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Complex symplectic moduli spaces (I)

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1. Introduction

<u>Theorem</u>
Let X be a compact Kähler manifold with vanishing Ricci curvature. Then there is a finite étale covering $X' \to X$ and a decomposition $X' \cong Y_1 \times \ldots \times Y_s$, where each factor Y_i is either a complex torus or a Calabi Yau or an irreducible hyperkähler manifold.

A hyperkähler metric is a differential-geometric notion. It is closely related to the complex-analytic notion of a holomorphic symplectic structure.

The precise relation between these notions and the general theory of hyperkähler manifolds will be dealt with in the lectures of D. Huybrechts. The more modest purpose of the present lecture series is to introduce the known examples of symplectic manifolds and to discuss aspects of their topology and geometry.

All these examples are, or are related to, moduli spaces of semistable sheaves on K3 or abelian surfaces, which justifies the title of this lecture series.

1.1. Symplectic manifolds

<u>Definition</u> Let X be a complex manifold. A (holomorphic) symplectic structure on X is a closed holomorphic 2-form σ such that the induced skew-symmetric pairing $\mathcal{T}_X \times \mathcal{T}_X \longrightarrow \mathcal{O}_X$ is nowhere degenerate.

Remarks 1. If σ is a symplectic structure on X, then X is necessarily even-dimensional, say

$$\dim_{\mathbb{C}} X = 2n$$

Then $\sigma^n = \sigma \wedge \ldots \wedge \sigma$ is a global section of the canonical line bundle $K_X = \Omega_X^{2n}$. In fact, a holomorphic 2-form σ is non-degenerate if and only if σ^n has no zeroes, so that in particular the canonical bundle is trivial,

$$K_X \cong \mathcal{O}_X$$

2. The condition that σ be closed is automatically satisfied if X is compact Kähler.

Examples 1. $X = \mathbb{C}^{2n}$, $\sigma = \sum_{i=1}^{n} dz_i \wedge dz_{i+n}$

2. Let Y be an arbitrary complex manifold and let $X := T^*Y \longrightarrow Y$ be the cotangent bundle. Then there is a tautological 1-form $\theta \in \Gamma(X, \Omega_X^1)$, and $\sigma = d\theta$ is a symplectic structure.

It is much harder to find compact examples.

Definition An irreducible symplectic manifold is a simply connected compact Kähler manifold with a symplectic structure σ such that $H^0(\Omega_X^2) = \mathbb{C} \cdot \sigma$.

1.2 K3-surfaces

The lowest dimensional possible examples of irreducible symplectic manifolds are surfaces. The only Kähler surfaces with trivial canonical bundle in Kodaira's classification are K3-surfaces and 2-dimensional tori.

Definition A smooth compact complex surface X is a K3-surface, if $K_X \cong \mathcal{O}_X$ and $H^1(\mathcal{O}_X) = 0$.

these assumptions completely determine the topology of X:

- It follows from $H^1\mathcal{O}_X = 0$ that $H^0\Omega^1_X = 0$ and hence $b_1 = h^{0,1} + h^{1,0} = 0$. By Poincaré duality one has $b_3 = 0$ as well.
- Since $K_X \cong \mathcal{O}_X$, we have $H^0\Omega^2_X = 0$ and by Serre-duality $H^2\mathcal{O}_X = 0$. Thus the holomorphic Euler characteristic of X is $\chi(\mathcal{O}_X) = h^{0,0} - h^{0,1} + h^{0,2} = 2$.
- Noether's formula shows that
  $c_2(X) = c_2(X) + c_1(X)^2 = 12 \cdot \chi(\mathcal{O}_X) = 24$.
  In particular, the topological Euler characteristic of X is 24, and since $b_1 = b_3 = 0$, one gets $b_2 = 22$.
- Since $b_1 = 0$, $H_1(X;\mathbb{Z})$ is a finite torsion group. But any non-trivial element would give rise to a finite étale covering $X' \to X$, where X' would again be a K3-surface but with topological Euler characteristic > 24, which is absurd. Hence $H_1(X;\mathbb{Z}) = 0$, and $H^2(X;\mathbb{Z})$ is a torsion free $\mathbb{Z}$-module, hence a lattice of rank 22.
- Since $w_1 = 0$ and $w_2 \equiv c_1(X) \bmod 2 = 0$, the parity of the quadratic form $(H^2(X;\mathbb{Z}), \cup)$ is even. Its signature can be obtained from Hirzebruch's theorem: $\sigma = \frac{1}{3}\left(c_1(X)^2 - 2c_2(X)\right) = -16$.
  From the classification theorem of unimodular even integral lattices it follows that
  $$(H^2(X;\mathbb{Z}), \cup) \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3} \oplus \left(-E_8\right)^{\oplus 2}.$$
- Now the Hodge numbers are completely determined, the Hodge diamond looks like
  $$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Typical examples of K3-surfaces are smooth hypersurfaces $X \subset \mathbb{P}^3$ of degree 4. These are simply-connected by Lefschetz' theorem ~~of~~ on hyperplane sections.

One can show that any K3-surface can be deformed into such a smooth quartic. As a consequence, all K3-surfaces are diffeomorphic and in particular simply-connected.

Finally, that every K3-surface can be endowed with a Kähler metric is a difficult theorem due to Siu.

1.3. Two-dimensional tori

The second group of candidates of symplectic surfaces, the two dimensional tori

$$A = \mathbb{C}^2/\Gamma \ , \quad \Gamma \subset \mathbb{C}^2 \text{ lattice},$$

fail for the obvious reason that $\pi_1(A) = \Gamma$ is non trivial. However, if z_1, z_2 are coordinates on $\mathbb{C}^2$, then $\sigma = dz_1 \wedge dz_2$ is a translation invariant form ~~on~~ $\mathbb{C}^2$ and descends to a symplectic structure on A. Moreover, the cohomology of A, including the Hodge decomposition, is given by

$$H^*(A;\mathbb{C}) \cong \Lambda^*(\mathbb{C}^2 \oplus \overline{\mathbb{C}^2})$$

so that the Hodge diamond looks like

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

Even though these tori are not themselves irreducible symplectic manifolds they will enter in the construction of higher dimensional examples of symplectic manifolds.

As a warm-up, recall the construction of Kummer varieties:

$\mathbb{Z}/2$ acts on $A = \mathbb{C}^2/\Gamma$ by the involution

$$\iota : A \longrightarrow A \, , \quad x \longmapsto -x \, .$$

This action is free except for the 16 2-torsion points $\frac{1}{2}\Gamma/\Gamma \subset A$, that yield singularities in

$$Y := A/\iota$$

of type A_1. Near $\bar{0} \in Y$, we have local coordinates

$$a = z_1^2 \, , \quad b = z_1 z_2 \, , \quad c = z_2^2 \, ,$$

which satisfy $ac - b^2 = 0$. Let $X \xrightarrow{\pi} Y$ be the blow-up of Y in the 16 singularities. The exceptional fibres are (-2)-curves.

What happens to the symplectic structure of A? Now, as $\sigma = dz_1 \wedge dz_2$ is ι-invariant it descends to a symplectic structure $\bar{\sigma}$ on Y_{reg}. Near $\bar{0}$, we can write it as

$$\bar{\sigma} = \frac{da \wedge db}{2a} = \frac{db \wedge dc}{2c}$$

In local coordinates a and β (with $b = a\beta$, $c = a\beta^2$), the pull-back of $\bar{\sigma}$ is given by

$$\pi^*(\bar{\sigma}) = \frac{1}{2}\, da \wedge d\beta$$

We see that $\pi^*(\bar{\sigma})$ extends over the exceptional divisors without zeroes and hence is a symplectic structure on X! Of course, X is again a K3-surface.

2. Hilbert schemes

The first higher dimensional example of an ~~i~~irreducible symplectic manifold was given by Fujiki: the blow-up of the diagonal in $S^2(X) = X^2/\mathfrak{S}_2$ for a K3-surface X. This example was generalised by Beauville. He showed that all Hilbert schemes of n-tuples of points on a K3-surface are irreducible symplectic manifolds and that one obtains a second series of so-called generalised Kummer varieties by modifying Hilbert schemes of abelian surfaces.

2.1. The Quot-scheme

Let X be a projective scheme with an ample divisor H. If F is a coherent sheaf, the function

$$P(F): \quad n \mapsto \chi(F \otimes \mathcal{O}_X(nH))$$

is a polynomial. Now let P be a given polynomial. There are too many sheaves F with $P(F) = P$ to be parametrised by a scheme of finite type. However, this is possible if restrict ourselves to sheaves F that appear as quotients of a fixed sheaf G.

Formally, we get a functor

$$\underline{\mathrm{Quot}}_{X,H}(G, P) : (\mathrm{Schemes})^{\circ} \longrightarrow (\mathrm{Sets})$$

$$S \mapsto \left\{ \mathcal{O}_S \otimes G \twoheadrightarrow \mathcal{F} \;\middle|\; \mathcal{F} \text{ } S\text{-flat}, \; P(\mathcal{F}_s) = P \; \forall s \in S \right\}$$

<u>Theorem</u> (Grothendieck)

The functor $\underline{\mathrm{Quot}}_{X,H}(G, P)$ is represented by a projective scheme $\mathrm{Quot}_{X,H}(G, P)$.

this means that there is a functorial bijection

$$\underline{\mathrm{Quot}}_{X,H}(G,P)\,(S) = \mathrm{Mor}\,(S, \mathrm{Quot}_{X,H}(G,P))$$

In particular, closed points in $\mathrm{Quot}_{X,H}(G,P)$ correspond to isomorphism classes of quotients
$q : G \twoheadrightarrow F$ with $P(F) = P$.

Taking $S = \mathrm{Spec}\,\mathbb{C}[\varepsilon]$, $\mathbb{C}[\varepsilon] = \mathbb{C}[t]/t^2$, we get an intrinsic description of the Zariski tangent space at a point $[q : G \to F]$. Tangent vectors correspond to flat deformations of q over $\mathbb{C}[\varepsilon]$, hence to commutative diagrams

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & K \arrow[r] \arrow[u] & G \arrow[r, "q"] \arrow[u] & F \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \tilde{K} \arrow[r] \arrow[u] & G \otimes \mathbb{C}[\varepsilon] \arrow[r, "\tilde{q}"] \arrow[u] & \tilde{F} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \varepsilon K \arrow[r] \arrow[u] & \varepsilon G \arrow[r] \arrow[u] & \varepsilon F \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $K = \mathrm{Ker}(q)$. Now $\tilde{q}$ is determined by $\tilde{K}$. As the latter module always contains εK, it is fixed by giving a homomorphism $K \to \varepsilon F$. It follows that

$$T_{[q]}\mathrm{Quot}_{X,H}(G,P) = \mathrm{Hom}_{\mathcal{O}_X}(K,F)$$

Quotient schemes will reappear in the context of moduli spaces of sheaves. For the moment we specialise in a different direction:

2.2. Hilbert schemes

Let X be a projective surface and $n \in \mathbb{N}_0$ a natural number, that we consider as a constant polynomial.

Definition $\mathrm{Hilb}^n(X) := \mathrm{Quot}_{X,H}(\mathcal{O}_X, n)$ is called the n-th Hilbert scheme of X.

Closed points in $\mathrm{Hilb}^n(X)$ correspond to exact sequences

$$0 \longrightarrow I_\xi \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_\xi \longrightarrow 0,$$

where $\xi \subset X$ is a zero-dimensional closed subscheme of length $\ell(\xi) := \dim_{\mathbb{C}} \mathcal{O}_\xi$. Thus any set of n pairwise disjoint points $x_1, \dots, x_n \in X$ defines an element in $\mathrm{Hilb}^n(X)$, but more complicated things can happen when some of the points collide.

Theorem (Fogarty)
If X is a smooth irreducible surface, then $\mathrm{Hilb}^n(X)$ is a smooth irreducible variety of dimension $2n$.

Proof It essentially suffices to show that

$$T_{[\xi]} \mathrm{Hilb}^n(X) = \mathrm{Hom}_{\mathcal{O}_X}(I_\xi, \mathcal{O}_\xi)$$

has constant dimension $2n$. Identify $\mathrm{Hom}(I_\xi, \mathcal{O}_\xi) = \mathrm{Ext}^1(\mathcal{O}_\xi, \mathcal{O}_\xi)$. Since $\mathrm{Hom}(\mathcal{O}_\xi, \mathcal{O}_\xi) \cong H^0(\mathcal{O}_\xi) \cong \mathbb{C}^n$ and $\mathrm{Ext}^2(\mathcal{O}_\xi, \mathcal{O}_\xi) = \mathrm{Hom}(\mathcal{O}_\xi, \mathcal{O}_\xi \otimes K)^{\vee} \cong \mathbb{C}^n$, it remains to show that

$$\chi(\mathcal{O}_\xi, \mathcal{O}_\xi) = \sum_{i=0}^{2} (-1)^i \dim \mathrm{Ext}^i(\mathcal{O}_\xi, \mathcal{O}_\xi) = 0.$$

This follows from Riemann-Roch.

There is a so-called Hilbert-Chow morphism

$$\rho_n : \mathrm{Hilb}^n(X) \longrightarrow S^n X = X^n / \mathfrak{S}_n$$
$$\xi \longmapsto \sum_{x \in X} \ell(\mathcal{O}_{\xi,x}) \cdot x$$

If $\sum n_i x_i \in S^n X$ with disjoint points $\{x_i\}$ and multiplicities n_i, $\sum n_i = n$, then

$$\rho_n^{-1}\left(\sum_{i=1}^{s} n_i x_i\right) = \rho_{n_1}^{-1}(n_1 x_1) \times \dots \times \rho_{n_s}^{-1}(n_s x_s)$$

In order to understand the fibres of ρ_n, it therefore suffices to consider

$$H_n := \rho_n^{-1}(nx) \quad \text{for some } x \in X.$$

Clearly, H_n does not depend on X or x, up to isomorphism.

If $n = 1$, $H_1 = \{\mathfrak{m}\}$, $\mathfrak{m} \subset \hat{\mathcal{O}}_{X,x}$ maximal ideal.

$n = 2$, $H_2 = \{ I \subset \hat{\mathcal{O}}_{X,x} \mid \dim_{\mathbb{C}} \hat{\mathcal{O}}_{X,x}/I = 2 \}$

$= \{ I/\mathfrak{m}^2 \subset \mathfrak{m}/\mathfrak{m}^2 \} \cong \mathbb{P}^1$.

More generally, if $n \geq 3$, there will be two types of subschemes ξ of length n with support at x:

- either $\dim T_x \xi = 2$,
- or $\dim T_x \xi = 1$, in which case ξ is contained in a smooth curve through x. These form an $\mathbb{A}^{n-2}$-bundle over $\mathbb{P}^1$ and are called curvilinear.

<u>Theorem (Briançon)</u>
The curvilinear schemes are dense in H_n. In particular, H_n is irreducible and of dimension $n-1$

<u>Exercise</u> H_3 is isomorphic to the projective cone over a rational cubic curve in $\mathbb{P}^3$.

Consider the diagram

$$\begin{array}{ccc} & & X^n \\ & & \downarrow{\scriptstyle \pi} \\ \mathrm{Hilb}^n(X) & \xrightarrow[\rho_n]{} & S^n(X) \end{array},$$

where π is the quotient map for the $\mathfrak{S}_n$-action. Let $(X^n)_*$, $S^n(X)_*$ and $\mathrm{Hilb}^n(X)_*$ denote the open subsets of tuples with at most one double point (and no worse coincidences). The complements of these open subsets have codimensions 4, 4 and 2, respectively (which follows from Briançon's thm.) Restricted to these subsets we can complete the diagram as follows:

$$\begin{array}{ccc} \mathcal{B}_\Delta(X^n)_* & \xrightarrow{\rho'} & (X^n)_* \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Hilb}^n(X)_* & \xrightarrow{\rho_n} & S^n(X)_* \end{array}$$

where π' is the quotient map for the induced $\mathfrak{S}_n$-action and ρ' is the blowing-up of $(X^n)_*$ along the two-codimensional submanifold

$$\Delta = \bigcup_{i<j} \Delta_{ij} \ , \quad \Delta_{ij} = \{(x_1,\dots,x_n) \in X^n_* \mid x_i = x_j\}$$

Moreover, restricted to $S^n(X)_*$, ρ_n is the blowing-up of $S^n(X)_*$ along $\pi(\Delta)$ with exceptional divisor

$$E = \rho_n^{-1}(\pi(\Delta)) = \pi'(\rho'^{-1}(\Delta)) \ .$$

<u>Theorem</u> (Beauville) $n \geq 2$.

i) $\pi_1(\mathrm{Hilb}^n(X)) \cong \pi_1(X)^{ab}$

ii) $H^2(\mathrm{Hilb}^n(X);\mathbb{C}) \cong H^2(X;\mathbb{C}) \oplus \Lambda^2 H^1(X;\mathbb{C}) \oplus \mathbb{C}[E]$
(as Hodge structures of weight 2)

iii) If $X = K3$, then $\mathrm{Hilb}^n(X)$ is an irreducible holomorphic symplectic manifold with
$\dim = 2n \geq 4$, $\quad b_2 = 23$.

Proof The homomorphisms induced by inclusion

$$\pi_1\left(B_\Delta(X^n)_* \setminus \varrho^{-1}(\Delta)\right) \longrightarrow \pi_1\left((X^n)_* \setminus \Delta\right) \longrightarrow \pi_1 X^n \cong (\pi_1 X)^n$$

and

$$\pi_1\left(\mathrm{Hilb}^n(X)_*\right) \longrightarrow \pi_1\left(\mathrm{Hilb}^n(X)\right)$$

are isomorphisms. The group $\mathfrak{S}_n$ acts freely on $B_\Delta(X^n)_* \setminus \varrho^{-1}(\Delta)$. One can show that the resulting covering sequence

$$1 \longrightarrow \pi_1\left(B_\Delta(X^n)_* \setminus \varrho^{-1}(\Delta)\right) \longrightarrow \pi_1\left(\mathrm{Hilb}^n(X)_* \setminus E\right) \longrightarrow \mathfrak{S}_n \longrightarrow 1$$

splits, so that

$$\pi_1\left(\mathrm{Hilb}^n(X)_* \setminus E\right) \cong \left(\pi_1(X)\right)^n \rtimes \mathfrak{S}_n,$$

where $\mathfrak{S}_n$ permutes the factors of the normal subgroup $\pi_1(X)^n$. Now glue in a tubular neighbourhood of E. An application of Seifert-van Kampen shows that this introduces a transposition in $\mathfrak{S}_n$ as relation. The normal subgroup generated by this transposition contains all of $\mathfrak{S}_n$. The assertion now follows from the algebraic lemma:

Lemma G any group, $n \geq 2 \Rightarrow G^n \rtimes \mathfrak{S}_n / \langle\langle \mathfrak{S}_n \rangle\rangle \cong G/[G,G]$.

Similarly, the inclusions $X^n_* \to X^n$, $\mathrm{Hilb}^n(X)_* \to \mathrm{Hilb}^n(X)$ induce isomorphisms for H^i, $i \leq 2$. Blowing-up adds direct summands $\mathbb{C}[\varrho^{-1}(\Delta_{ij})]$ to H^2. Hence

$$H^2\left(B_\Delta(X^n)_*\right) = \bigoplus_{i=1}^n pr_i^* H^2(X;\mathbb{C}) \oplus \bigoplus_{i<j}\left(pr_i^* H^1(X;\mathbb{C}) \otimes pr_j^* H^1(X;\mathbb{C}) \oplus \mathbb{C}[\varrho^{-1}\Delta_{ij}]\right)$$

In order to pass to $\mathrm{Hilb}^n(X)_*$ we need to take $\mathfrak{S}_n$-invariants. Note that the $\mathfrak{S}_n$-action introduces signs for $H^1 \otimes H^1$, whenever the two factors are exchanged. This yields:

$$H^2\left(\mathrm{Hilb}^n(X)\right) \cong H^2(X;\mathbb{C}) \oplus \Lambda^2 H^1(X;\mathbb{C}) \oplus \mathbb{C}[E].$$

If $X = K3$, then $\pi_1 X = 0 \Rightarrow \pi_1 \mathrm{Hilb}^n X = 0$.
Moreover, $H^1(X;\mathbb{C}) = 0 \Rightarrow H^2(\mathrm{Hilb}^n(X)) = H^2(X) \oplus \mathbb{C}[E]$.
and $H^{2,0}(\mathrm{Hilb}^n(X)) = H^{2,0}(X) \cong \mathbb{C}$.

Let σ be a symplectic structure on X. Then the symplectic structure $\sum pr_i^* \sigma$ on X^n is $\mathfrak{S}_n$-invariant and hence descends to a symplectic structure on $S^n(X)_* \setminus \pi(\Delta)$. A local computation similar to the one for Kummer varieties shows that this form extends to a symplectic structure on $\mathrm{Hilb}^n(X)_*$. The rest follows from the observation:

<u>Lemma</u> Let Z be a subscheme of codimension ≥ 2 in a manifold Y. Then any symplectic structure σ on $Y - Z$ extends uniquely to a symplectic structure on Y.

<u>Proof</u> By Hartog's theorem (in the analytic category) or normality (arguing algebraically), σ extends uniquely to a 2-form on Y. The degeneracy locus of σ is the vanishing locus of the section σ^n in K_Y and hence a divisor. This would have to be contained in Z, which is absurd.

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There are general results about the structure of H^* for irreducible symplectic manifolds (see the lectures of Huybrechts). For Hilbert schemes of K3-surfaces, much more is known:

<u>Theorem</u> (Göttsche - Soergel ; Nakajima)

$$\bigoplus_{n=0}^{\infty} \bigoplus_{i=0}^{4n} H^i(\mathrm{Hilb}^n(X);\mathbb{C})\, t^i q^n = S^*\Big(\bigoplus_{m=1}^{\infty} \bigoplus_{j=0}^{4} H^j(X;\mathbb{C})\, t^{2m+j-2} q^m\Big)$$

(Lehn - Sorger) computed also the ring structure.