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WORKSHOP ON THEORETICAL PLASMA ASTROPHYSICS

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Hydrodynamics and Magnetohydrodynamics

E.N. Parker

Dept. of Physics, University of Chicago, USA

These are preliminary lecture notes, intended only for distribution to participants.

Preliminary Remarks

Concerning hydrodynamics & magnetohydrodynamics in nature, where no one applies external electrical potentials

Basic Point: The dynamics of gases and magnetized plasmas is described by the equations of Newton and Maxwell.

Consider the large-scale bulk motion of gases, plasmas, and magnetic field

It is often stated that:

1. Hydrodynamics does not apply to collisionless gas.

False: As we shall see, the hydrodynamic momentum equation has nothing to do with collisions.

2. MHD does not apply to a collisionless plasma or to a partially ionized gas, largely because the scalar Ohm's law does not apply.

False: Ohm's law is not the basis for MHD.

3. The magnetic field is caused by the current, so the electric current is the basic quantity.

False: The current is created and driven by $\mathbb{B}$. Newton & Maxwell provide field equations in terms of $\mathbb{B}$ and $\mathbf{v}$, not $\mathbf{j}$ and $\mathbb{E}$.

4. The overall dynamical behavior can be represented by an equivalent electric circuit

False: The current flows in the frame of reference in which $\mathbb{E}'=0$.

5. It must be shown that the ions and electrons move in such a way as to satisfy Ampere.

False: Newton's eqns. and Maxwell's eqns are fundamental laws of nature. So they are automatically compatible.

You cannot construct a physical situation in which they are contradictory.

Subsequent viewgraphs will display lots of equations. They are intended as background. and should not distract attention from the <u>basic physical facts and relations</u>.

Hydrodynamics

Consider the large-scale bulk motion of a gas. The individual particles may, or may not, collide with eachother.

Assume that the particles are so numerous that the local gas density can be accurately defined.

Denote the characteristic scale of the gas dynamics by L. The local scale $\ell \ll L$ defines a characteristic volume $V = \ell^3$. N is the local number density. Require that $N\ell^3 \gg 1$ so that the statistical fluctuations $O(N\ell^3)^{\frac{1}{2}}$ are sufficiently small.

There is then a well defined local mean gas velocity V_i.

The hydrodynamic equations are merely the statement of conservation of particles and momentum.

The time rate of change of a local density is equal to the negative divergence of the flux of that density.

Thus

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial x_i}(N v_i)$$

and

$$\frac{\partial}{\partial t} N M v_i = -\frac{\partial}{\partial x_j} N M v_i v_j - \frac{\partial p_{ij}}{\partial x_j} + F_i$$

Let u_i = velocity of individual particle, with

$$u_i = v_i + w_i$$

v_i = local mean bulk velocity

w_i = thermal velocity relative to mean.

Compute mean over local volume $V = \ell^3$

$$N = \frac{1}{V}\sum_V \ , \ Nv_i = \frac{1}{V}\sum_V u_i = \frac{1}{V}\sum_V (v_i + w_i)$$

$$\sum_V w_i = 0$$

Particle density N, particle flux Nv_i

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial x_k} N v_k \ , \ \frac{dN}{dt} = -N\frac{\partial v_k}{\partial x_k}$$

Momentum density

$$\frac{1}{V}\sum_V M u_j = NMv_j$$

Flux of momentum density

$$\frac{1}{V}\sum_V M u_i u_j = \frac{1}{V}\sum_V M v_i v_j + \frac{1}{V}\sum M w_i w_j$$

Flux of momentum density

$$NMv_iv_j + p_{ij} \qquad\qquad p_{ij} = \frac{1}{V}\sum_V M w_i w_j$$

p_{ij} is pressure tensor = flux of momentum density carried by thermal motions

Hence

$$\frac{\partial}{\partial t} NMv_i = -\frac{\partial}{\partial x_j} NMv_iv_j - \frac{\partial p_{ij}}{\partial x_j} + F_i$$

where F_i is force applied from without.

$$NM\left(\frac{\partial v_i}{\partial t} + v_i\frac{\partial v_j}{\partial x_j}\right) = -\frac{\partial p_{ij}}{\partial x_i} + F_i$$

Compute p_{ij}

Interparticle collisions do not conserve p_{ij}

In the absence of scattering by collisions and plasma waves, density of momentum flux is

$$\frac{1}{V}\sum_{V} M u_i u_j = N M v_i v_j + p_{ij}$$

Flux of this density is

$$\frac{1}{V}\sum_{V} M u_i u_j u_k = \frac{1}{V}\sum_{V} M(v_i+w_i)(v_j+w_j)(v_k+w_k)$$

$$= N M v_i v_j v_k + v_i p_{jk} + v_j p_{ik} + v_k p_{ij} + T_{ijk}$$

with

$$T_{ijk} = \frac{1}{V}\sum_{V} M w_i w_j w_k$$

This is the heat flow tensor, representing the flux of p_{ij} transported by thermal motions.

The time rate of change of the momentum flux density is the negative divergence of the total flux of momentum flux density.

This relation can be reduced to

$$\frac{dp_{ij}}{dt} = \frac{\partial p_{ij}}{\partial t} + v_k \frac{\partial p_{ij}}{\partial x_k}$$

$$= -(v_i F_j + v_j F_i) - \frac{\partial T_{ijk}}{\partial x_k}$$

$$- p_{ij}\frac{\partial v_k}{\partial x_k} - p_{ik}\frac{\partial v_j}{\partial x_k} - p_{jk}\frac{\partial v_i}{\partial x_k}$$

Examples: Let $F_i = \frac{\partial T_{ijk}}{\partial x_k} = 0$

Adiabatic, collisionless

$$\frac{\partial v_1}{\partial x_1} = \frac{1}{\tau}, \quad \frac{\partial v_2}{\partial x_2} = \frac{\partial v_3}{\partial x_3} = 0, \quad \frac{\partial v_i}{\partial x_j} = 0 \text{ for } i \neq j$$

$$\frac{dp_{11}}{dt} = -\frac{3}{\tau} p_{11}, \quad \frac{dN}{dt} = -\frac{1}{\tau} N, \quad \frac{dp_{22}}{dt} = -\frac{1}{\tau} p_{22}$$

$$p_{11} \sim N^3, \quad p_{22}, p_{33} \sim N$$

$$\frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_2} = \frac{1}{\tau}, \quad \frac{\partial v_3}{\partial x_3} = 0$$

$$\frac{dp_{11}}{dt} = -\frac{4}{\tau} p_{11}, \quad \frac{dN}{dt} = -\frac{2}{\tau} N$$

$$p_{11}, p_{22} \propto N^2, \quad p_{33} \sim N$$

$$\frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_2} = \frac{\partial v_3}{\partial x_3} = \frac{1}{\tau}$$

$$\frac{dp_{11}}{dt} = -\frac{5}{\tau} p_{11}, \quad \frac{dN}{dt} = -\frac{3}{\tau} N$$

$$p_{11}, p_{22}, p_{33} \sim N^{5/3}$$

In the presence of collisions, add the scattering

$$\frac{dp_{11}}{dt} = -\frac{1}{\tau}(2p_{11} - p_{22} - p_{33}) + \cdots,$$

$$\frac{dp_{22}}{dt} = -\frac{1}{\tau}(2p_{22} - p_{33} - p_{11}) + \cdots,$$

$$\frac{dp_{33}}{dt} = -\frac{1}{\tau}(2p_{33} - p_{11} - p_{22}) + \cdots.$$

For collision dominated gas $p_{11} = p_{22} = p_{33} \equiv p$

$$\frac{1}{\gamma - 1}\frac{dp}{dt} + \frac{\gamma}{\gamma - 1} p \frac{\partial u_k}{\partial x_k} = K \nabla^2 T$$

The off diagonal terms of p_{ij} can be represented by a suitable viscosity.

Magnetohydrodynamics

MHD is based on the concept that the magnetic field is transported bodily by the fluid

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \text{dissipation terms}$$

Consider a gas with enough free electrons and ions that it cannot support any significant electric field $\mathbf{E}'$ in its own frame of reference.

$$\mathbf{E}' = \mathbf{E} + \underline{v} \times \mathbf{B}$$

If $\mathbf{E}' = 0$, then

$$\mathbf{E} = -\underline{v} \times \mathbf{B}$$

and

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

The result is MHD, regardless of the details

Note that $\mathbb{E}$ plays no significant dynamical role.

$$\frac{\mathbb{E}^2}{8\pi} = O\left(\frac{v^2}{c^2}\right)\frac{B^2}{8\pi}$$

Note, too, that the existence of $\mathbb{E}$ depends upon what frame of reference the calculation uses.

See example in V. Vasyliunas, 2001, Geophys. Res. Letters, 28, 2177.

Similarly $\mathbf{j}$ plays no dynamical role because it has no energy and no strength.

Note that in any real gaseous medium, $\mathbf{j}$ is driven by a weak $\mathbb{E}'$, pulling energy out of the magnetic field.

$\mathbb{B}$ causes $\mathbf{j}$, not vice versa.

Note that $\mathbf{j}$ is driven by the relation

$$\frac{\partial \mathbb{E}}{\partial t} = c\nabla \times \mathbb{B} - 4\pi \mathbf{j}$$

To determine the dynamical role of $\mathbb{E}, \mathbb{B}$ consider Poynting's theorem for a collection of particles with mass distribution $\rho(\mathbf{r}, t)$ and associated charge distribution $s(\mathbf{r}, t)$. The individual particle has velocity $\mathbf{v}$ and

$$\mathbf{j} = \mathbf{v}\, s(\mathbf{r}, t)$$

$$\rho \frac{d\mathbf{v}}{dt} = s\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)$$

$$= \frac{\mathbb{E}\,\nabla\cdot\mathbb{E}}{4\pi} + \mathbf{j}\times\mathbb{B}/c$$

Using Maxwell's equations, it can be shown that

$$\rho \frac{dv_i}{dt} = \frac{\partial M_{ij}}{\partial x_j} - \frac{\partial}{\partial t}\left(\frac{P_i}{c^2}\right)$$

or

$$\frac{\partial}{\partial t}\left(\rho v_i + \frac{P_i}{c^2}\right) = \frac{\partial M_{ij}}{\partial x_j} + \frac{\partial R_{ij}}{\partial x_j}$$

where

$$M_{ij} = -\delta_{ij}\frac{E^2+B^2}{8\pi} + \frac{E_iE_j + B_iB_j}{4\pi} \qquad \text{Maxwell stress tensor}$$

$$R_{ij} = -\rho v_i v_j \qquad \text{Reynolds stress tensor}$$

$$\mathbb{P} = c\,\frac{\mathbf{E}\times\mathbb{B}}{4\pi} \qquad \text{Poynting vector}$$

Note that, if initially each particle is represented by a localized spike in ρ and S, it is an easy matter to average over a local volume V because the equations are all linear in ρ and S.

Note that if

$$\mathbb{E} = -v \times \mathbb{B}/c$$

then

$$\mathbb{P} = v_\perp \frac{B^2}{4\pi}$$

where $v_\perp$ is the velocity $\perp \mathbb{B}$.

The particle kinetic energy varies as

$$\frac{\partial}{\partial t}\frac{1}{2}\rho v^2 = S v \cdot \mathbb{E} = \mathbb{j} \cdot \mathbb{E}$$

which can be rewritten as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \frac{B^2}{8\pi}\right) + \nabla \cdot \mathbb{P} = 0$$

with the electromagnetic energy flux given by

$$\mathbb{P} = v_\perp \frac{B^2}{4\pi}$$

again.

Note that the dynamics is all in terms of

$$\rho = NM, \quad p = NkT, \quad \mathbf{v}, \text{ and } \mathbf{B}$$

when $\mathbf{E} = -\mathbf{v} \times \mathbf{B}/c$

$\mathbf{E}$ and $\mathbf{j}$ are secondary passive quantities.

Consider the role of the neglected $\mathbf{E}'$.

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - c\,\nabla \times \mathbf{E}'$$

For a collision dominated plasma, $\mathbf{j} = \sigma \mathbf{E}'$.
Hence $\mathbf{E}' = \frac{c}{4\pi\sigma} \nabla \times \mathbf{B}$ and

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B})$$

$$\eta = \frac{c^2}{4\pi\sigma} \sim 0.5 \times 10^{13} / T^{3/2} \quad \mathrm{cm^2/sec.}$$

Magnetic Reynolds number

$$N_R = \frac{vL}{\eta}$$

For N_R the principal effect is bulk transport of $\mathbf{B}$.

For a partially ionized gas

N = number density of neutral atoms
n = number density of ions/electrons

v = mean bulk velocity of neutral gas

w = mean bulk velocity of ions

u = mean bulk velocity of electrons

τ_i = ion-neutral collision time

τ_e = electron-neutral collision time

τ = ion-electron collision time

p = pressure of neutral gas

$$NM \frac{dv}{dt} = -\nabla p + \frac{nM(w-v)}{\tau_i} + \frac{nm(u-v)}{\tau_e}$$

Consider a slightly ionized gas, $n \ll N$.

Neglect ion and electron pressures.

$$m\frac{d\mathbf{v}}{dt} = -e\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) - \frac{m(\mathbf{v}-\mathbf{V})}{\tau_e} - \frac{m(\mathbf{v}-\mathbf{W})}{\tau}$$

$$M\frac{d\mathbf{w}}{dt} = +e\left(\mathbf{E} + \frac{\mathbf{w}\times\mathbf{B}}{c}\right) - \frac{M(\mathbf{w}-\mathbf{V})}{\tau_i} + \frac{m(\mathbf{v}-\mathbf{W})}{\tau}$$

$$\mathbf{j} = ne(\mathbf{w}-\mathbf{v})$$

From Ampere's law

$$\mathbf{v} = \mathbf{w} - \frac{c}{4\pi ne}\nabla\times\mathbf{B}$$

Neglect the electron and ion inertia. The sum of the two eqns. of motion gives

$$\frac{nM(\mathbf{w}-\mathbf{V})}{\tau_i} + \frac{nm(\mathbf{v}-\mathbf{V})}{\tau_e} = en(\mathbf{w}-\mathbf{v})\times\frac{\mathbf{B}}{c}$$

$$= \mathbf{j}\times\mathbf{B}/c = \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi}$$

Hence, for the neutral atoms

$$NM\frac{d\mathbf{V}}{dt} = -\nabla p + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi}$$

which is the usual MHD momentum eqn.

Note that

$$w = v + \frac{(\nabla\times B)\times B}{4\pi n Q} + \frac{cm/\tau_e}{4\pi n e Q}\,\nabla\times B$$

$$u = v + \frac{(\nabla\times B)\times B}{4\pi n Q} - \frac{cM/\tau_i}{4\pi n e Q}\,\nabla\times B$$

where

$$Q \equiv \frac{M}{\tau_i} + \frac{m}{\tau_e} \cong \frac{M}{\tau_i}$$

Then

$$E = -\frac{v\times B}{c} - \frac{[(\nabla\times B)\times B]\times B}{4\pi n c Q} + \frac{M/\tau_i - m/\tau_e}{4\pi n e Q}(\nabla\times B)\times B$$

$$+ \frac{c}{4\pi n e^2}\left[\frac{(M/\tau_i)(m/\tau_e)}{Q} + \frac{m}{\tau}\right]\nabla\times B$$

Define

$$\alpha \equiv \frac{cB}{4\pi n e}\,\frac{M/\tau_i - m/\tau_e}{M/\tau_i + m/\tau_e}$$ Hall coefficient

$$\beta \equiv \frac{B^2}{4\pi n Q}$$ Pedersen coefficient

$$\eta \equiv \frac{c^2}{4\pi n e^2}\left[\frac{(M/\tau_i)(m/\tau_e)}{M/\tau_i + m/\tau_e} + \frac{m}{\tau}\right]$$ Ohmic coeff.

Write $\mathbf{b} = \mathbf{B}/B$, so that

$$\mathbf{E} = \frac{B}{c}\left[-\mathbf{v}\times\mathbf{b} - \beta\left[(\nabla\times\mathbf{b})\times\mathbf{b}\right]\times\mathbf{b} + \alpha(\nabla\times\mathbf{b})\times\mathbf{b} + \eta\nabla\times\mathbf{b}\right]$$

$$\mathbf{E}' = \frac{B}{c}\left[\eta\nabla\times\mathbf{b} + \alpha(\nabla\times\mathbf{b})\times\mathbf{b} - \beta\left[(\nabla\times\mathbf{b})\times\mathbf{b}\right]\times\mathbf{b}\right]$$

The induction equation

$$\partial\mathbf{B}/\partial t = -c\nabla\times\mathbf{E}$$

becomes

$$\frac{\partial\mathbf{b}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{b}) - \nabla\times(\eta\nabla\times\mathbf{b})$$
$$+ \nabla\times\left\{\beta\left[(\nabla\times\mathbf{b})\times\mathbf{b}\right]\times\mathbf{b} - \alpha(\nabla\times\mathbf{b})\times\mathbf{b}\right\}$$

This is the usual MHD eqn. with two extra terms

In terms of the nondimensional Lorentz force

$$\mathbf{f} = \frac{(\nabla\times\mathbf{b})\times\mathbf{b}}{4\pi}$$

$$\frac{\partial\mathbf{b}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{b}) - \nabla\times(\eta\nabla\times\mathbf{b}) + \nabla\times\left[\beta\mathbf{f}\times\mathbf{b} - \alpha\mathbf{f}\right]$$

The magnetic energy equation can be written

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2} b^2\right) + \nabla\cdot\left[v_\perp b^2 + \eta f + \beta b^2 f_\perp + \alpha f \times b \right]$$

$$= - v \cdot f - \eta (\nabla \times b)^2 - \beta f^2$$

The right hand side represents the dissipation of magnetic energy. The term in square brackets represents the transport of magnetic energy

Consider the Hall effect with $\beta = \eta = \nabla\alpha = 0$

$$\frac{\partial b}{\partial t} = \nabla \times (v \times b) - 4\pi\, \alpha\, \nabla \times f$$

$$\frac{\partial v}{\partial t} + \omega \times v = -\frac{\nabla p}{NM} - \nabla\left(\tfrac{1}{2} v^2\right) + 4\pi\, C^2 f$$

$$C^2 = B^2/(4\pi NM) \;, \quad \omega = \nabla \times v$$

Then

$$\frac{\partial \omega}{\partial t} = \nabla \times (v \times \omega) + 4\pi C^2 \nabla \times f$$

Hence

$$\frac{\partial}{\partial t}\left(b + \frac{\alpha}{C^2}\omega\right) = \nabla \times \left[v \times \left(b + \frac{\alpha}{C^2}\omega\right)\right]$$

Note that the Hall (vorticity) contribution is smaller $O(\frac{1}{L})$ compared to the magnetic field. And that makes it the same order as resistive diffusion.

$$\frac{\alpha}{\eta} \sim \Omega_e \tau_e \qquad \Omega_e = \frac{eB}{mc}$$

The Hall effect is a small-scale effect.

See JGR, 101, 10587–10625, (1996)

If the ion and electron pressures, inertia, and other applied forces $\mathbb{L}_i$, $\mathbb{L}_e$ per unit mass are included, then

$$\frac{\partial \mathbb{b}}{\partial t} = \nabla\times(\mathbb{v}\times\mathbb{b}) - \nabla\times\left[\eta\nabla\times\mathbb{b} - \beta\mathbb{f}\times\mathbb{b} + \alpha\mathbb{f}\right]$$

$$- \frac{cm/\tau_e}{eBQ}\nabla\times\left[\frac{\nabla p_i}{n} + M\left(\frac{d\mathbb{w}}{dt} - \mathbb{L}_i\right)\right]$$

$$+ \frac{cM/\tau_i}{eBQ}\nabla\times\left[\frac{\nabla p_e}{n} + m\left(\frac{d\mathbb{u}}{dt} - \mathbb{L}_e\right)\right]$$

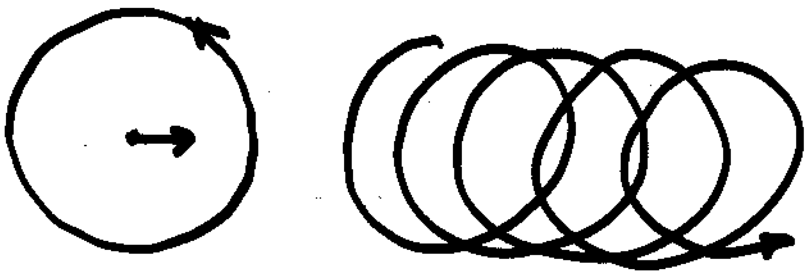
$$- \frac{1}{Q}\nabla\times\left\{\left[\frac{\nabla(p_i+p_e)}{n} + M\left(\frac{d\mathbb{w}}{dt} - \mathbb{L}_i\right) + m\left(\frac{d\mathbb{u}}{dt} - \mathbb{L}_e\right)\right]\times\mathbb{b}\right\}$$

These extra terms include thermoelectric effects, the Biermann battery, the Eddington-Sweet effect, etc. which are all negligible under ordinary $\underbrace{\text{}}_{\text{large-scale}}$ circumstances, in astrophysical settings.

But watch out for the small scales arising in tangential discontinuities, rapid reconnection, etc.

Compatibility of Newton and Maxwell

Consider a collisionless plasma, made up of equal numbers of electrons and singly charged ions. Calculate the electron and ion motions using the guiding center approximation



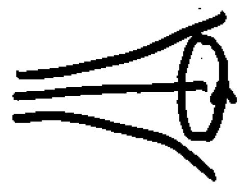
Write
$$u = c\frac{\mathbf{E}\times\mathbf{B}}{B^2}, \quad \mathbf{E}_\perp = -\frac{u\times \mathbf{B}}{c}$$

The motion of the guiding center is

$$v = u + \frac{1}{2}\frac{M w_\perp^2 c}{e B^4}\,\mathbf{B}\times\nabla\tfrac{1}{2}B^2 + \frac{M w_\parallel^2 c}{e B^4}\,\mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}]$$

Note that

$$\left(\frac{dv}{dt}\right)_\parallel = -\frac{w_\perp^2}{2B^4}\,\mathbf{B}\{\mathbf{B}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{B}]\}$$



Define

$$p_\perp = \sum_i \tfrac{1}{2} M w_\perp^2, \quad p_\parallel = \sum_i M w_\parallel^2$$

The current density is

$$\mathbf{j}_\perp = \frac{c}{B^2}\,\mathbf{B}\times\left[\nabla p_\perp - \left[\frac{p_\parallel - p_\perp}{B^2}\right](\mathbf{B}\cdot\nabla)\mathbf{B} + NM\frac{d\mathbf{u}}{dt}\right]$$

and Ampere's law becomes

$$NM\frac{d\mathbf{u}}{dt} = -\nabla_\perp\left(p_\perp + \frac{B^2}{8\pi}\right) + \left[\frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi}\right]_\perp\left\{1 + \frac{p_\perp - p_\parallel}{B^2/4\pi}\right\}$$

So Ampere's law is automatically satisfied if the bulk velocity $\mathbf{u}$ satisfies Newton's equation.

See <u>Phys. Rev.</u> 107, 924 (1957).

Chew-Goldberger-Low Approximation

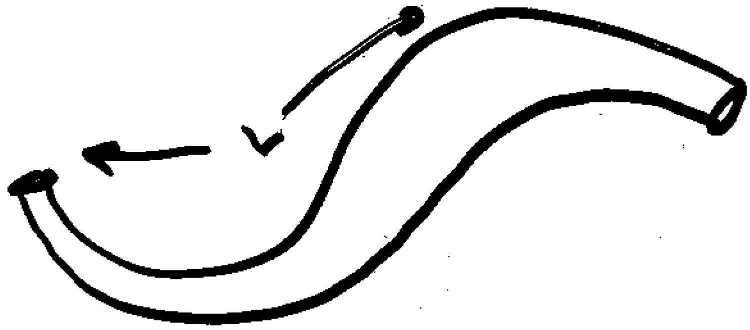
Let L denote scale of plasma and field in the direction along the field. There are, then, four invariants

$L w_{\parallel} = \text{constant}$

$A B = \text{constant}$

$A L N = \text{constant}$

$w_{\perp}^2 / B = \text{constant}$



So

$$\frac{d}{dt}\left(\frac{p_{\perp}}{NB}\right) = 0, \quad \frac{d}{dt}\left(\frac{p_{\parallel} B}{N^3}\right) = 0$$