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Self-Organization & Singular Perturbations in Plasma Flows

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These are preliminary lecture notes, intended only for distribution to participants.

Self-organization & Singular Perturbations in Plasma Flows

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Hierarchy of relaxed states

3 rd — Double Beltrami field

$$\boldsymbol{B} = C_+\boldsymbol{G}_+ + C_-\boldsymbol{G}_-$$
$$\boldsymbol{V} = C'_+\boldsymbol{G}_+ + C'_-\boldsymbol{G}_-$$
$$(\nabla \times \boldsymbol{G}_\pm = \lambda_\pm \boldsymbol{G}_\pm)$$

↑ ← Flow

2 nd — Beltrami field

$$\boldsymbol{B} = C\boldsymbol{G} \quad (\nabla \times \boldsymbol{G} = \lambda \boldsymbol{G})$$

↑ ← Current

1 st — Harmonic field

$$\boldsymbol{B} = \boldsymbol{B}_h \quad (\nabla \times \boldsymbol{B}_h = 0)$$

↑ ← Magnetic field

0 th — Vacuum

$$\boldsymbol{B} = 0$$

Hierarchy of relaxed states

3 rd — **Double Beltrami field**

$$(curl - \lambda_+)(curl - \lambda_-)\boldsymbol{B} = 0$$

↑ ← Flow

2 nd — **Beltrami field**

$$(curl - \lambda)\boldsymbol{B} = 0$$

↑ ← Current

1 st — **Harmonic field**

$$curl\,\boldsymbol{B} = 0$$

↑ ← External field

0 th — **Vacuum**

$$\boldsymbol{B} = 0$$

Topology of Function Spaces

 $\xleftrightarrow{\text{Topology}}$ 

fine structure — coarse grain

strong topology $\xrightarrow{\text{coercive}}$ weak topology

fragile $\xleftarrow{\text{dissipation}}$ robust

Mathematical Exercise

$$\begin{cases} \mathcal{G}(u) = \int_\Omega |\nabla u|^2 dx \\ H(u) = \int_\Omega |u|^2 dx \end{cases} \qquad \left(\begin{array}{l} u|_{\partial\Omega} = 0 \\ \Omega \subset \mathbb{R}^N;\ \text{bounded} \end{array} \right)$$

(i) find a minimizer of $\mathcal{G}(u)$ under $H(u)=1$

$$\delta[\mathcal{G}(u) - \lambda H(u)] = 0$$

$$\rightarrow \begin{cases} -\Delta u = \lambda u \\ u|_{\partial\Omega} = 0 \end{cases}$$

$$\rightarrow \quad u = a\varphi_j \quad (-\Delta\varphi_j = \lambda\varphi_j)$$

$$H(u) = |a|^2 = 1 \quad \rightarrow a = 1$$

$$\mathcal{G}(u) = \lambda_j \quad \rightarrow \boxed{u = \varphi_1}$$

(ii) find a minimizer of $H(u)$ under $\mathcal{G}(u)=1$

$$\delta[H(u) - \mu\mathcal{G}(u)] = 0$$

$$\rightarrow \begin{cases} -\Delta u = \mu^{-1} u \\ u|_{\partial\Omega} = 0 \end{cases}$$

$$\rightarrow u = a\varphi_j$$

$$\mathcal{G}(u) = |a|^2\lambda_j = 1 \quad \rightarrow a = \lambda_j^{-1/2}$$

$$H(u) = 1/\lambda_j \quad \rightarrow \boxed{u = 0}$$

I. Self-organization

Essential characteristics

- irreversible process, but <u>not</u> dominated by dissipation
- generation of stable equilibrium with global simplicity (coarse grain)
- <u>not</u> always

conventional models

(1) Dirichlet principle (diffusion process)

$$\partial_t u = -\partial G \qquad (G : \text{convex functional})$$

ex. $\partial_t u = \nabla\cdot(D(x)\nabla u) + f(x)$

minimizes

$$G = \frac{1}{2}\int D(x)|\nabla u|^2 dx - \int f(x)\, u\, dx$$

- Exact, if we have G.
- predicts "trivial" equilibrium.

(2) Selective dissipation

$H_0, H_1, \cdots, H_\nu$ constants of motion (ideal)

minimize H_0 with keeping $H_1 \cdots H_\nu$ constant.

ex. minimize $\int B^2 dx - \mu \int A\cdot B\, dx$

$\rightarrow$ Taylor relaxed state.

- Hypothetical model
- H_0 must be coercive
- always ?
- may derive unstable solution

(3) Statistical distribution

$H_0, H_1, \cdots, H_n$ constants of motion (ideal)

maximize S. on the ensemble
(entropy)

- Ensemble must be a closed manifold
- second quantization

(4) Minimum entropy production

Minimize "entropy production" with keeping $H_1 \cdots H_n$ const.

ex. minimize $\int \eta (\nabla \times B)^2 dx - \mu_1 \int B^2 dx - \mu_2 \int A \cdot B dx$

- may derive non-equilibrium (unstable) solutions.

A new framework

Target functional (F)

- measure of fluctuations and dissipation (turbulence)
- coercive functional

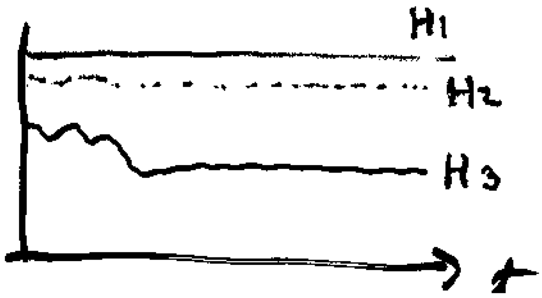
Constraints ($H_1, \cdots, H_\nu$)

- constants of motion of ideal dynamics
- adjusted by a small dissipation
- continuous functionals in the topology of F.

Self-organization

$$\delta \left[F - \mu_1 H_1 - \mu_2 H_2 \cdots \mu_\nu H_\nu \right] = 0$$

succeeds when $H_1 \cdots H_\nu$ are adjusted to produce a stable equilibrium so that F is minimized.



Self-organization in two-fluid MHD

$$\partial_t \mathbb{W}_j + \nabla\times(\mathbb{W}_j \times \mathbb{V}_j) = 0$$

$$\begin{cases} \mathbb{W}_1 = \mathbb{B}, & \mathbb{V}_1 = V_e = \mathbb{V} - \nabla\times\mathbb{B} \quad \text{(electrons)} \\ \mathbb{W}_2 = \mathbb{B} + \nabla\times\mathbb{V}, & \mathbb{V}_2 = \mathbb{V} \quad \text{(ions)} \end{cases}$$

Constants of motion

$$\begin{cases} H_0 = \|\mathbb{V}\|^2 + \|\mathbb{B}\|^2 = \int |\mathbb{V}|^2 + |\mathbb{B}|^2 \, dx \\ H_1 = (\mathrm{curl}^{-1} \mathbb{W}_1, \mathbb{W}_1) = \int A\cdot\mathbb{B} \, dx \\ H_2 = (\mathrm{curl}^{-1} \mathbb{W}_2, \mathbb{W}_2) = \int (A+\mathbb{V})\cdot(\mathbb{B}+\nabla\times\mathbb{V}) \, dx \end{cases}$$

(Note: H_0 is <u>not</u> coercive to controle H_2.
H_2 is <u>not</u> convex)

measure of fluctuations and dissipation

$$F = \cancel{\|\nabla\times\mathbb{W}_2\|^2} \ \|\mathbb{W}_2\|^2 \quad \text{(ion enstrophy)}$$

Self-organization

$$\delta\left[F - \mu_0 H_0 - \mu_1 H_1 - \mu_2 H_2\right] = 0$$

Euler - Lagrange equation

$$(\mathrm{curl} - \lambda_1)(\mathrm{curl} - \lambda_2)(\mathrm{curl} - \lambda_3)\,\mathbb{U} = 0 \qquad (T)$$

$$(\mathbb{U} = \mathbb{B} \text{ or } \mathbb{V})$$

→ "triple Beltrami" solution

$$\mathbb{U} = c_1 \mathbb{G}_1 + c_2 \mathbb{G}_2 + c_3 \mathbb{G}_3$$

$$(\mathrm{curl}\, \mathbb{G}_j = \lambda_j \mathbb{G}_j)$$

→ $\mathbb{U}$ is a stable equilibrium if $\mathbb{U}$ is a "double Beltrami" with small enough λ_1 and λ_2.

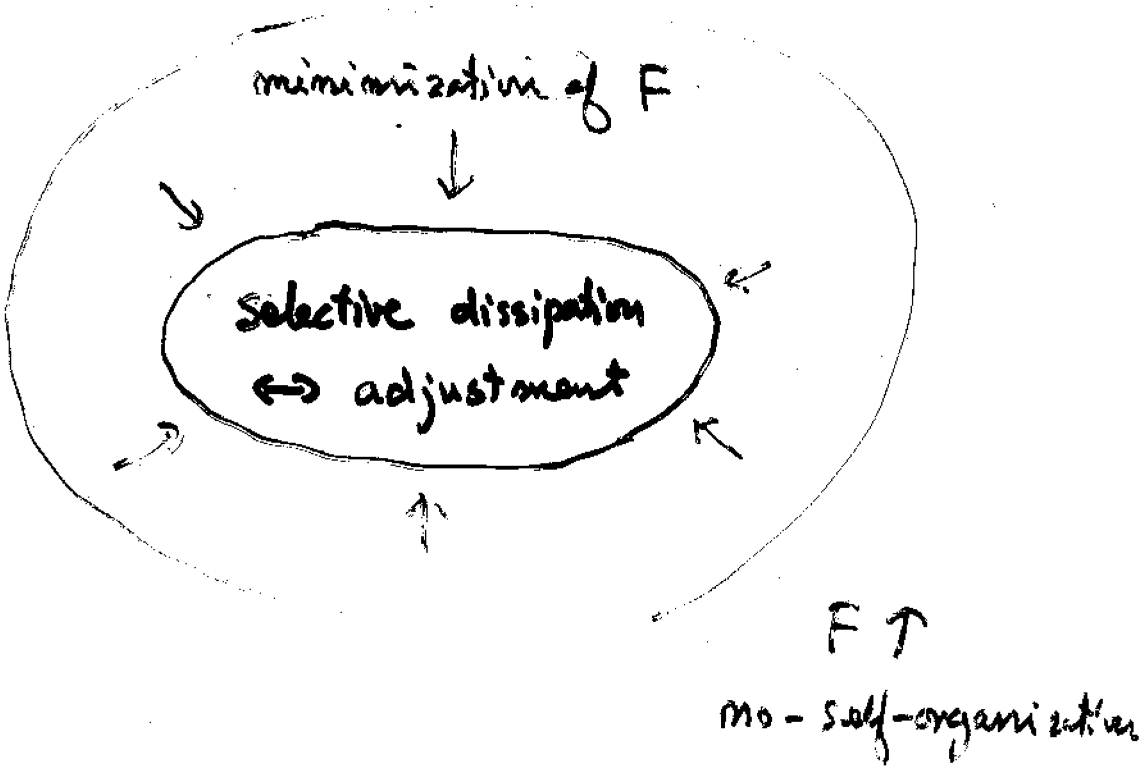
→ This occurs when H_0, H_1, H_2 satisfies a relation

$$H_0 - \mu_1' H_1 - \mu_2' H_2 = 0$$

$$\left(\Leftrightarrow \ \delta [H_0 - \mu_1' H_1 - \mu_2' H_2] = 0 \right)$$

$$\Leftrightarrow (\mathrm{curl} - \lambda_1)(\mathrm{curl} - \lambda_2)\,\mathbb{U} = 0 \qquad (D)$$

— View in the topology of F —



Singular Perturbation

$$\text{small scale} \longleftrightarrow \text{large scale}$$

DB model

$$\mathbb{U} = C_+ \mathbb{G}_+ + C_- \mathbb{G}_- \qquad (\mathbb{U} = \mathbb{B} \text{ and } \mathbb{V})$$

$$\nabla \times \mathbb{G}_\pm = \lambda_\pm \mathbb{G}_\pm$$

$$\lambda_+ \gg \lambda_-$$

$$\begin{array}{llll} \text{microscopic model} : & \varepsilon\text{-model} & \rightarrow & f_\varepsilon \\ & & & \downarrow \lim_{\varepsilon \to 0} ? \\ \text{macroscopic model} : & 0\text{-model} & \rightarrow & f_0 \end{array}$$

- ε-model = 2-fluid MHD

$$\begin{cases} \partial_t \mathbb{A} = (\mathbb{V} - \underline{\varepsilon} \nabla \times \mathbb{B}) \times \mathbb{B} - \nabla(\phi - \underline{\varepsilon} p_e) \\ \partial_t (\underline{\varepsilon} \mathbb{V} + \mathbb{A}) = \mathbb{V} \times (\mathbb{B} + \underline{\varepsilon} \nabla \times \mathbb{V}) - \nabla(\underline{\varepsilon} \frac{V^2}{2} + \underline{\varepsilon} p_i + \phi) \end{cases}$$

$$\left(\varepsilon = \frac{c/\omega_{pi}}{l} \right)$$

- 0-model = MHD

$$\begin{cases} \partial_t \mathbb{B} + (\mathbb{V} \cdot \nabla) \mathbb{B} - (\mathbb{B} \cdot \nabla) \mathbb{V} = 0 \\ \partial_t \mathbb{V} + (\mathbb{V} \cdot \nabla) \mathbb{V} - (\mathbb{B} \cdot \nabla) \mathbb{B} = -\nabla(p + \frac{B^2}{2}) \end{cases}$$

DB field (static sol. of the ε-model)

$$\partial_t \Omega_j - \nabla\times(\mathbb{V}_j \times \Omega_j) = 0$$

$$(e)\begin{cases} \Omega_1 = \mathbb{B} \\ \mathbb{V}_1 = \mathbb{V} - \varepsilon\nabla\times\mathbb{B} \end{cases} \qquad (i)\begin{cases} \Omega_2 = \mathbb{B} + \varepsilon\nabla\times\mathbb{V} \\ \mathbb{V}_2 = \mathbb{V} \end{cases}$$

Beltrami field

$$\begin{cases} \Omega_1 = a\,\mathbb{V}_1 \\ \Omega_2 = b\,\mathbb{V}_2 \end{cases}$$

$$\rightarrow \begin{cases} \mathbb{B} = C_+ G_+ + C_- G_- \\ \mathbb{V} = (a^{-1} + \varepsilon\lambda_+) C_+ G_+ + (a^{-1} + \varepsilon\lambda_-) C_- G_- \end{cases}$$

$$\lambda_\pm = \frac{1}{2\varepsilon}\left[(b - a^{-1}) \pm \sqrt{(b - a^{-1})^2 - 4(1 - b/a)}\right]$$

writing $b/a = 1 + \delta$

$$\begin{cases} \lambda_- \simeq \frac{\delta}{2}\left(\frac{1}{a} - a\right)^{-1} & \rightarrow O(1) \\ \lambda_+ \simeq \frac{-1}{2}\left(\frac{1}{a} - a\right) & \rightarrow \infty \end{cases}$$

How is the small scale created ?

Characteristics of the O-model

$$(\nabla\varphi)^2\ (\partial_t\varphi + \mathbb{V}\cdot\nabla\varphi)\,(\partial_t\varphi + (\mathbb{V}+\mathbb{B})\cdot\nabla\varphi)^2\,(\partial_t\varphi + (\mathbb{V}-\mathbb{B})\cdot\nabla\varphi)^2 = 0$$

elliptic — flow — Alfvén

Quasi-statics : $\partial_t = 0$

$$(\nabla\varphi)^2\ (\mathbb{V}\cdot\nabla\varphi)\ ((\mathbb{B}-\mathbb{V})\cdot\nabla\varphi)^2\,((\mathbb{B}+\mathbb{V})\cdot\nabla\varphi)^2 = 0$$

① non-integrability (chaos) of characteristics (3D)

② non linearity (fields are unknowns)

intersecting characteristics

$\updownarrow$ ($\rightarrow$ shocks)

"structures" $\leftarrow$

dispersion = singular perturbation

Summary

- ∞ - dim vector space

  different norms induce different topology

- coercive forms

  - measure fluctuations and dissipation

  - controle self-organization

    const. of motion must be adjusted

    ($\leftarrow$ self-organiation)

  - controle stability (Lyapunov function)

D B model

Mahajan – Y., PRL **81** (1998), 4863.

Variational Principle

Y. – Mahajan, PRL. **88** (2002), 095001.

Beltrami Fields

Y. – Giga, Math. Z. **204** (1990), 235.

Y., J. Math. Phys. **33** (1992), 1252.

Y. – Mahajan, J. Math. Phys. **40** (1999), 5080.