham cohomology groups $H^k(\mathfrak{X})$ can be calculated as the cohomology groups of the global de Rham complex $\big(\Gamma(\mathfrak{X}, \Omega^*), d\big)$.*
- *the compact support cohomology groups $H^k_c(\mathfrak{X})$ can be calculated using the global complex $\big(\Gamma_c(\mathfrak{X}, \Omega^*), d\big)$.*

If $\mathfrak{X}$ is proper, we also have:

- *these two complexes are equal, i.e., for every q we have*

$$\Gamma_c(\mathfrak{X}, \Omega^q) = \Gamma(\mathfrak{X}, \Omega^q) ,$$

- *for every k we have*

$$H^k(\mathfrak{X}) = H^k_c(\mathfrak{X}) ,$$

  in particular, there exists an integral

$$[\texttt{inte}] \quad \int_{\mathfrak{X}} : H^n(\mathfrak{X}) \longrightarrow \mathbb{R}, \tag{10}$$

- *the induced pairing*

$$H^k(\mathfrak{X}) \otimes H^{n-k}(\mathfrak{X}) \longrightarrow \mathbb{R}$$

  is perfect.

Let us denote the structure map of an atlas X_0 admitting a partition of unity by $\pi : X_0 \to \mathfrak{X}$. With this notation, we may write the integral (10) as follows:

$$\int_{\mathfrak{X}} \omega = \int_{X_0} \rho\, \pi^* \omega .$$