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SPRING SCHOOL ON SUPERSTRINGS AND RELATED MATTERS

NONCOMMUTATIVE SOLITONS

Part I

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Noncommutative Solitons – Trieste Lectures

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Abstract

These are pedagogical lectures on solitons in noncommutative field theories.

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Contents

1. The Context

Here we will try to provide the context in which the study of noncommutative field theories and their classical solutions assume importance.

The Importance of Open Strings: The understanding of the role of open strings in string theory via D-branes, has proven to be a development of overwhelming importance. This understanding was instrumental in correctly counting black hole microstates, one of the dramatic successes of string theory. One of the surprises was the manner in which a purely gravitational phenomenon, like black hole entropy, was described in terms of open strings, which at least classically don't contain closed string excitations like the graviton.

This connection between open and closed strings was sought to be further exploited in the Matrix theory proposal for a DLCQ description of M-Theory. But it's most striking manifestation was the AdS/CFT duality of Maldacena relating large N gauge theories to pure closed string theories. This conjecture is a reflection of an underlying duality between open and closed strings which is yet to be completely understood.

Decoupling limits: In the AdS/CFT duality, one takes a certain scaling limit of open string theories living on D-branes in which only the massless gauge theory modes survive and are described by a (super) Yang-Mills lagrangian. The massive open string states are effectively decoupled by taking the string scale to infinity. This scaling limit of open string theories is conjectured to describe pure closed strings propagating in the near horizon geometry of the D-branes. The fact that one can gain nontrivial information from studying a simple field theory limit of string theory has led one to examine more closely the various

decoupling limits of string theory. (Cf. Kutasov's lectures.) Taking decoupling limits of different sorts also help one to focus more sharply on various aspects of string theory. The idea is to retain enough complexity nevertheless being easier to analyse than the full theory.

Noncommutativity and String Theory: In parallel with these developments, and at first sight unrelated to it, is an ambitious program initiated by Sen which attempts, among other things, to understand closed strings in terms of open strings. The idea is to use the formulation of open string interactions in terms of a cubic string field theory as a complete description of string theory. This formulation relies on a representation of open string interactions which consists of gluing them in a fundamentally noncommutative way [1]. This defines an associative but noncommutative product of string fields in terms of which the string field action is expressed. D-branes are nontrivial classical solutions of this action while closed strings could arise as some kind of quantum excitations.

Since noncommutativity is in some sense intrinsic to string theory (not just a property of some backgrounds) and perhaps plays a crucial role in understanding the notions that replace classical geometry, it is worthwhile to try and understand it better. When one takes the conventional field theory limit of open string theory, the remnant of the noncommutativity is the rather trivial matrix algebra of the Chan-Paton indices. It does not involve the noncommutativity that comes from the extended nature of the open string.

Noncommutative Field theories: One might therefore ask if there is a limit of string theory which has the relative simplicity of keeping only a field theoretic number of degrees of freedom yet displaying the extended nature of a string. It turns out that the answer is yes. One can obtain a nonlocal deformation of field theories by taking a decoupling limit of open strings in a large magnetic field [2], [3]. The massive string modes decouple leaving a kind of elastic dipole object.

These resulting noncommutative field theories will be the main topic of these lectures.

Noncommutative solitons: More specifically, we will study the classical limit of these noncommutative field theories and find finite energy soliton solutions that have no counterpart in local field theories. Among the nice features of these solitons is that they are fairly universal and more or less insensitive to the details of the theory. They exhibit various novel features like nonabelian enhancement of symmetry when they are coincident.

In fact, these solitons are really the D-branes of string theory manifested in a field theory. This is somewhat surprising as it does not happen that you can find D-branes as finite energy excitations in a conventional field theory limit of string theory. The simplicity of noncommutative solitons implies that one can study many properties of D-branes very explicitly in this context.

Therefore the motivation for studying these solitons will be to use them as a simple set of probes of stringy behaviour in a well controlled manner. Much of the applications have been in the context of issues of tachyon condensation in open string theory. We can however also use these solitons to probe issues of how D-branes see space time, for instance.

Finally, the field theoretic aspects of these solitons are interesting in themselves and might perhaps have applications in very different contexts such as in the Quantum Hall effect.

2. Strings in a large Magnetic Field

As a prelude to studying strings in a large magnetic field, let us look at point particles in a large magnetic field.

The action for (nonrelativistic) point particles reads as

$$S = \int dt \left(\frac{1}{2} m \dot{x}_\mu \dot{x}^\mu + e B_{\mu\nu} x^\mu \dot{x}^\nu \right). \tag{2.1}$$

The conjugate momentum Π_μ to x^μ is

$$\Pi_\mu = m \dot{x}_\mu + e B_{\mu\nu} x^\nu. \tag{2.2}$$

In the limit where the energy $\omega \ll \frac{e|B|}{m}$, the canonical commutation relations become simply

$$[x^\mu, x^\nu] = i(B^{-1})^{\mu\nu} \frac{m}{e}. \tag{2.3}$$

Thus at energies much less than the cyclotron frequency $\frac{e|B|}{m}$, when one is in the lowest Landau level, one effectively has noncommuting coordinates. This is why the physics of the quantum hall effect displays some features of noncommutativity.

Now write the action for an open string in a constant magnetic field. We assume that the open string ends on a p brane in some of whose worldvolume directions the magnetic field is switched on.

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \left(g_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu - 2\pi i \alpha' B_{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \right). \tag{2.4}$$

The additional term involving B is really a boundary term which couples to the charges at the end of the open string like a constant magnetic field.

It leads to boundary conditions in the directions along the brane which are mixed.

$$(g_{\mu\nu} \partial_n X^\nu + 2\pi i \alpha' B_{\mu\nu} \partial_t X^\nu)|_{\partial\Sigma} = 0. \tag{2.5}$$

One can write down the Green's functions on the disc worldsheet with these boundary conditions. What we will need is the particular case when the X's are at on the boundary of the disc (parametrised by τ).

$$< X^\mu(\tau) X^\nu(\tau') >= -\alpha' G^{\mu\nu} \ln(\tau - \tau')^2 + \frac{i}{2} \Theta^{\mu\nu} \epsilon(\tau - \tau'). \tag{2.6}$$

Here

$$\begin{aligned} G^{\mu\nu} &= \left(\frac{1}{g + 2\pi\alpha' B} g \frac{1}{g - 2\pi\alpha' B} \right)^{\mu\nu} \\ \Theta^{\mu\nu} &= -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B} \right)^{\mu\nu} \end{aligned} \tag{2.7}$$

are usually called the open string metric and the noncommutativity parameter[3]. The open string metric is what determines the mass shell condition for open string states. Θ is called the noncommutativity parameter since the above OPE essentially implies that

$$[X^\mu(\tau), X^\nu(\tau)] = i\Theta^{\mu\nu}. \tag{2.8}$$

Note that Θ has dimensions of length^2.

There is one more ingredient, namely that the effective coupling of open string modes is also rescaled by a factor that depends on the magnetic field. We will not need the exact expression until later.

The noncommutativity parameter leads to an extra term in the OPE of vertex operators $e^{ik\cdot X}$:

$$e^{ik_1\cdot X}(\tau)e^{ik_2\cdot X}(\tau') \sim (\tau-\tau')^{2\alpha' G^{\mu\nu}k_{1\mu}k_{2\nu}}e^{-i\frac{1}{2}\Theta^{\mu\nu}k_{1\mu}k_{2\nu}}e^{i(k_1+k_2)\cdot X}(\tau') + \dots \tag{2.9}$$

The additional term $e^{-i\frac{1}{2}\Theta^{\mu\nu}k_{1\mu}k_{2\nu}}$ can be understood in position space as giving a nonlocal interaction which is expressed in terms of the Moyal product.

$$(f\star g)(x) = e^{i\frac{1}{2}\Theta^{\mu\nu}\partial_\mu\partial'_\nu}f(x)g(x')|_{x=x'}. \tag{2.10}$$

In general, there will be such a phase factor for all vertex operators implying that the effect of the magnetic field on the effective action in spacetime is completely captured by replacing all local products by the Moyal products, if we additionally remember to make all metric contractions with the open string metric.

We can now take the equivalent of the limit of a large magnetic field, namely take $\alpha'|B| \gg 1$. Here $|B|^2 = B_{\mu\nu}B_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$. We will in addition demand that this limit is taken keeping the open string metric $G^{\mu\nu}$ and $\Theta^{\mu\nu}$ finite. This requires taking the string scale to infinity ($\alpha' \to 0$). In the absence of the magnetic field this would mean decoupling all the massive string modes giving a field theory of the zero mode (if we keep the coupling constant finite).

With the magnetic field, as we have seen the only effect is to replace local products with the moyal product. The terms involving massive modes (both open and closed) then decouple for the same reason as in the case without a magnetic field. The lowest string modes then interact via a nonlocal deformation of ordinary field theory.

3. Scalar Noncommutative Solitons

With these motivations we will start our study of semiclassical noncommutative field theories. The simplest example is a theory of a single scalar field in $2+1$ dimensions with noncommutativity in the two spatial directions. We will parametrize the spatial R^2 by complex coordinates $z, \bar{z}$. The energy functional

$$E = \frac{1}{g^2}\int d^2z\,(\partial_z\phi\partial_{\bar{z}}\phi + V(\phi)_\star)\,, \tag{3.1}$$

where $d^2z = dxdy$. Fields in the action are multiplied using the Moyal star product (which reads in complex form as),

$$(f\star g)\,(z,\bar{z}) = e^{\frac{\theta}{2}(\partial_z\partial_{\bar{z}'}-\partial_{z'}\partial_{\bar{z}})}f(z,\bar{z})g(z',\bar{z}')|_{z=z'}. \tag{3.2}$$

Note that since $\int f\star g = \int fg$, the moyal product drops out of the quadratic term in the action.

Before we look for classical solutions to this action, let us recall that the scalar theory without noncommutativity does not have any lump solutions. This is actually true for any bounded potential in spatial dimension greater than one, and follows from a simple scaling argument of Derrick [4]. If $\phi_0(x)$ be an extremum of the energy functional (3.1) (with $\theta = 0$), then consider the energy of the field configurations $\phi_\lambda(x) = \phi_0(\lambda x)$.

$$\begin{aligned} E(\lambda) =& \frac{1}{g^2}\int d^Dx\left(\frac{1}{2}(\partial\phi_0(\lambda x))^2 + V(\phi_0(\lambda x))\right) \\ =& \frac{1}{g^2}\int d^Dx\left(\frac{1}{2}\lambda^{2-D}(\partial\phi_0(x))^2 + \lambda^{-D}V(\phi_0(x))\right). \end{aligned} \tag{3.3}$$

Since $\phi_0(x)$ is an extremum, we require $\frac{\partial E(\lambda)}{\partial\lambda}|_{\lambda=1} = 0$. that is,

$$\int d^Dx\left(\frac{1}{2}(D-2)(\partial\phi_0(x))^2 + DV(\phi_0(x))\right) = 0$$

. For spatial dimension $D \geq 2$, for a potential bounded from below by zero, the only way this can be true is for the kinetic and the potential terms to separately vanish. There are therefore no nontrivial configurations. Note that this argument fails once one includes higher derivative terms.

We now seek finite energy (localized) solitons of (3.1) for nonzero θ. Since no solutions exist for $\theta = 0$ (3.3) , we begin our search in the limit of large noncommutativity, $\theta \to \infty$. It is useful to non-dimensionalize the coordinates $z \to z\sqrt{\theta}$, $\bar{z} \to \bar{z}\sqrt{\theta}$. As a result, the $\star$ product will henceforth have no θ; i.e. it will be given by (3.2) with $\theta = 1$. Written in rescaled coordinates, the dependence on θ in the energy is entirely in front of the potential term:

$$E = \frac{1}{g^2} \int d^2z \left(\frac{1}{2}(\partial\phi)^2 + \theta V(\phi)_\star \right) \tag{3.4}$$

In the limit $\theta \to \infty$, with V held fixed, the kinetic term in (3.4) is negligible in comparison to $V(\phi)$, at least for field configurations varying over sizes of order one in our new coordinates.

Our considerations apply to generic potentials $V(\phi)$, but we will, for definiteness, mostly discuss those of polynomial form

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \sum_{j=3}^{r} \frac{b_j}{j}\phi^j. \tag{3.5}$$

3.1. Solutions in the $\theta = \infty$ Limit

After neglecting the kinetic term, the energy

$$E = \frac{\theta}{g^2} \int d^2z V(\phi)_\star, \tag{3.6}$$

is extremised by solving the equation

$$\left(\frac{\partial V}{\partial \phi} \right)_\star = 0. \tag{3.7}$$

For instance, for a cubic potential one has to solve an equation of the form

$$m^2\phi + b_3 \phi \star \phi = 0 \tag{3.8}$$

If $V(\phi)$ were the potential in a commutative scalar field theory, the only solutions to (3.7) would be the constant configurations

$$\phi = \lambda_i, \tag{3.9}$$

where $\lambda_i \in \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ are the various real extrema of the function $V(x)$. The derivatives in the definition of the star product allow for more interesting solutions of (3.7).

In order to find all solutions of (3.7) we will exploit the connection between Moyal products and quantization. Given a C^∞ function $f(q,p)$ on R^2 (thought of as the phase space of a one-dimensional particle), there is a prescription which uniquely assigns to it an operator $O_f(\hat{q},\hat{p})$, acting on the corresponding single particle quantum mechanical Hilbert space, $\mathcal{H}$. It is convenient for our purposes to choose the Weyl or symmetric ordering prescription

$$O_f(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2} \int d^2k \tilde{f}(k) e^{-i(k_q\hat{q}+k_p\hat{p})}, \tag{3.10}$$

where

$$\tilde{f}(k) = \int d^2x e^{i(k_q q + k_p p)} f(q,p), \tag{3.11}$$

and

$$[\hat{q},\hat{p}] = i. \tag{3.12}$$

With this prescription, it may be verified that

$$\frac{1}{2\pi} \int dp dq f(q,p) = \mathrm{Tr}_{\mathcal{H}} O_f, \tag{3.13}$$

and that the Moyal product of functions is isomorphic to ordinary operator multiplication

$$O_f \cdot O_g = O_{f\star g}. \tag{3.14}$$

In order to solve any algebraic equation involving the star product, it is thus sufficient to determine all operator solutions to the equation in $\mathcal{H}$. The functions on phase space corresponding to each of these operators may then be read off from (3.10). We will now employ this procedure to find all solutions of (3.7).

It is easy to see that $O = \lambda_i P$ is a solution to $V'(O) = 0$, if P is an arbitrary projection operator on some subspace of $\mathcal{H}$ and if λ_i is an extremum of $V(x)$. The energy of this solution is, using (3.13) ,

$$E = \frac{2\pi\theta}{g^2} \mathrm{Tr} V(O_\phi) = \frac{2\pi\theta}{g^2} V(\lambda_i) \mathrm{Tr} P. \tag{3.15}$$

Thus the energy is finite if P is projector onto a finite dimensional subspace of $\mathcal{H}$.

In fact, you can convince yourself that the most general solution to (3.7) takes the form

$$O = \sum_j a_j P_j \tag{3.16}$$

where $\{P_j\}$ are mutually orthogonal projection operators onto one dimensional subspaces,

$$P_i P_j = \delta_{ij} P_j; \qquad Tr_{\mathcal{H}} P_i = 1, \tag{3.17}$$

with a_j taking values in the set $\{\lambda_i\}$ of real extrema of $V(x)$.

From now on we will restrict ourselves to a potential with one nontrivial minimum λ other than the one at the origin.

We have a huge infinity of solutions of the form λP. To see what they mean, let us translate them into position space. It will be convenient for this purpose to choose a particular basis in $\mathcal{H}$. Let $|n\rangle$ represent the energy eigenstates of the one dimensional harmonic oscillator whose creation and annihilation operators are defined by

$$a = \frac{\widehat{q} + i\widehat{p}}{\sqrt{2}}; \quad a^\dagger = \frac{\widehat{q} - i\widehat{p}}{\sqrt{2}}. \tag{3.18}$$

Note that $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Any operator may be written as a linear combination of the basis operators $|m\rangle\langle n|$'s, which, in turn, may be expressed in terms of a and $a^\dagger$ as

$$|m\rangle\langle n| =: \frac{a^{\dagger m}}{\sqrt{m!}} e^{-a^\dagger a} \frac{a^n}{\sqrt{n!}} : \tag{3.19}$$

where double dots denote normal ordering. We will first describe operators of the form (3.16) that correspond to radially symmetric functions in space. As $a^\dagger a \approx \frac{r^2}{2}$, operators corresponding to radially symmetric wavefunctions are functions of $a^\dagger a$. From (3.19), the only such operators are linear combinations of the diagonal projection operators $|n\rangle\langle n| = \frac{1}{n!} : a^{\dagger n} e^{-a^\dagger a} a^n :$. Hence all radially symmetric solutions of (3.7) correspond to operators of the form $O = \lambda \sum a_n |n\rangle\langle n|$, where the numbers a_n can take values 0 or 1.

It is not difficult to translate these operators back to position space [5]. One finds

$$|n\rangle\langle n| = \frac{1}{(2\pi)} \int d^2k e^{\frac{-k^2}{4}} L_n(\frac{k^2}{2}) e^{-i(k_{\bar{z}} a + k_z a^\dagger)} \tag{3.20}$$

where $L_n(x)$ is the n^{th} Laguerre polynomial. The field $\phi_n(x, y)$ that corresponds to the operator $O_n = |n\rangle\langle n|$ is, therefore,

$$\phi_n(r^2 = x^2 + y^2) = \frac{1}{(2\pi)} \int d^2k e^{\frac{-k^2}{4}} L_n(\frac{k^2}{2}) e^{-ik.x} = 2(-1)^n e^{-r^2} L_n(2r^2). \tag{3.21}$$

Note that $\phi_0(r^2)$ is the simple gaussian $2e^{-r^2}$. In summary, (3.7) has an infinite number of real radial solutions, given by

$$\sum_{n=0}^{\infty} a_n \phi_n(r^2) \tag{3.22}$$

where $\phi_n(r^2)$ is given by (3.21) and each a_n takes values either 0 or 1. These solutions will have finite energy if only a finite number of the a_n are nonzero, as is evident from (3.15).

We also see from (3.15) that the action at $\theta = \infty$ has a large symmetry $O_\phi \to U O_\phi U^\dagger$, where U is any unitary operator acting on $\mathcal{H}$. This $U(\infty)$ global symmetry generates new nonradially symmetric solutions out of the radially symmetric ones. The most general projection operator $O = \lambda P$, of rank k, is unitarily related to a projection operator which is diagonal (in the SHO basis), that is of the form $\lambda(\sum_{i=0}^{k-1} |i><i|)$. And the corresponding solutions are all degenerate in energy. In fact, their energy $E = \frac{2\pi k\theta}{g^2} V(\lambda)$ is k times the energy of the minimal energy soliton $k = 1$. This suggests an interpretation as k solitons which will become clearer as we proceed.

It is remarkable that the energy of the soliton is completely insensitive to the value of the scalar potential at any point except $\phi = \lambda$. Thus the mass of the soliton is unchanged if the height of the barrier in $V(\phi)$ (between $\phi = \lambda$ and $\phi = 0$, see Fig. 4.) is taken to infinity while $V(\lambda)$ is kept fixed. This is true even though $\phi_0(r)$, the solitonic field configuration corresponding to $\lambda|0\rangle\langle 0|$, decreases continuously from $\phi = 2\lambda$ at $r = 0$ to $\phi = 0$ at $r = \infty$! It is also striking that the form of the solutions themselves are remarkably universal too, more or less independent of the details of the potential.

3.2. Stability and Moduli Space at $\theta = \infty$

Because of the $U(\infty)$ symmetry it suffices to examine the stability of radial solutions of the form

$$\phi(r^2) = \lambda \sum_{n=0}^{k-1} \phi_n(r^2) \tag{3.23}$$

to small fluctuations. Since any $U(\infty)$ rotation does not change the energy of our solution (3.23), it is sufficient to study the stability to radially symmetric fluctuations. These are most conveniently parameterized as deformations of the eigenvalues. The energy for an arbitrary radially symmetric state $\phi(r^2) = \sum_{n=0}^{\infty} c_n \phi_n(r^2)$ is

$$E = \frac{2\pi\theta}{g^2} \sum_{n=0}^{\infty} V(c_n).$$

The solutions with $c_n \in \{\lambda, 0\}$ are manifestly local minima of E, as λ and 0 are minima of the function $V(x)$. Thus the solution of the form (3.23) (and all solutions unitarily related to it) are stable to small fluctuations. (If any of the c_n took the value of a local maximum of $V(x)$, then it is equally easy to see that while the corresponding $\phi(r^2)$ would be a solution to (3.7) it is not stable to small radial fluctuations.)

The stability of the gaussian soliton $\lambda\phi_0(r^2)$ may qualitatively be understood as follows. Since $\lambda\phi_0(r^2) = 2\lambda e^{-r^2}$ is a Gaussian of height 2λ, far away from the origin, $\phi_0(x) = 0$, but near $x = 0$, it is in the vicinity of the second vacuum. In other words, the static solution corresponds to a bubble of the "false" vacuum. The area of the bubble is of order one (or θ in our original coordinates), the non-commutativity scale. In a commutative theory such a bubble would decay by shrinking to zero size. Noncommutativity prevents the bubble from shrinking to a spatial size smaller than $\sqrt{\theta}$. In order to decay, ϕ_0 actually has to scale to zero - but that process involves going over the hump in the potential and so is classically forbidden.

The $U(\infty)$ symmetry of (3.7) results in there being an infinite number of zero modes for a given solution with energy $2\pi k V(\lambda)$. This infinite dimensional moduli space can be mathematically characterised as follows. The rank

k hermitian projection operators on $\mathcal{H}$ (or equivalently, the k-dimensional hyperplanes in $\mathcal{H}$) form a manifold known as the Grassmannian $\mathrm{Gr}(k, \mathcal{H})$, which can also be described as the coset space

$$\frac{\mathrm{U}(\infty)}{\mathrm{U}(k) \times \mathrm{U}(\infty - k)}, \tag{3.24}$$

where $\mathrm{U}(\infty)$ acts on the entire space, while $\mathrm{U}(\infty - k)$ acts only on the orthogonal complement of a k-dimensional hyperplane.

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