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Orbifold cohomology and quantum cohomology of orbifolds

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Orbifold cohomology and quantum cohomology of orbifolds

ICTP, Trieste, September 2002

lectures by Dan Abramovich

Part 1: Moduli of twisted stable maps

- with Angelo Vistoli
- other construction by W. Chen and Y. Ruan

Work over $\mathbb{C}$

recall that

nice moduli problems are typically Deligne-Mumford stacks admitting projective coarse moduli spaces.

Classic Examples:

- $\overline{\mathcal{M}}_{g,n}$; $\overline{\mathcal{M}}_{g,n}(X,\beta)$

Notation: $\overline{\mathcal{M}}_{g,n}(X,\beta) =: \mathcal{K}_{g,n}(X,\beta)$

- $BG = \{pt\}/G$

= moduli of principal homogeneous G-spaces

- the quotient stack $[V/G]$

Stacks have been introduced as "moduli objects" for families of schemes, sheaves etc.

People have now come to accept them as basic algebraic geometry objects.

Is $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ likely to be interesting for a DM stack $\mathcal{X}$?

examples:

1. $\mathcal{X} = \overline{\mathcal{M}}_{\gamma,\nu} \rightsquigarrow$

$\mathcal{K}_{g,n}(\mathcal{X})$ = moduli of fibered surfaces: families of stable curves over stable curves

2. $\mathcal{X} = \mathbf{B}G \rightsquigarrow$ moduli of principal G bundles on stable curves

recall:

Definition. A *stable n-pointed map*

$$(C, \ldots x_i \ldots, f : C \to X)$$

of genus g and image class β is a morphism from a *prestable n-pointed curve* $(C, x_1, \ldots, x_n)$ to a variety X, such that

- the group $\mathrm{Aut}_X(f : C \to X, x_i)$ of automorphisms of f fixing x_i is finite

-

Now let $\mathcal{X}$ be a Deligne–Mumford stack.

Preliminary Definition. A *stable n-pointed map*

$$(C, x_i, f : C \to \mathcal{X})$$

is a morphism from a *prestable n-pointed curve* (C, x_i) to the stack $\mathcal{X}$ such that

- $\mathrm{Aut}_{\mathcal{X}}(f : C \to \mathcal{X})$ is finite

-

PROBLEM: NOT COMPLETE!!

e.g. consider degeneration of a smooth curve of genus 2,

and map to BG, with $G = (\mathbb{Z}/r)^4$.

smooth fiber C_η has connected principal G bundles, one for each isomorphism

$$H_1(C_\eta, \mathbb{Z})/r \ \tilde{\to}\ G$$

C_s has $H_1(C_\eta, \mathbb{Z}) = \mathbb{Z}^3$, so no connected G bundle!!

Claim: this problem is resolved if we allow the degenerate curve to be a *stack.*

Consider the example above:

$$\begin{array}{ccc} P_\eta & \hookrightarrow & ? \\ \downarrow & & \downarrow \\ C_\eta & \hookrightarrow & C \\ \downarrow & & \downarrow \\ \{\eta\} & \hookrightarrow & S \end{array}$$

If we try to extend directly (e.g. normalization of C in P_η) we typically get ramification index r over C_s.

(i) after base change of order r may assume P_η extends over C_{sm}:

$$\begin{array}{ccccc} P_\eta & \subset & P_{\text{sm}} & & \\ \downarrow & & \downarrow & & \\ C_\eta & \subset & C_{\text{sm}} & \subset & C \end{array}$$

(Purity of Branch Locus)

(ii) at node p we have in local analytic coordinates $C_p : xy = t^r$

We can pick a "uniformization" $V_p \to C_p$

$V_p : \{\xi\eta = t\}$ where $u = \xi^r,\ v = \eta^r$

V_p nonsingular. Purity $\Longrightarrow$ the bundle extends:

$$P_p \to V_p$$

There is an action of μ_r on V_p which lifts to the bundle P_p.

(iii) $V_p \to C_p$ with μ_r action is a **chart** for an **orbifold structure** $\mathcal{C} \to C$,
namely a DM stack $\mathcal{C}$ with moduli space C

moreover we have a morphism $\mathcal{C} \to BG$

Twisted curves ($n = 0$)

Definition. A twisted (pre-stable) curve is a Deligne–Mumford stack $\mathcal{C}$ with coarse moduli space C such that

1. C is a pre-stable curve,
2. the map $\mathcal{C} \to C$ to the coarse moduli curve is an isomorphism away from singularities of C, and
3. $\mathcal{C}$ has at most nodes as singularities

Local description in a family

$$C : xy = f(t)^r$$

$$\mathcal{C} = [V/\mu_r], \quad V : \xi\eta = f(t), \quad x = \xi^r, y = \eta^r$$

Action of μ_r :

$$(\xi, \eta) \quad \mapsto \quad (\zeta_r \cdot \xi \,,\, \zeta_r^a \cdot \eta)$$

with $\gcd(a, r) = 1$.

Terminology:

$r = 1$: **untwisted**; $r > 1$: **twisted**.

$a \equiv -1 \mod r$: **balanced**; otherwise **unbalanced**.

Note: if $f(t) \neq 0$ then balanced.

$n > 0$: Twisted curves at a marking

One way to arrive at marked curves is by normalizing a nodal curves and "separating" a node in two markings.

Local description via parameters

$$C : \ x$$

$$\mathcal{C} = [V/\mu_r], \quad V : \xi, \quad x = \xi^r$$

Action of μ_r : $\quad \xi \quad \mapsto \quad \zeta_r \cdot \xi$

Global description: a marking is a closed substack which is a **gerbe** banded by μ_r.

NOT NECESSARILY A SECTION

Definition

Let $\mathcal{X}$ be a D-M stack, $\mathcal{X} \to X$ projective coarse moduli scheme.

An n-pointed **twisted stable map** $f : \mathcal{C} \to \mathcal{X}$ to $\mathcal{X}$ of genus g, class β, is a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \\ C & \xrightarrow{\bar{f}} & X \end{array}$$

where

(i) $\mathcal{C}$ twisted curve with coarse moduli space C

(ii) $\mathcal{C} \to \mathcal{X}$ representable

(no unnecessary branchings)

(iii) $C \to X$ stable n-pointed map of genus g and class β

($\Leftrightarrow$ automorphism group is finite)

Theorem: the moduli of n-pointed twisted stable maps of genus g and class β to $\mathcal{X}$ is a proper D-M stack $\mathcal{K}_{g,n}(\mathcal{X},\beta)$ admitting a projective coarse moduli scheme $\mathbf{K}_{g,n}(\mathcal{X},\beta)$.

In fact, there is a commutative diagram of finite maps (not cartesian)

$$\begin{array}{ccc} \mathcal{K}_{g,n}(\mathcal{X},\beta) & \to & \mathbf{K}_{g,n}(\mathcal{X},\beta) \\ \downarrow & & \downarrow \\ \mathcal{K}_{g,n}(X,\beta) & \to & \mathbf{K}_{g,n}(X,\beta) \end{array}$$

Issues coming in proof:

(i) existence of bottom line in diagram

(ii) valuative criterion for properness like for $\mathbf{B}G$

(iii) annoying technicalities

Problem: a stack is a category, but this is a-priori a 2-category.

Claim: the 2-category of twisted curves is *equivalent* to the associated category.

Lemma Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks over a scheme S. Assume that there exists a dense open representable substack (i.e. an algebraic space) $U \subseteq \mathcal{X}$ and an open representable substack $V \subseteq \mathcal{Y}$ such that F maps U into V. Then any automorphism of F is trivial.

proof aside

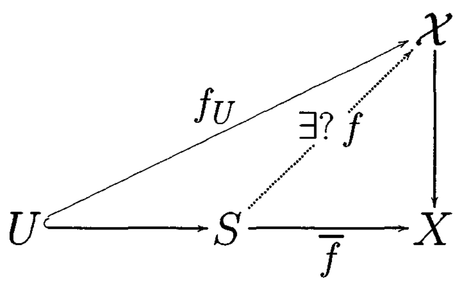
in fact we have:

Proposition: the category of twisted curves is an algebraic stack.

Purity Lemma

S a nonsingular surface, $U \subset S$, $\quad U = S \smallsetminus \{pt\}$.

Given a diagram



Then the f_U extends uniquely to $f : S \to \mathcal{X}$

(up to unique isomorphism...)

proof aside

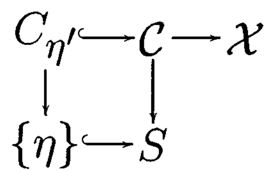
Valuative criterion for properness (one case):

Suppose $S = \mathrm{Spec} R$, the spectrum of a discrete valuation ring;

Generic point $\eta = \mathrm{Spec} K$, special point s.

Suppose C_η / K is an untwisted curve, $C_\eta \to \mathcal{X}$ a (twisted) stable map.

Then, after a base change $S' \to S$, there is an extension



Steps:

(i) Kontsevich: extend the map to the coarse moduli space X.

(ii) Properness of $\mathcal{X}$: extend the map $C_\eta \to \mathcal{X}$ over an open set containing the generic points of C_s

(iii) Purity Lemma: the map extends over the smooth locus of C.

(iv) local argument as for BG: lift the map to $[V_p/\mu_r] \to \mathcal{X}$

(v) descent: replace $[V_p/\mu_r]$ by something representable.

Global description of a twisted marking

Recall that a marking $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ is a gerbe. In fact, there is a nice canonical way to recover $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ from $\Sigma^{C} \subset C$, away from nodes.

The point is that the pullback of Σ^{C} is $r \cdot \Sigma^{\mathcal{C}}$.

Definition. Let V be a scheme, L an invertible sheaf, s a global section of L. Define a stack $\sqrt[r]{(L,s)}$ whose objects over a scheme T are

$$(f, M, t, \phi),$$

where

- $f : T \to V$ is a morphism
- M is an invertible sheaf on V
- t is a section of M
- $\phi : M^{\otimes r} \xrightarrow{\sim} f^*L$ is an isomorphism, such that
- $\phi(t^{\otimes r}) = s$,

and arrows are fiber diagrams.

Proposition Near Σ^C, the twisted curve $\mathcal{C}$ is isomorphic to $\sqrt[r]{(L,s)}$ where L is $O_C(\Sigma^C)$ and s is the defining section.

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Part 2: Algebraic Orbifold Quantum Products

- Work over $\mathbb{C}$
- This work with Tom Graber and Angelo Vistoli
- Quantum products based on W. Chen + Y. Ruan
- global quotient case by Fantechi + Göttche.

Suppose that $\mathcal{X}$ is an algebraic stack, with moduli space X. Let $\mathcal{C}$ be a twisted curve

Contraction Lemma: Given (possibly unstable)

$$\begin{array}{ccc} \mathcal{C} & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ C & \to & X \end{array}$$

Then if $C \to X$ can be stabilized,

so does $\mathcal{C} \to \mathcal{X}$.

Applications:

(a) functoriality in $\mathcal{X} \to \mathcal{Y}$

(b) forgetful maps for untwisted markings and universal curve.

proof aside

Recall: Gromov-Witten theory

Let X be a smooth projective variety. Consider the correspondence

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n+1}(X,\beta) & \xrightarrow{e_{n+1}} & X \\ \Big\downarrow{\scriptstyle e_1\times\cdots\times e_n} & & \\ X^n & & \end{array}$$

Define

$$\begin{aligned} H^*(X)^n &\longrightarrow H^*(X) \\ \gamma_1 \times \cdots \times \gamma_n &\mapsto \langle \gamma_1, \ldots, \gamma_n, * \rangle_{g,\beta}\,, \end{aligned}$$

where

$$\langle \gamma_1, \ldots, \gamma_n, * \rangle_{g,\beta} =$$

$$(e_{n+1})_* \left(e^*_{1,\ldots,n}(\gamma_1 \times \cdots \times \gamma_n) \cap [\overline{\mathcal{M}}_{g,n+1}(X,\beta)]^v \right)$$

In genus zero these *GW invariants* satisfy *WDVV relations*, giving *associativity*.

"Small" quantum product is defined by

$$\gamma_1 * \gamma_2 = \sum_{\beta \in H_2(X)} \langle \gamma_1, \gamma_2, * \rangle_{0,\beta}\; q^\beta.$$

(grading of q^β is $2c_1(X) \cdot \beta$)

Associativity (assuming nothing odd):

$$\sum_{\beta_1+\beta_2=\beta} \big\langle \langle \gamma_1, \gamma_2, * \rangle_{0,\beta_1}, \gamma_3, * \big\rangle_{0,\beta_2} =$$

$$\sum_{\beta_1+\beta_2=\beta} \big\langle \langle \gamma_1, \gamma_3, * \rangle_{0,\beta_1}, \gamma_2, * \big\rangle_{0,\beta_2}.$$

A key step in proving associativity is a morphism from

$$\overline{\mathcal{M}}_{0,3}(X,\beta_1) \underset{X}{\times} \overline{\mathcal{M}}_{0,3}(X,\beta_2)$$

to a "boundary divisor" in

$$\overline{\mathcal{M}}_{0,4}(X,\beta_1+\beta_2)$$

$$\begin{array}{ccccc} & & & \mathcal{M}_2 & \to X \\ & & & \swarrow \quad \searrow & \\ \mathcal{M}_1 & & \to X & & X \\ \swarrow \quad \searrow & & & & \\ X \qquad X & & & & \end{array}$$

$$\begin{array}{ccccc} \mathcal{M}_1 \times_X \mathcal{M}_2 & \hookrightarrow & D_{(12|34)}(X) & \longrightarrow & \overline{\mathcal{M}}_{0,4}(X,\beta) \\ & & \downarrow & & \downarrow \\ & & D_{(12|34)} & \hookrightarrow & \overline{\mathcal{M}}_{0,4} \end{array}$$

In order to generalize this picture, we will need

1. an analogue of $H^*(X)$ or $A^*(X)$ for a smooth Deligne–Mumford stack,
2. an analogue of the evaluation maps e_i and e_{n+1},
3. an analogue of the virtual fundamental class, and
4. an analogue of the fiber product description of the boundary divisors.

We'll start with the **last**

Interpretation of boundary: suppose $C = C_1 \overset{p}{\cup} C_2$

Then

$$Hom(C, X) =$$

$$Hom(C_1, X) \underset{Hom(\{p\},X)}{\times} Hom(C_2, X)$$

Now say $\mathcal{C}$ is a nodal **twisted** curve.

$$\mathcal{C} = \mathcal{C}_1 \overset{\wp}{\cup} \mathcal{C}_2,$$

where

$$\wp \simeq B(\mu_r)$$

$$Hom(\mathcal{C}, \mathcal{X}) =$$

$$Hom(\mathcal{C}_1, \mathcal{X}) \underset{Hom(\wp,\mathcal{X})}{\times} Hom(\mathcal{C}_2, \mathcal{X})$$

So evaluation lands in

$$Hom\,(\,B(\mu_r)\,,\;\mathcal{X}\,)$$

(and not just in $\mathcal{X}$)

Inertia stack

Here

$$\mathcal{X}_1 = \bigcup_r HomRep(\mathcal{B}\boldsymbol{\mu}_r, \mathcal{X})$$

$$= \left\{ (x, H, \chi) \;\middle|\; \begin{array}{c} x \in Ob(\mathcal{X}), \\ H \subset \mathrm{Aut}\ x, \\ \chi : H \tilde{\to} \boldsymbol{\mu}_r \end{array} \right\}$$

$$= \left\{ (x, g) \;\middle|\; \begin{array}{c} x \in Ob(\mathcal{X}), \\ g \in \mathrm{Aut}\ x \end{array} \right\}$$

$$= \mathcal{I}(\mathcal{X})$$

The points of the coarse moduli space $I(X)$ are

$$(\,[x]\,,\ (g)\,)$$

where $[x] \in X$ is the isom. class of an object x and
(g) is the *conjugacy* class of an element

$$g \in \mathrm{Aut}(x)$$

Let $\mathcal{X}$ be a *smooth* Deligne–Mumford stack of dimension d with projective coarse moduli space X

WE WORK WITH

$$H^*_{orb}(\,\mathcal{X}\,) \quad = \quad H^*(\,\mathcal{X}_1\,)$$

Age

Let $([x], (g)) \in I(X)$

Can write locally $\mathcal{X} = [V / \Gamma]$,
where $\Gamma = \mathrm{Aut}(x)$, and V smooth.

Diagonalize the action of g on the tangent $T_x(V)$, with eigenvalues

$$e^{(2\pi i)\cdot r_j}, \quad j = 1, \ldots, d$$

with $0 \leq r_j < 1$.

Definition: the age

$$a(x, g) = \sum_{i=1}^{d} r_j.$$

Claim: $a(x, g)$ is constant on every connected component $Z \subset I(X)$ (notation $a(Z)$)

Claim: $a(x, g) + a(x, g^{-1}) = \dim X - \dim Z$

GRADING

Definition: The degree of an element $\alpha \in H^i(Z)$ in $H^*_{orb}(\mathcal{X})$ is defined to be

$$i + 2a(Z)$$

Note: $\dim H^i_{orb}(\mathcal{X}) = \dim H^{d-i}_{orb}(\mathcal{X})$.

Evaluation

Unless $g = 0, n = 3, \beta = 0$, the markings of $\mathcal{K}_{g,n}(X, \cdot)$ are nontrivial grbes.

Define

$\overline{\mathcal{M}}_{g,n}(X, \cdot)$ = twisted stable maps with sections

= fibered product of universal gerbes over $\mathcal{K}_{g,n}(X, \cdot)$

Have evaluation maps

$$\overline{\mathcal{M}}_{g,n}(X, \cdot) \xrightarrow{e_i} \mathcal{X}_1$$

recall that in

$$\mathcal{C} = \mathcal{C}_1 \overset{\wp}{\cup} \mathcal{C}_2,$$

we can naturally identify $B(\mu_r) \simeq \wp \subset \mathcal{C}_2$,

via the action of μ_r on the normal bundle to the marking in $\mathcal{C}_2$.

but then the action on the $\mathcal{C}_1$ side is opposite!

$$(\xi, \eta) \mapsto (\zeta_r \cdot \xi, \zeta_r^{-1} \cdot \eta)$$

Define involution

$$\iota : \mathcal{X}_1 \to \mathcal{X}_1$$

via

$$g \mapsto g^{-1}$$

Define:

$$\check{e}_i = \iota \circ e_i$$

Deformations / Obstructions: general case

(i) Twisted curves are unobstructed:

$$Ext^2(\Omega^1_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}}) = 0$$

(ii) Obstructions of $f : \mathcal{C} \to \mathcal{X}$ lie in

$$H^1(\mathcal{C}\,,\ f^*T_{\mathcal{X}}\,)$$

(iii) Relative infinitesimal deformations are

$$H^0(\mathcal{C}\,,\ f^*T_{\mathcal{X}}\,)$$

Behrend - Fantechi, Kresch $\Longrightarrow$

virtual fundamental class $[\mathcal{K}_{g,n}(X,\cdot)]^v$

Because of sections need to use

$[\overline{\mathcal{M}}_{g,n}(X,\cdot)]^w \;=\; r_1\cdots r_n\,[\overline{\mathcal{M}}_{g,n}(X,\cdot)]^v$

Special case: every deformation is **un**obstructed.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ \pi\downarrow & & \\ \overline{\mathcal{M}} & & \end{array}$$

then

$$[\overline{\mathcal{M}}]^v \;=\; c_{top}(E) \;=\; c_{top}\left(\mathbb{R}^1\pi_*\, f^*T_{\mathcal{X}}\,\right)$$

If, moreover, $\mathcal{C} \;=\; [D/H]$ with $\tilde{f} : D \to \mathcal{X}$, then

$$E \;=\; (\,\mathbb{R}^1\tilde{\pi}_*\,\tilde{f}^*T_{\mathcal{X}}\,)^H$$

3-point orbifold G–W invariants

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,3}(\mathcal{X},\beta) & \stackrel{\check{e}_3}{\to} & \mathcal{X}_1 \\ e_1 \times e_2 \downarrow & & \\ \mathcal{X}_1 \times \mathcal{X}_1 & & \end{array}$$

for $\gamma_i \in H^*_{orb}(\mathcal{X})$ Define

$\langle \gamma_1, \gamma_2, * \rangle_\beta =$

$$(\check{e}_3)_* \left(e_1^*(\gamma_1) e_2^*(\gamma_2) \cap [\overline{\mathcal{M}}_{0,3}(X,\beta)]^w \right)$$

"Small" quantum product is again defined by

$$\gamma_1 * \gamma_2 = \sum_{\beta \in H_2(\boldsymbol{X})} \langle \gamma_1, \gamma_2, * \rangle_{0,\beta} \; q^\beta.$$

Theorem: this is a graded skew-commutative associative ring.

grading of q^β is $2(c_1(\boldsymbol{\mathcal{X}}) \cdot \beta)$

Will only prove when every deformation is unobstructed, and curves smooth freely

Associativity (assuming nothing odd):

$$\sum_{\beta_1+\beta_2=\beta} \big\langle \langle \gamma_1, \gamma_2, *\rangle_{0,\beta_1}, \ \gamma_3, \ * \big\rangle_{0,\beta_2} =$$

$$\sum_{\beta_1+\beta_2=\beta} \big\langle \langle \gamma_1, \gamma_3, *\rangle_{0,\beta_1}, \ \gamma_2, \ * \big\rangle_{0,\beta_2}.$$

$$\begin{array}{ccccccc} & & & & \mathcal{M}_2 & \to & \mathcal{X}_1 \\ & & & \swarrow & & \searrow & \\ & \mathcal{M}_1 & \to & \mathcal{X}_1 & & \mathcal{X}_1 & \\ \swarrow & & \searrow & & & & \\ \mathcal{X}_1 & & \mathcal{X}_1 & & & & \end{array}$$

The product diagram

$$\begin{array}{ccc} \mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 & \to & \mathcal{M}_2 \\ \downarrow & & \downarrow \\ \mathcal{M}_1 & \to & \mathcal{X}_1 \end{array}$$

can be expanded:

$$\begin{array}{ccccc} \mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 & \to & \mathcal{M}_2 & \to & \mathcal{X}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_1 \times \mathcal{M}_2 & \to & \mathcal{X}_1 \times \mathcal{M}_2 & \to & \mathcal{X}_1 \times \mathcal{X}_1 \\ \downarrow & & \downarrow & & \\ \mathcal{M}_1 & \to & \mathcal{X}_1 & & \end{array}$$

It follows that the coefficient in $(\gamma_1 * \gamma_2) * \gamma_3$ is

$$r_\times^2 \;\; e^\times_{4*}\left(\left(\prod_{i=1}^{3} e_i^{\times\,*}(\gamma_i) \right) \cdot \mathfrak{C} \right)$$

where

$$\mathfrak{C} \;=\; c_{top}(\, p_1^* E_1 \;\oplus\; p_2^* E_2 \,)\; c_{top}(\, N_\delta / N_{p_1 \times p_2} \,)$$

Here

E_i = obstruction bundle of $f_i : \mathcal{C}_i \to \mathcal{X}$,

$N_\delta / N_{p_1 \times p_2}$ = excess normal bundle

Can we make this "symmetric"?

The divisor diagram

$$\begin{array}{ccccc}
 & & & & \mathcal{M}_1 \underset{\mathcal{X}_1}{\times} \mathcal{M}_2 \\
 & & & & \downarrow \\
\mathcal{M}_{0,4}(X) & \leftarrow & D \underset{\mathbf{M}_{0,4}}{\times} \mathcal{M}_{0,4}(X) & \supset & D(X) \\
\downarrow & & \downarrow & & \\
\mathbf{M}_{0,4} & \supset & D & &
\end{array}$$

So enough to show

$$\mathfrak{C} = gl^* c_{top}(E)$$

E = obstruction bundle of $f_{0,4} : \mathcal{C}_{0,4} \to \mathcal{X}$.

We consider the normalization sequence

$$0 \to f^*T\mathcal{X} \to \nu_* f'^* T\mathcal{X} \to (f^*T\mathcal{X})|_\Sigma \to 0.$$

Pushing forward to $\mathcal{Y}$ gives

$$0 \to \pi_* f^*T\mathcal{X} \to \pi'_* f'^* T\mathcal{X} \to (\pi_*(f^*T\mathcal{X})|_\Sigma)$$

$$\to gl^*E \to p_1^*E_1 \oplus p_2^*E_2 \to 0.$$

Let Q be the bottom kernel.

Need: $Q =$ excess bundle.

Have on top

$$0 \to \ T\mathcal{Y} \ \to \ p_1^*T\mathcal{M}_1 \oplus p_2^*T\mathcal{M}_2$$

$$\to \ \pi_*((f^*T\mathcal{X})|_\Sigma) \ \to \ Q \ \to 0$$

So we got

$$0 \to N_{p_1 \times p_2} \to \pi_*((f^*T\mathcal{X})|_\Sigma) \to Q \to 0$$

So the following suffices:

Lemma.

$$\pi_*\left((f^*T\mathcal{X})|_\Sigma\right) \ \simeq \ e_\times^* \ T\mathcal{X}_1$$

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Part 3: Stringy Orbifold Cohomology: Theory and Examples

- Work over $\mathbb{C}$
- This work with Tom Graber and Angelo Vistoli
- Quantum products based on W. Chen + Y. Ruan
- global quotient case by Fantechi + Göttsche.

Equivariant Riemann-Roch

Let $\mathcal{C}$ be a smooth twisted curve, $\mathcal{E}$ a vector bundle on $\mathcal{C}$.

Let x be a twisted marking, locally $\mathcal{C}$ is $[V/\mu_r]$,

V has coordinate z,

Say μ_r acts on $T_{V,x}$ via $v \mapsto \zeta_r \cdot v$.

(Then μ_r ants on z via $z \mapsto \zeta_r^{-1} \cdot z$.)

say μ_r acts on $\mathcal{E}_x$ with eigenvalues $e^{(2\pi i)\cdot r_i}$, with $0 \leq r_i < 1$.

the **age** is

$$a(\mathcal{E}, x) = \sum r_i$$

Theorem.

$$\chi(\mathcal{C}, \mathcal{E}) = d(\mathcal{E}) - rk(\mathcal{E})(g-1) - \sum_x a(\mathcal{E}, x)$$

Exercise: quantum product is graded.

Stringy Orbifold cohomology:

We defined

$$\gamma_1 * \gamma_2 = \sum_{\beta \in H_2(X)} \langle \gamma_1, \gamma_2, * \rangle_{0,\beta} \; q^\beta.$$

Now we can plug in $q^\beta \mapsto 0$ and obtain an associative ring structure on $H^*_{orb}(\mathcal{X}, \mathbb{Q})$

Exercise: This ring has unit $1 \in H^*(\mathcal{X}, \mathbb{Q})$.

Defined $\langle \gamma_1, \gamma_2, * \rangle_{0,\beta}$ using $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, \beta)$

Claim: when $\beta = 0$ the three gerbes are trivial, and there are evaluation maps

$$\mathcal{K}_{0,3}(\mathcal{X}, \beta) \to \mathcal{X}_1.$$

These maps are **representable**

So the above lifts to a product

$$\gamma_1 \smile \gamma_2$$

on $H^*_{orb}(\mathcal{X}, \mathbb{Z})$

Theorem [ℵGV]: this is well defined for arbitrary smooth separated Deligne–Mumford stack, independent of choices and associative.

Proof of associativity is much more delicate.

The Fantechi-Göttsche ring

In case $\mathcal{X} = [Y/G]$ write

$$\widetilde{Y} = \bigsqcup_{g \in G} Y^g$$

The group G acts by sending $y \in Y^g$ to $h(y) \in Y^{hgh^{-1}}$. So we have $\mathcal{I}(\mathcal{X}) \simeq [\widetilde{Y}/G]$

Notation:

$$H^*(Y, G) = H^*(\widetilde{Y}, \mathbb{Q}).$$

Theorem [Fantechi-Göttsche]:

1. There is a (natural) associative (noncommutative) ring structure on $H^*(Y, G)$, with an action of G.

2. There is a ring isomorphism $H^*(Y, G)^G \simeq H^*_{orb}(Y)$

The construction and proof is very much analogous to the one for $H^*_{orb}(Y)$

Example 0

$$\mathcal{X} = BG$$

$$\mathcal{I}(\mathcal{X}) = [G \overset{adj}{/} G] = \bigsqcup_{(g)} B(C(g)) =: \bigsqcup_{(g)} \mathcal{X}_{(g)}$$

$$H^*_{orb}(BG, \mathbb{Z}) = \bigoplus_{(g)} H^*(B(C(g)), \mathbb{Z})$$

$$T_{BG} = 0 \implies a(Z) = 0$$

Component $\mathcal{K}_{g,h}$ of $\mathcal{K}_{0,3}(BG,0)$ with $g, h, (gh)^{-1}$:

$$\mathcal{K}_{g,h} \simeq B(C(g) \cap C(h))$$

product:

$$x_{(g)} \smile x_{(h)} = \sum_{(g',h')} \left| \frac{C(g'h')}{C(g') \cap C(h')} \right| x_{(gh)}$$

$$H^*_{orb}(BG, \mathbb{Q}) = C(\mathbb{Q}[G]) = (\mathbb{Q}[G])^G$$
$(H^*(\{pt\}, G) = \mathbb{Q}[G])$

$$H^*_{orb}(B(\mu_r), \mathbb{Z}) = \mathbb{Z}[s,t]/(s^r - 1, rt)$$

Example 1

$$\mathcal{X} = [\mathbb{A}^1 / \mu_r]$$

$$\mathcal{I}(\mathcal{X}) = \mathcal{X} \sqcup \coprod_{k=1}^{r-1} B(\mu_r)$$

$$H^*(\mathcal{X}_i, \mathbb{Z}) = \mathbb{Z}[t_i]/(rt_i)$$

Age of X_i is i/r

Components:

$$\mathcal{K}_{i,j} = \begin{cases} \mathcal{X} & i = j = 0 \\ B(\mu_r) & \text{otherwise} \end{cases}$$

$[\mathcal{K}_{i,j}]^v \;=\; [\mathcal{K}_{i,j}]$ whenever $i+j \le r$ by grading.

Product:

$$x_i \smile x_j = \begin{cases} x_{i+j} & i+j < r \\ t_0 & i+j = r \end{cases}$$

$$t_0 \smile x_i = t_i$$

$$H^*_{orb}(\mathcal{X}, \mathbb{Z}) = \mathbb{Z}[x_1]/(r x_1^r)$$

Example 2 The wighted projective stack

$$\mathbb{P}^1[a,b] = [\,(\mathbb{A}^2 \smallsetminus \{0\})\ /\mathbb{G}_m\,]$$

with the action $(x,y) \mapsto (t^a x, t^b y), \quad a, b > 0.$

Let $g = \gcd(a,b),\ \ l = ab/g.$

$$\mathcal{I}(\mathcal{X}) = \coprod_{i=0}^{g-1} \mathcal{X} \sqcup \coprod_{i=0}^{a-g} B(\mu_a) \sqcup \coprod_{i=0}^{b-g} B(\mu_b)$$

$$= \coprod_{i=0}^{g-1} X_i \sqcup \coprod_{\substack{i=1 \\ \gcd(i,a/g)=1}}^{a-1} A_i \sqcup \coprod_{\substack{i=1 \\ \gcd(i,b/g)=1}}^{b-1} B_i$$

$$H^*(X, \mathbb{Z}) = \mathbb{Z}[t]/(abt^2)$$

Age of A_i is ig/a, of B_i is ig/b

Ring:

$H^*_{orb}(\mathcal{X}, \mathbb{Z}) =$

$$\frac{\mathbb{Z}[X, A, B, T]}{(X^g - 1, aAT, bBT, A^{\frac{a}{g}} - bXT, B^{\frac{b}{g}} - aXT, AB)}$$

Exercise: Find $H^*_{orb}(\mathcal{X}, \mathbb{Q})$ and $H^*_{orb}(\mathcal{X}, \mathbb{Z})$ when $\mathcal{X} = \mathbb{P}^2[1, 1, a]$

Example:

$$\mathcal{X} \;=\; [\,V^n \,/\, \mathcal{S}_n\,]$$

Notation:

$\sigma \in \mathcal{S}_n$ has $l(\sigma) =$ length,

$o(\sigma) =$ number of cycles $= n \,-\, l(\sigma)$.

Also $\dim V = d$.

$$\mathcal{I}(\mathcal{X}) \;=\; \bigsqcup_{(\sigma)} [\,(V^n)^\sigma \,/\, C(\sigma)]$$

$$(V^n)^\sigma \simeq V^{o(\sigma)} \quad \Longrightarrow \quad \dim \mathcal{X}_{(\sigma)} = d \cdot o(\sigma)$$

Age: $(\sigma) = (\sigma^{-1})$

$$\implies a(\mathcal{X}_{(\sigma)}) = \frac{d \cdot l(\sigma)}{2}$$

Components:

$$\mathcal{K}_{(\sigma,\tau)} = \left[(V^n)^{\langle\sigma,\tau\rangle} / C(\sigma) \cap C(\tau)\right]$$

Notation: $\mathcal{O}(\sigma, \tau)$ = set of orbits of $\langle\sigma, \tau\rangle$.

$W = \mathbb{C}^n$ standard representation of $\mathcal{S}_n$.

W_I = representation of $\langle\sigma, \tau\rangle$ on the subspace with coordinates in $I \subset \{1, \dots, n\}$, $I \in \mathcal{O}(\sigma, \tau)$.

Have $\mathcal{C} \xrightarrow{\pi} \mathcal{K}_{(\sigma,\tau)} \to \mathcal{X}$

$f^*T_{\mathcal{X}}$ = "$\bigoplus_{I\in\mathcal{O}(\sigma,\tau)}(W_I \otimes \pi^*T_{V_I})$"

$$H^0(\mathcal{C}, W_I \overset{G}{\otimes} \mathcal{O}_{\mathcal{C}}) = 1$$

$$\chi(\mathcal{C}, W_I \overset{G}{\otimes} \mathcal{O}_{\mathcal{C}}) =$$
$$|I| - a(g, W_I) - a(h, W_I) - a((gh)^{-1}, W_I)$$

so $r_I = H^1_I = \frac{|I|+1-o_I(g)-o_I(h)-o_I(gh)}{2}$.

So W_I contributes a factor

$$\begin{cases} 1 & \text{if } r_i = 0 \\ c_{top}(T_V) & \text{if } r_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Finally there is a constant factor of $\left|\frac{C(gh)}{C(g)\cap C(h)}\right|$.