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Equilibrium of Stochastic Magnetic Fields

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EQUILIBRIUM OF STOCHASTIC MAGNETIC FIELDS

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Turbulent $\alpha\omega$-dynamos create large-scale mean fields with internal stochastic lines of force.

An example is the bipolar magnetic field above an active region on the Sun.

Consider the static equilibrium of a field with complicated interweaving of the field lines.

Consider and initial uniform field B_0 in the z-direction, extending through a nonresistive fluid from $z = 0$ to $z = L$.

Introduce the 2-D incompressible fluid motion

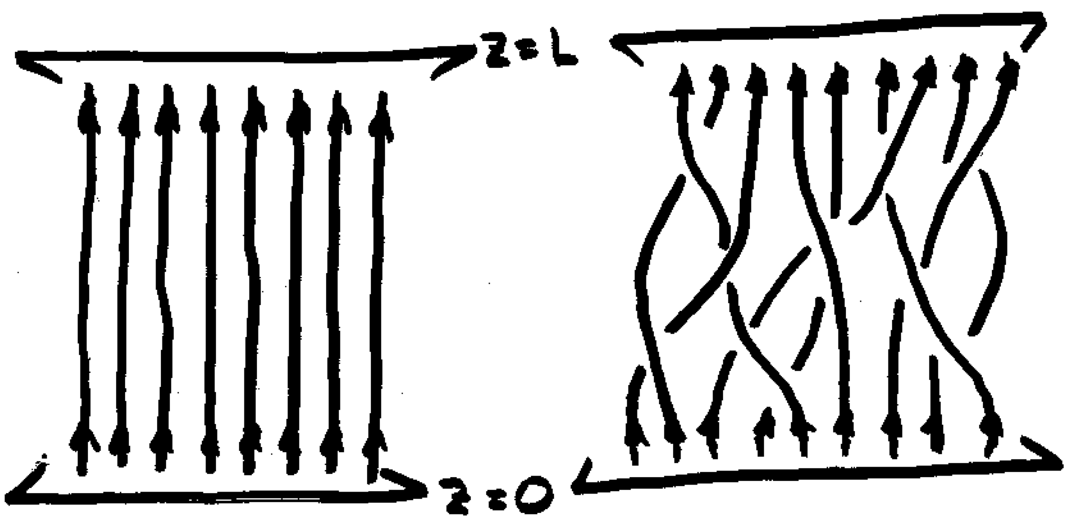
$$v_x = +kz\frac{\partial\psi}{\partial y}, v_y = -kz\frac{\partial\psi}{\partial x}, v_z = 0$$

where the arbitrary function $\psi(x,y,kzt)$ and its derivatives are continuous, bounded, and generally well behaved.

After a time t the field is given by

$$B_x = +B_0 kt\frac{\partial\psi}{\partial y}, B_y = -B_0 kt\frac{\partial\psi}{\partial x}, B_z = B_0$$

Note that the field is everywhere continuous.



Hold the footpoints of the field fixed at $z = 0$, L and release the fluid throughout $0 < z < L$ so that the field can relax to the lowest available energy state. (Introduce a small viscosity)

With the field lines unbreakable and tied at each end, the field line topology is preserved during the relaxation.

It is obvious that there exists a lowest energy state.

With uniform fluid pressure applied at $z = 0$, L the fluid pressure is uniform throughout. ($\mathbf{B} \bullet \nabla p = 0$)

Therefore, the lowest energy state represents the Maxwell stresses of the field in equilibrium with themselves.

$$M_{ij} = -\delta_{ij} \frac{B^2}{8\pi} + \frac{B_i B_j}{4\pi}$$

$$\frac{\partial M_{ij}}{\partial x_j} = 0$$

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$$

Hence

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}.$$

This well known force free equilibrium equation has mixed characteristics, so that it is unlike the partial differential equations with which we are familiar. The curl yields

$$\mathbf{B} \times \nabla \alpha = \nabla^2 \mathbf{B} + \alpha^2 \mathbf{B}$$

which looks like a quasilinear elliptic equation, (two sets of complex characteristics). However, the divergence yields

$$\mathbf{B}\bullet\nabla\alpha = 0,$$

showing the field lines to make up a family of real characteristics. So, $\nabla\times\mathbf{B} = \alpha\mathbf{B}$ has the novel property of mixed characteristics.

To see what the mixed characteristics imply, consider the idealized situation in which the field line swirling has a transverse (x,y) correlation length λ << L. Suppose that the swirling consists of a succession of n distinct random patterns, each extending for a distance L/n >> λ along the field. The swriling of *m*th pattern is uncorrelated with the $m + 1$ and $m - 1$ patterns.

Now it is obvious that the winding of the field lines around each other in any given pattern determines the torsion coefficient α if we suppose the fields to be continuous.

The mutual wrapping of the field lines may vary along a given line within a pattern, and certainly varies along the line from one pattern to the next.

Hence a continuous field in the presence of successive wrapping patterns requires a different α for each pattern. However, with

$$\mathbf{B}\bullet\nabla\alpha = 0,$$

a change in α is not possible.

Yet a solution to $\nabla\times\mathbf{B} = \alpha\mathbf{B}$ exists in every case.

The resolution of the dilemma is the formation of surfaces of tangential discontinuity. The field direction is discontinuous across the surface of discontinuity, while the field magnitude is continuous.

The surface of discontinuity represents the surface of contact between the regions of continuous field on either side. Neither **B** nor α is defined on the surface of discontinuity, so the difference in field direction across the discontinuity is not subject to $\mathbf{B}\bullet\nabla\alpha = 0$.

Elliptic equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

$$\psi = f(x+iy),\ h(x-iy)\ ;\ f(a),\ h(b)$$

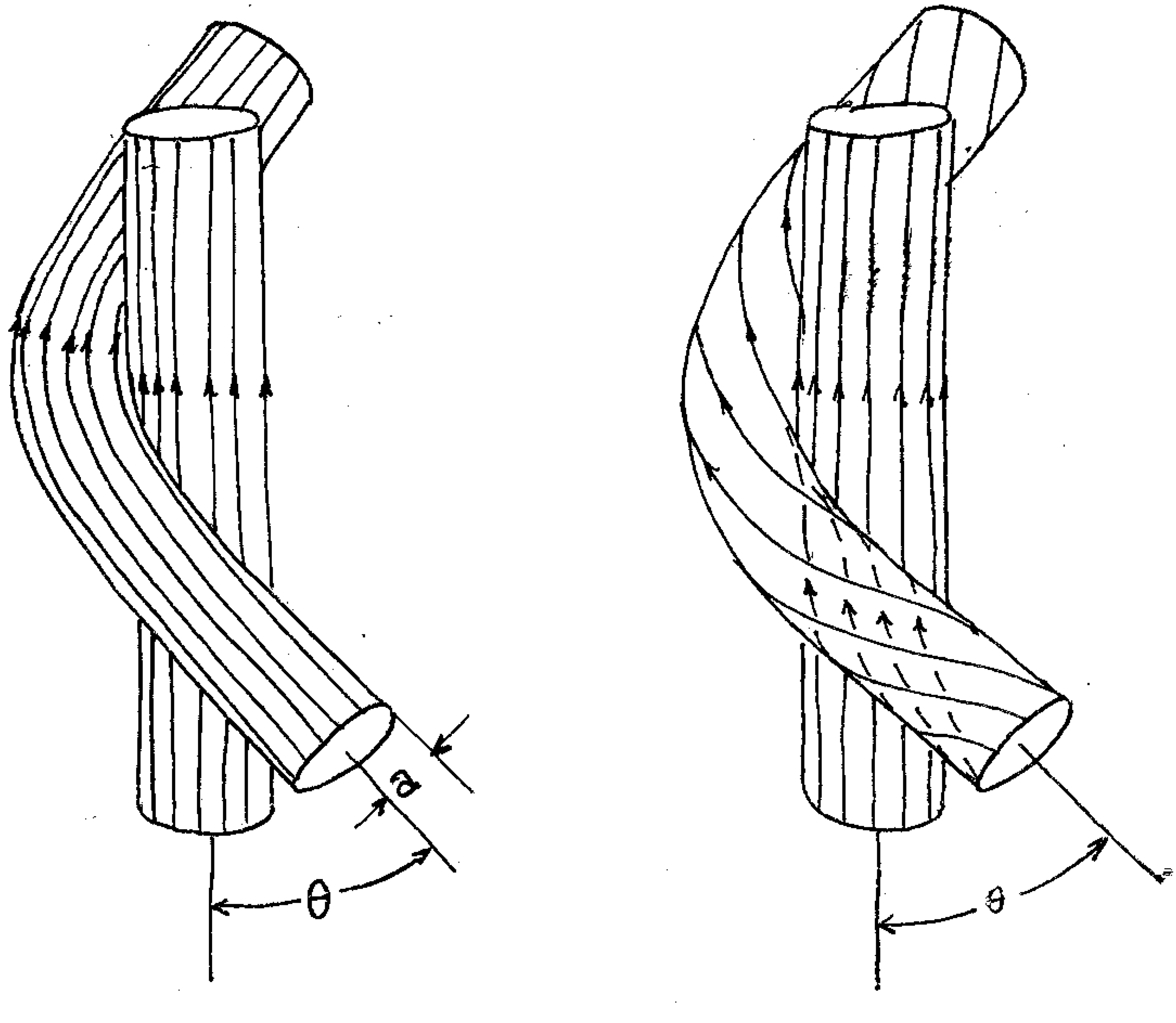
$$x+iy=a,\ x-iy=b$$

Hyperbolic equation

$$\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}=0$$

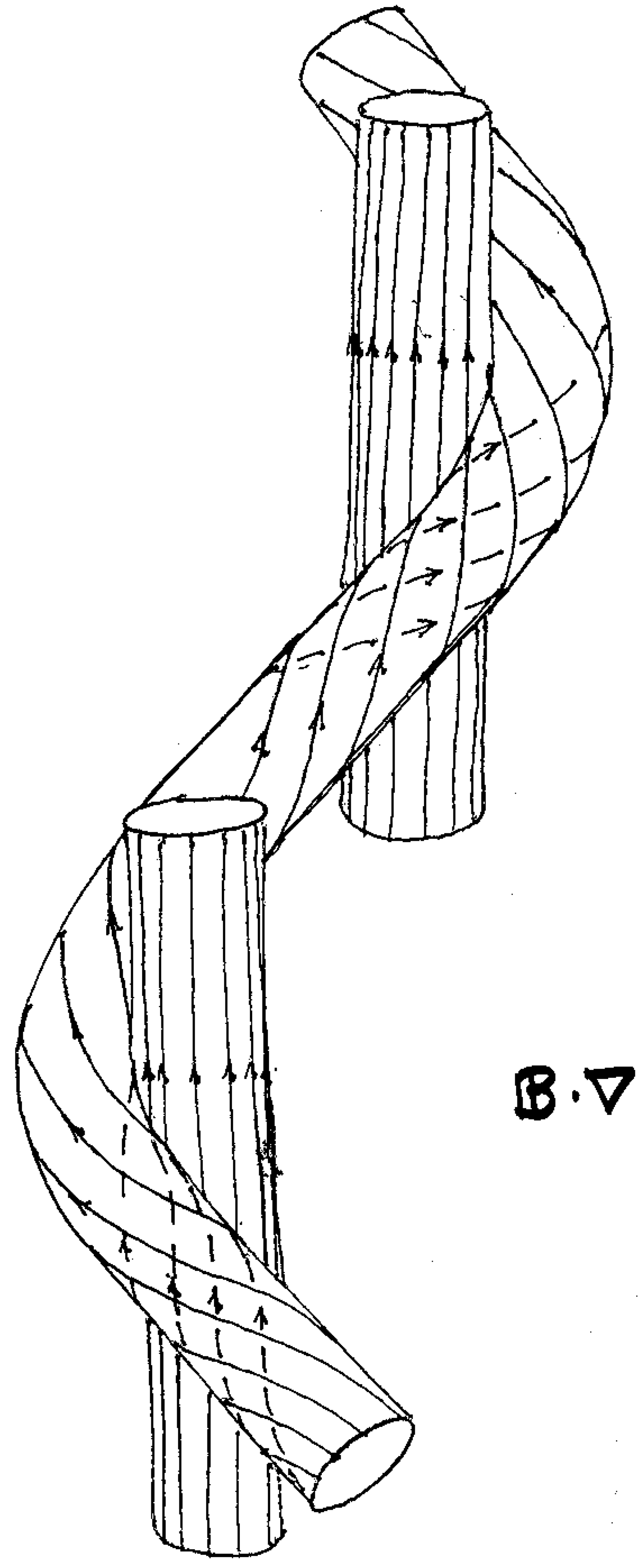
$$\psi = f(x+y),\ h(x-y)\ \ ;\ \ f(a),\ h(b)$$

$$x+y=a,\ x-y=b$$



Inclination θ at the surface of a circular cross section implies a spiral wavelength

$$\lambda = \frac{2\pi a}{\theta}$$



$$\mathbf{B}\cdot\nabla\alpha = 0$$

Take another approach. Dilate the space in the z-direction by a uniform factor $D >> 1$. The transverse field components B_x, B_y are reduced by the factor $1/D$, while B_z is affected but little in the final equilibrium.

Write

$$\mathbf{B} = \mathbf{e}_z B + \varepsilon B \mathbf{b}(x,y,z)$$

$$\alpha = \varepsilon a,$$

where $\varepsilon = 1/D$.

The field lines are inclined to the z-direction by angles of the order of ε, and $\partial/\partial z$ is small $O(\varepsilon)$ compared to $\partial/\partial x$, $\partial/\partial y$. Equilibrium requires

$$\frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z} = \varepsilon a b_x$$

$$\frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x} = \varepsilon a b_y$$

$$\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y} = a$$ [illegible]

$$\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} = O(\varepsilon)$$

Let

$$b_x = +\frac{\partial\varphi}{\partial y}, b_y = -\frac{\partial\varphi}{\partial x}$$

Then

$$a = -\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right)$$

The requirement **B**•∇α=0 becomes

$$\frac{\partial a}{\partial \varsigma} = \frac{\partial\varphi}{\partial x}\frac{\partial a}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial a}{\partial x}$$

where $\zeta = \varepsilon z$.

This equation for a is identical in form with the 2D vorticity equation.

$$v_x = +\frac{\partial\theta}{\partial y}, v_y = -\frac{\partial\theta}{\partial x}$$

$$\omega = -\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right)$$

$$\frac{\partial\omega}{\partial t} = \frac{\partial\theta}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\theta}{\partial y}\frac{\partial\omega}{\partial x}$$

So a evolves with ζ in the same way that ω evolves with t. The vorticity suffers the same restriction as the torsion, being unable to change magnitude and sign as the stream line or field line extends into a different swirling pattern.

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}$$

$$\mathbf{B} \cdot \nabla \alpha = 0$$

$$\mathbf{B} \times \nabla \alpha = \nabla^2 \mathbf{B} + \alpha^2 \mathbf{B}$$

Two families of complex characteristics and one family of real characteristics, viz the field lines.

Specify $\mathbf{B}(x,y,0)$ on the lower boundary $z = 0$.

$$\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} + \alpha B_y$$

$$\frac{\partial B_y}{\partial z} = \frac{\partial B_z}{\partial y} - \alpha B_x$$

$$\frac{\partial B_z}{\partial z} = -\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}$$

$$\alpha = \frac{1}{B_z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$B(x,y,z) = B(x,y,0) + \left(\frac{\partial B}{\partial z}\right)_0 z + \frac{1}{2}\left(\frac{\partial^2 B}{\partial z^2}\right)_0 z^2 + \ldots$$

Note that B(x,y,0) determines B(x,y,L).

We do not know B(x,y,0). We know only that

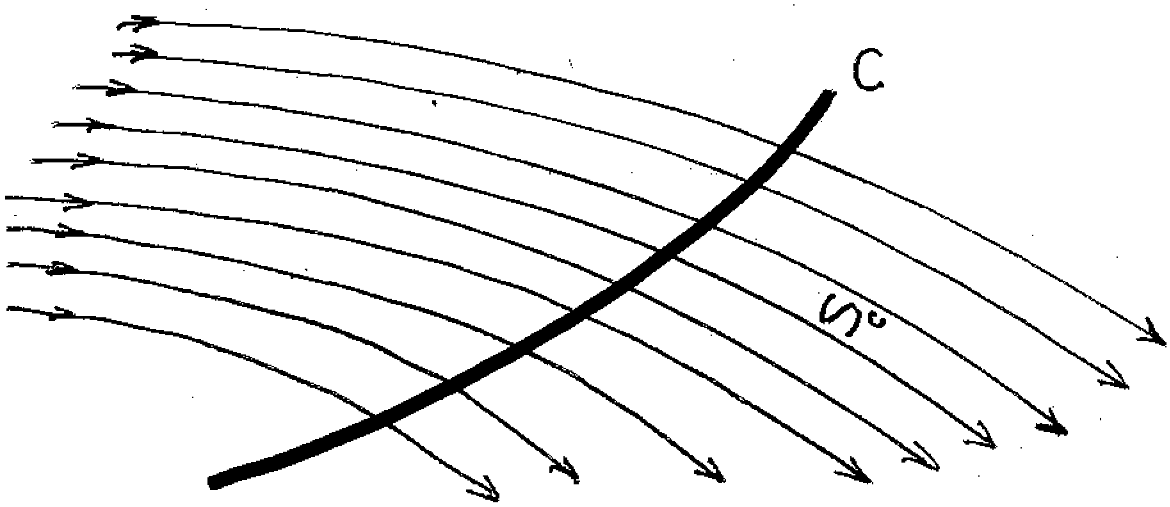
$$B_z(x,y,0) = B_0.$$

However, we know from the physics that there exists a unique lowest energy state in every case.

The essential information must be contained in the precise topology (interlacing) of the field lines, determined by ψ.

How does a field line extend through the field?

Flux surface S_C defined by the curve C



The 2-D magnetic field in S_C has $\nabla\times\mathbf{B} = 0$, $\mathbf{B} = -\nabla\phi$

$$B\frac{dX_i}{ds} = -\frac{\partial\phi}{\partial X_i}, (\nabla\phi)^2 = B^2$$

The optical ray path for the wave $\exp i\Phi$, index of refraction n, is

$$n\frac{dx_i}{ds} = +\frac{\partial\Phi}{\partial x_i}, (\nabla\Phi)^2 = n^2$$

The field line follows the equivalent optical ray path with n = B.

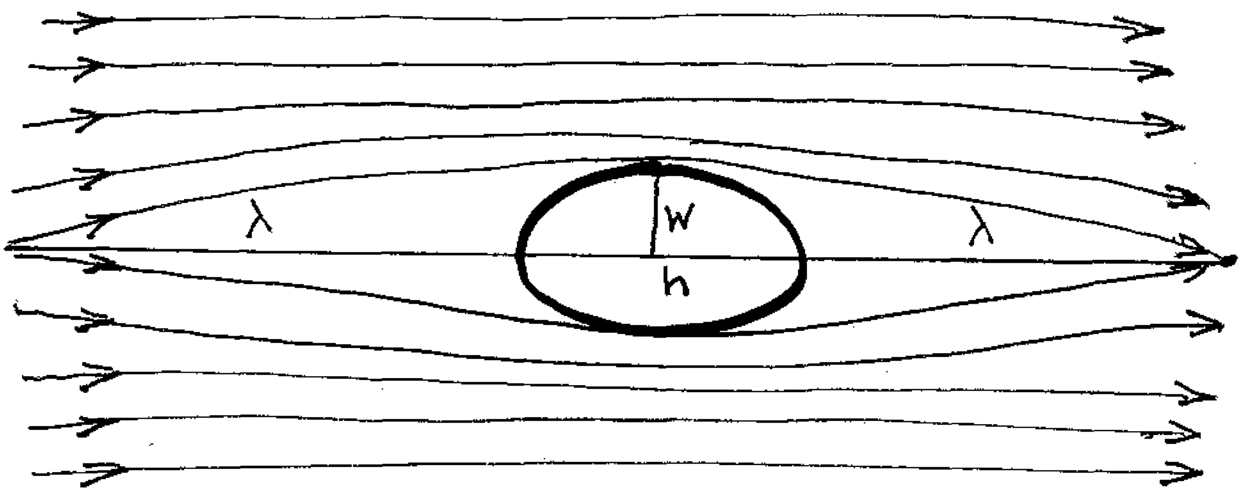
Fermat's principle

$$\delta\int_1^2 ds B = 0$$

Euler equation in a flat surface

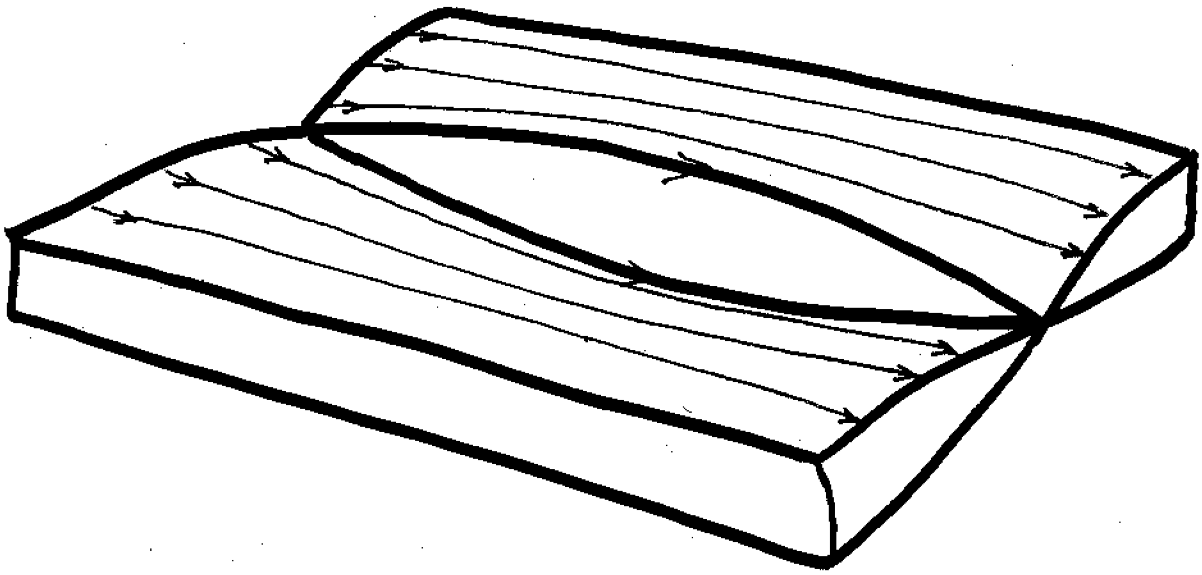
$$\frac{d^2y/dx^2}{\left[1+(dy/dx)^2\right]} = \frac{\partial \ln B}{\partial y} - \frac{dy}{dx}\frac{\partial \ln B}{\partial x}$$

Consider a localized maximum B + ΔB



Path across maximum , hΔB; path around maximum, Bw^2/λ. Gap in flux surface when

$$\frac{\Delta B}{B} > \frac{w^2}{h\lambda}$$

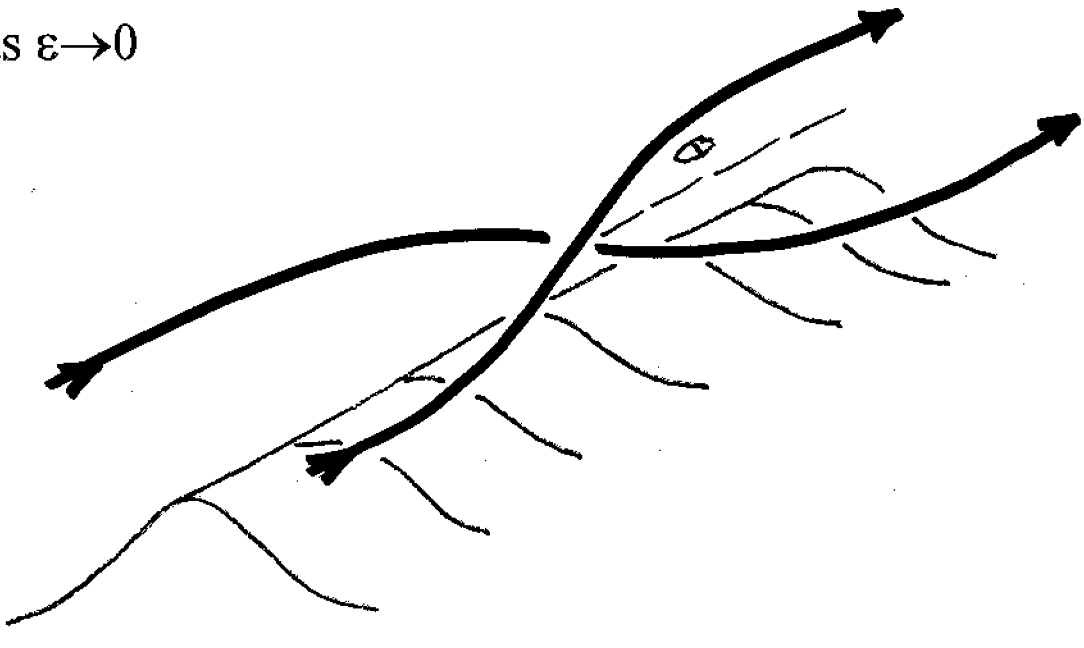


Consider a ridge of enhance $\Delta B(x)$ along $x = 0$.

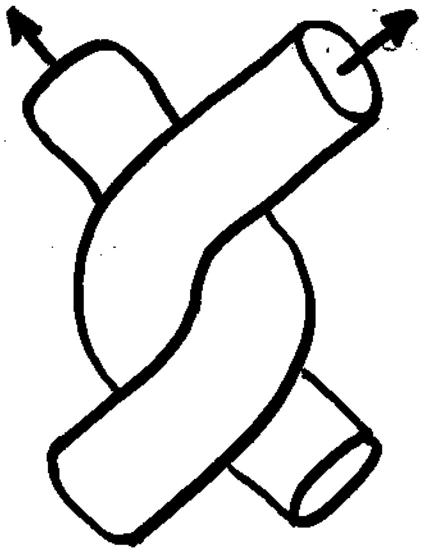
$$(B + \Delta B)\cos\theta = B\sin\varepsilon.$$

$$\theta \cong \left(\frac{2\Delta B}{B}\right)^{1/2}$$

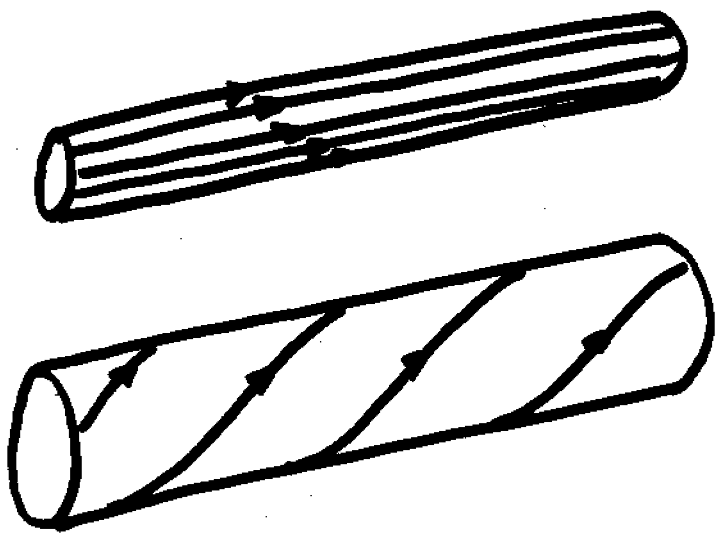
as $\varepsilon \to 0$



Interlacing flux bundles wraps one flux bundle around another, creating a localized maximum in ΔB and a gap in the local flux surfaces.



Twisting a flux bundle causes the bundle to expand against its neighbors, creating ridges of maximum ΔB and refractive discontinuities in the magnetic field.



Consider the increase of B in a flux tube confined to the radius $\varpi = R$ as the bundle is subjected to a uniform twisting.

Initially $B_z = B_0$. With one revolution of the field in length $2\pi a$, it follows that

$$\frac{B_\varphi}{B_z} = \frac{\varpi}{a}$$

Static equilibrium is described by the generating function $f(\varpi)$, with

$$B_z^2 = f + \frac{\varpi}{2}\frac{df}{d\varpi}, \quad B_\varphi^2 = -\frac{\varpi}{2}\frac{df}{d\varpi}$$

Conservation of total flux requires

$$R^2 B_0 = 2\int_0^R d\varpi\,\varpi B_z(\varpi)$$

It follows from the ratio of B_φ/B_z that

$$f(\varpi) = \frac{C}{\left(1 + \varpi^2/a^2\right)}$$

Conservation of flux requires that the constant of integration C have the value

$$C = B_0^{2}\left(\frac{R}{a}\right)^4 \frac{1}{\left[\ln\left(1+R^2/a^2\right)\right]^2}$$

The magnetic pressure at the surface is

$$P = \frac{f(R)}{8\pi}$$

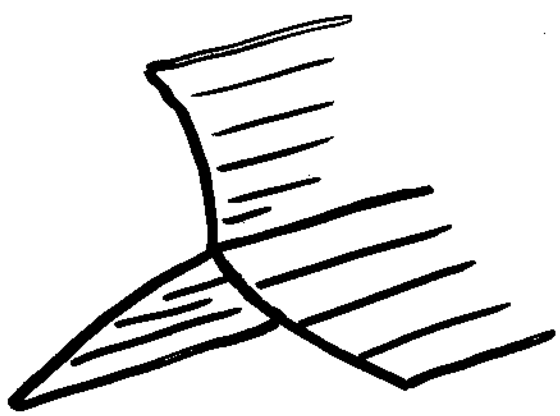
$$= \frac{B_0^2}{8\pi}\left(\frac{R}{a}\right)^4 \frac{1}{\left(1+R^2/a^2\right)\left[\ln\left(1+R^2/a^2\right)\right]^2}$$

$$\cong \frac{B_0^2}{8\pi}\left[1+\frac{1}{12}\left(\frac{R}{a}\right)^4+\ldots\right]$$

The tangential discontinuity represents the surface of contact between two regions of continuous field.

Hence, it contains a surface current but no magnetic field.

Tangential discontinuities extend without end along the field, and generally have no edges, except where they intersect other surfaces of tangential discontinuity



The result is TD's throughout $0<z<L$ wherever magnetic flux is twisted or wrapped.

With a small resistivity the Maxwell stress cannot come into equilibrium, but continues trying to do so, producing rapid dissipation and reconnection.

The reconnection continues until the field topology is reduced to so simple a form that TD's are no longer an intrinsic part of static equilibrium.

The lowest available energy state of almost all magnetic topologies has the remarkable reconnective property of reducing the topology to a primitive form over a modest period of time, no matter how small the resistivity.

Reconnection occurs in astronomical settings wherever both ends of an external magnetic field are held in dense convecting plasma. The convection continually drives the interlacing, so reconnection in the extended field (low beta)is a continuing process.

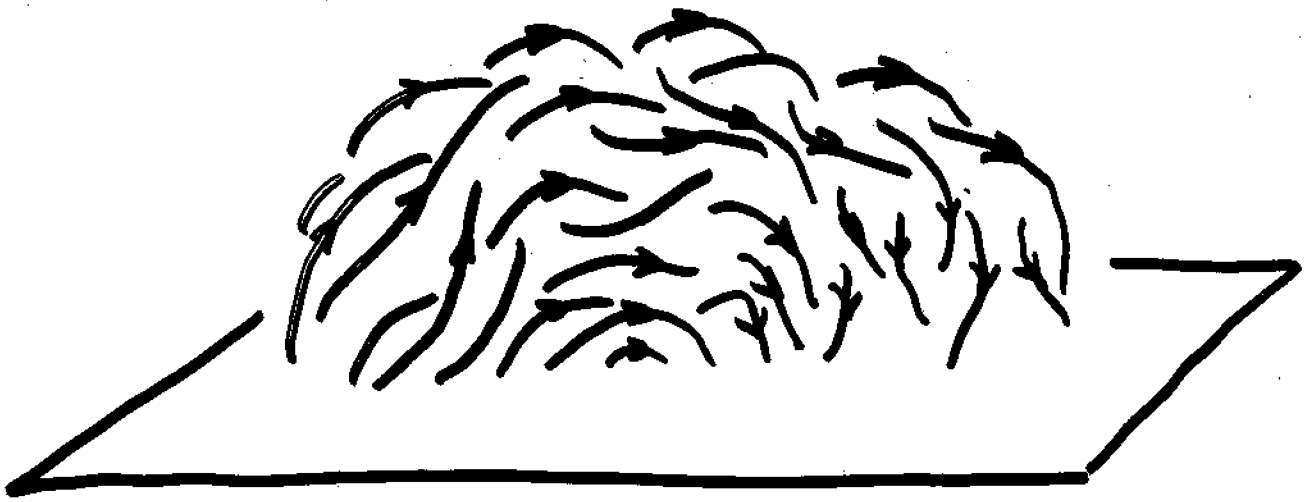
Laboratory magnetic fields confining plasmas may have small initial misalignments that provide weak reconnection, altering the field topology after a time so as to provide rapid reconnection.

Solar X-ray corona

10^7 ergs/cm^2 sec, $T = 1\text{-}5 \times 10^6$ K, $N \sim 10^{10}$/cm^3

The emitting gas is trapped in bipolar magnetic fields of 10^2 gauss

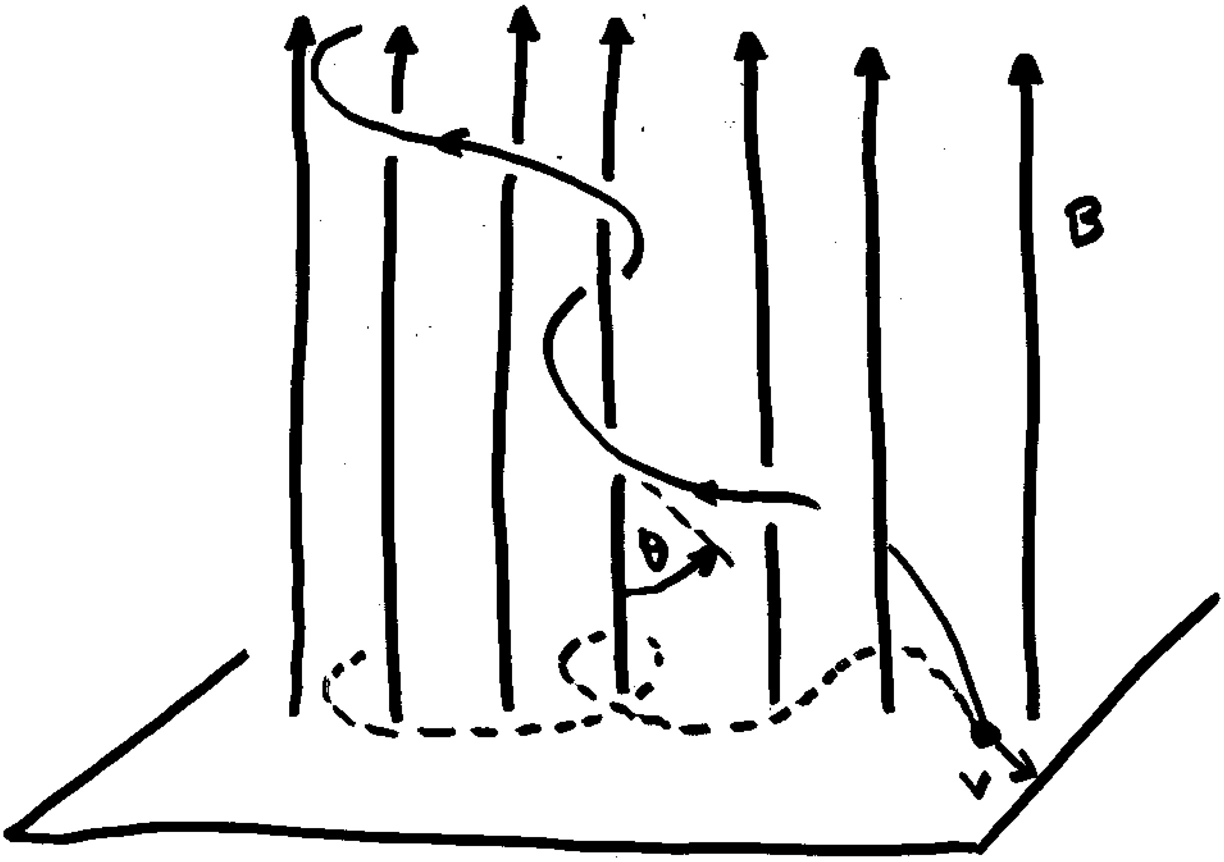
The photospheric footpoints move at random. $v \sim 1$ km/sec.



Power input $\sim \dfrac{B \Delta B}{4\pi} v$ $\qquad \dfrac{\Delta B}{B} \sim \dfrac{vt}{L}$

$$\frac{\Delta B}{B} \sim \frac{\text{Power input}}{v\, B^2/4\pi} \sim 0.1$$

Nanoflares $\sim 10^{22} - 10^{24}$ ergs



$$\tan\theta = \frac{vt}{L} \qquad \frac{\Delta B}{B} = \tan\theta = \frac{vt}{L}$$

Power input, ergs/cm² sec

$$P = \frac{B\Delta B}{4\pi} v$$

$$\frac{\Delta B}{B} = \frac{P}{v\, B^2/4\pi}$$

Spontaneous Current Sheets in Magnetic Fields

Oxford University Press, New York, 1994.