united nations educational, scientific and cultural organization
international atomic energy agency

the
**abdus salam**
international centre for theoretical physics

H4.SMR/1304-8

"COLLEGE ON SOIL PHYSICS"

12 March - 6 April 2001

# *Saturated and Unsaturated Flow*

*M. Kutilek* and *D. R. Nielsen*

Elsevier
Soil and Tillage Research (Journal)
Prague
Czech Republic

---

College on Soil Physics
ICTP, TRIESTE, 12-29 March, 2001

# LECTURE NOTES

## Saturated and Unsaturated flow

Extended text of the textbook
Soil Hydrology, 1994 by M. Kutílek and D.R. Nielsen

Miroslav Kutílek

Professor Emeritus
*Nad Patankou 34, 160 00 Prague 6, Czech Republic*
*Fax/Tel. +420 2 311 6338*
E-mail: kutilek@ecn.cz

# 5 HYDRODYNAMICS OF SOIL WATER

We introduced the concept of soil water potential in Chapter 4 where the system was studied at the equilibrium state. It was characterized by a driving force equal to zero and the value of the total potential being equal at all locations in the soil. Under such conditions, the water flux density is zero and both the soil water content $\theta$ and the soil water pressure head $h$ are invariant with time. Here we derive equations describing the water flux when the soil system, not in equilibrium, manifests a total potential $\Phi$ not being constant in space.

## 5.1 BASIC CONCEPTS

The flow of water in soil can be described microscopically and macroscopically. On the microscopic scale, the flow in each individual pore is considered and for each defined continuous pore, the Navier-Stokes equations apply. For their solution we lack detailed knowledge of the geometrical characteristics of individual pores to obtain a solution for the REV. Even with this knowledge, a tremendous effort would be required necessitating voluminous calculations for even a relatively small soil domain. Nevertheless, this type of procedure is often applied in some theoretical investigations where the basic laws of fluid mechanics are invoked. In such studies the real porous system is usually defined by a model assuming great simplification of reality.

The macroscopic or phenomenological approach of water transport relates to the entire cross-section of the soil with the condition of an REV being satisfied. The rate of water transport through the cross section of the REV (the representative elementary area REA) is the flux. In order to emphasize the fact that water does not flow through the entire macroscopic areal cross section, the term flux density (or flux ratio, macroscopic flow rate et al.) is used to describe the flow realized through only that portion of the area not occupied by the solid phase and, by the air phase eventually when we deal later on with unsaturated soil. Moreover, we use the term flux density understanding that we actually mean the volumetric water flux density having the dimensions of velocity $[LT^{-1}]$.

Inasmuch as the principal equation derived for this macroscopic approach is Darcy's equation, the scale for which this approach is valid is often denoted as the Darcian scale. For soils, the area of this scale is usually in the range of $cm^2$ to $m^2$. Beyond this scale in either direction, larger or smaller, Darcian scale equations may not be realistic. Unless we state otherwise, equations will be derived and solved mainly for the Darcian scale related to a particular REV.

On the Darcian scale, water flow in soils is comparable to other transport processes such as heat flow, molecular diffusion etc. when the appropriate driving force is defined. For example, when the distant ends of a metal rod are kept at different temperatures, heat flow exists. Similarly, molecular diffusion depends upon a difference of concentration in two mutually interconnected

$$|h| = a A_m (w - 2w_m)^{-b} \tag{4.41}$$

where $a$ and $b$ are physically based constants defined in a model where the soil is represented by individual particles, $A_m$ the specific surface of the particles, $w$ the mass water content and $w_m$ the mass water content when a monomolecular layer of water exists on the particles. The advantage of (4.41) is its physical interpretation of the coefficients in (4.40) that applies in the region $-\infty < h < -0.3$ MPa.

After an exhaustive study of experimentally determined SWRC on many soils, Brooks and Corey (1964) rewrote (4.40) to be

$$\theta_E = \left(\frac{h_A}{h}\right)^{\lambda} \tag{4.42}$$

where the effective water content $\theta_E = (\theta - \theta_r)/(\theta_S - \theta_r)$ and $\lambda$, the pore size distribution index, is a characteristic of the soil with values approximately equal to 2 to 5. The value of $\lambda$ is large for soils having a uniform pore size distribution and small for soils with a wide range of pore sizes.

Among the less frequently used expressions of $\theta(h)$ are hyperbolic, error function or exponential equations of other authors. For the convenience of analytic or approximate solutions of some elementary hydrologic processes, the SWRC can be formulated by still another equation which is well suited to the mathematical development of the solution (Broadbridge and White, 1988), however its practical applicability for experimental data has not yet been proved.

Inasmuch as (4.42) does not offer a satisfactory description of the SWRC in the wet region, especially for soils not exhibiting a distinct value of $h_A$ or $h_W$, van Genuchten (1980) proposed the equation

$$\theta_E = \left[1 + (\alpha |h|)^n\right]^{-m} \tag{4.43}$$

where $\alpha$, $n$ and $m$ are fitting parameters with their limitations being $\alpha > 0$, $n > 1$, $|h| \geq 0$ and $0 < m < 1$. Values of $n$ occur between 1.2 and 4 and those of $\alpha$ between $10^{-3}$ and $10^{-2}$ cm$^{-1}$. Because of computational expediency, values $m$ have arbitrarily been taken equal to $(1 - 1/n)$. van Genuchten and Nielsen (1985) have proposed for pragmatic reasons to merely consider $\theta_r$ and $\theta_S$ as empirical fitting parameters. Note that the physically real residual water content on the SWRC in Fig. 4.15 was denoted by $\theta_{Wr}$. Equation (4.43) can be adopted to describe each of the branches of the hysteretic loop. The detailed procedure for expressing MDC, MWC and the scanning curves by a modified (4.43) is described by Luckner et al. (1989). Usually, $\alpha_w \approx 2\alpha_d$ where $w$ denotes wetting and $d$ is for drainage. Equation (4.43) does not allow the existence of $h_A$ and (4.42) is not suitable if an inflection point exists on the SWRC. Hence, a compromise description of the SWRC is achieved when $\theta_E$ is replaced by $\theta_e$, especially if the SWRC is further used for the determination of the unsaturated hydraulic conductivity (Šír et al., 1985). The definition of $\theta_e$ requires that $\theta$ be replaced by $\theta_e = (\theta - \theta_b)/(\theta_a - \theta_b)$, and hence, (4.43) becomes

$$\theta_e = \left[1 + (\alpha |h|)^n\right]^{-m} \tag{4.44}$$

pools. Soil water flow is conditioned by the existence of a driving force stemming from a difference of total potentials between two points in the soil. Laymen mistakenly suppose that the driving force of water flow in an unsaturated soil is related to differences in soil water content. This supposition, valid only for a few specified conditions, generally leads to erroneous conclusions.

Here, we first formulate basic flow equations for the simplest case of flow in a saturated, inert rigid soil. Afterwards, we deal with water flow in a soil not fully saturated with water. This latter type of flow is commonly called unsaturated flow while the former is called saturated flow. To be more precise, we should distinguish the former from the latter flows as those occurring at positive and negative soil water pressures, respectively. If the flow of both air and water in the soil system is simultaneously considered, we speak of two phase flow. Initially, we assume that the concentration of the soil solution does not affect the soil water flow. Subsequently, our discussion is extended to swelling and shrinking soils. Finally, we examine linked or coupled flows together with some specific phenomena of transport at temperatures below 0°C.

All equations that we derive are supposed to be applicable to not only analytical and approximate mathematical solutions of the components of the soil hydrological system but to all deterministic models of soil hydrology.

## 5.2 SATURATED FLOW

We assume that water is flowing in all pores of the soil under a positive pressure head $h$. In field situations the soil rarely reaches complete water saturation. Usually it is quasi-saturated with the soil water content $\theta_W = mP$ where $m$ has values of 0.85 to 0.95 at $h \geq 0$, and $P$ is the porosity. Entrapped air occupies the volume $P(1 - m)$. And for this discussion of saturated flow, the impact of entrapped air is not considered.

### 5.2.1 Darcy's Equation

For the derivation of Darcy's equation we shall discuss a simple experiment demonstrated in Fig. 5.1. The soil is placed in a horizontal cylinder connected on both sides with vessels containing water maintained at a constant level in each vessel by an overflow valve. If the water level on the left side is higher than that on the right side, water flows to the right. The rate of discharge $Q = V/t$ is simply measured by the volumetric overflow $V$ in time $t$. The flux density $q$ $[LT^{-1}]$ (macroscopic flow rate) is

$$q = \frac{V}{At} \tag{5.1}$$

where $A$ is the cross-sectional area of the soil column perpendicular to the direction of flow. Sometimes, the term $q$ is also called the Darcian flow rate. The mean water flow rate (velocity) in the soil pores $v_p$ is

$$v_p = q / P . \tag{5.2}$$

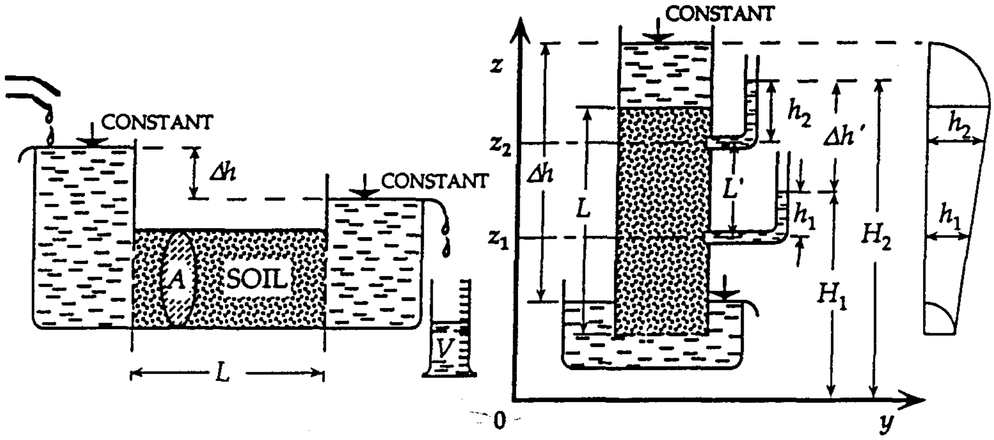


*Figure 5.1. Simple steady flow experiments on saturated soil columns. On the right, the soil placed in a cylinder is provided with piezometers for measuring pressure heads $h_1$ and $h_2$ at depths $z_1$ and $z_2$, respectively. The total potential head $H = h + z$.*

In 1856, Darcy experimentally demonstrated for columns of sand a linear relationship between the flux density $q$ and the hydraulic gradient $I_h$. In our experiment shown on the right side of Fig. 5.1

$$q = K_S \frac{\Delta h}{L} = K_S \frac{\Delta h'}{L'} = K_S I_h \qquad (5.3)$$

where $\Delta h/L$ or $\Delta h'/L'$ is the hydraulic gradient $I_h$, $\Delta h$ the difference between water levels on both ends of the soil column of length $L$ and $\Delta h'$ the difference between water levels in the piezometers separated by the distance $L'$ in the direction of flow. Both $\Delta h$ and $\Delta h'$ are considered the hydraulic head drop along the soil. Inasmuch as $\Delta h/L$ is dimensionless, $K_S$ has the dimension of $q$ [$LT^{-1}$]. When we read piezometer levels $h_1$ and $h_2$ at elevations $z_1$ and $z_2$, respectively, we have in terms of the total potential $H$

$$q = - K_S \left( \frac{H_2 - H_1}{z_2 - z_1} \right) \qquad (5.4)$$

where the total potential head $H$ (= $h + z$) is related to a unit weight of water. In a more general way (5.4) becomes

$$q = - K_S \, grad \, H . \qquad (5.5)$$

Equation (5.5) states that the flux density is proportional to the driving force of the water flow which is the gradient of the potential. Inasmuch as $K_S$ is a constant for a given soil, we write $\phi^* = K_S H$, and hence,

$$q = - grad \, \phi^* \qquad (5.6)$$

where $\phi^*$ is $K_S H$. The negative sign in the above equations means that water flows in the direction of decreasing potential or against the positive direction of z in Fig. 5.1. The value of $K_S$ depends upon the nature of the soil and is numerically equal to the flow rate when the hydraulic gradient is unity. Values of $K_S$ commonly range from less than 0.1 cm·day$^{-1}$ ($10^{-8}$ m·s$^{-1}$) to more than $10^2$ cm·day$^{-1}$ ($10^{-5}$ m·s$^{-1}$).

In layered soils we have to specify the direction of the flow relative to that of the layering. When the flow is parallel to the layers, the total flux is the sum of the fluxes for each of the individual layers, see Fig. 5.2. Hence, $Q = Q_1 + Q_2 + Q_3$ and

$$q(b_1 + b_2 + b_3)d = (q_1 b_1 + q_2 b_2 + q_3 b_3)d. \tag{5.7}$$

For a column of width $d = 1$, length $L$ and thickness $b$ composed of three layers each of thickness $b_i$, the total flux density for a hydraulic head drop $\Delta h$ is

$$q = \left( \frac{K_1 b_1 + K_2 b_2 + K_3 b_3}{b_1 + b_2 + b_3} \right) \frac{\Delta h}{L}$$

or

$$q = K_S' \frac{\Delta h}{L}. \tag{5.8}$$

Here, the apparent hydraulic conductivity $K_S'$ is the arithmetic mean of the individual values for each layer.

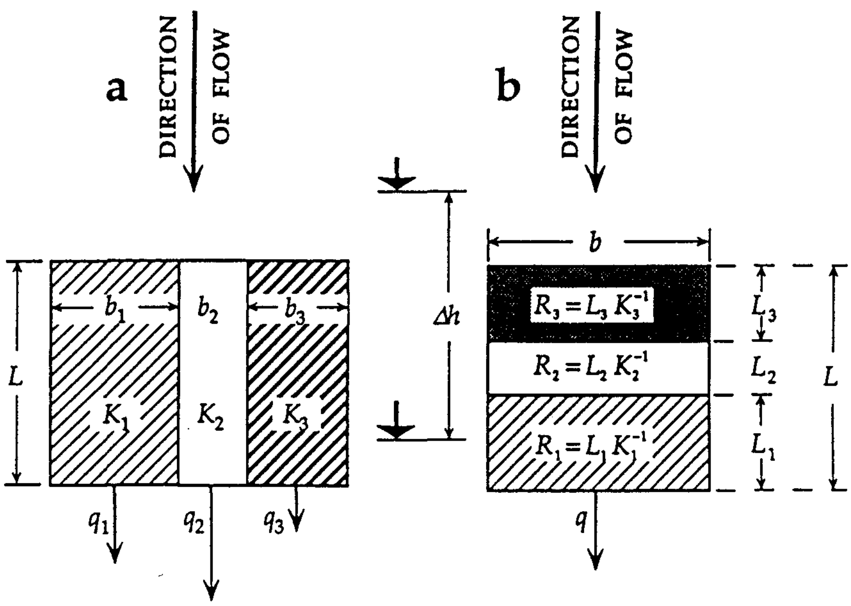


*Figure 5.2. Flux density q in layered soils: a. Direction of flow is parallel to layering. b. Direction of flow is perpendicular to layering.*

When water flows perpendicular to the layering, we introduce, analogous to an electrical resistance, the hydraulic resistance of each layer $R_i = L_i / K_{Si}$ having units of time. In Fig. 5.2 the flow combined from the three layers is

$$q = \frac{\Delta h}{R_1 + R_2 + R_3}. \quad (5.9)$$

With the total resistance of the system $R = \Sigma R_i$ we obtain the harmonic mean or the apparent hydraulic conductivity $K_S^* = L/R$.

When the flow is at an angle < 90° to the layers, the difference of $K_S$ in each of the layers causes a change of the direction of streamlines (Zaslavsky and Sinai, 1981, and Miyazaki, 1990), see Fig. 5.3.

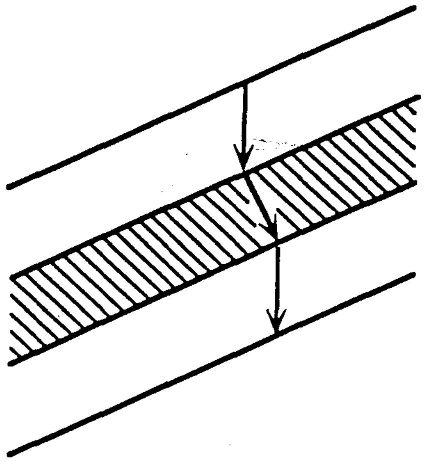


*Figure 5.3. Change of the direction of flow in a soil with inclined direction of layers.*

### 5.2.2 Saturated Hydraulic Conductivity

Inasmuch as the soil water potential $H$ can be expressed in three modes, the dimension of the hydraulic conductivity is not necessarily [$LT^{-1}$]. From (5.5) we obtain for the three dimensions of $H$ three different dimensions of $K_S$. Although expressing $K_S$ in units of velocity is usually more convenient, any one of the following sets of units is occasionally preferred.

| *H* | | *grad H* | | $K_S$ | |
|---|---|---|---|---|---|
| $J \cdot kg^{-1}$ | [$L^2T^{-2}$] | $J \cdot kg^{-1} \cdot m^{-1}$ | [$LT^{-2}$] | s | [T] |
| Pa | [$ML^{-1}T^{-2}$] | $Pa \cdot m^{-1}$ | [$ML^{-2}T^{-2}$] | $m \cdot s^{-1} \cdot \rho_W^{-1} \cdot g^{-1}$ | [$M^{-1}L^3T$] |
| m | [L] | dimensionless | | $m \cdot s^{-1}$ | [$LT^{-1}$] |

The empirical, intuitive derivation of Darcy's equation (5.5) can be theoretically justified from Navier-Stokes equations applied to an REV of a model of a porous medium and scaled with a characteristic length. In order to obtain (5.5), inertial effects were neglected and the density and viscosity of water were assumed invariant (Bear, 1972; Whitaker, 1986). Scheidegger (1957) showed that $K_S$ should be considered a scalar quantity for isotropic soils, and a tensor of rank 2 for anisotropic soils with the value of $K_S$ dependent upon the direction of flow. When the tensor $K_S$ is assumed to be symmetric, its principal axes, defined by six values, are identical to those of an ellipsoid of conductivity. If the gradient of the potential is not in the direction of a principal axis, the direction of flow is different from that of the gradient.

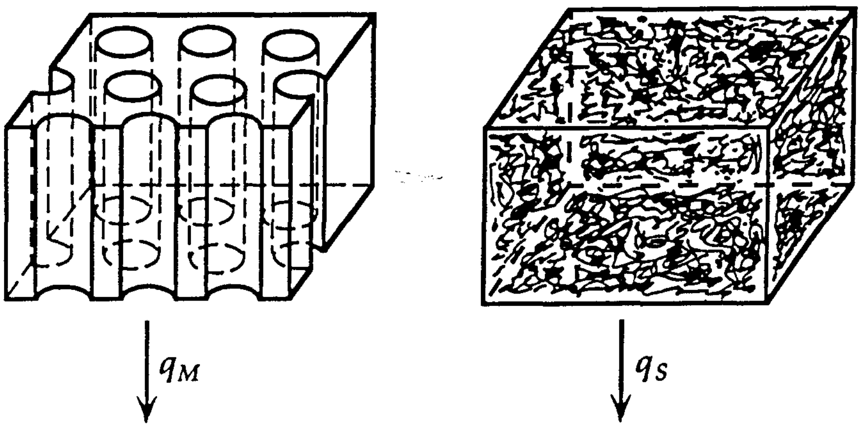


*Figure 5.4. In Kozeny's model, the complicated soil porous system (right) is represented by a bundle of parallel capillary tubes of uniform radius (left). The flux density q, saturated hydraulic conductivity $K_S$, porosity P and surface area of pores $A_m$ are the same in both the model and the soil.*

From a theoretical treatment we can obtain a physical interpretation of the hydraulic conductivity. We develop here a modified and simplified model of Kozeny (cf. Scheidegger, 1957) consisting of a bundle of parallel capillary tubes of uniform radius. We assume that the soil and the model are identical with respect to porosity $P$, specific surface $A_m$ [$L^{-1}$] and water flux density $q$ [$LT^{-1}$], see Fig. 5.4. The mean flow rate $v_p$ in a capillary of radius $r$ is described by Hagen-Poiseuille's equation

$$v_p = \frac{\rho_W g r^2}{8\mu} I_h \tag{5.10}$$

where $g$ is the acceleration of gravity [$LT^{-2}$], $\rho_W$ the density of water [$ML^{-3}$], $\mu$ the dynamic viscosity [$ML^{-1}T^{-1}$] and $I_h$ the hydraulic gradient [dimensionless]. With $n$ being the number of capillaries of unit length $x$, the porosity of the model is

$$P = n\pi r^2 x / V_u \tag{5.11}$$

where $V_u$ is the unit volume and the specific surface is

$$A_m = 2n\pi r x / V_u. \tag{5.12}$$

From (5.11) and (5.12) we obtain

$$r = \frac{2P}{A_m}. \tag{5.13}$$

And, from (5.2) and (5.10) we obtain

$$q = \frac{1}{2}\frac{\rho_W g}{\mu}\frac{P^3}{A_m^2} I_h. \qquad (5.14)$$

Because soil pores are irregularly shaped and mutually interconnected, a shape factor $c$ replaces 1/2 in (5.14). Letting

$$K_p = \frac{cP^3}{A_m^2} \qquad (5.15)$$

we obtain

$$q = K_p \frac{\rho_W g}{\mu} I_h \qquad (5.16)$$

which is identical to (5.3). Because the term $K_p$ relates to the flow of any fluid through a soil, it is called the permeability [$L^2$]. The unusual dimension of $K_p$ represents the cross-sectional area of an equivalent pore. Although now almost obsolete, the historical unit of 1 Darcy = 1 $\mu m^2$ was used for describing permeability.

Inasmuch as flow channels in the soil are curved compared with those of a capillary model, a tortuosity factor $\tau$ introduced in (5.15) yields the Kozeny equation

$$K_p = \frac{cP^3}{\tau A_m^2}. \qquad (5.17)$$

The tortuosity $\tau$ is the ratio between the real flow path length $L_e$ and the straight distance $L$ between the two points of the soil. Because $L_e > L$, $\tau > 1$. In a monodispersed sand manifesting a value of $\tau \approx 2$, the flow path forms approximately a sinusoidal curve (Corey, 1977).

Equations identical or of similar type to (5.17) have been derived by many authors. If a model of parallel plates is used instead of capillary tubes and the slits are oriented in the direction of the laminar flow, we obtain the mean flow rate

$$v_p = \frac{\rho_W g d^2}{3\mu} I_h \qquad (5.18)$$

where $2d$ is the distance between the plates. When $B$ is the width of the plates, $P = 2ndBx/V_u$ and $A_m = 2nx(2d + B)/V_u$. Taking $x = 1$ and $B = 1$, we obtain $d = P(A_m - 2P)$ and hence,

$$K_p = \frac{cP^3}{\tau (A_m - 2P)^2}. \qquad (5.19)$$

Looking at (5.17) and (5.19), we are reminded of the Kozeny-Carman equation

$$K_p = \frac{P^3}{5A_m^2 (1-P)^2}. \qquad (5.20)$$

Its derivation was shown in detail by Scheidegger (1957).

From (5.3) and (5.16) the relationship between $K_S$ and any formulation of $K_p$ is

$$K_S = K_p \frac{\rho_W g}{\mu}. \qquad (5.21)$$

Kozeny's equation shows that $K_S$ is sensitive to porosity. However, in his model the pore radii are considered uniform while those in real soils have broad distributions. For real soils, we subdivide the pores according to their radii into $j$ categories each having an equivalent radius $r_j$. For $r_j > r_{j+1}$ the flux in each category $q_j(r_j^4, n_j)$ where $n_j$ is the percentage of the $j$-th category in the whole soil, $q_j >> q_{j+1}$ .Total flux $q = \Sigma q_j$ as shown for parallel layering. Thus, let us assume for $j = 1$, the percentage of the category of largest pores is eliminated by compaction. Although the porosity may be only marginally reduced, the value of $K_S$ may be reduced by orders of magnitude. It is logical, therefore, that aggregation of a soil may increase $K_S$ by orders of magnitude, yet the porosity may remain nearly the same. And, vice versa, soil dispersion or disaggregation substantially decreases $K_S$. For example, in a loess soil, the saturated hydraulic conductivity of its surface after a heavy rain decreases 3 to 4 orders of magnitude compared with its original value owing mainly to two processes - disaggregation and the blockage of pores by the released clay particles (McIntyre, 1958). Compaction of soil in the A-horizon and in the bottom of the plowed sub horizon causes a much greater decrease of $K_S$ than that predicted from a decrease of porosity in the simple Kozeny equation because compaction reduces primarily the content of large soil pores associated with values of pressure head $h = 0$ to -100 cm.

Although the textural class of a soil may have a large influence on the value of $K_S$, any attempt to establish a correlation between the two attributes usually fails. Only for those soils and soil horizons of the same genetic development occurring in the same region and being similarly managed will a correlation between texture and $K_S$ be manifested. On the other hand, a few generalities may exist. For example, the smallest values of $K_S$ in each of the main textural classes can be approximated. In sandy soils, the minimum value of $K_S$ is about 100 cm·day$^{-1}$, in silty loams about 10 cm·day$^{-1}$ and in clays about 0.1 cm·day$^{-1}$. In peats, $K_S$ decreases with an increasing degree of decomposition of the original organic substances. When the degree of decomposition of a peat is about 40 to 50%, the value of $K_S$ diminishes to values of $K_S$ typical of unconsolidated clays. Extreme drainage and concomitant drying of peat soils causing compaction and an increase of soil bulk density also reduce the magnitude of $K_S$. Moreover, because this drying increases hydrophobism, entrapment of air during wetting is enhanced and contributes even further to the decrease of $K_S$.

In loams and clays, the nature of the prevalent exchangeable cation plays an important role relative to the value of $K_S$, see Fig. 5.5. In vertisols, an increase of the percentage of exchangeable sodium (ESP) is accompanied by a decrease in $K_S$ when the ESP reaches 15 to 20%, provided that the soluble salt content of the soil water is small. For example, with the electrical conductivity of the soil paste EC being 1 mS·cm$^{-1}$ or less, the value of $K_S$ can decrease two or three orders of magnitude. On the other hand, even for the same soil having a large ESP, if the concentration of soluble salts is increased substantially to an EC value of about 8 mS·cm$^{-1}$ or more, the $K_S$ value is not significantly affected. The value of ESP is closely related to the sodium adsorption ratio SAR of percolating water [SAR = $Na/(Ca + Mg)^{1/2}$]. If Ca is completely absent (i.e. only Mg appears

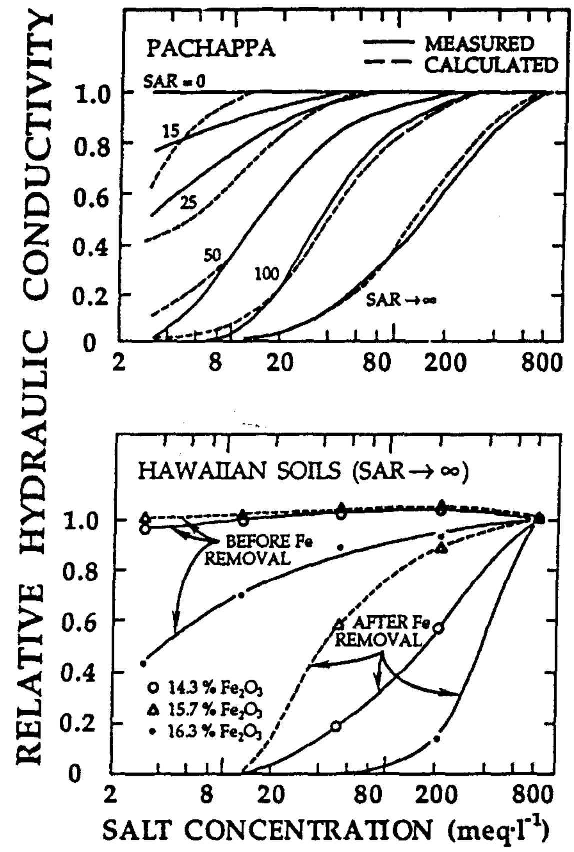


*Figure 5.5. The influence of sodium adsorption ratio SAR and salt concentration upon the value of the saturated hydraulic conductivity is very strong in aridisols, ustolls and vertisols (see top), while in oxisols with a large accumulation of free iron oxides the influence of SAR and salt concentration is weak (see bottom) (McNeal et al., 1968).*

in the denominator), the value of $K_S$ is more greatly reduced than when Mg is absent (McNeal et al., 1968). When monovalent cations are considered in addition to Na, we find that potassium leads to a decrease of $K_S$ but its influence is not as strong as that of exchangeable Na. It has been shown that large organic cations such as pyridinium cause the value of $K_S$ to increase by several orders of magnitude in montmorillonitic clay while their impact on the value of $K_S$ of kaolinitic clays is less significant (Kutílek and Salingerová, 1966). These variations are closely related to the degree of flocculation or peptization of the soil colloidal particles that can be quantified with the value of the $\zeta$-potential derived from double layer theory. Applying this theory, the decrease in $K_S$ owing to the action of rain water (very small EC) is easily predicted for soils having large SAR values. These predictions are not necessarily successful for soils that differ pedologically. For example, a solution of high SAR value percolating through an Oxisol does not decrease the value of $K_S$ even after reducing the solute concentration because the abundant free iron oxides prevent peptization and disaggregation of the soil particles. The value of $K_S$ also depends upon the composition of the clay fraction. It decreases in the order kaolinite, illite and montmorillonite. And, soil organic matter has a profound impact upon the magnitude of $K_S$, owing to its cementing action that promotes aggregate stability. Bacteria and algae may reduce the value of hydraulic conductivity in long term laboratory tests and in some field conditions when sewage treatment effluents are either used for irrigation or disposed on special lands. These same effects can also exist in an irrigated soil well supplied with plant nutrients and sunshine. The process is not necessarily restricted to only anaerobic conditions inasmuch as some aerobic bacteria may cause a reduction in the value of $K_S$. The reduction is partly attributed to cells of bacteria and algae mechanically clogging the soil pores and to slimy, less pervious products of microbial activity being deposited on the walls and necks of the soil pores. For long term laboratory experiments, bactericides are commonly used to prevent the value of $K_S$ from decreasing. In general, there are many factors influencing the value of $K_S$ that are usually not considered in simplified models.

Soils classified according to their values of $K_S$ are

| | |
|---|---|
| *very low permeability* | $K_S < 10^{-7}$ m·s$^{-1}$ |
| *low permeability* | $10^{-7} < K_S < 10^{-6}$ m·s$^{-1}$ |
| *medium permeability* | $10^{-6} < K_S < 10^{-5}$ m·s$^{-1}$ |
| *high permeability* | $10^{-5} < K_S < 10^{-4}$ m·s$^{-1}$ |
| *excessive permeability* | $K_S > 10^{-4}$ m·s$^{-1}$ |

Geological materials are similarly classified as

| | |
|---|---|
| *compacted clays* | $10^{-11} < K_S < 10^{-9}$ m·s$^{-1}$ |
| *gravel* | $10^{-1} < K_S < 10^{1}$ m·s$^{-1}$ |

All such classification schemes above are problematical. For soils in a certain region, a more appropriate classification would be based upon the frequency distribution of $K_S$. Based upon that frequency distribution, we can identify sub regions where a particular range of $K_S$ is expected.

When values of $K_S$ are considered relative to their position within a soil profile, soils are grouped into these seven classes.

1. $K_S$ does not change substantially in the profile.
2. $K_S$ of the A-horizon is substantially greater than that of the remaining soil profile and no horizon of extremely low $K_S$ exists.
3. $K_S$ gradually decreases with soil depth without distinct minima or maxima.
4. $K_S$ manifests a distinct minimum value in the illuvial horizon or in the compacted layer just below the plow layer.
5. Soil of high permeability with its development belonging to one of the first four classes covering the underlying soil of very low permeability.
6. Soil of very low permeability with its development belonging to one of the first four classes covering the underlying soil of very high permeability.

7. $K_S$ changes erratically within the profile owing to extreme heterogeneity in the soil substrata.

The influence of the temperature upon the value of $K_S$ can be examined with (5.21). Inasmuch as $\rho_W$ is negligibly influenced by temperature, changes of $K_S(T)$ depend primarily upon the viscosity $\mu(T)$.

### 5.2.3 Darcian and Non-Darcian Flow

We have already mentioned that Darcy's equation is valid only for small rates when the inertial terms of the Navier-Stokes equations are negligible. For engineering purposes the upper limit of the validity of Darcy's equation given by (5.3) through (5.6) is indicated by the critical value of Reynolds' number for porous media

$$Re = \frac{q d \rho}{\mu} \tag{5.22}$$

where $d$ denotes length. In sands, $d$ is the effective diameter of the particles, or with some corrections, the effective pore diameter. Sometimes $d$ is related to the permeability of the sand, e.g. $d = K_p^{1/2}$. However, in all soils other than sands, $d$ is not at all definable and hence, (5.22) is not applicable. The difficulty in defining $d$ is manifested by controversy in the literature regarding the assignment of critical values of $Re$. Most frequently, critical values of $Re$ have been reported to range from 1 to 100. In this post linear region, the flow is often described by the Forchheimer equation (Bear, 1972)

$$\frac{dH}{dx} = aq + bq^2 \tag{5.23}$$

where $a$ is the material constant analogous to $K_S$ and $b$ is functionally dependent upon the water flux density. This non linearity is caused primarily by inertia and by turbulence starting only at very large values of flux density, see Fig. 5.6. A more detailed theoretical discussion is given by Cvetkovič (1986).

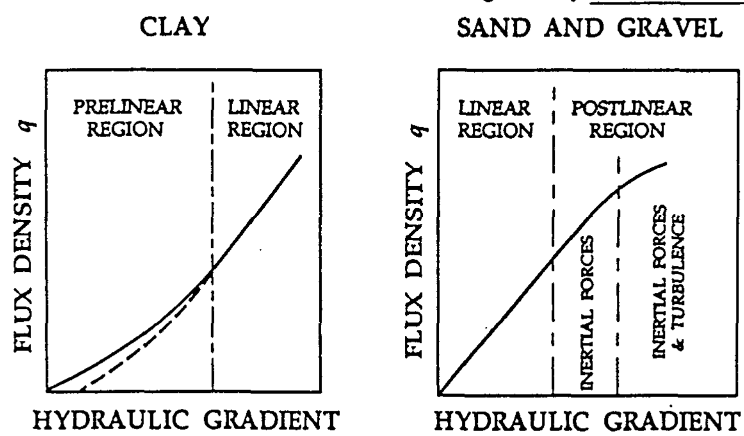


*Figure 5.6. Deviations from the linearity of Darcy's equation.*

Deviations from Darcy's equation have also been observed in laboratory experiments for very small flux density values. We define, therefore, the pre linear region of flow where $q$ increases more than proportionally with $I_h$, see Fig. 5.6. This deviation from Darcy's equation, most often observed within pure clay having very large specific surfaces (e.g. $10^2\ m^2 \cdot g^{-1}$), has been explained by the action of three factors: a) Clay particles shift and the clay paste consolidates owing to the imposed hydraulic gradient and the flow of water, b) It is theoretically assumed that the viscosity of water close to the clay surfaces is different than that of bulk water or that in the center of the larger soil pores. According to Eyring's molecular model where the viscosity depends upon the activated Gibbs' free energy, the first two to five molecular layers have a distinct increased viscosity. Owing to the great value of the specific surface in clays, the contribution of the first molecular layers to the alteration of averaged viscosity may not be negligible. c) The coupling of the transfer of water, heat, solutes et al. may also contribute to the existence of the pre linear region (Swartzendruber, 1962; Kutílek, 1964 and 1972; Nerpin and Chudnovskij, 1967).

Deviations from Darcian flow are not frequently described or observed, and the post linear region is only rarely reached in sands and gravelly sands. There is not yet any field experimental evidence of the existence of a pre linear region. Darcy's equation is, therefore, either exact or at least a very good approximation entirely adequate for soil hydrology.

### 5.2.4 Measuring $K_S$

Saturated hydraulic conductivity is one of the principal soil characteristics and for its determination, only direct measurement is appropriate. Indirect methods, derived from soil textural characteristics which are sometimes combined with aggregate analyses, generally do not lead to reliable values. Considering soil texture as an example, soil water flow is totally independent from the laboratory procedure of dispersing, separating and measuring the percentage of "individual" soil particles which do not even exist "individually" in natural field soils. It has been shown in section 5.2.2 that the value of $K_S$ is closely related physically to the porous system within a soil. Inasmuch as a quantitative description of this porous system is much more difficult than the measurement of $K_S$, direct measurement of $K_S$ is preferred. When $K_S$ is ascertained by water flux density and potential gradient measurements, we will speak about the determination of $K_S$. In order to avoid misunderstanding, when additional assumptions are used to evaluate these two quantities somewhat less directly, we will speak about the estimation of $K_S$.

Measuring is realized either in the laboratory on soil core samples previously taken from the field, or directly in the field without removing a soil sample. Field methods are preferred. They provide data that better represent the reality of water flow in natural conditions. Their main disadvantage is the lack of rigorous quantitative procedures for measuring soil attributes in the majority of field tests. For laboratory measurements, the size of the REV should be theoretically estimated in order that an appropriate soil core sampler be selected. In practice, because the REV is rarely determined, a standard core or cylinder

## 5.3 UNSATURATED FLOW IN RIGID SOILS

By the term rigid soil we designate soils that do not change their bulk volume with a change of water content. We assume that unsaturated flow in soils is governed by the same laws that apply to saturated flow. For unsaturated flow we must consider the fact that a portion of the soil pores filled by air could indeed resaturate or drain. In our discussion of unsaturated flow, capillarity will be quoted as well as the term capillary rise frequently used in the literature. However, general mathematical formulations of physical phenomena should be independent of such simplifying ideas as soil capillaries and consequently, when we mention capillarity, it is just for the sake of modeling approximately some effects occurring in real soils.

### 5.3.1 Darcy-Buckingham Equation

A simple example of unsaturated flow demonstrated in Fig. 5.8 is analogous to the examples of experiments with saturated flow. The cylinder containing the soil has small openings within its walls leading to the atmosphere. Semipermeable membranes, permeable to water but not to air, separate the soil from free water on both sides of the cylinder. The pools of water are connected to the cylinder with flexible tubes. Full saturation of the soil is first achieved when both pools, lifted to the highest point of the soil, displace the soil air through the openings on the top side of the cylinder. At this moment, there is

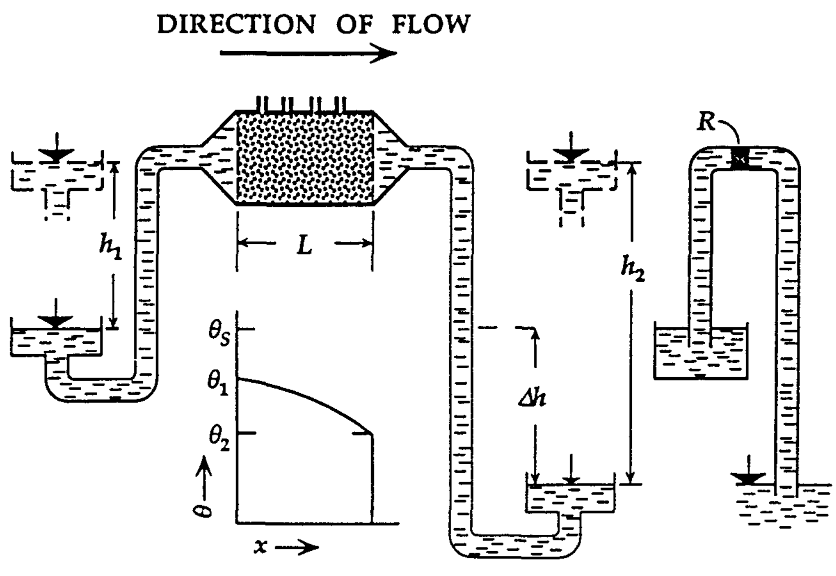


*Figure 5.8. A simple steady flow experiment on an unsaturated soil column. At the right, an analogy of flow in a syphon with an installed resistance R.*

no flow in the system and the soil is assumed water saturated. With the pool on the left side of the cylinder lowered to the position $h_1$ and the pool on the right side to position $h_2$, air enters into the soil through the openings as the soil starts to drain in a manner similar to a soil placed on a tension plate apparatus. The soil on the left side of the cylinder will be drained to a lesser extent than that on the right side with the soil water content distribution from left to right being nonlinear. Although water flows from the left pool to the right pool, the rate of flow is reduced significantly compared with that when the soil is water saturated. If the water level in each of the pools is kept at a constant elevation with time, steady flow will eventually be reached with the water content at each point within the soil remaining invariant. At this time, the flux density $q$ will depend upon the hydraulic gradient and be governed by an equation similar to (5.3)

$$q = -K \frac{\Delta h}{L} \tag{5.30}$$

where $K$ is the unsaturated hydraulic conductivity $[LT^{-1}]$. Inasmuch as the soil is not saturated and flow occurs primarily in those pores filled with water, the value of $K$ will be smaller than that of $K_S$ for the same soil. As for saturated flow we commonly take the potential related to the weight of water, i.e. in units of pressure head. For the majority of practical problems, all components of the total potential except those of gravity and soil water are neglected. Hence, (5.30) rewritten to allow the hydraulic conductivity to be a function of the soil water potential head $h$ is

$$q = - K(h)\frac{dH}{dz} \tag{5.31}$$

and for two and three dimensional problems

$$q = -K(h) grad H. \tag{5.32}$$

Equation (5.33) is equivalent to Darcy's equation, and because Buckingham (1907) was the first to describe unsaturated flow dependent upon the potential gradient, equations such as (5.31) and (5.32) are called Darcy-Buckingham equations. The unsaturated hydraulic conductivity $K$ is physically dependent upon the soil water content $\theta$ because water flow is realized primarily in pores filled with water. Because the relationship $\theta(h)$ is strongly influenced by hysteresis, $K(h)$ is strongly hysteretic. On the other hand, it follows from percolation theory that $K(\theta)$ is only mildly hysteretic.

Examples of $K(\theta)$ and $K(h)$ demonstrated in Fig. 5.9, show that the more permeable soil at saturation does not necessarily keep its greater permeability throughout the entire unsaturated region. It is also evident in Fig. 5.9 that the hysteretic behavior of $K(h)$ demands that, for a given value of $h$, the value of $K$ is greater for drainage than for wetting.

The Darcy-Buckingham equation is adequate for describing unsaturated flow only if the soil water content is not changing in time. Unfortunately, this is seldom the case. When $\theta$ and $q$ alter in time, we must combine (5.31) with the equation of continuity. The equation of continuity relates the time rate of change of $\theta$ to the spatial rate of change of $q$ in a small elemental volume of soil. The resulting differential equation is strongly non-linear and its solution even for simple conditions is most difficult. Generally, (5.31) is in itself not satisfactory for the solution of such hydrologically important processes as evaporation, infiltration, drainage, subsurface flow etc. Exceptional situations or highly simplified flow conditions are usually the only problems described by the sole use of (5.31).

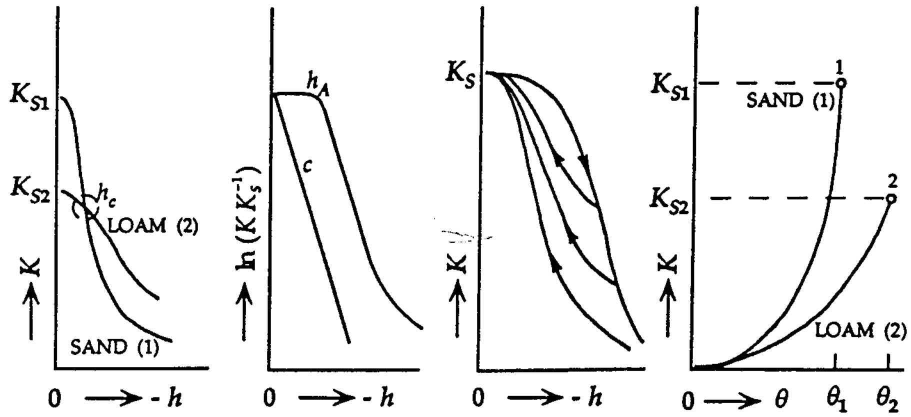


*Figure 5.9. Dependence of unsaturated hydraulic conductivity K upon negative pressure head h (strongly hysteretic) and upon soil water content θ.*

## 5.3.2 Unsaturated Hydraulic Conductivity

We distinguish two approaches for a physical interpretation of the measured hydraulic conductivity $K$. The first is based on the direct application of the Kozeny equation. The second uses the soil water retention curve to quantify the pore size distribution. With this quantification the Kozeny equation is used for sub-groups of pores. In addition to these two physical approaches, empirical formulations of $K(h)$ are used to merely express observed relationships.

Let us first apply the Kozeny equation. Inasmuch as only a portion of the pores is filled with water in an unsaturated soil, we replace the porosity $P$ by the soil water content $\theta$. We assume that the value of the tortuosity $\tau$ is described by Corey (1954, quoted by Corey, 1977) as

$$\frac{\tau_S}{\tau(\theta)} \approx \left(\frac{\theta - \theta_r}{P - \theta_r}\right)^2 \qquad (5.33)$$

which is valid for sands where $\tau_S$ is the tortuosity in the saturated soil, $\tau(\theta)$ the tortuosity in the soil having water content $\theta$ and $\theta_r$ a residual water content. When it is assumed that the tortuosity owing to a change of soil water content is ignored in (5.17), Leibenzon (1947) derived the following expression

$$\frac{K}{K_S} = \left(\frac{\theta - \theta_r}{P - \theta_r}\right)^n \qquad (5.34)$$

where the exponent $n$ should have values ranging from 3.3 to 4. Averianov (1949) proposed that $n = 3.5$ is a good robust estimate. The value of exponent $n$ is related to the pore size distribution and thus to the soil water retention curve SWRC, see Brooks and Corey (1964) who recommended $n = 2/\lambda + 3$ with $\lambda$ read from (4.42) when $P$ is replaced by $\theta_S$ in (5.34). Later on, Russo and Bresler (1980) found that values of 1 or 2 fit better than 3 in the exponent $n$. Childs and Collis-George (1950) obtained the equation

$$K = \alpha \frac{\theta^3}{A_m^2} \tag{5.35}$$

which is comparable to that of Deryaguin et al. (1956)

$$K = \alpha \frac{A_m d^3}{2\mu} \tag{5.36}$$

provided that we assume smooth walls. At small soil water contents and in soils having rough walls, the power function $K \alpha \theta^n$ remains but the exponent $n \neq 3$. A physical interpretation can be obtained using a fractal model of wall roughness (Toledo et al., 1990) The identity of both equations (5.35) and (5.36) is reached when the average thickness of the water film $d$ is taken as functionally dependent upon $\theta$ for a given specific surface $A_m$. In both equations $\alpha$ is an empirical coefficient.

Inasmuch as $\theta(h)$ exists, the dependence of $K$ upon $h$ is also deducible with many empirical formulae quoted in the literature.

Gardner (1958) modified Wind's (1955) empirical proposal

$$K = a h^{-m} \tag{5.37}$$

to the relationship

$$K = \frac{a}{|h|^m + b} \tag{5.38}$$

applicable to $h = 0$ where $a$, $b$ and $m$ are empirical coefficients. Note that for $h = 0$, $a/b = K_S$.

Gardner's exponential relationship (1958)

$$K = K_S \exp(ch) \tag{5.39}$$

is frequently used in analytical solutions. If $K/K_S$ is plotted against $h$ on semi-log paper, a straight line is obtained. This relation usually fits the experimental data well in the range from $h = 0$ ($\theta = \theta_S$) to a certain $h_{lim}$, see Fig. 5.9. ~~(top right)~~ For soils manifesting a distinct air entry value $h_A$, Gardner and Mayhugh (1958) modified (5.39) to

$$K = K_S \exp\left[c\left(h - h_A\right)\right]. \tag{5.40}$$

The value of the empirical coefficient $c$ with dimension [L$^{-1}$] is related to soil texture. and most frequently, $c = 0.1$ to $0.01$ cm$^{-1}$. For $\delta$-function soils in the Green and Ampt approximation of infiltration the value of $c$ is numerically equivalent to the soil water pressure head $|h_f|$ at the wetting front, see Chapter 6. Both (5.39) and (5.40) have been broadly used in analytic and semi-analytic solutions , especially for steady flow problems as we show with some examples in Chapter 6 and as was fully reviewed by Pullan (1990). The reciprocal of $c$ ($= \lambda_c^{-1}$) is sometimes used as one of the soil hydraulic characteristics. In such cases,

$\lambda_c$ is denoted as a microscopic capillary length (Bouwer, 1966; White, 1988).

Because (5.39) and (5.40) are valid in the wet range, (5.38) might be preferred in the dry range. For $h < h_{lim}$, we must use $c_2$ different from $c$ to extend the applicability of the equation to the dry range. $K(h)$ is often defined as a composite function. For example, the range of $h_A > h > h_{lim}$, (5.39) applies and for $h < h_{lim}$, (5.37) or (5.38) applies in order to simulate the entire soil water regime in some instances.

From studies of capillarity in sands, Brooks and Corey (1964) obtained the frequently used relationship

$$\frac{K}{K_S} = \left(\frac{h_A}{h}\right)^m \tag{5.41}$$

where $m$ depends upon the pore size distribution. Usually, $m$ = 3 to 11.

Physical interpretation of $K(\theta)$ or $K(h)$ must include in addition to the total porosity, the distribution of the pore sizes. Recognizing from (5.10) that the flow rate in a cylindrical capillary is $v_p$ $(r^2)$, drainage of the largest pores drastically reduces the value of $K$ in spite of the relatively small volume of those pores. Childs and Collis-George (1950) were the first to propose a method relating $K(\theta)$ to a pore size distribution function $f(r)$. Using the soil water retention curve to reflect $f(r)$, they obtained (5.35) as a simplified result. Their general approach attracted attention and was further developed and modified. We show those developments here.

In its simplest form the porous system is composed of $j$ categories of pores with $j$ = 1 for the category of smallest pores. In each category the pore radii are in ranges $r_{j-1}$ to $r_j$. In each category the flux is $q_j(\bar{r}_j^4, n)$ where $\bar{r}_j$ is the mean radius and $n$ is the percentage of the category and frequently $q_{j-1} < q_j$ even if $n_{j-1} > n_j$. Assuming $\nabla H = -1$, the unsaturated hydraulic conductivity $K = \Sigma q_j$. When the soil is only partially saturated with water, contributions of fluxes $q_j$, $q_{j-1}$ et al. from the larger , empty pores of radii $j$, ($j$-1) et al. do not exist, see Fig. 5.10. In a more exact derivation, we start with the mean flow rate $v_p$ in pores of radius $r$ according to the Hagen-Poiseuille equation

$$v_p(r) = ar^2 I_h \tag{5.42}$$

where $a = \rho w g / 8\mu$, see (5.10). The flux density in a porous system with a continuous distribution function of pores $f(r)$ and a tortuosity $\tau$ is

$$q = \frac{1}{\tau}\int_0^r v_p(r) f(r) dr. \tag{5.43}$$

Or, with (5.42) for $I_h = 1$, $q = K$ and

$$K = \frac{1}{\tau}\int_0^r ar^2 f(r) dr. \tag{5.44}$$

When $f(r)dr$ is approximated by $d\theta_E(h)$, i.e. by the derivative of the SWRC and for the relation between the pore radius and the pressure head ($r = c/h$), we obtain

$$K = \frac{ac}{\tau}\int_0^{\theta_E} \frac{1}{h^2(\theta_E)} d\theta_E(h). \tag{5.45}$$

For relative hydraulic conductivity $K_r$

$$K_r = K / K_S \tag{5.46}$$

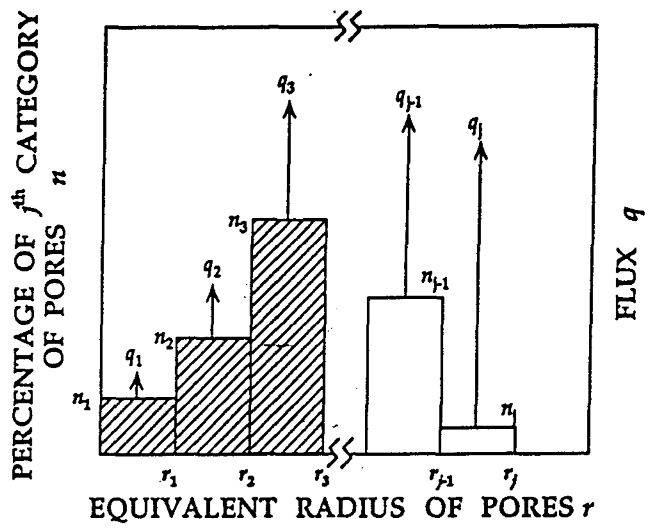


*Figure 5.10. Contributions of soil pore categories with $r_1 < r_2 < \cdots r_j$ and their fluxes $q_1, q_2, \cdots q_j$ to the total flux $q = q_1 + q_2 + \cdots + q_j$. If grad $H = 1$, $q = K$ and $K_{r1 \cdots 3} \ll K_{rj}$.*

and with the tortuosity from (5.33) modified to

$$\tau_S / \tau = \theta_E^b, \tag{5.47}$$

we have

$$K_r = \theta_E^b \int_0^{\theta_E} \frac{d\theta_E}{h^2(\theta_E)} \Bigg/ \int_0^1 \frac{d\theta_E}{h^2(\theta_E)}. \tag{5.48}$$

Various authors have not found a unique interpretation for exponent b in the above equation. Marshall (1958) and Millington and Quirk (1961) defined b as the probability of occurrence of continuous pores. For isotropic and homogeneous media $b = 2P_f$ with $P_f$ denoting that portion of the porosity within which water is flowing. Marshall assumed $P_f = 1$ and hence, $b$ was 2. Millington and Quirk used $P_f$ as 2/3 and hence, $b$ was 4/3. Burdine (1953) interpreting the tortuosity with (5.33) evaluated $b$ as 2.

Inasmuch as the microscopic pore size distribution is used to characterize the macroscopic flux in a soil, Mualem (1976) classified such models as microscopic models. After evaluating about 50 soils on a macroscopic scale, Mualem decided that $b$ was 0.5 and (5.48) is modified to

$$K_r = \theta_E^{0.5} \left[ \int_0^{\theta_E} \frac{d\theta_E}{h(\theta_E)} \Bigg/ \int_0^1 \frac{d\theta_E}{h(\theta_E)} \right]^2. \tag{5.49}$$

If the van Genuchten soil water retention curve (4.43)

$$\theta_E = \frac{1}{\left[1 + \left(\alpha |h|\right)^n\right]^m} \tag{5.50}$$

and (5.48) are combined, we obtain

$$K_r(\theta_E) = \theta_E^b\left[1-\left(1-\theta_E^{1/m}\right)^m\right]^a \tag{5.51}$$

with $m = 1 - c/n$, and $n > 1$. These specific relations are suggested for simple evaluation of the integrals in (5.49). For the model of Burdine, $a = 1$, $b = 2$ and $c = 2$. For the model of Mualem, $a = 2$, $b = 0.5$ and $c = 1$. Let us note that Mualem's database consisted mainly of repacked laboratory soils and $b = 0.5$ does not hold for all field soils where the deviation may vary from less than -10 to more than 10 (van Genuchten et al., 1989).

Mualem's model of $K(h)$ is

$$\frac{K(h)}{K_S} = \frac{\left\{1-(\alpha|h|)^{n-1}\left[1+(\alpha|h|)^n\right]^{-m}\right\}^2}{\left[1+(\alpha|h|)^n\right]^{m/2}}. \tag{5.52}$$

Similarly, using the soil water retention curve $\theta_E = (h_A/h)^\lambda$ of Brooks and Corey (4.42) the relative hydraulic conductivity is

$$\frac{K(\theta_E)}{K_S} = \theta_E^{\,b+a/\lambda} \tag{5.53}$$

and

$$\frac{K(h)}{K_S} = \left(\frac{h_A}{h}\right)^{a+b/\lambda}. \tag{5.54}$$

For the model of Childs and Collis-George, $a = 2$ and $b = 2$. For that of Burdine, $a = 2$ and $b = 3$. For that of Mualem, $a = 2$ and $b = 2.5$.

The exponent $(a + b/\lambda)$ in (5.54) is identical to the exponent $m$ in (5.41). Although the value of the exponent should theoretically be in a narrow range between 2.5 and 4.5, experimental data yield values that extend to about 11. This discrepancy can be explained by the over-simplification of the porous body in the model. In the derivation of the above equations, several approximations were made. First, the soil porous system was modeled by a bundle of cylindrical capillary tubes. Second, the pore size distribution function was approximated from the soil water retention curve. And third, the value of $b$ was empirically evaluated. However, in spite of these approximations for the derivation of $K(\theta_E)$ and $K(h)$, the most problematic is the proper interpretation of the soil water retention curve close to $\theta_S$.

A formal sensitivity analysis of (5.52) by Wösten and van Genuchten (1988) showed that differences in $K_r$ increase with a decrease in $h$ (i.e. with the soil drying) as the parameter $\alpha$ is altered, see Fig. 5.11. On the other hand, the influence of the exponent $n$ also brings about a great potential error in the wet region. In Fig. 5.11, $\theta_E$ has been replaced by $\theta$ with the soil water retention curve (4.43) having the form

$$\theta = \theta_r + \frac{\theta_S - \theta_r}{\left[1+\alpha|h|^n\right]^m}. \tag{5.55}$$

Additional sensitivity analyses made by Šír et al. (1985) and Vogel and Císlerová (1988) show the role of an error $\delta h(\theta_E)$ in the experimental determination of

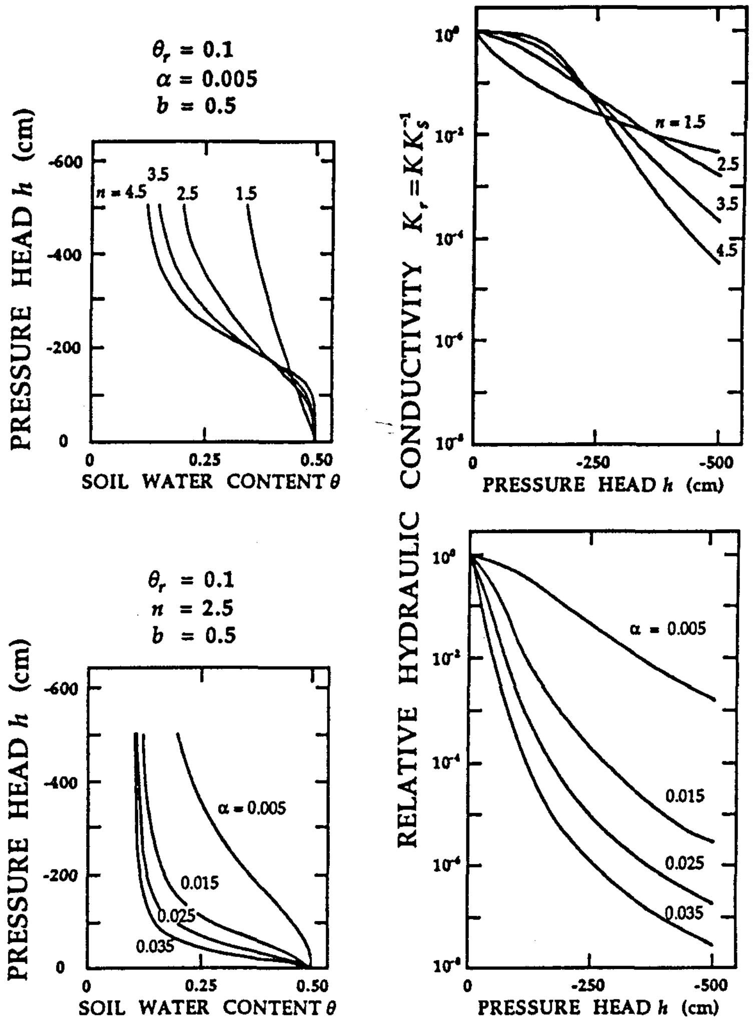


*Figure 5.11. Sensitivity analysis of (5.50) and (5.52) by Wösten and van Genuchten (1988) shows the dependence of $h(\theta)$ and $K(h)$ upon parameters $\alpha$ and $n$.*

$h(\theta_E)$. If $\delta h(\theta_E)$ is a constant in the range $0 \leq \theta_E \leq 1$, the absolute error of $K(\theta)$ rises steeply with an increase of $\theta_E$.

These derived equations are valid for the laboratory Darcian scale assuming microscopic homogeneity of the porous system. The assumption of microscopic homogeneity does not hold if the soil is aggregated, penetrated by plant roots and earthworms or dissected by fissures. In such cases the Darcian scale must be adapted to the pedon scale, i.e. to the reality of a field soil. On the pedon scale, microscopic homogeneity of the porous system may or may not exist.

The hydraulic conductivity function $K(h)$ of soil aggregates is about two orders of magnitude less than that of the bulk soil. The difference in $K(h)$ between aggregates of the soil and bulk soil decreases only slowly in the wet interval for $h > -800$ cm for well developed aggregates (Gunzelmann et al., 1987). In aggregated soils we deal with two different domains of velocity fields. One domain is related to interpedal pores and characterized by an accelerated flux. The second domain conducts water and solutes at relatively small flow rates and is found in intrapedal pores.

When the soil porous system is characterized by a bi-modal pore size distribution curve (Fig. 2.4), the relation $K(h)$ shows two distinct regions. For $0 > h > h_1$, only the by-pass pores belonging to the secondary peak are considered with Mualem's model applied to the soil water retention curve of the interpedal (by-pass) pores. For $h < h_1$ we use the remaining portion of the SWRC representing only the intrapedal (matric) pores, see Fig. 5.12 and Othmer et al. (1991). Hence, two matching factors are needed. For the region $0 > h > h_1$ the matching factor is $K_S$. For the region $h < h_1$ it is a measured value of $K(h < h_1)$. Although this mechanistic separation of the two porous systems uses the same basic equations, the accelerated fluxes through the by-pass pores are conveniently described.

Up to now we have discussed the problems related to $K(\theta)$ in a wet soil. In a dry soil, the probable errors in modeling $K(\theta)$ are related to $\theta_r$. The residual soil water content $\theta_r$ in $\theta_E$ of (5.50) and further on in other $K(\theta)$ models leads to $K(\theta \leq \theta_r) = 0$. This zero value of hydraulic conductivity for $\theta > 0$ is in agreement with our description of SWRC in Section 4.3 where we used $\theta_{Wr}$ to denote the boundary between coherent and incoherent water phase distributions. However, we have shown in the same Section 4.3 that $\theta_r$ is obtained as a fitting parameter which we are not allowed to interpret physically. Thus, the physically observed $\theta_{Wr}$ in $K(\theta)$ may not coincide with $\theta_r$ obtained by fitting (4.42) or (4.43) to experimental SWRC data. A simple method for independently estimating $\theta_{Wr}$ for $K(\theta)$ models has not yet been proposed and tested on a broad scale.

Attempts to physically interpret the unsaturated hydraulic conductivity function lead to more realistic models of porous media than the parallel capillary tube model considered up to now. From soil morphology and from macroscopic measurements of the SWRC, the topological structures within a porous system can be deduced by either percolation theory or procedures of fractal geometry – or by a combination of both. The soil water flux is then related to the flow within individual pores and their fractal dimensions. Subsequently, the flow is formulated by the Hagen-Poiseuille equation with a procedure formally analogous to that developed by Childs and Collis-George (see e.g. Rieu and Sposito, 1991).

In addition to capillary pores, soil contains macropores where water is not influenced by meniscus forces. Such macropores originate owing to the growth and decay of plant roots, activities of soil edaphon and shrinkage in loam and clay soils.

Macropores play a special role in the flow of water especially during infiltration. When the soil water pressure is positive or when an unsaturated soil is ponded with water, water flows in the so-called "macropores". The

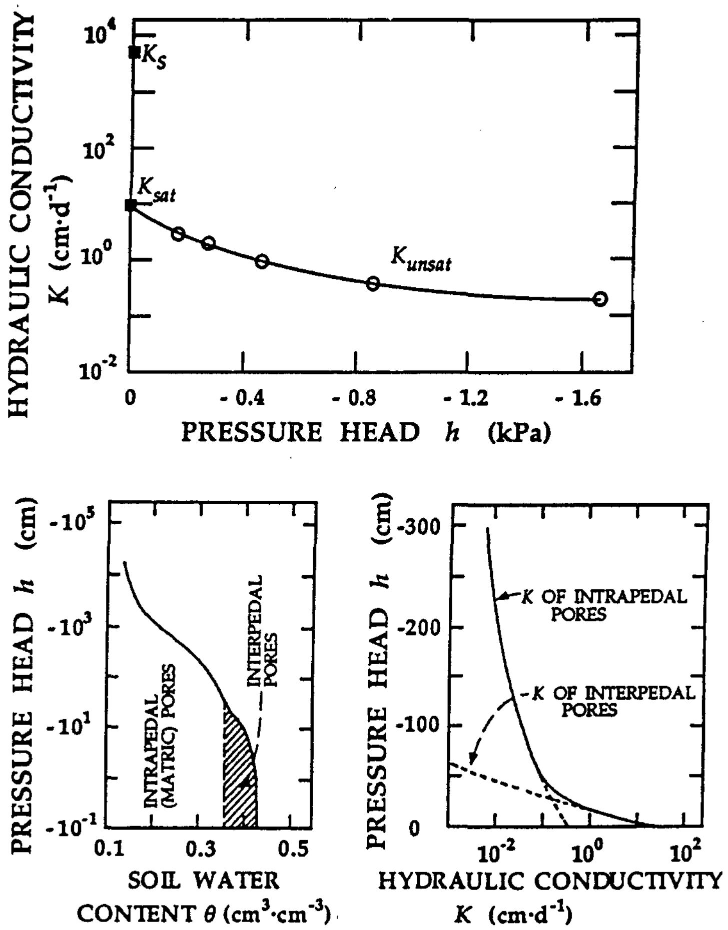


*Figure 5.12. Saturated hydraulic conductivity of a soil with macropores $K_S$ and saturated hydraulic conductivity with macropores excluded $K_{sat}$ which smoothly continues to K(h). Top graph is for Fluvaquent clay (Boolting et al., 1991). Relative hydraulic conductivity $K_r(h)$ of an unsaturated soil frequently has two branches owing to a bi-modal porosity. The example given in the bottom graphs is from the Bt horizon of a loamy soil (Othmer et al., 1991).*

mechanism of the flow in this case may be different from that of the capillary porous system. Water may flow either along the walls of the pores like a thick film, or through the entire cross-sectional area of the pore. When water conduction in cracks is combined with absorption, the kinetic wave approximation (German and Beven, 1985) can be used. The flux density is restricted just to macropores and it is generally reduced by absorption. The theory describes the transformation of both flux density and front velocity when water is transported in macropores as a pulse. It is applicable only under the provision that the macropores do not change during the transport of water. The theory cannot be used for modeling water flow in the fissures of shrinking-swelling soils.

When the soil is fully water-saturated and the flux exists at positive pressure, the saturated hydraulic conductivity comprises the flux in macropores together with flow in the soil matrix. If the flux in macropores is hindered, or if macropores are absent with the soil matrix not changed and remaining still water-saturated, the resulting value of the hydraulic conductivity may be decreased by two to four orders of magnitude (Boolting et al., 1991; Liu et al., 1994).

In this book, we use the term macropores only for pores without capillarity. In some of the literature a confusion exists inasmuch as coarse capillary pores and by-pass pores are also called macropores just to emphasize the large flux in those pores. However, if capillarity is manifested with the flow realized by the gradient of the negative soil water pressure, the Darcy-Buckingham equation is still appropriate with no need to replace it.

Up to this point we have assumed that Darcy's equation is fully applicable to unsaturated flow. However, when the validity of Darcy's equation is doubted for saturated flow in clays, non-Darcian pre linear flow should be even more pronounced for unsaturated flow in clays. Experiments indicating this possibility (Swarzendruber, 1963) have been theoretically explained (Bolt and Groenevelt, 1969).

The influence of the temperature upon $K(\theta)$ is usually expressed by $\mu_W(T)$ in

$$K(\theta) = K_r(\theta)\, K_p \rho_W g / \mu_W. \qquad (5.56)$$

However, Constanz (1982) provided experimental evidence that in some instances (5.56) was only approximate.

The influence of the concentration of the soil solution and of the exchangeable cations is similar to that already mentioned for $K_S$. Dane and Klute (1977) reported that a decrease of the concentration in the soil solution when the SAR was kept constant resulted in roughly the same decrease of $K$ in the whole range of $\theta$, see Fig. 5.13. It is also expected that the function $K(\theta)$ would change with ESP (Kutílek, 1983).

Measuring techniques for determining $K(\theta)$ are usually related to the solution of specified unsteady flow processes,

## 5.3.3 Richards' Equation

Equation (5.32) is fully applicable to steady unsaturated flow when $\nabla \cdot q = 0, dq/dt = 0$ and $d\theta/dt = 0$. In practical situations, unsteady flow frequently exists with $d\theta/dt \neq 0$. In these situations, two equations are needed to describe the flux density and the rate of change of $\theta$ in time. The flux density is described by the Darcy-Buckingham equation and the rate of filling or emptying of the soil pores is described by the equation of continuity. Consider the prism element having edges of length $\Delta x$, $\Delta y$ and $\Delta z$ given in Fig. 5.14. The difference between the

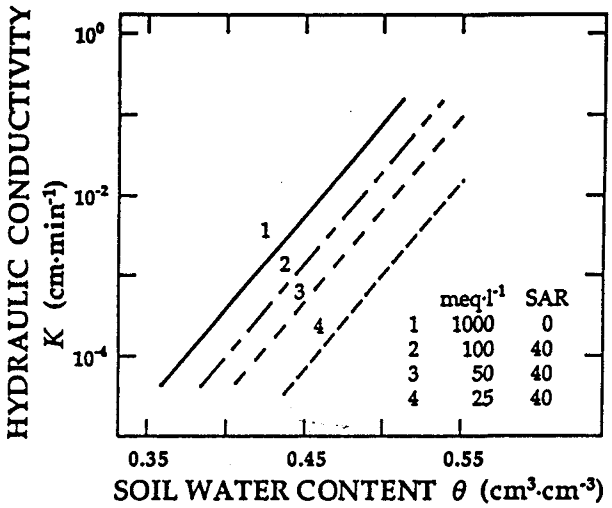


*Figure 5.13. The influence of SAR and the concentration of the soil solution upon K(θ) according to Dane and Klute (1977).*

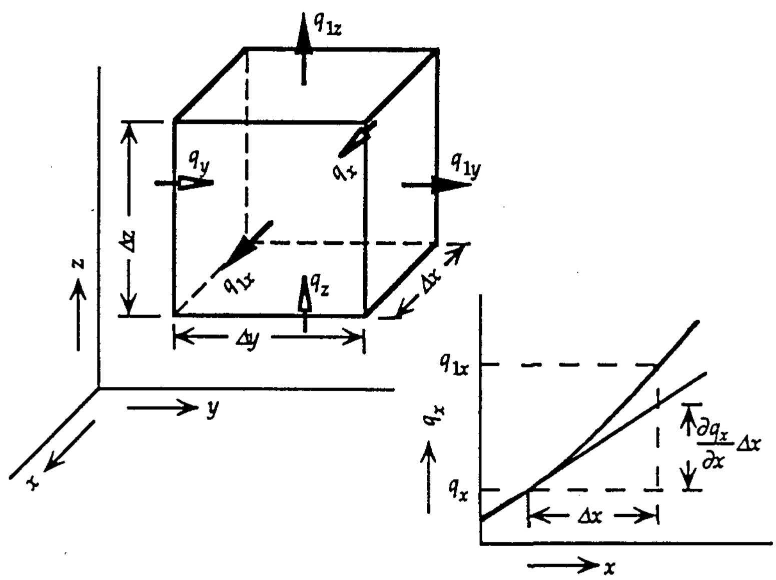


*Figure 5.14. Derivation of the equation of continuity (5.62).*

volume of water flowing into the element and that flowing out of the element is equal to the difference of water content in the element in time $\Delta t$. The rate of inflow (macroscopic) in the direction of the $x$ axis is $q_x$. If we assume the change in $q_x$ is continuous, the rate of outflow is $[q_x + (\partial q_x / \partial x)\Delta x]$. The inflow volume is $q_x \Delta y \Delta z \Delta t$ and the outflow volume is $[q_x + (\partial q_x / \partial x)\Delta x]\Delta y \Delta z \Delta t$. The difference between inflow and outflow volumes is

$$\left\{q_x \Delta y \Delta z \Delta t - \left[q_x + (\partial q_x / \partial x)\Delta x\right]\Delta x \Delta y \Delta z\right\} \tag{5.57}$$

or

$$-\left(\frac{\partial q_x}{\partial x}\right)\Delta x \Delta y \Delta z \Delta t. \tag{5.58}$$

Similarly in the direction of the $y$ axis, the difference between the inflow and the outflow volumes is

$$-\left(\frac{\partial q_y}{\partial y}\right)\Delta x \Delta y \Delta z \Delta t \tag{5.59}$$

and that in the direction of the $z$ axis

$$-\left(\frac{\partial q_z}{\partial z}\right)\Delta x \Delta y \Delta z \Delta t. \tag{5.60}$$

The sum of the above differences equals the change of the water content of the element. Provided that $\theta(t)$ has a continuous derivative for $t > 0$,

$$\frac{\Delta \theta}{\Delta t}\Delta x \Delta y \Delta z \Delta t = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)\Delta x \Delta y \Delta z \Delta t . \tag{5.61}$$

Taking the limit as $t \to 0$, we obtain the equation of continuity

$$\frac{\partial \theta}{\partial t} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right). \tag{5.62}$$

If we insert for $q_x$, $q_y$ and $q_z$ from (5.32), we have

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[K(h)\frac{\partial H}{\partial x}\right] + \frac{\partial}{\partial y}\left[K(h)\frac{\partial H}{\partial y}\right] + \frac{\partial}{\partial z}\left[K(h)\frac{\partial H}{\partial z}\right] \tag{5.63}$$

provided that the soil is isotropic. In one-dimensional form for $H = h + z$ the above equation becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[K(h)\frac{\partial h}{\partial z}\right] + \frac{\partial K}{\partial z}. \tag{5.64}$$

Equations (5.63) and (5.64) are called Richards' equations in the name of the author who first derived them (1931).

If the soil is either wetting or drying, $\theta$ will be uniquely dependent upon only $h$ and

$$\frac{\partial \theta}{\partial t} = \frac{d\theta}{dh}\frac{\partial h}{\partial t}. \tag{5.65}$$

Hence, the capacitance form of Richards' equation is obtained as

$$C_W(h)\frac{\partial h}{\partial t} = \frac{\partial}{\partial z}\left[K(h)\frac{\partial h}{\partial z}\right] + \frac{\partial K}{\partial z} \tag{5.66}$$

where soil water capacity $C_W = d\theta/dh$ [$L^{-1}$] is illustrated in Fig. 5.15. An alternative development using

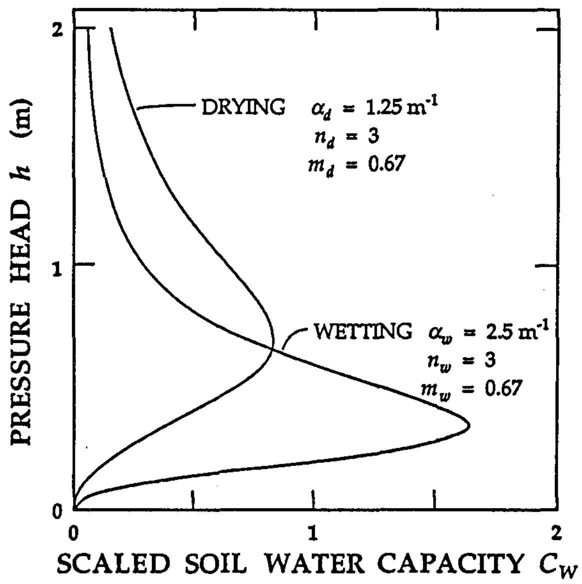


*Figure 5.15. Soil water capacity $C_W$ as a function of pressure head h for drying and wetting (Luckner et al., 1989).*

$$\frac{\partial h}{\partial z} = \frac{dh}{d\theta}\frac{\partial \theta}{\partial z} \tag{5.67}$$

leads to the diffusivity form of Richards' equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] + \frac{dK}{d\theta}\frac{\partial \theta}{\partial z} \tag{5.68}$$

where the Darcy-Buckingham equation has the diffusivity form

$$q = -D(\theta)\frac{\partial \theta}{\partial z} + K(\theta) \tag{5.69}$$

and the soil water diffusivity $D$ is the term derived from

$$D(\theta) = K(\theta)\frac{dh}{d\theta}. \tag{5.69a}$$

The main reason for the derivation of either the capacitance equation (5.66) or the diffusivity equation (5.68) is the reduction of the number of variables from 4 to 3.

Both equations (5.66) and (5.68), strongly non-linear owing to functions $C_W(h)$, $K(h)$ and $D(\theta)$, are sometimes called Fokker-Planck equations. The name of (5.68) was derived from its resemblance (when its second term on the right hand side is omitted) to that for molecular diffusion. The units of $D$ in (5.68) are identical to those of the diffusion coefficient. Many analytical and semi-analytical solutions for the diffusivity equation for various boundary conditions are known from the theories of diffusion (Crank, 1956) and heat flow (Carslaw and Jaeger, 1959). They have been profitably applied for the solution of many processes of unsaturated flow in soils. Whenever there is a region of positive pressure in the soil (5.68) is not applicable and (5.66) should be used.

Sometimes, Kirchhoff's transformation

$$U = \int_{h_0}^{h} K(h)\, dh \tag{5.70}$$

is used with (5.64) to yield

$$\frac{C_W(h)}{K(h)}\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial z^2} - \frac{1}{K(h)}\frac{dK}{dh}\frac{\partial U}{\partial z} \tag{5.71}$$

or

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 U}{\partial z^2} - \frac{\partial K}{\partial z}. \tag{5.72}$$

Because the last term of (5.64), (5.66) and (5.68) originated from the gravitational component $z$ of the total potential $H$, it is frequently referred to as the gravitational term of the Richards' equations. The first term of the right hand side of each of those equations expresses the flow of water in the soil owing to the gradient of the soil water (matric) potential component $h$. In some instances when the gravitational term is neglected, the solution of the resulting non-linear diffusion equation with its non-constant diffusivity

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] \tag{5.73}$$

offers approximate results. If the flow is horizontal, solutions of (5.73) are exact.

### 5.3.4 Soil Water Diffusivity

The most common $D(\theta)$ relationship is demonstrated in Fig. 5.16. With the exception of the region of very small soil water contents less than $\theta_H$ ($h < -10^5$ cm), the curve steeply rises with $\theta$. Soil water diffusivity $D(\theta)$ in the wet range above $\theta_H$ is typically less steep in its relation to $\theta$ as compared with $K(\theta)$. In this wetter range of $\theta$, $D$ changes about five orders of magnitude compared with seven orders of magnitude for $K$.

In the dry region of $0 \le \theta < \theta_H$ with a great portion of pores filled with air, water vapor flow is enhanced while liquid water flow is limited to that of very thin water films on the soil solid surfaces. The rate of liquid flow, strongly dependent upon the thickness of the film, has already been demonstrated by (5.36). Here, the vapor flux exceeds the liquid flux. A more detailed discussion on water vapor flux will be given in Section 5.3.5. Now, we shall study in detail the monotonically rising part of $D(\theta)$, i.e. for $\theta > \theta_H$.

Among the well known and frequently used empirical equations is the exponential form (Gardner and Mayhugh, 1958)

$$D = D_o \exp\left[\beta(\theta - \theta_o)\right] \tag{5.74}$$

where $D_o$ corresponds to $\theta_o$ and $\beta$ ranges approximately between 1 and 30. Or,

$$D = \alpha \exp\left[\beta(\theta_E^*)\right] \tag{5.75}$$

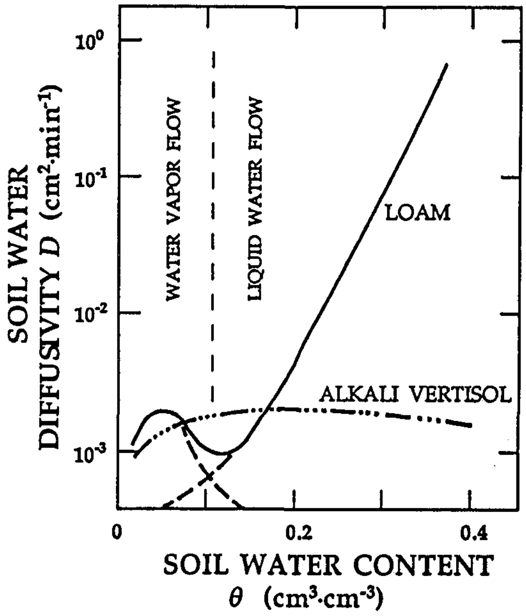


*Figure 5.16. Dependence of soil water diffusivity D upon the soil water content θ. For liquid flow the function is nearly exponential for the majority of soils except for swelling clays. The secondary peak in the dry region near θ = 0 is owing to the dominance of water vapor flow.*

where $\theta_r$ is replaced by $\theta_H$ in $\theta_E^*$ and at $\theta_H$, $D = \alpha$. A physically more exact equation should be derived from the soil water retention curve and from $K(\theta)$. Using (4.43) and (5.51) in (5.69a), van Genuchten (1980) obtained

$$D(\theta_E)=\frac{K_S(1-m)\theta_E^{1/2-1/m}}{\alpha m\left(\theta_S-\theta_r\right)}\left[\left(1-\theta_E^{1/m}\right)^{-m}+\left(1-\theta_E^{1/m}\right)^{m}-2\right]. \quad (5.76)$$

If the simpler (4.42) is used instead of (4.43) we have

$$D(\theta_E)=\frac{K_S h_A \theta_E^{(b-1)+(a-1)/\lambda}}{\lambda\left(\theta_S-\theta_r\right)} \quad (5.77)$$

with the values of $a$ and $b$ being those given earlier for (5.53) and (5.54).

In some clays, mainly alkali Vertisols, the value of $D$ decreases with an increase of $\theta$, if the soil is confined and not allowed to swell, see Fig. 5.16 (Kutílek, 1983, 1984). For some undisturbed soils as well as for disturbed repacked soil columns in the laboratory (Clothier and White, 1981), $D$ does not vary as strongly with $\theta$ as discussed above. If $(D_{max} - D_{min})$ is less than one-half an order of magnitude, the linearized form of (5.73)

$$\frac{\partial\theta}{\partial t} = \overline{D}\,\frac{\partial^2\theta}{\partial x^2} \tag{5.78}$$

serves as an excellent approximation where the mean-weighted diffusivity $\overline{D}$ for the wetting process (Crank, 1956) is

$$\overline{D} = \frac{5}{3(\theta_o-\theta_i)^{5/3}}\int_{\theta_i}^{\theta_o}(\theta-\theta_i)^{2/3}D(\theta)d\theta \tag{5.79}$$

and for the drainage process is

$$\overline{D} = \frac{1.85}{(\theta_o-\theta_i)^{1.85}}\int_{\theta_i}^{\theta_o}(\theta-\theta_i)^{0.85}D(\theta)d\theta \tag{5.80}$$

where $\theta_i$ is the initial soil water content and $\theta_o$ is $\theta$ at $x = 0$ for $t > 0$.

Soils manifesting values of $D$ that are constant or nearly so are called "linear soils" because (5.78) is a linear equation. If the Brooks and Corey soil water retention curve (4.42) is used, $K_r(\theta_E)$ is described by (5.53) and $D(\theta_E)$ by (5.77). The condition of a "linear soil" is satisfied if in these equations [$\lambda = -(a-1)/(b-1)$] or [$a = b = 1$]. If the first condition is applied to the Burdine equation, we obtain $h = h_A\ \theta_E^2$ and $K_r = \theta_E^{-1}$. Neither of these equations describe physical reality. Similarly, equations of Childs and Collis-George or those of Mualem lead to unacceptable results. The second condition leads to (Kutílek et al., 1985)

$$h = h_A\,\theta_E^{-1/\lambda} \tag{5.81}$$

$$K_r = \theta_E^{1/\lambda+1} \tag{5.82}$$

and

$$D = -\frac{h_A}{\lambda}\frac{K_S}{(\theta_S-\theta_r)} \tag{5.83}$$

This discussion shows the restrictions in the definition of a strictly linear soil when we require that $D$ has a constant value and $K_r$ is linearly dependent upon $\theta_E$. If the second condition is not satisfied, we speak of "linear" soils. In general, there exists a family of "linear soils" described by the above equations. If $\lambda = 1$, the hydraulic conductivity function (5.82) is quadratic and meets the requirements of the solutions of Burgers' equation (Clothier et al., 1981).

Concluding, we should keep in mind that the soil water diffusivity is used in Richards' equation in order to reduce the number of variables. It has no direct physical meaning and is only defined mathematically, see (5.69). Moreover, inasmuch as $D(\theta)$ is dependent upon the derivative of the soil water retention curve, it has different values for wetting and drying processes. The temperature dependence of $D(\theta)$ is in accordance with changes of surface tension and viscosity with $T$. However, its prediction is only approximate owing to some not well understood phenomena that associates the temperature dependence of $h(\theta)$ and $K(\theta)$.

### 5.3.5 Diffusion of Water Vapor

In section 5.3.4 we have already shown that the relative maximum in the $D(\theta)$ relationship in the dry region is caused by water vapor flow. Indeed, the soil water diffusivity $D$ contains two components: $D_L$ the diffusivity of liquid water, and $D_G$ the diffusivity of water vapor, i.e. the gaseous phase. Hence, $D = D_L + D_G$ (Philip, 1957a). Jackson (1964) derived $D_G$ as analogous to the earlier introduced soil water diffusivity

$$D_G = D_p \frac{d\rho_{rG}}{d\theta} \tag{5.84}$$

where $\rho_{rG}$ is the relative density (concentration) of water vapor and $D_p$ the diffusion coefficient of water vapor in soil which is approximated by

$$D_p = D_a \alpha (P - \theta)^{\mu} \tag{5.85}$$

where $D_a$ is the diffusion coefficient of water vapor in free air and $\alpha$ and $\mu$ are factors that account for the tortuosity and complexity of the soil porous system. Detailed information about (5.85) is provided by Currie (1960).The term $d\rho_{rG}/d\theta$ is actually the slope of the adsorption isotherm and its inflection point corresponds to the relative maximum of $D_G(\theta)$. The water vapor diffusivity rises to this maximum from $\theta = 0$ and reaches this value at a relative water vapor pressure $p/p_o = 0.3$ to 0.4 in the majority of soils. The maximum value of $D_G$, having a wide range between $10^{-4}$ to $10^{-3}$ $cm^2 \cdot s^{-1}$, depends upon soil texture, mineralogy of the clay fraction and organic matter content, see Fig. 5.17. At greater soil water contents $D_G$ decreases as $D_L$ increases. Values of $D_L$ exceed those of $D_G$ at $p/p_o \approx 0.5$ to 0.8. In terms of the average thickness of the adsorbed water films on the soil solid surface, $D_G$ reaches a maximum value after the first molecular layer is completed and before or at least when the second molecular layer is formed. $D_L$ exceeds $D_G$ when about 4 to 6 molecular layers of adsorbed water exist.

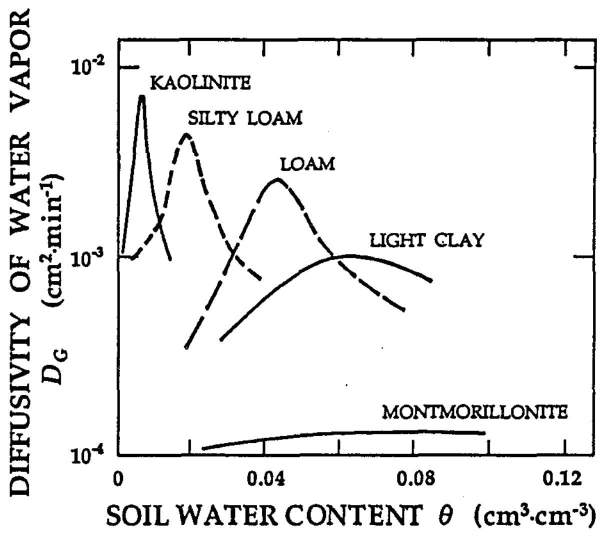


*Figure 5.17. Diffusivity of soil water vapor versus soil water content close to $\theta$ = 0 for soils and clay minerals (Kutílek, 1966).*

For structured soils, (5.84) and (5.85) are far from reality. For more realistic descriptions 2- or 3-modal porosity models should be considered (Currie and Rose, 1985). The effect of such consideration is similar to that of a bi-modal porosity model upon $K(h)$ in the wet region.

It is worthwhile to repeat here at the end of this chapter the principal theoretical gain of all discussions. Without a knowledge of the hydraulic functions of soils – $h(\theta)$, $K_S$, $K(\theta)$ and $D(\theta)$ – a quantitative description of water flow in soils is not feasible.

## 5.4 TWO PHASE FLOW

Up to now we have assumed that the presence of air in soil pores offers no resistance to water flow. Because the fluidity of the soil air is greater than that of water by an order of magnitude, its motion relative to that of water is implicitly neglected. The assumption of one phase flow – water only – is not appropriate when the free entrance or escape of air is blocked by small passages owing to either a layer of a greater water content or of lesser permeability. If the entrapped air is continuous, it forms a "barrier" against the flow of liquid water and the flux density of water is reduced. In such cases the flows of both air and water should be solved as mutually influencing each other (Elrick, 1961). For the sake of simpicity we are here neglecting water vapor flow. In wet soil where the flow of air can play an important role, the contribution of water vapor flux to the liquid water flux is negligibly small. In a dry soil at about $h < -10^5$ cm where the water vapor flux is important, the influence of air flow upon water flow is not expected. We are still dealing with isothermal conditions. We discuss therefore only the interactions of the flow of the liquid water phase and air as a complex gaseous phase.

In the theory of two phase flow, the phases (air and water) are assumed to behave like two immiscible liquids. The basic flow equations of Darcy-Buckingham rewritten to fit both phases (air and water) express the pressure gradient as the driving force are

$$q_A = -\lambda_A\left(\frac{\partial p_A}{\partial z} - \rho_A g\right) \tag{5.86}$$

and

$$q_W = -\lambda_W\left(\frac{\partial p_W}{\partial z} - \rho_W g\right) \tag{5.87}$$

where the indices $A$ and $W$ denote air and water, respectively, $p$ is pressure $[ML^{-1}T^{-2}]$, $\rho$ the density $[ML^{-3}]$ and $g$ the acceleration of gravity $[LT^{-2}]$. The fluid mobility $\lambda$ is

$$\lambda = \frac{K_p K_r}{\mu} \tag{5.88}$$

where $K_r$ is the relative conductivity of the particular fluid, $K_p$ the permeability $[L^2]$ and $\mu$ the dynamic viscosity. The dependence of both $K_{rA}$ and $K_{rW}$ upon the water content of a sand determined experimentally by Touma and Vauclin (1986) is given in Fig. 5.18. The curve $K_{rA}(\theta)$ is not a mirror image of $K_{rW}(\theta)$. The

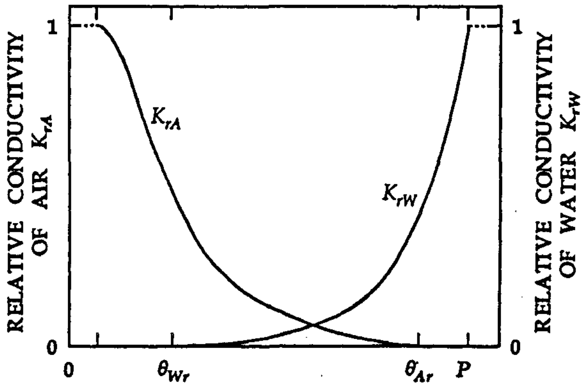


*Figure 5.18. Relative conductivities of air $K_{rA}$ and $K_{rW}$ as related to the soil water content $\theta$ (Touma and Vauclin, 1986).*

decrease of $K_{rW}$ with a decrease in $\theta$ is more rapid than the decrease of $K_{rA}$ with an increase in $\theta$. At $\theta_{Wr}$, the value of $K_{rW}$ is zero. Similarly, at $\theta_{Ar} < P$, the value of $K_{rA}$ is zero. The continuity equation for air is

$$\frac{\partial(\rho_A\theta_A)}{\partial t} + \frac{\partial(\rho_A q_A)}{\partial z} = 0 \qquad (5.89)$$

with $\theta_A$ (= $P - \theta$) the volumetric air content. The continuity equation for water is

$$\frac{\partial(\rho_W\theta)}{\partial t} + \frac{\partial(\rho_W q_W)}{\partial z} = 0 \ . \qquad (5.90)$$

The difference between the pressure of the two fluids is the capillary pressure $p_c = p_A - p_W$, and for water taken as the incompressible fluid

$$\rho_W = \rho_{W_o},$$

we have

$$\frac{\rho_A}{\rho_{A_o}} = \frac{p_A}{p_{A_o}} \qquad (5.91)$$

where the index $o$ denotes the value at the reference atmospheric pressure. Inasmuch as the pressure head $h$ was more convenient for one-phase flow, we use here

$$h_A = \frac{p_A - p_{A_o}}{\rho_W g} \qquad (5.92)$$

and

$$h_W = \frac{p_W - p_{A_o}}{\rho_W g} \qquad (5.93)$$

where $h_c = h_A - h_W$. Hence, (5.86) and (5.87) transcribe to the more familiar forms

$$q_A = -K_A(\theta)\left(\frac{\partial h_A}{\partial z} - \frac{\rho_A}{\rho_W}\right) \tag{5.94}$$

and

$$q_W = -K_W(\theta)\left(\frac{\partial h_W}{\partial z} - 1\right) \tag{5.95}$$

with $h_A - h_W = h_c \equiv h$. The air conductivity $K_A$ ($= \rho_W g \lambda_A$) and $K_W$ [identical to the earlier notation $K(\theta)$] each have the dimension [$LT^{-1}$]. Analogous to Richards' equations (Touma and Vauclin, 1986) we have for air

$$\frac{\partial(\rho_A \theta_A)}{\partial t} = \frac{\partial}{\partial z}\left[\rho_A K_A(\theta)\left(\frac{\partial h_A}{\partial z} - \frac{\rho_A}{\rho_W}\right)\right]. \tag{5.96}$$

When we substitute $\theta_A = P - \theta$ with

$$\rho_A = \rho_{A_o}\left(1 + h_A / h_o\right)$$

and $C_W = d\theta/dh_c$ we obtain the capacitance equation

$$\rho_A C_W \frac{\partial h_W}{\partial t} + \left[(P-\theta)\frac{\rho_{A_o}}{h_o} - \rho_A C_W\right]\frac{\partial h_A}{\partial t} = \\ = \frac{\partial}{\partial z}\left[\rho_A K_A(\theta)\left(\frac{\partial h_A}{\partial z} - \frac{\rho_A}{\rho_W}\right)\right]. \tag{5.97}$$

Similarly, for water

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[K_W(\theta)\left(\frac{\partial h_W}{\partial z} - 1\right)\right] \tag{5.98}$$

and

$$C_W\left(\frac{\partial h_A}{\partial t} - \frac{\partial h_W}{\partial t}\right) = \frac{\partial}{\partial z}\left[K_W(\theta)\left(\frac{\partial h_W}{\partial z} - 1\right)\right]. \tag{5.99}$$

When the air is continuously connected to and has the same pressure as that of the external atmosphere (taken as the reference), $h_A = 0$, $h_c = h_W$ and (5.97) is identical to (5.66).

The above theory offers reliable results if the pressure (or potential) drop of the water across the less permeable barrier is small, or if a barrier limiting the air flow is not reached by a wetting front in the case of infiltration. If a substantially less permeable layer exists for water, a steep water pressure gradient develops across the less permeable layer. In this circumstance, (5.66) suffices.

The solution or dissolution of air in soil water should follow the behavior of the individual gases of the air mixture according to Henry's law

$$C_i = k_i p_i \tag{5.100}$$

where $C_i$ is the concentration of the $i$-th gas, $k_i$ a constant dependent upon the temperature and the nature of the gas and $p_i$ the partial pressure of the $i$-th gas. Considering the numerical values of $k_i$ and $p_i$ for gases composing the soil air, the solubility of nitrogen and oxygen are important. When the pressure of the soil water varies by an order of magnitude, the concentrations of dissolved $N_2$ and $O_2$ in the soil water vary substantially. For example, with an abrupt decrease in soil water pressure as water passes through a less permeable layer, its originally dissolved gases are released. Consequently, small air bubbles usually accumulate on the bottom boundary of a less permeable soil layer. By this mechanism the hydraulic resistance of the less permeable layer increases and is time dependent. A detailed quantitative understanding based upon experimental data is yet to be reported in the literature.

## 5.5 FLOW IN NON-RIGID (SWELLING) SOILS

When a soil swells or shrinks owing to a water content change, the previously developed equations must be modified. A complete theory has not yet been developed for flow accompanied by three-dimensional volume changes resulting in an opening of cracks when a soil dries or their closing when a soil wets. The theory developed up to the present time (Smiles and Rosenthal, 1968; 1969 b Philip, 1973a) deals only with one-dimensional deformation. It is not applicable to three-dimensional volumetric changes associated with non steady flow of water that lead to the formation and closing of cracks. The one-dimensional theory describes wetting of artificially repacked soil columns in the laboratory. For a draining or drying process even in laboratory conditions, it is not applicable because the formation of cracks and transport in the cracked medium are not described. Owing to restrictions and the artificial character of processes described by the theory, we are not presenting it here except for the introduction of the concept of Lagrangian coordinates.

When a soil swells during wetting, Darcy's equation should be modified according to Gersevanov (1937) to relate the rate of water flow to the solid phase. Instead of Euler's coordinate system, Lagrange's coordinates need to be considered. For a one-dimensional treatment of swelling, the original proposal of Gersevanov was redefined by Smiles and Rosenthal (1968) to yield

$$\frac{dm}{dx} = \frac{1}{1+e}. \qquad (5.101)$$

Upon integration the above equation becomes

$$m = \int_{-\infty}^{x} \frac{dx}{1+e} \qquad (5.102)$$

or

$$m = \int_{-\infty}^{x} (1-P)\,dx. \qquad (5.103)$$

Equation (5.101) states that the ratio of the material coordinate $m$ to the Eulerian coördinate equals the ratio of the volume of the solid phase to the total volume of the soil. Fig. 5.19 demonstrates the material coordinate associated with the experiment that follows. A dry soil originally packed into a tube to the height denoted by $x = 0$, has its surface flooded with water. As water infiltrates into the swelling soil, the soil surface rises. After a certain time of infiltration we denote the depth of the wetting front $x_f$ from the original position $x = 0$. Simultaneously, we measure the soil water content and the soil bulk density. From those measurements the porosity $P$ is plotted against negative values of $x$

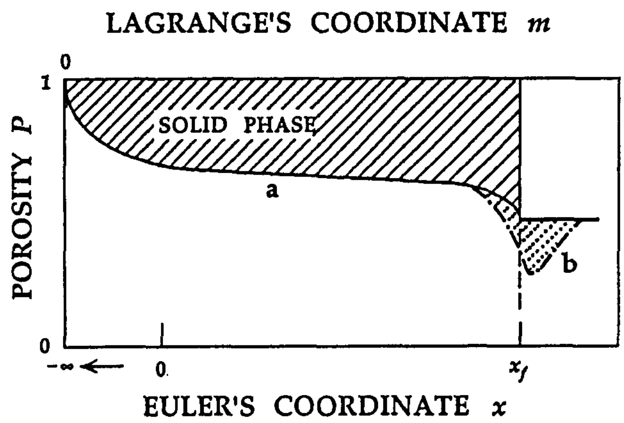


*Figure 5.19. Material coordinate m in the solution for one-dimensional infiltration of water into: a. pre consolidated clay and b. the same clay without pre consolidation. The wetting front position is at $x_f$.*

inasmuch as the soil surface rose compared to its original position at $x = 0$. The material coordinate $m$ expresses the volume of the solid phase wetted by the water. Because we cannot predict the elevation to which the soil will swell, we define the lower limit of integration as $-\infty$. Note that for $P = 1$, the integral (5.103) yields a value of $m = 0$. If the soil is pre consolidated, we can assume that the value of $P = e(1 + e)$ is constant for $x > x_f$. If the soil is not pre consolidated, the soil is compressed ahead of the wetting front by the swelling pressure. This relative minimum $P_{min}$ moves at the same rate as the wetting front with the value of $P_{min}$ decreasing, i.e. the value of $(P_i - P_{min})$ increases with time and with surface load (Kutílek, 1984a). All of these effects are especially distinct in montmorillonitic clays of Vertisols and increase with an increase of ESP and a decrease of EC.

Darcy's equation has to be modified for Lagrangian coordinates and the hydraulic conductivity redefined, especially when we deal with non steady flow. Owing to a lack of an appropriate theory and the extremely complicated character of the system, we recommend simplified approaches related to individual cases of elementary hydrologic processes.

## 5.6 NON-ISOTHERMAL FLOW

Up to now we have described unsaturated flow for only isothermal conditions. Typically, such conditions are rare in soil hydrology. Nonetheless, most modeling of soil hydrology utilizes equations formulated for isothermal conditions as a good approximation for non-isothermal situations. However, for non-isothermal situations the equations must account for thermally driven processes.

Under non-isothermal conditions differences in temperature across a distance z produce a heat flux that alters water transport. More generally, each flow process influences every other flow process and hence, these coupled processes are studied with the theory of irreversible thermodynamics.

### 5.6.1 Coupled Processes

Coupled flow phenomena are described by phenomenological equations of Onsager (1931). The first flux $J_1$ is described by

$$J_1 = L_{11} X_1 + L_{12} X_2 + L_{13} X_3 + \cdots + L_{1n} X_n \qquad (5.104)$$

and similarly for the other flow processes,

$$J_i = \sum_{k=1}^{n} L_{ik} X_k \qquad (5.105)$$

where $X_i$ is the conjugate driving force which produces the $i$-th flow and $L_{ii}$ is the phenomenological "direct" coefficient which relates the flux to the driving force. The above equation shows that the flux $J_1$ may also be driven by forces $X_2$, $X_3$, ··· $X_n$ if the coupling or cross coefficients $L_{12}$, $L_{13}$, ··· $L_{1n}$ differ from zero. In general, $L_{1k}$ shows the contribution of the $k$-th flux $J_k$ to the first flux $J_1$ when a driving force $X_k$ exists. On the other hand, if the $i$-th flow process is not related to any of the other $n$ flow processes, $L_{ik} = 0$ for $k \neq i$ and we obtain

$$J_i = L_{ii} X_i \qquad (5.106)$$

which is the equation of heat flux, molecular diffusion or that of Darcy (5.5). The linearity holds only for slow processes not too distant from an equilibrium state. The coefficients $L_{ik} = J_i / X_k$ are flows per unit force (or conductances) and $L_{ik} = L_{ki}$. In this chapter when we deal with simultaneous fluxes of water and heat $J_T$, (5.105) becomes

$$J_W = L_{WW} X_W + L_{WT} X_T \qquad (5.107)$$

and

$$J_T = L_{TT} X_T + L_{TW} X_W. \qquad (5.108)$$

Note that the first term on the right hand side of (5.107) is the Darcy-Buckingham equation and the following term accounts for the additional contribution of water flow owing to the heat flow.

Equation (5.105) has validity for all coupled phenomena in soils when $n$ flow processes occur simultaneously. The physical interpretation of $L_{ik}$ and $X_k$ in soils have been discussed in detail in the literature (e.g. Bolt and Groenevelt, 1972, and Groenevelt an Bolt, 1972).

### 5.6.2 Flow in Non-isothermal Conditions

Because the mechanisms of the influence of a temperature gradient upon the liquid and vapor water phases are not identical, separate terms are required to describe the flux of liquid and that of vapor. Here we follow a slightly modified procedure of Philip and de Vries (1957) to describe these coupled flows.

For the flux density of water vapor $q_G$ we have

$$q_G = -D_{G\theta} \nabla\theta - D_{GT} \nabla T. \tag{5.109}$$

The dimension of $D_{GT}$ is $[L^2T^{-1}grad^{-1}]$, usually in units $[cm^2{\cdot}s^{-1}{\cdot}K^{-1}]$. For the flux density of liquid water $q_L$ is

$$q_L = -D_{L\theta} \nabla\theta - D_{LT} \nabla T - K. \tag{5.110}$$

The dimension of $D_{LT}$ is identical to that of $D_{GT}$.

The flux density of all of the pore water, both liquid and vapor, is

$$q = -D_\theta \nabla\theta - D_{GT} \nabla T - D_{LT} \nabla T - K \tag{5.111}$$

where $D_\theta = (D_{L\theta} + D_{G\theta})$ and $D_{G\theta}$ here is identical to $D_G$ in (5.90). Such equivalent relations for the coupled coefficients $D_{GT}$ and $D_{LT}$ are not available. Combining (5.111) with the equation of continuity for a one-dimensional system with soil depth $z > 0$ measured from the soil surface, we obtain an equation similar to (5.68)

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(D_\theta \frac{\partial\theta}{\partial z}\right) + \frac{\partial}{\partial z}\left(D_{GT} \frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial z}\left(D_{LT} \frac{\partial T}{\partial z}\right) - \frac{\partial K}{\partial z}. \tag{5.112}$$

The temperature is described by

$$C_T \frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(\lambda_T \frac{\partial T}{\partial z}\right) + \rho_G \chi \frac{\partial}{\partial z}\left(D_{GT} \frac{\partial T}{\partial z}\right) \tag{5.113}$$

where $\lambda_T$ is the thermal conductivity $[W{\cdot}m^{-1}{\cdot}K^{-1}]$, $\chi$ the latent heat of vaporization $[J{\cdot}kg^{-1}]$ and $C_T$ the volumetric heat capacity $[J{\cdot}K^{-1}{\cdot}m^{-3}]$.

Solutions of such practical problems of non-isothermal flow in layered soils require equations that are transcribed into forms analogous to the capacitance equation (5.66) with pressure head gradient replacing the soil water content gradient. Hence, (5.109) for one-dimension becomes

$$q_G = -D_{Gh} \frac{\partial h}{\partial z} - D_{GT} \frac{\partial T}{\partial z} \tag{5.114}$$

and (5.110) is

$$q_L = -\left(K_{Lh} \frac{\partial h}{\partial z} - 1\right) - D_{LT} \frac{\partial T}{\partial z}. \tag{5.115}$$

Note that $K_{Lh} = K(h)$ and $D_{Gh}$ is not identical to $D_{G\theta}$. The equation of continuity is also modified and $C_T$ and $\lambda$ of (5.113) are specified once for the vapor phase and once for the liquid phase. For details of modifying the equations for numerical procedures, see e.g. Passerat de Silans et al. (1989).

### 5.6.3 Flow at Temperature T< 0°C

Although the transport of water at temperatures less than 0°C is generally described with theories outlined in the previous chapters, the hydraulic functions are greatly modified by the formation and presence of ice in soils.

In a frozen soil not far below 0°C we distinguish ice, liquid water and air within the mixture. The fundamental thermodynamic equation relates the relative Gibbs free energy to the freezing point as demonstrated in Fig. 5.20. Remembering that the definition of soil water potential is the equivalent of Gibbs free energy, the SWRC should be equivalent to the graph demonstrating

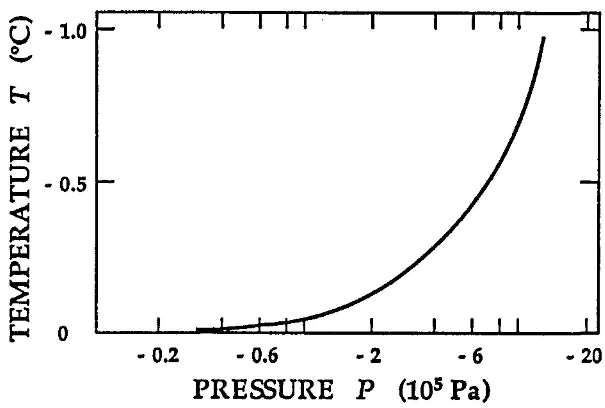


*Figure 5.20. Theoretical relation between the temperature of water freezing and the free energy expressed as negative pressure.*

the soil unfrozen water content $\theta_u(T)$. However, we must modify this simple consideration owing to the fact that frozen soils manifest varying degrees of rigidity. Soils rigid in their unfrozen state start to behave as non-rigid soils at $T < 0°C$. In the hydrostatics of frozen soils we must include a component in the total potential that accounts for the equilibrium ice pressure $p_i$. Moreover, the mechanism of the development of micro- and macro-lenses of ice is still not well known. Additionally, hysteretic concepts require modification owing to the deformation of the soil solid matrix and to the hysteretic behavior of liquid water and ice lenses. Hence, any final equilibrium state is substantially

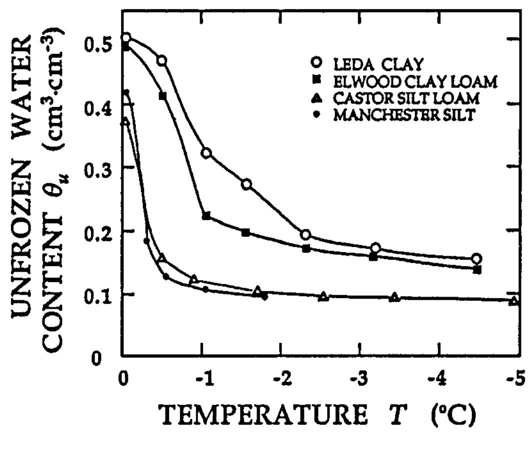


*Figure 5.21. Unfrozen water content $\theta_u$ for various soils as a function of temperature (Patterson and Smith, 1981).*

influenced by the path and direction of the temperature variation. Thus far, measurements of soil behavior show quantitative agreement with basic thermodynamic equations. For example, the content of unfrozen water $\theta_u$ decreases strongly with a drop of temperature below 0°C in the range of -0.5 to -2.0°C, see Fig. 5.21. With further decreases of temperature below this threshold range, the value of $\theta_u$ changes only slightly. Consequently, for the solution of problems of water flow below 0°C we need to directly measure $\theta_u(T)$. Additionally, the classical SWRC and $\theta_u(T)$ should be combined analogous to the procedure used for non-rigid soils at $T > 0$°C (Groenevelt and Kay, 1977). The temperature dependence of the hydraulic conductivity at $T < 0$°C is then similar to the water dependence of the unsaturated hydraulic conductivity function, roughly $K_u(\theta_u^3)$. Therefore, the steepest decrease of $K_u$ occurs at temperatures only a few tenths of a °C just below 0°C, see Fig. 5.22.

Unsteady flow in partially frozen, unsaturated soil is accompanied by many side phenomena such as formation of ice lenses, frost heaving etc. Hence, an objective formulation of the basic transport equations analogous to Richards' equation remains unresolved except for some well-defined simple (or simplified) problems. Present-day equations of mass transport which consider heat flow and heat consumption and release are considered with a combination of Darcian and microscopic scales.

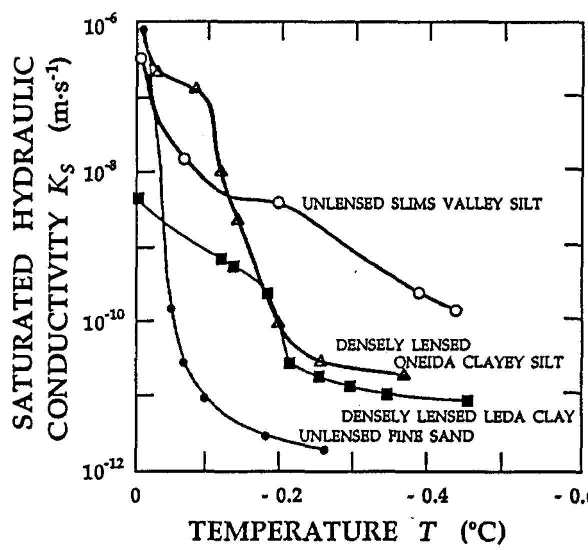


*Figure 5.22. Hydraulic conductivity for various soils as a function of temperature. Before the temperature was decreased, the soil was saturated with water (Burt and Williams, 1976).*

Problems

~~5.6 Non-isothermal flow~~

# PROBLEMS

1. The permeability of a soil $K_p = 0.3\ \mu m^2$. What is the water flux density when the potential head gradient *grad* $H = 2$?
2. In the previous problem, what is the equivalent pore radius in the soil approximated by Kozeny's model?
3. Formulate the functional relationship $K(\theta)$ if the model of the soil leads to $K_p$ described by (5.19)?
4. What is the direction of flow if the tensiometer readings of pressure head $h_p$ are: $h_p$ = -150 cm at soil depth z = 20 cm and $h_p$ = -140 cm at z = 50 cm?
5. Compute the flux density in the previous problem if $K(h)$ is expressed by (5.39) with $K_S = 20$ cm·day$^{-1}$ and $c = 0.05$ cm$^{-1}$.
6. If $K/K_S = \theta_E^m$ and $D = \alpha\theta_E^n$, what should be the relationship between $m$ and $n$ in order to obtain a physically realistic $\theta(h)$ retention curve? Interpret the physical meaning of the coefficient $\alpha$.
7. If $q_z > q_{1z}$ in Fig. 5.14, insert <, > or = for:
$$\frac{\partial q}{\partial z}\ldots 0 \qquad \text{and} \qquad \frac{\partial \theta}{\partial t}\ldots 0$$
Do the same if $q_z = q_{1z}$.
8. What are the dimensions of the terms in (5.62) and (5.71)?
9. For fluid mobilities $\lambda_A$ and $\lambda_W$ in (5.86) and (5.87), insert <,> or = in $\lambda_A \ldots \lambda_W$. Do the same for the conductivities at $\theta/\theta_S = 0.5$, $K_A \ldots K_W$.
10. Derive the dimensions in (5.86).
11. Derive the value of the Lagrangian coordinate $m$ for the compressed part of the unconsolidated soil at the wetting front in a swelling soil (see Fig. 5.19). How is $m$ related to $x$?
12. Consider that steady state water flow conditions exist in a 100-cm long horizontal column of homogeneous soil. A pressure head $h = 0$ is maintained at the left end of the column ($x = 0$) while $h$ = -100 cm is maintained at the other end. Calculate the direction and magnitude of the flux density and graph $h(x)$ for each of the following $K(h)$ functions: a. $K$ = 2, $K = (2 + 0.0199h)$, and $K = 2\exp(0.053h)$ cm·hr$^{-1}$.
13. Same as problem 12 except the column is in a vertical position with the pressure head $h = 0$ at the top ($z = 0$) and $h$ = -100 cm at the bottom of the column.
14. For problems 12 and 13, show that the product of $K$ and the derivative of h is everywhere along the column equal to a constant – the value of $K(h)$.
15. Consider that steady state water flow conditions exist in a vertical, homogenous soil column with a water table maintained at depth $z$ = 300 cm. Assuming that $K(h)$ is described by (5.38) with values of $a$, $b$ and $m$ being 200 cm$^3$·d$^{-1}$, 100 cm$^2$ and 2, respectively, calculate and graph $h(z)$ for evaporation rates of 0.0054, 0.005 0.002 and 0.001 cm·d$^{-1}$ and for infiltration rates of 0.005, 0.001, 0.01, 0.1 and 1 cm·d$^{-1}$. What is the maximum rate of evaporation when $h \to -\infty$?

To be shifted to page 218, after (=below) Problem 14 there.

## Extension of 5.3.2.

**Extension of Eq. (5.34) and the paragraph related to it:**

Brooks and Corey (1964) eq. is:

$$\frac{K}{K_S} = (\theta_E)^n \text{ where } n = 2/\lambda + 3 \text{ and } K/K_S = K_r \qquad (5.34a)$$

However, this equation is frequently called either Campbell's eq. (1974), or Clapp-Hornberger eq. (1978) with just a change of applied symbols, or with $\theta_r = 0$ in $\theta_E$. It is obvious that such a procedure is against scientific ethics. When fractal theories are applied, a general relationship is (Giménez et al., 1997)

$$K_r \propto \theta^n \text{ where } n = 2/(3 - D) + 2 + \alpha \qquad (5.34b)$$

Where $D$ is fractal dimension and $\alpha$ is a pore interaction parameter that accounts for pore connectivity and tortuosity of flow path. However, its value is fluctuating in great ranges and practically it is a fitting parameter again. An other development derived from SWRC by Fuentes et al. (1996)

$$\theta \propto h^{D-3} \qquad (5.34c)$$

equation for relative hydraulic conductivity

$$K_r \propto \theta^n \text{ with } n = 2/(3 - D) + 2D/3 \qquad (5.34e)$$

**Extension of Eq. (5.54) and the related equations:**

When fractal theories are applied, a general relationship is derived first for tortuosity and then for $K(h)$ relationships.

**Extension of Fig. 5.12 and the related discussion on p. 110:**

If just one matching point of $K_S$ is used for prediction of the unsaturated hydraulic conductivity $K$ in soil exhibiting bi-modal porosity, the difference between the directly measured $K$ and the computed $K$ according to Mualem-van Genuchten model is by one order of magnitude and even more, Fig. 5.12a. If two matching points are applied, i.e. $K_S$ for $K$ close to saturation and the measured $K$ at a certain $h$ , a substantial agreement is reached for measured and computed $K(h)$, see Fig. 5.12b.

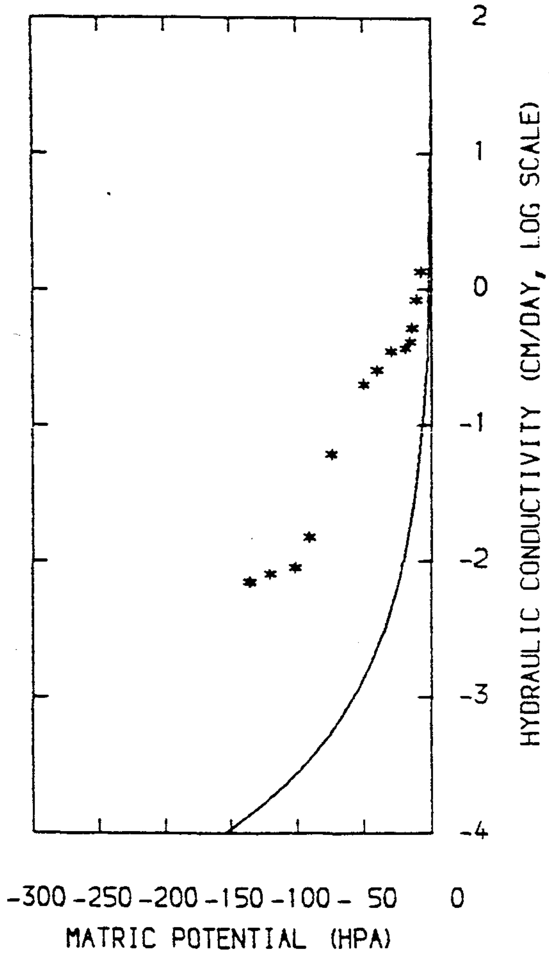


*Figure 5.12a: Unsaturated conductivity with $K_S$ as matching point*

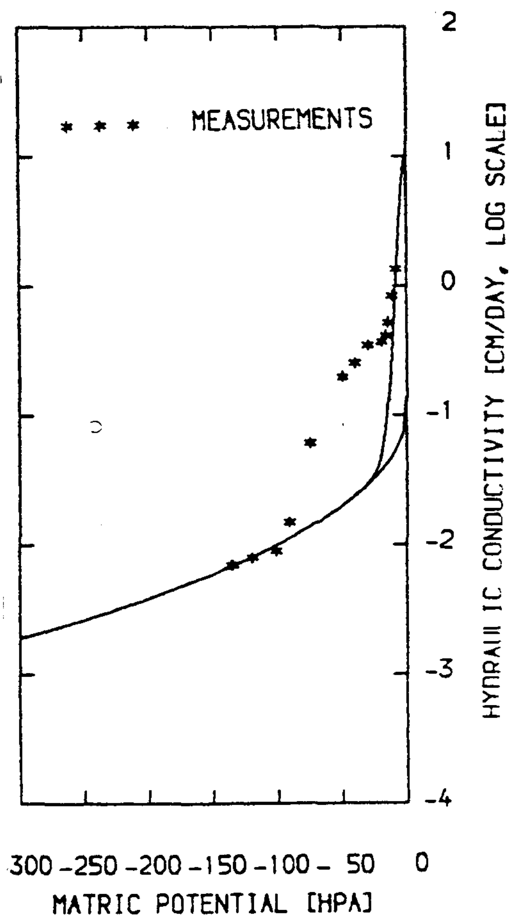


*Figure 5.12b: Unsaturated conductivity with two matching points*

Compaction of the soil results in the reduction of $K_S$ even by orders of magnitude due to the reduction of the volume of interaggregate pores, minimizing it often to zero. This reduction results not only in the decrease of $K_S$ , but unsaturated conductivity is reduced for the whole range of $h$ related to interaggregate porosity, generally for $h$ *between 0* and about −100 cm Compaction is permanent in layer below the tillage depth. See the Fig. 5.12c.

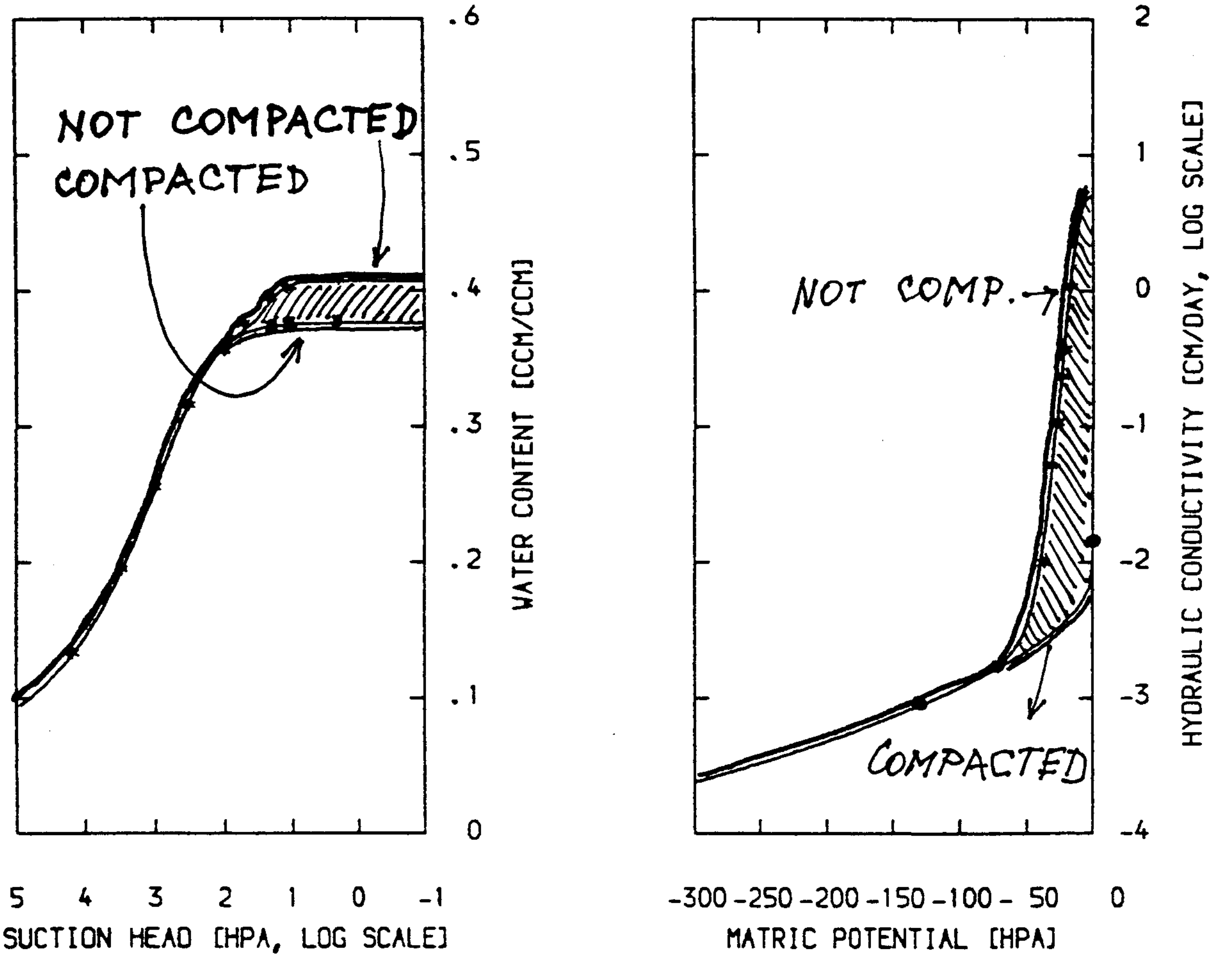


***Figure 5.12c. SWRC and unsaturated hydraulic conductivity in compacted and not compacted subsoil***

Unsaturated hydraulic conductivity function is not constant in various seazons, example is in next figure.

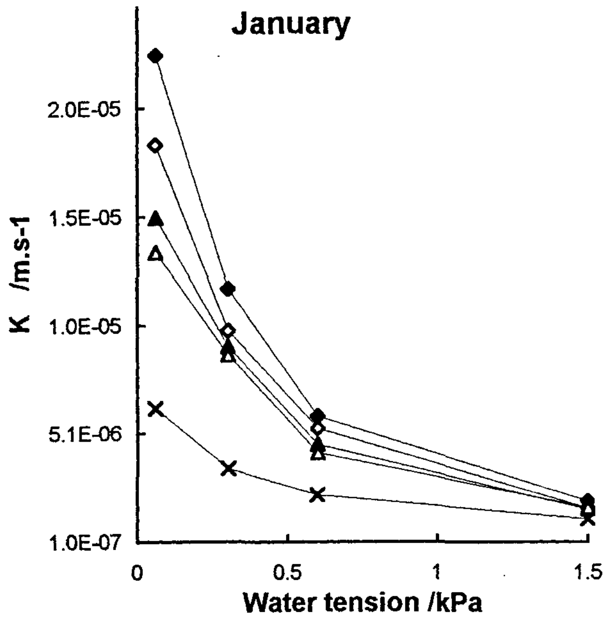


*Figure 5.12d: Seazonal variation of K in semiarid climate in soils with various cultivation methods (Kribaa et al, 2001).*