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SPRING SCHOOL ON SUPERSTRINGS AND RELATED MATTERS

INTRODUCTION TO LITTLE STRING THEORY

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Introduction to Little String Theory

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1. Introduction

Much has been learned over the years by studying string dynamics near various kinds of "impurities." Examples include string propagation on orbifolds [1], where one finds "twisted sectors" corresponding to fundamental strings trapped at the orbifold singularities, and vacua with D-branes which contain localized excitations corresponding to open strings ending on the branes.

In both of these examples, the states localized at the impurity couple to the bulk – *e.g.* two open strings ending on a D-brane can fuse into a closed string that can leave the brane. It is sometimes possible to decouple the physics of the localized modes from bulk dynamics by taking a low energy limit, $E << m_s$, where $m_s = 1/\sqrt{\alpha'}$ is the string scale, associated with the tension of the fundamental string $T = 1/2\pi\alpha'$.

Whenever this limit gives rise to an interacting theory, it corresponds to a local quantum field theory (QFT), such as the non-abelian gauge theories found on branes. This embedding of field theoretic dynamics into string theory led in recent years to many insights into field theory and string theory (see *e.g.* [2,3] for reviews).

The purpose of these lectures is to describe another class of impurities – Neveu-Schwarz fivebranes [4], or equivalently singularities of Calabi-Yau manifolds and other spaces[1]. One of the striking features of the dynamics of $NS5$-branes is that it can be decoupled from the bulk without taking the low energy limit $\alpha' \to 0$. The decoupled theory of $NS5$-branes is known as Little String Theory[2] (LST). It has the following properties:

(1) The theory is non-local. For example, upon compactification on tori, LST exhibits T-duality.

(2) It has a Hagedorn density of states at high energies, $\rho(E) \sim E^\alpha \exp(\beta_H E)$.

(3) The theory can be defined in six or fewer spacetime dimensions. It has super – Poincare invariant vacua with sixteen or fewer supercharges.

[1] Orbifolds are examples of such singularities, but in [1] they are in fact resolved by a finite expectation value of a non-gravitational modulus – the B field [5]. We will be interested below in situations where this v.e.v. is zero or at least very small.

[2] A name due to [6].

(4) LST is a non-gravitational theory: there is no massless spin two particle in the spectrum.

(5) The theory appears to have well defined off-shell Green functions, unlike (closed) critical string theory, where it is believed that only on-shell observables can be studied.

Note that while properties (1) and (2) are reminiscent of critical string theory, properties (3), (4) and (5) are different in the two cases.

The main purpose of these lectures is to describe in more detail some of the above properties and the techniques that were used to study them. Most of these results were obtained by using holography, and this is the approach that will be followed here. In particular, I will not describe an alternative approach to LST based on a discrete light-cone quantization (DLCQ) of the theory, which utilizes a certain $1+1$ dimensional sigma model [7,8,9]. For a review of that approach and LST in general as of 1999, see [10].

There are several reasons why I think LST is of some interest. Among them:

(1) In most (compactified) supersymmetric string theories one finds moduli spaces of vacua. For generic values of the moduli the perturbative description is non-singular, but one can often tune the moduli so that a singularity appears somewhere on the compact manifold. The dynamics near the singularity is described by LST. Thus LST is part of the dynamics of rather conventional looking string vacua at special points in the moduli space. Furthermore, when supersymmetry is broken, it is possible that the theory is dynamically driven to such singular points in moduli space.

(2) LST is relevant for the study of strongly coupled gauge theories, which can be realized on $NS5$-branes wrapped around Riemann surfaces or D-branes stretched between fivebranes (see [2] for a review). There are also applications to matrix theory [11], which in fact provided some of the original motivation for the construction of this theory [12,13].

(3) It was recently proposed that LST might be phenomenologically relevant for brane world scenarios with a relatively low string scale [14].

More generally, LST appears to be a structure that is intermediate in complexity between local QFT and critical string theory. It has the non-locality and

Hagedorn spectrum characteristic of critical string theory, but not the complications associated with gravity. A better understanding of its structure might shed light on string theory, strongly coupled gauge theory (QCD strings), holography and other matters.

The plan of these lectures is as follows. We start in section 2 by describing the limit in which the dynamics of $NS5$-branes decouples from bulk physics. In section 3 we discuss the holographic description of this limit and some of the properties of LST mentioned above. In particular, we exhibit some of the observables and the physical states of the theory.

In section 4 we discuss the high energy thermodynamics of LST. We show that the spectrum has a Hagedorn growth and compute the Hagedorn temperature and the first subleading term in the entropy which shows that the thermodynamics is unstable. In section 5 we introduce and study a class of vacua of LST which can be analyzed in a controlled weak coupling expansion.

2. The decoupling limit of flat $NS5$-branes

Consider a vacuum of type II string theory which contains N parallel $NS5$-branes, which are extended in the directions $(x^1, \cdots, x^5)$ and are pointlike in $(x^6, \cdots, x^9)$. We will initially take the fivebranes to be at the same point and will later examine the deformations that separate them in the directions $(6,7,8,9)$.

The presence of the fivebranes breaks the Lorentz symmetry:

$$SO(9,1) \to SO(5,1) \times SO(4) \tag{2.1}$$

¿From the fivebrane worldvolume point of view, $SO(5,1)$ is the Lorentz symmetry, while $SO(4)$ is an R symmetry. The fivebranes also break half of the supersymmetry, reducing the number of unbroken supercharges from thirty two to sixteen. In terms of six dimensional supersymmetry along the fivebranes, IIA fivebranes preserve a chiral $(2,0)$ supersymmetry[3], while IIB fivebranes preserve $(1,1)$ supersymmetry.

[3] I.e. two complex supercharges in the **4** of Spin(5,1).

Since $NS5$-branes are dynamical objects, like D-branes, one expects to find a rich spectrum of excitations on the branes. To decouple the dynamics on the fivebranes from the bulk, consider the limit

$$g_s \to 0; \quad \frac{E}{m_s} = \text{fixed} \tag{2.2}$$

Processes in which modes that live on the fivebranes are emitted into the bulk as closed strings are suppressed in this limit, since the corresponding amplitudes are proportional to g_s and thus go to zero. At the same time, the dynamics on the $NS5$-branes does not become free in this limit. One way to see this is to consider the low energy limit of the resulting theory and to show that it is not free.

Consider first the low energy limit of N $NS5$-branes in type IIB string theory. S-duality relates this to N $D5$-branes; thus the low energy theory is a six dimensional gauge theory with $(1,1)$ supersymmetry and gauge group $U(N)$. The gauge coupling of the theory on the $D5$-branes is

$$\frac{1}{g_D^2} = \frac{m_s^2}{g_s} \tag{2.3}$$

Using the transformation of g_s and m_s under S-duality one finds that the gauge coupling on the $NS5$-branes is

$$\frac{1}{g_N^2} = m_s^2 \tag{2.4}$$

Thus in the limit (2.2) the gauge coupling remains fixed. Since the gauge theory in question is non-renormalizable, the gauge coupling g_N in fact changes with the scale, approaching zero at long distances and growing at short distances. At energies of order m_s the gauge theory description breaks down and more data needs to be supplied to define the theory. As we will see, there are in fact additional degrees of freedom in the theory at (roughly) that scale, and the full density of states is much larger than that in any local QFT. At any rate, since the dynamics at scales $E < m_s$ is not free, the full theory must be interacting.

Note that the above arguments are only valid for $N > 1$ fivebranes. The low energy theory on a single $NS5$-brane *is* free. Indeed, we will see later that LST is interacting only for $N > 1$.

The infrared dynamics of N IIA $NS5$-branes is more involved. One finds in this case a non-trivial IR fixed point with $(2,0)$ superconformal symmetry [15]. To see that something special is happening in the IR imagine separating the fivebranes in the $(6,7,8,9)$ directions. In the IIB theory, one then finds D-strings stretched between the fivebranes; their masses go to zero as the fivebranes approach each other. The resulting massless states are the off-diagonal $U(N)$ gauge bosons on the fivebranes.

The analogous process for IIA involves $D2$-branes stretched between the fivebranes. The ends of the $D2$-branes are strings bound to the fivebranes. Their tension goes to zero when the fivebranes coincide [16]. These tensionless strings signal the interacting nature of the low energy limit of the IIA fivebrane theory – the $(2,0)$ superconformal field theory.

Thus, we conclude that the limit (2.2) corresponds to an interacting theory on the $NS5$-branes decoupled from the bulk. What sort of theory is it? Already at the level of the present discussion there are a few hints of non-local/stringy behavior. Let us mention two:

(1) T-duality: Compactify some or all of the dimensions $(1,2,3,4,5)$ on circles. $NS5$-branes are known to transform to themselves under T-duality along their worldvolume. Since the limit (2.2) commutes with T-duality, inversion of the radius of a single circle ($R \to 1/m_s^2 R$) exchanges the IIA and IIB LST's, while inversion of an even number of radii is a symmetry of the theory.

(2) The theory contains strings with tension $T = 1/2\pi\alpha'$, which can be interpreted as fundamental strings bound to the fivebranes. In the IIB case[4], these strings can be constructed in the low energy gauge theory as instanton solutions, which are extended (say) in $(0,1)$ and localized in $(2,3,4,5)$. The tension of these strings is proportional to the instanton action, $1/g_N^2$, which indeed (2.4) is the fundamental string tension. Of course, this construction gives rise to long strings, and it is not clear what are the properties of short strings which actually govern the dynamics, but it suggests that LST is a theory of strings. Later we will see further evidence that supports this.

[4] A similar construction can be performed in the IIA case.

It is instructive to compare the decoupling limit (2.2) with the limits studied in D-brane physics. Usually, to decouple the physics of D-branes from the bulk one considers the low energy limit

$$\frac{E}{m_s} \to 0; \quad g_s = \text{fixed} \tag{2.5}$$

and the decoupling from the bulk is the standard low energy decoupling of QFT from gravity. In contrast, the limit (2.2) for D-branes gives rise in general to a free theory on the branes, since g_s determines both the open and the closed string couplings.

A limit for N D-branes which is more analogous to (2.2) is

$$g_s \to 0; \quad \lambda = g_s N = \text{fixed}; \frac{E}{m_s} = \text{fixed} \tag{2.6}$$

The open string coupling λ is fixed; hence the theory on the D-branes remains interacting. Since $g_s \to 0$, the closed string sector decouples, despite the fact that a low energy limit has not been taken. The resulting theory is an open string theory without closed strings; it has some things in common with LST although there are differences as well.

3. A holographically dual description of LST

The construction described in the previous section is useful for establishing the existence of LST, but it does not provide efficient techniques for studying the theory. To proceed, we will use a holographically dual description proposed in [17] (see also [18,19]). This duality is a generalization of the AdS/CFT correspondence [3]; it postulates that LST is equivalent to ten dimensional string theory in the background of the fivebranes, in the limit (2.2). In this section I will describe the fivebrane geometry and will briefly discuss the duality of [17].

The geometry, dilaton and NS B-field around N $NS5$-branes in type II string theory are [4]:

$$\begin{aligned} ds^2 &= dx_\mu dx^\mu + (1+\frac{N\alpha'}{r^2})dx^i dx^i \\ e^{2\Phi} &= g_s^2(1+\frac{N\alpha'}{r^2}) \\ H_{ijk} &= -\epsilon_{ijkl}\partial^l\Phi \end{aligned} \tag{3.1}$$

where $\mu = 0,1,2,\cdots,5$ are worldvolume coordinates and $i,j,k,l = 6,7,8,9$ are transverse ones.

To take the limit (2.2) one must send $r \to 0$ at the same rate as g_s. Defining $r = g_s \exp\sigma$ we have in this limit

$$\begin{aligned} ds^2 &= dx_\mu dx^\mu + N\alpha'(d\sigma^2 + d\Omega_3^2) \\ \Phi &= -\sigma \end{aligned} \tag{3.2}$$

and we suppress the B-field (3.1). String propagation in this geometry corresponds to an "exact conformal field theory" [4]:

$$\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times SU(2)_N \tag{3.3}$$

$\mathbb{R}^{5,1}$ is the worldvolume of the fivebranes. $\mathbb{R}_\phi$ is the real line labeled by $\phi = \sqrt{N\alpha'}\sigma$. The dilaton goes like (3.2):

$$\Phi = -\frac{Q}{2}\phi;\ \ Q = \frac{2}{\sqrt{N\alpha'}} \tag{3.4}$$

The last factor in (3.3) describes the angular three-sphere in (3.2). The B-field (3.1) is precisely such that the CFT on the three-sphere, whose radius is

$$R_{\rm sphere} = \sqrt{N\alpha'} \tag{3.5}$$

is described by a level N WZW model. We see that the number of fivebranes N determines the slope of the linear dilaton, Q, and the level of $SU(2)$ current algebra. More precisely, since (3.3) is a background for the superstring, the worldsheet theory contains ten free fermions: ψ^μ, $\mu = 0,1,2,\cdots,5$, the superpartners of x^μ; ψ^i, $i = 3,+,-$, the superpartners of the $SU(2)$ currents J^i; and ψ^ϕ, the superpartner of ϕ. The total level N of the $SU(2)$ current algebra receives a contribution of $N-2$ from the worldsheet bosons, and 2 from the fermions ψ^i, which transform in the adjoint of the total $SU(2)$ current algebra. The total central charge of the worldsheet theory (3.3) is

$$(6+\frac{1}{2}\times 6) + (1+\frac{6}{N}+\frac{1}{2}) + (\frac{3(N-2)}{N} + 3\times\frac{1}{2}) = 15 \tag{3.6}$$

which is the correct value for the superstring.

The background (3.3) is thus expected to be holographically dual to the LST on the fivebranes. We next discuss some features of this duality. First note that while the string coupling (3.4) vanishes far from the fivebranes (*i.e.* as $\phi \to \infty$), it diverges as one approaches the branes ($\phi \to -\infty$, or $r \to 0$ in (3.1)). The $NS5$-branes have the remarkable property that quantum effects near the branes cannot be turned off no matter how small the string coupling is far from the branes [4]. This makes it clear that LST is not a free theory[5], as argued above, but it raises the question whether one can analyze the physics of the string background (3.3), (3.4) perturbatively. We will return to this question below.

As is familiar from the AdS/CFT correspondence, on-shell observables in the "bulk" theory – string theory on (3.3) – correspond to off-shell observables in the "boundary" theory – the LST corresponding to N $NS5$-branes. More precisely, off-shell observables in LST correspond to *non-normalizable* observables in string theory on (3.3), whose wavefunctions are supported near the "boundary" at $\phi \to \infty$. This can be explained as follows (in analogy with the AdS case).

Consider (say) a scalar field on the manifold (3.3). As $\phi \to \infty$, its wavefunction $\Psi(\phi, x^\mu)$ (assuming for simplicity a profile constant on the angular S^3) behaves as:

$$\Psi(\phi, x^\mu) \sim \sum_k C_k e^{\lambda_k\phi} e^{ik_\mu x^\mu} \tag{3.7}$$

[5] For $N \geq 2$ fivebranes. Note that for $N = 1$, the bosonic $SU(2)$ current algebra has formally a negative level, $N - 2 = -1$, and the construction breaks down.

where

$$\lambda_k^2 = k_\mu k^\mu + C \tag{3.8}$$

C is a constant which depends on the mass of the scalar field and on N. Choosing the positive root of (3.8), we see that the mode (3.7) is non-normalizable and thus the coefficients C_k do not fluctuate – they are not integrated over in the process of integrating over all field configurations in the path integral [20]. Thus, we can think of the C_k as fixed sources. The string partition sum with the fixed boundary conditions (3.7) as $\phi \to \infty$, $Z_{\rm bulk}(C_k)$, can be interpreted as the generating functional of off-shell Green functions in the six dimensional LST via:

$$Z_{\rm bulk}(C_k) = \langle \exp\left(-\sum_k C_k \theta(k)\right)\rangle_{LST} \tag{3.9}$$

where $\theta(k_\mu)$ is the off-shell observable which couples to the source C_k. More qualitatively, modes that are non-normalizable in the "near-horizon" geometry (3.3) are nothing but bulk modes in the full geometry (3.1); they are supported at finite r. Thus, they are not part of the LST but rather are fixed background sources (in the limit (2.2)), which couple to the brane modes via couplings like (3.9).

Similarly, *normalizable modes* in the geometry (3.3) correspond to *states* in LST, since in the full geometry (3.1) they correspond to modes localized at the fivebranes (*i.e.* at $r \to 0$). To illustrate all this, we next give an example each of off-shell observables and states in LST, as described in the holographically dual picture.

3.1. Example 1: Chiral operators in LST

As discussed above, the low energy limit of IIB LST is a $U(N)$ gauge theory with $(1,1)$ supersymmetry. This theory contains four scalar fields in the adjoint of $SU(N)$, X^i, $i = 6,7,8,9$, which parametrize the locations of the N fivebranes in $(6,7,8,9)$. The gauge invariant off-shell operators

$$\mathrm{Tr} X^{i_1} X^{i_2} \cdots X^{i_n}; \quad n = 2,3,4,\cdots N \tag{3.10}$$

where we only take the completely symmetric and traceless combination in $(i_1, \cdots, i_n)$, are lowest components of short multiplets of supersymmetry. Writing the $SO(4)$ symmetry in (2.1) as

$$SO(4) \simeq SU(2)_L \times SU(2)_R \tag{3.11}$$

the operators (3.10) transform in the spin $(n-1, n-1)$ representations. In string theory on (3.3) these chiral operators are described as follows. The $SU(2)_L \times SU(2)_R$ symmetry on (3.11) corresponds to the left and right moving $SU(2)$ symmetries in the $SU(2)_N$ WZW model in (3.3). Physical primaries of this symmetry are $V_{j;m,\bar{m}}$ with the same spin $(2j = 0,1,2,\cdots, N-2)$ under both $SU(2)$'s. $(m, \bar{m})$ are the eigenvalues of $(J_3, \bar{J}_3)$.

The lowest lying observables have the form (in the -1 picture)

$$\xi_{\alpha\beta}\psi^\alpha \bar{\psi}^\beta e^{\beta\phi} e^{ik_\mu x^\mu} \tag{3.12}$$

where $\alpha, \beta = 0,1,2,\cdots 9$ and $\xi_{\alpha\beta}$ is a polarization tensor satisfying the usual physical state conditions. One can show that (3.10) correspond to[6]

$$(\psi\bar{\psi}V_j)_{j+1} e^{\frac{2j}{\sqrt{N\alpha'}}\phi} \tag{3.13}$$

where ψ stands for the three fermions associated with the $SU(2)$ WZW and the brackets mean that ψ, which has spin 1 under $SU(2)_L$, is coupled with V_j into a spin $j+1$ combination (and similarly for the right movers). Thus, the non-normalizable operators (3.13) transform as

$$(j+1, j+1); \quad 2j = 0,1,2,\cdots, N-2 \tag{3.14}$$

in exact agreement with what was found for (3.10) above. Applying the spacetime supercharges gives the other members of the supermultiplets. Thus, the sets of short representations of supersymmetry in LST and in string theory on (3.3) agree.

[6] We set k_μ to zero for simplicity.

3.2. Example 2: Normalizable states

A large set of normalizable states is obtained by considering vertex operators of the form

$$V(\phi) \sim e^{(-\frac{Q}{2}+i\lambda)\phi} \tag{3.15}$$

on $\mathbb{R}_\phi$. Recall that the vertex operators are related to the wavefunctions (3.7) by a factor of g_s, which here is a function of ϕ (3.4). Thus, (3.15) actually corresponds to a wavefunction

$$\Psi(\phi) \sim e^{i\lambda\phi} \tag{3.16}$$

which is (δ-function) normalizable, and thus gives rise to states in LST. Since λ is arbitrary, there is in fact a continuum of such states. To compute their masses, consider the states (3.12) as an example. The mass shell condition reads:

$$k_\mu k^\mu - \beta(\beta+Q) = 0 \tag{3.17}$$

Plugging in $\beta = -\frac{Q}{2} + i\lambda$, we find

$$M^2 = \frac{1}{N\alpha'} + \lambda^2 \tag{3.18}$$

Thus, we find a continuum above the gap $m_s/\sqrt{N}$. The gap is given by a natural scale in LST; looking back at (2.4), we see that it is related to the 't Hooft coupling of the low energy super Yang Mills theory (for IIB fivebranes).

3.3. The strong coupling problem

As we have seen before, the background (3.3) has the property that the string coupling depends on ϕ; it goes to zero as $\phi \to \infty$ and diverges as $\phi \to -\infty$. In this subsection we would like to discuss the physical origin of this behavior and its implications. The strong coupling region $\phi \to -\infty$ corresponds to the vicinity of the brane. This corresponds to the low energy region in the theory on the branes [18].

The low energy behavior of LST is different for IIA and IIB fivebranes. In the IIB case, the low energy limit is a six dimensional $U(N)$ gauge theory,

which is weakly coupled in the IR. Thus, in the limit $\phi \to -\infty$, the dual string theory on (3.3) should reproduce the weakly coupled gauge theory on the branes. Since one does not expect to find two different weakly coupled description of the same physics, the "bulk" description should either be strongly coupled, or exhibit large curvatures (or both). Since in our case the curvature of (3.3) is small, it is natural to find that the string coupling is growing in the infrared region.

In the IIA case the infrared limit of LST is somewhat different. As discussed earlier, one finds in this case a non-trivial superconformal field theory with chiral (2,0) supersymmetry, the (2,0) theory. Thus, it is not obvious that one should run into any strong coupling problems in the dual description.

To see what is going on, recall that type IIA string theory can be thought of as an eleven dimensional theory, M-theory, compactified on a circle of radius R_{11}, which is related to the eleven dimensional Planck scale l_{11} and the string scale m_s and coupling g_s via

$$m_s R_{11} = l_{11}^3 m_s^3 = g_s \tag{3.19}$$

The eleven dimensional theory contains membranes and fivebranes (the $M2$ and $M5$-branes), which preserve half of the supersymmetry; the IIA $NS5$-branes are M_5-branes located at points on the circle. Thus, to study them using holography we should construct the background around N coincident $M5$-branes. Taking the limit (2.2), which corresponds to $R_{11}, l_{11} \to 0$ with m_s fixed, one finds the eleven dimensional metric

$$ds^2 = H^{-\frac{1}{3}}\left[dx_\mu dx^\mu + H(dx_{11}^2 + dr^2 + r^2 d\Omega_3^2)\right] \tag{3.20}$$

where

$$H = \sum_{n=-\infty}^{\infty} \frac{N l_{11}^3}{[r^2 + (x_{11} - 2\pi n R_{11})^2]^{\frac{3}{2}}} \tag{3.21}$$

x_{11} is a coordinate on the circle; it is periodic with period $2\pi R_{11}$. In the limit $r \to \infty$, the background (3.20) goes over to (3.3). The radius of the x_{11} circle goes to zero and one finds the linear dilaton behavior discussed above. As

$r \to 0$ only one term in the sum over n in (3.21) (say $n = 0$) contributes, and the metric reduces to the near-horizon background of N coincident $M5$-branes in eleven dimensions. This background, $AdS_7 \times S^4$, is known to be dual to the $(2,0)$ superconformal field theory via AdS/CFT [3]. If N is large, it can be studied using eleven dimensional supergravity; otherwise one needs the full M-theory, which is not understood for these backgrounds.

Thus, we see that the growth of the coupling and associated breakdown of string perturbation theory as $\phi \to -\infty$ in the background (3.3) have slightly different origins in the IIA and IIB cases. However, regardless of the origin of this problem, one can ask what is the dual description of LST good for in view of its existence? We have already seen two examples of applications of the formalism. Since off-shell observables correspond to non-normalizable wavefunctions supported in the region $\phi \to \infty$, we can classify the observables of LST by analyzing such wavefunctions; since the coupling is small at large ϕ, perturbative string theory is suitable for this. Also, any normalizable states that are supported in the weakly coupled asymptotic region, like those described in section *3.2*, can be studied using the formalism.

Correlation functions of the observables discussed above are in general difficult to analyze. Since the string coupling goes to zero as $\phi \to \infty$, disturbances on the boundary have to propagate to finite ϕ in order to interact. Thus, to compute correlation functions in LST one needs information about the strong coupling region. E.g. for IIA fivebranes, one has to understand M-theory in the background (3.20), (3.21) which seems difficult[7].

There are actually some situations in which the strong coupling problem can be avoided. In the next section we describe an example of such a situation, which is in fact of independent interest, the high energy density behavior of LST.

[7] For large N and energies much lower than m_s one can use classical eleven dimensional supergravity to compute correlation functions. See [21] for details.

4. High energy thermodynamics of LST

At very high energy density one expects the thermodynamics of fivebranes to be dominated by black brane states. Thus, in this section we will analyze the thermodynamics of near-extremal fivebranes and deduce from it the entropy-energy relation. We will find that the density of states has the Hagedorn behavior

$$\rho(E) \sim E^{\alpha} e^{\beta_H E} \left[1 + O\left(\frac{1}{E}\right)\right]. \qquad (4.1)$$

One of our main purposes is to compute β_H and α.

4.1. Thermodynamics of near-extremal fivebranes

The supergravity solution for N coincident near-extremal $NS5$-branes in the string frame is [22]:

$$ds^2 = -\left(1 - \frac{r_0^2}{r^2}\right) dt^2 + \left(1 + \frac{N\alpha'}{r^2}\right)\left(\frac{dr^2}{1 - \frac{r_0^2}{r^2}} + r^2 d\Omega_3^2\right) + dy_5^2, \qquad (4.2)$$

$$e^{2\Phi} = g_s^2 (1 + \frac{N\alpha'}{r^2}). \qquad (4.3)$$

$r = r_0$ is the location of the horizon, dy_5^2 denotes the flat metric along the fivebranes, and $d\Omega_3^2$ is the metric on a unit three-sphere, as before. The solution also involves a non-zero NS $B_{\mu\nu}$ field which we suppress. The configuration (4.2), (4.3) has energy per unit volume

$$\frac{E}{V_5} = \frac{1}{(2\pi)^5 \alpha'^3}\left(\frac{N}{g_s^2} + \mu\right), \qquad (4.4)$$

where

$$\mu = \frac{r_0^2}{g_s^2 \alpha'}. \qquad (4.5)$$

The first term in (4.4) is the tension of extremal $NS5$-branes and can be ignored for the thermodynamic considerations below (it is a ground state energy). μ measures the energy density above extremality and g_s is the asymptotic string coupling, which goes to zero in the decoupling limit.

The near-horizon geometry is obtained by sending $r_0, g_s \to 0$ keeping the energy density μ fixed. Changing coordinates to $r = r_0 \cosh\sigma$ and Wick rotating $t \to it$ to study the thermodynamics, one finds

$$ds^2 = \tanh^2\sigma dt^2 + N\alpha' d\sigma^2 + N\alpha' d\Omega_3^2 + dy_5^2, \tag{4.6}$$

$$e^{2\Phi} = \frac{N}{\mu\cosh^2\sigma}. \tag{4.7}$$

This background corresponds to the worldsheet CFT

$$H_3^+/U(1) \times SU(2)_N \times \mathbb{R}^5, \tag{4.8}$$

where

$$H_3^+ = \frac{SL(2,C)_N}{SU(2)_N} \tag{4.9}$$

is the Euclidean AdS_3 CFT which plays an important role in the AdS-CFT correspondence; the coset $H_3^+/U(1)$, parametrized by (σ, t) in (4.6), is a semi-infinite cigar [23]. The background (4.8) describes the high energy density thermodynamics of fivebranes; it should be compared to (3.3), which is dual to the zero temperature theory.

The absence of a conical singularity at the tip ($\sigma = 0$ in (4.6)) requires the circumference of the cigar to be

$$\beta_H = 2\pi\sqrt{N\alpha'}. \tag{4.10}$$

Thus, Euclidean time lives on a circle of radius $\sqrt{N\alpha'}$, and the temperature of the system is $T_H = 1/\beta_H$. In particular, the temperature is independent of the energy density μ, which determines the value of the string coupling at the tip of the cigar (4.7).

The fact that the temperature is independent of the energy means that the entropy is proportional to the energy. Therefore, the free energy is expected to vanish[8],

$$-\beta\mathcal{F} = S - \beta E = 0. \tag{4.11}$$

[8] See [24] for a related discussion in the low energy gravity approximation.

In general in string theory the free energy is related to the string partition sum via

$$-\beta\mathcal{F} \equiv \log Z(\beta) = Z_{\text{string}}, \tag{4.12}$$

where Z_{string} is the single string partition sum, given by a sum over connected Riemann surfaces [25]. The string path integral should be performed over geometries in which Euclidean time is compactified on a circle of radius $R = \beta/2\pi$ (asymptotically). For high energies one expects the thermodynamics to be dominated by the black brane geometry (4.2), (4.6) and thus the free energy is proportional to the partition sum of string theory in the background (4.8).

The string partition sum Z_{string} can be expanded as follows:

$$Z_{\text{string}} = e^{-2\Phi_0} Z_0 + Z_1 + e^{2\Phi_0} Z_2 + \cdots, \tag{4.13}$$

where $\exp(\Phi_0)$ is the effective string coupling in the geometry (4.6) and Z_h the genus h partition sum in the background (4.8). Although the string coupling varies along the cigar (see (4.7)), it is bounded from above by its value at the tip,

$$e^{2\Phi_0} = \frac{N}{\mu}. \tag{4.14}$$

Therefore, it is natural to associate (4.14) with the effective coupling in (4.13). We see that the string coupling expansion in the background (4.8) provides an asymptotic expansion of the free energy in powers of $1/\mu$.

The leading term in the free energy (4.12), (4.13) goes like

$$-\beta\mathcal{F} = \frac{\mu}{N} Z_0 \tag{4.15}$$

and corresponds to a free energy that goes like the energy (Z_0 is proportional to the volume of the fivebrane). This term is expected to vanish (see (4.11)), and therefore we conclude that the spherical partition sum in the background (4.8) should vanish. The fact that this is indeed the case follows from the results of [26]; we will not discuss it further here.

To compute $1/\mu$ corrections to the free energy we have to examine string loop effects in the background (4.8). We next turn to the one loop correction Z_1 (see (4.13)).

4.2. The leading $1/\mu$ correction to classical thermodynamics

As discussed above, one expects the entropy-energy relation to take the form (4.1)

$$S(E) = \beta_H E + \alpha \log \frac{E}{\Lambda} + O\left(\frac{1}{E}\right), \tag{4.16}$$

where Λ is a dimensionful constant (a UV cutoff) which we will not keep track of below. Consider the canonical partition sum

$$Z(\beta) = \int_0^\infty dE \rho(E) e^{-\beta E}. \tag{4.17}$$

Near the Hagedorn temperature one might expect $Z(\beta)$ to be dominated by the contributions of high energy states;[9] if this is the case, one can replace $\rho(E)$ by (4.1) and find,

$$Z(\beta) \simeq \int dE E^\alpha e^{(\beta_H - \beta)E} \simeq (\beta - \beta_H)^{-\alpha-1}. \tag{4.18}$$

The free energy (4.12) is thus given by

$$\beta\mathcal{F} \simeq (\alpha+1)\log(\beta - \beta_H). \tag{4.19}$$

The energy computed in the canonical ensemble is

$$E = \frac{\partial(\beta\mathcal{F})}{\partial\beta} \simeq \frac{\alpha+1}{\beta-\beta_H}; \tag{4.20}$$

thus the free energy (4.19) can be written as

$$-\beta\mathcal{F} \simeq (\alpha+1)\log E. \tag{4.21}$$

Comparing to the expansion (4.12) – (4.14) we see that the leading term in the free energy arises from the torus (one loop) diagram in the background (4.8), since it scales as μ^0, like Z_1 in (4.13).

[9] We will see that this assumption is valid slightly *above* the Hagedorn temperature, but is *not* valid slightly below it.

The torus partition sum in the background (4.8) is in fact divergent, since it is proportional to the infinite volume of the cigar, associated with the region far from the tip, $\phi \to \infty$. As is standard in other closely related contexts, we will regulate this divergence by requiring that

$$\phi \leq \phi_{UV}. \tag{4.22}$$

In the fivebrane theory, this can be thought of as introducing a UV cutoff. This makes the partition sum finite, but the bulk of the amplitude still comes from the region far from the tip of the cigar. For the purpose of computing this "bulk contribution" one can replace the cigar by a long cylinder with ϕ bounded on one side by the UV cutoff (4.22) and on the other by the location of the tip of the cigar. Combining (3.4) and (4.14) we find that

$$\frac{1}{Q}\log\frac{\mu}{N} \leq \phi \leq \phi_{UV}. \tag{4.23}$$

Thus, the length of the cut-off cylinder is

$$L_\phi = \phi_{UV} - \frac{1}{Q}\log\frac{\mu}{N} = -\frac{1}{Q}\log E + \text{const.} \tag{4.24}$$

Since we are only interested in the energy dependence, we suppress in (4.24) a large energy independent contribution. Any contributions to the torus partition sum from the region near the tip of the cigar can also be lumped into this constant. Note the minus sign in front of $\log E$ in (4.24). The length L_ϕ is of course positive; the minus sign simply means that L_ϕ decreases as E grows.

To recapitulate, for the purpose of calculating the bulk contribution to the torus partition sum, we can replace the background (4.8) by

$$\mathbb{R}_\phi \times S^1 \times SU(2)_N \times \mathbb{R}_5. \tag{4.25}$$

The linear dilaton direction is regulated as in (4.23). The circumference of the S^1 is β_H (4.10).

The background (4.25) is easy to analyze since it is very similar to that describing flat space at finite temperature (see *e.g.* [27,28,29]). The bosonic fields on the worldsheet are seven free fields, one of which (Euclidean time) is compact, and a level $N-2$ $SU(2)$ WZW model. The worldsheet fermions are free and decoupled from the bosons; their partition sum, and in particular the sum over spin structures, is the same as in the flat space analysis, which we briefly review next.

Collecting all the contributions to the thermal torus partition sum in the background (4.25) we find,[10]

$$Z_1 = \frac{\beta V_5 L_\phi}{4} \int_F \frac{d^2\tau}{\tau_2} \left(\frac{1}{4\pi^2\alpha'\tau_2}\right)^{7/2} \frac{1}{|\eta(\tau)|^{10}} Z_{N-2}(\tau) \times \\ \sum_{n,m\in Z} \sum_{\mu,\nu=1}^{4} \delta_\mu U_\mu(n,m) \delta_\nu U_\nu(n,m) \left(\frac{\vartheta_\mu(0,\tau)}{\eta(\tau)}\right)^4 \left(\frac{\vartheta_\nu(0,\bar\tau)}{\eta(\bar\tau)}\right)^4 e^{-S_\beta(n,m)}. \tag{4.26}$$

The modular integral runs over the standard fundamental domain F. Z_{N-2} is the partition sum of level $N-2$ $SU(2)$ WZW [11] (see for example [30]),

$$Z_{N-2}(\tau) = \sum_{m=0}^{N-2} \chi_m^{(N-2)}(q) \chi_m^{(N-2)}(\bar q) = \sum_{m=0}^{N-2} |\chi_m^{(N-2)}(q)|^2, \tag{4.27}$$

where $q = \exp(2\pi i\tau)$ and

$$\chi_m^{(N-2)}(q) = \frac{q^{\frac{(m+1)^2}{4N}}}{\eta(q)^3} \sum_{n\in Z} [1+m+2nN)] q^{n(1+m+Nn)}. \tag{4.28}$$

We note for future reference that Z_{N-2} is real and positive.

μ, ν denote the spin structure for left and right moving worldsheet fermions, respectively. $\delta_\mu = (\pm,-,+,-)$ are signs coming from the usual GSO projections for IIA and IIB superstrings at zero temperature; n, m are winding numbers of Euclidean time around the two non-contractible cycles of the torus. The soliton factor $S_\beta(n,m)$ is given by

$$S_\beta(n,m) = \frac{\beta^2}{4\pi\alpha'\tau_2}(m^2 + n^2|\tau|^2 - 2\tau_1 mn). \tag{4.29}$$

[10] We follow the conventions of [29], which should be consulted for additional details. We also drop the subscript H on β_H, and will reinstate it later.

[11] We choose the A series modular invariant; the D and E series modular invariants can also be studied and correspond to other vacua of LST [17].

$U_\mu(n,m)$ are additional signs that are associated with finite temperature. Their role is to implement the standard thermal boundary conditions, that spacetime bosons (fermions) are (anti-)periodic around the Euclidean time direction. One can show [29] that this requirement together with modular invariance leads to:

$$\begin{aligned} U_1(n,m) &= \frac{1}{2}\left(-1 + (-1)^n + (-1)^m + (-1)^{n+m}\right) \\ U_2(n,m) &= \frac{1}{2}\left(1 - (-1)^n + (-1)^m + (-1)^{n+m}\right) \\ U_3(n,m) &= \frac{1}{2}\left(1 + (-1)^n + (-1)^m - (-1)^{n+m}\right) \\ U_4(n,m) &= \frac{1}{2}\left(1 + (-1)^n - (-1)^m + (-1)^{n+m}\right). \end{aligned} \tag{4.30}$$

The terms with $\mu = 1$ in (4.26) vanish because of the presence of fermionic zero modes for the $(+,+)$ spin structure, or equivalently since $\vartheta_1(0,\tau) = 0$.

The torus partition sum (4.26) can be rewritten in a way that makes it manifest that the coefficient of $\beta V_5 L_\phi/4$ is positive,

$$Z_1 = \frac{\beta V_5 L_\phi}{4} \int_F \frac{d^2\tau}{\tau_2} \left(\frac{1}{4\pi^2\alpha'\tau_2}\right)^{7/2} \frac{1}{|\eta(\tau)|^{18}} Z_{N-2}(\tau) \times \\ \sum_{n,m\in Z} \left| \sum_{\mu=2}^{4} U_\mu(n,m) \delta_\mu \vartheta_\mu^4(0,\tau) \right|^2 e^{-S_\beta(n,m)}. \tag{4.31}$$

It is not difficult to check that the integral (4.31) is convergent at $\tau_2 \to \infty$, the only region where a divergence could occur.

To exhibit the interpretation of (4.31) as a sum over the free energies of physical string modes one can proceed as follows [25,27,28]. Using the modular invariance of the integrand and the covariance of (n,m), one can extend the integral from the fundamental domain to the strip

$$S: \quad -\frac{1}{2} \le \tau \le \frac{1}{2}; \quad \tau_2 \ge 0, \tag{4.32}$$

while restricting to configurations with $n = 0$ in (4.31). This leads to

$$Z_1 = \frac{\beta V_5 L_\phi}{4} \int_S \frac{d^2\tau}{\tau_2} \left(\frac{1}{4\pi^2\alpha'\tau_2}\right)^{7/2} \frac{1}{|\eta(\tau)|^{18}} Z_{N-2}(\tau) \times \\ \sum_{m=-\infty}^{\infty} \left| \sum_{\mu=2}^{4} U_\mu(0,m) \delta_\mu \vartheta_\mu^4(0,\tau) \right|^2 e^{-S_\beta(0,m)}. \tag{4.33}$$

The integral over τ_1 projects on physical states (*i.e.* those with $L_0 = \bar{L}_0$), while τ_2 plays the role of a Schwinger parameter. Because of the Jacobi identity $\vartheta_2^4(0,\tau) - \vartheta_3^4(0,\tau) + \vartheta_4^4(0,\tau) = 0$, and the fact that $U_2(0,m) = (-)^m$, $U_3(0,m) = U_4(0,m) = 1$, the sum over m in (4.33) can be restricted to odd integers. It is not difficult to check in this representation too that the integral over τ_2 is convergent.

We are now ready to determine the parameter α in (4.16), (4.21). Using the relation (4.12) between the free energy $\mathcal{F}$ and the string partition sum, as well as (4.21), we see that Z_1 should be proportional to $\log E$. This is indeed the case in (4.33) since the length L_ϕ goes like $-\log E$ (see (4.24)). Combining these relations we find that

$$\begin{aligned} \alpha + 1 = -\frac{\beta V_5}{4Q} \int_S \frac{d^2\tau}{\tau_2} \left(\frac{1}{4\pi^2\alpha'\tau_2}\right)^{7/2} \frac{1}{|\eta(\tau)|^{18}} Z_{N-2}(\tau) \times \\ \sum_{m=-\infty}^{\infty} \left| \sum_{\mu=2}^{4} U_\mu(0,m)\delta_\mu \vartheta_\mu^4(0,\tau) \right|^2 e^{-S_\beta(0,m)}. \end{aligned} \tag{4.34}$$

We see that $\alpha + 1$ is negative, as stated above.[12] Physically, it is clear that it is counting the free energy of the perturbative string modes which live in the vicinity of the black brane. An interesting point which was mentioned in [31,32] is that α is an extensive quantity – it is proportional to the volume of the fivebrane V_5, in contrast, say, to the one particle free energy in critical string theory, where the analogous quantity is of order one.

The integral (4.34) appears in general to be rather formidable and we do not know whether it can be performed exactly. In the remainder of this section we will compute it in the limit $N \to \infty$, where the computation simplifies.

For large N the partition sum corresponding to the three-sphere, $Z_{N-2}(\tau)$, simplifies significantly. Indeed, for $N \gg 1$ (4.27) can be approximated as

$$Z_{N-2}(\tau) = \frac{1}{|\eta(q)|^6} \sum_{p=0}^{\infty} |q|^{\frac{(p+1)^2}{2N}} (p+1)^2. \tag{4.35}$$

[12] Of course, since the r.h.s. of (4.34) is proportional to V_5 which is assumed to be very large, we can neglect the $+1$ on the left hand side.

Returning to the evaluation of α, (4.34), we have

$$\begin{aligned} \alpha + 1 = -\frac{\beta V_5}{4Q} \left(\frac{1}{4\pi^2\alpha'}\right)^{7/2} \int_S \frac{d^2\tau}{\tau_2^{9/2}} \left| \frac{1}{\eta(\tau)} \right|^{24} \times \\ \sum_{m \in 2Z+1} \sum_{p=0}^{\infty} e^{-\frac{(p+1)^2\tau_2}{2N}} (p+1)^2 e^{-\frac{\beta^2 m^2}{4\pi\alpha'\tau_2}} \left| \vartheta_2^4 + \vartheta_3^4 - \vartheta_4^4 \right|^2 (0,\tau). \end{aligned} \tag{4.36}$$

At this point it is useful to recall that the inverse temperature β in (4.36) is in fact the Hagedorn temperature of LST, (4.10). In the large N limit, $\beta_H \sim \sqrt{N}$ becomes large (or, equivalently, the Hagedorn temperature is small in string units) and the exponential term in (4.36) suppresses the amplitude, unless τ_2 is large as well (of order N). Therefore, the τ integral in (4.36) is dominated by the large τ_2 region, which corresponds to the free energy of the supergravity modes. To compute the integral we recall the asymptotic forms of the ϑ and η functions at large τ_2 (see *e.g.* [33])

$$\begin{aligned} \vartheta_2(0,\tau) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2} = 2q^{\frac{1}{8}}(1 + q + \ldots) \\ \vartheta_3(0,\tau) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} = 1 + 2q^{\frac{1}{2}} + \ldots \\ \vartheta_4(0,\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} = 1 - 2q^{\frac{1}{2}} + \ldots \\ \eta(\tau) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = q^{\frac{1}{24}} + \ldots . \end{aligned} \tag{4.37}$$

Plugging in (4.36) and using the definition of the modified Bessel function

$$K_\nu(z) = \frac{1}{2}\left(\frac{2}{z}\right)^\nu \int_0^\infty t^{\nu-1} e^{-\frac{z^2}{4t} - t} dt, \tag{4.38}$$

we find

$$\begin{aligned} \alpha + 1 = -\frac{8V_5}{\pi^6(N\alpha')^{5/2}} \sum_{k,p=0}^{\infty} \left(\frac{2\pi(2k+1)^2}{(p+1)^2}\right)^{-7/4} (p+1)^2 \times \\ K_{-\frac{7}{2}}(\sqrt{2\pi}(p+1)(2k+1)) \simeq -4.08 \cdot 10^{-4} V_5 (N\alpha')^{-5/2} \equiv -a_1 V_5. \end{aligned} \tag{4.39}$$

Note that, as expected, a_1 is positive. Of course, as is clear from (4.36), we can write $\alpha + 1$ as $-a_1 V_5$ with a_1 a positive constant for all N, but in general a_1 receives contributions from massive string modes and is thus given by a complicated modular integral. The large N behavior of a_1 is simpler and is given by (4.39).

The fact that α goes like $N^{-5/2}$ for large N was found in a different way in [31], by analyzing the deformation of the classical solution (4.6) at one string loop. Our analysis determines the coefficient of $N^{-5/2}$, and in particular its sign, which is important for the thermodynamics.

In the discussion above, the fivebrane was assumed to be effectively non-compact. It is interesting to study the thermodynamics of fivebranes wrapped around compact manifolds, and in particular the dependence of α on the size and shape of the manifold. As an example of the sort of dependence one can expect, consider compactifying the fivebrane on $(S^1)^5$ where all five circles have the same radius R. It is sufficient to consider the case $R \geq \sqrt{\alpha'}$ since smaller radii give rise to the same physics due to T-duality.

As is standard in string theory, the effect of this is to replace the contribution of the non-compact zero modes on R^5 by the momentum and winding sum on $(S^1)^5$:

$$\frac{V_5}{(4\pi^2\alpha'\tau_2)^{5/2}} \longrightarrow \left(\sum_{l,p\in Z} q^{\frac{\alpha'}{4}\left(\frac{l}{R}+\frac{pR}{\alpha'}\right)^2} \bar{q}^{\frac{\alpha'}{4}\left(\frac{l}{R}-\frac{pR}{\alpha'}\right)^2} \right)^5 . \qquad (4.40)$$

Consider for simplicity the limit $N \to \infty$ discussed above. As mentioned after eq. (4.36), since the Hagedorn temperature is very low, the modular integral is dominated in this case by $\tau_2 \sim N$. If the radius R is much larger than $\sqrt{N\alpha'}$, the sum over momenta on the r.h.s. of (4.40) can be approximated by an integral and gives the same contribution as in the non-compact case (namely the l.h.s. of (4.40)). For $R \sim \sqrt{N\alpha'}$ one has to include a few low lying momentum modes – this is a transition region. For $\sqrt{\alpha'} < R \ll \sqrt{N\alpha'}$ one

can neglect all contributions of momentum (and winding) modes just like one is neglecting the contributions of oscillator states. Thus, we get in this case

$$\alpha + 1 = -\frac{\beta}{2Q}\left(\frac{1}{4\pi^2\alpha'}\right)\int_0^\infty \frac{d\tau_2}{\tau_2^2} \cdot 1024 \sum_{k,p=0}^{\infty} e^{-\frac{\beta^2(2k+1)^2}{4\pi\alpha'\tau_2} - \frac{(p+1)^2\tau_2}{2N}} =$$
$$-\frac{256}{\pi}\sum_{k,p=0}^{\infty}\left(\frac{2\pi(2k+1)^2}{(p+1)^2}\right)^{-1/2}(p+1)^2 K_{-1}(\sqrt{2\pi}(p+1)(2k+1)) \simeq -3.693. \qquad (4.41)$$

Interestingly, we find that for small fivebranes α is independent of the number of fivebranes N in the $N \to \infty$ limit. Note also that in this case it is important to keep the +1 on the l.h.s. of (4.41), since α is of order one.

To summarize, the power α that appears in the high energy density of states (4.1) exhibits an interesting dependence on the size of the spatial manifold that the fivebranes are wrapping. For manifolds of size much larger than the characteristic scale of LST, $\sqrt{N\alpha'}$, α is proportional to the volume of the manifold, while for sizes much smaller than this charateristic scale, it saturates at a finite value, which is independent of N (for large N), (4.41). If the density of states (4.1) is due to strings confined to the fivebranes, then these strings belong to a new universality class, with typical configurations not exceeding the size $\sqrt{N\alpha'}$. It would be interesting to understand this universality class better (see also [31]).

4.3. Comments on the near-Hagedorn thermodynamics of LST

The main result of our discussion so far is that the thermodynamics corresponding to non-extremal fivebranes is unstable. The temperature-energy relation has the form (4.20), with α given by (4.36) or for large N by (4.39), (4.41). Since it is negative, the temperature is above the Hagedorn temperature, and the specific heat is negative. This raises two immediate questions:

(1) What is the thermodynamics for temperatures slightly below the Hagedorn temperature?

(2) What is the nature of the instability above the Hagedorn temperature?

Consider first the behavior well below the Hagedorn temperature, $\beta \gg \beta_H$. In this regime, the thermodynamics is expected to reduce to that corresponding to the extreme IR limit of LST, which is the $(2,0)$ six dimensional SCFT for type IIA LST, or six dimensional $(1,1)$ SYM for IIB. From the point of view of the holographic description, this regime corresponds to the strong coupling region of the near-horizon geometry of the fivebranes, and thus should not be visible in the perturbative theory on the cigar (4.6).

What happens as the temperature approaches T_H from below? One might expect that due to the Hagedorn growth in the density of states (4.1), the high energy part of the spectrum dominates as $\beta \to \beta_H$, and the partition sum is well approximated by (4.18). What actually happens depends on the value of α, as we discuss next.

Consider first the case of large V_5 ($R \gg \sqrt{N\alpha'}$ in the discussion at the end of section 3). In this case, $|\alpha|$ is large, and the contribution to the partition sum of the high energy part of the spectrum, (4.18), goes rapidly to zero as $\beta \to \beta_H$. The integral over E is dominated by states with moderate energies, whose contribution to the partition sum is analytic at β_H. It is clear that the mean energy remains finite as we approach the Hagedorn temperature from below, and that thermodynamic fluctuations are suppressed (by a factor of the volume V_5). Since the Hagedorn temperature is reached at a finite energy, it corresponds to a phase transition.

As V_5 decreases, α decreases as well, until it reaches the value (4.41). The fluctuations in energy in the canonical ensemble increase with decreasing α. To see that, consider the case $R \ll \sqrt{N\alpha'}$ in the discussion at the end of section 3. Since $-5 < \alpha < -4$ in that case, the expectation values $\langle E^n \rangle$ with $n \geq 4$ in the canonical ensemble diverge as

$$\langle E^n \rangle \sim (\beta - \beta_H)^{-\alpha-n-1}. \qquad (4.42)$$

In such situations, one is instructed to pass to the microcanonical ensemble, in which the energy is fixed and the temperature is defined by

$$\beta = \frac{\partial \log \rho}{\partial E} = \beta_H + \frac{\alpha}{E} + \cdots \qquad (4.43)$$

The perturbative evaluation of β in (4.43) gives a temperature *above* the Hagedorn temperature. This of course does not imply that LST cannot be defined at

temperatures below T_H; instead, it means that to study the theory at such temperatures one must compute $S(E)$ to all orders in $1/E$, include non-perturbative corrections, and solve the equation (4.43) to find the energy E corresponding to a particular $\beta > \beta_H$. From the form of the leading terms in $S(E)$ it is clear that the solution of this equation will correspond to finite E. We are led again to the conclusion that the Hagedorn temperature is reached at a finite energy and thus is associated with a phase transition.

Since the study of the non-extremal fivebrane geometry in the previous sections is perturbative in $1/E$, it is not useful for studying the regime $\beta > \beta_H$. Nevertheless, it seems clear that the specific heat is positive there (this is certainly the case for the infrared theory on the fivebranes). Furthermore, since the energy – temperature relation is such that the Hagedorn temperature is reached at a finite energy, we are led to the second question raised in the beginning of this section: what is the nature of the high temperature phase of LST?

The perturbative analysis of the near-extremal fivebrane, which is valid for β slightly below β_H, predicts that the thermodynamics is unstable. Usually, in such situations the instability is associated with a negative mode in the Euclidean path integral (a tachyon). Examples include the instability of flat space at finite temperature in Einstein gravity [34], and the thermal tachyon that appears above the Hagedorn transition in critical string theory. The one loop instability found above leads one to believe that a similar negative mode should appear in LST above the Hagedorn temperature.

In [35] it was shown that there is a natural candidate for this, a mode that lives near the tip of the cigar and is classically massless. It is likely that one loop corrections give a tachyonic correction to the mass of this state above the Hagedorn temperature, but this has not been proven and we will not discuss the detailed properties of this state here.

5. Weakly Coupled Little String Theory

In the previous section we saw that the high energy thermodynamics of LST can be analyzed reliably using the holographically dual description, since at large energy density the strongly coupled region on $\mathbb{R}_\phi$ is eliminated, and the coupling never exceeds (4.14), a value that can be made arbitrarily small by increasing the energy density. In this section we will describe another situation where something similar happens at zero temperature, by studying the theory away from the origin of its moduli space of vacua.

Recall that the theory of N fivebranes contains four scalars in the adjoint of $U(N)$, X^i, $i = 6,7,8,9$, parametrizing motions in $(6,7,8,9)$. IIA fivebranes have one more scalar X^{11}, which is compact, but we will not discuss it here. The moduli space of vacua of LST is $\mathbb{R}^{4N}/S_N$ for IIB and $(\mathbb{R}^4 \times S^1)^N/S_N$ for IIA. The origin corresponds to coincident fivebranes; other points are labeled by relative separations of the fivebranes.

The four scalars X^i can be parametrized by two complex $N \times N$ matrices,

$$\begin{aligned} A &\equiv X^8 + iX^9 \\ B &\equiv X^6 + iX^7 \end{aligned} \tag{5.1}$$

Consider a point on the moduli space where

$$\begin{aligned} \langle A \rangle =& 0 \\ \langle B \rangle =& r_0 \mathrm{diag}(1, e^{\frac{2\pi i}{N}}, e^{\frac{4\pi i}{N}}, \cdots, e^{\frac{2\pi i(N-1)}{N}}) \; . \end{aligned} \tag{5.2}$$

This corresponds to fivebranes symmetrically distributed around a circle of radius r_0 in the $(6,7)$ plane. The gauge invariant characterization of this vacuum is

$$\langle \mathrm{Tr}\, B^N \rangle = r_0^N \tag{5.3}$$

with all other v.e.v.'s of the operators (3.10) set to zero. Since for a single fivebrane the worldvolume dynamics is trivial, in order to get a non-trivial result in the limit (2.2), we have to tune $r_0 \to 0$ as we take the limit. E.g., in the IIB case the masses of D-strings stretched between $NS5$-branes

$$M_W \sim \frac{r_0 m_s^2}{g_s} \tag{5.4}$$

must be kept finite in the limit. This leads one to consider the double scaling limit

$$g_s \to 0; \; r_0 m_s \to 0 \tag{5.5}$$

with M_W (5.4) held fixed.

Distributing the branes on a circle as in (5.2) breaks the $SO(4)$ R-symmetry

$$SO(4) \to SO(2) \times Z_N \tag{5.6}$$

We will next show that is also eliminates the strong coupling singularity at $\phi \to -\infty$ discussed above.

The first thing we have to understand is how to describe the vacuum (5.3) in the holographically dual theory. In section *3.1* we found the vertex operators corresponding to the gauge invariant operators (3.10). It is not difficult to see that

$$\mathrm{Tr}\, B^N \leftrightarrow \psi^+ \bar{\psi}^+ V_{\frac{N}{2}-1;\frac{N}{2}-1,\frac{N}{2}-1} \exp\left[\frac{2}{\sqrt{N\alpha'}}(\frac{N}{2}-1)\phi\right] \tag{5.7}$$

Adding the vertex operator (5.7) to the worldsheet action is equivalent, via the prescription (3.9), to adding the operator Tr B^N to the action of LST. In order to turn on a v.e.v. of Tr B^N instead, as in (5.3), we have to use the same vertex operator but replace the charge β in (3.12) by

$$\beta \to -Q - \beta \tag{5.8}$$

Thus, to describe the vacuum (5.3) we must study the worldsheet Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \lambda G_{-\frac{1}{2}} \bar{G}_{-\frac{1}{2}} \psi^+ \bar{\psi}^+ V_{\frac{N}{2}-1;\frac{N}{2}-1,\frac{N}{2}-1} e^{-\sqrt{\frac{N}{\alpha'}}\phi} + \mathrm{c.c.} \tag{5.9}$$

where we explicitly wrote the worldsheet supercharges which are needed to turn a $(-1,-1)$ picture vertex operator to a $(0,0)$ picture one (the appropriate picture for a term in the worldsheet Lagrangian). λ is a coupling related to r_0. The precise relation will be (indirectly) determined below. $\mathcal{L}_0$ is the free Lagrangian describing string propagation on (3.3). Since the coupling λ breaks explicitly the $SU(2)_L \times SU(2)_R$ symmetry, it is convenient to analyze its effect by rewriting the background (3.3) as

$$\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times \left(S^1 \times \frac{SU(2)}{U(1)}\right)/Z_N \tag{5.10}$$

where $SU(2)/U(1)$ is an $N=2$ minimal model, and S^1 a circle of radius $\sqrt{N\alpha'}$. Denoting the coordinate along the circle by Y, one can show that the interaction in (5.9) can be written as

$$\delta\mathcal{L} = \lambda G_{-\frac{1}{2}} \bar{G}_{-\frac{1}{2}} e^{-\frac{2}{\alpha' Q}(\phi+iY)} + \text{c.c} \tag{5.11}$$

This interaction is familiar in CFT as the $N=2$ Liouville interaction. Thus, we find that to describe the vacuum (5.3), we must replace the infinite cylinder $\mathbb{R}_\phi \times S^1$ in (5.10) by the $N=2$ Liouville model. Note that:

(1) The fact that the interaction (5.9), (5.11) preserves $N=2$ superconformal invariance is related to the fact that spacetime supersymmetry remains unbroken along the moduli space of LST.

(2) The interaction (5.11) grows as $\phi \to -\infty$. One can show that it resolves the strong coupling singularity discussed in section 3. We will see this directly momentarily.

To study $N=2$ Liouville theory, it is convenient to use a dual description of this background. It was argued in [36] that $N=2$ Liouville is equivalent via strong-weak coupling duality on the worldsheet to CFT on the cigar, $H_3^+/U(1)$, which was discussed in section 2. The parameter N which enters the definition of $N=2$ Liouville (5.11) via Q is mapped under the duality to the level of the underlying $SL(2)$ current algebra.

I will not describe the duality or the evidence for it here, but rather will use it to conclude that the vacuum (5.2), (5.3) is dual to

$$\mathbb{R}^{5,1} \times \left(\frac{SL(2)}{U(1)} \times \frac{SU(2)}{U(1)}\right)/Z_N \tag{5.12}$$

Note that the unbroken R-symmetry $SO(2) \times Z_N$ of the vacuum (5.3) is manifest in the description (5.12). The $SO(2)$ symmetry corresponding to rotations in

the $(8,9)$ plane is realized as the $U(1)$ translation symmetry around the cigar. The rotation symmetry in the $(6,7)$ plane, which is broken to Z_N by the v.e.v. of B, corresponds to winding number around the cigar. This quantum number is not conserved, since winding can slip off the tip of the cigar. The Z_N orbifold in (5.12) leads to a Z_N remnant of it (since it allows fractional windings $\in Z/N$).

The radius of the circle on which the fivebranes lie, r_0 in (5.2), is related to the value of the string coupling at the tip of the cigar, g_{cigar}. The precise relation can be determined by noting that D-branes stretched between fivebranes, whose mass is given by (5.4), correspond in (5.12) to D-branes at the tip of the cigar, whose mass is m_s/g_{cigar}. This implies that

$$g_{\text{cigar}} \simeq \frac{m_s}{M_W} \tag{5.13}$$

Thus, the theory is weakly coupled when $M_W >> m_s$; as M_W decreases, we recover the original strongly coupled theory described holographically by (3.3).

The weakly coupled nature of the theory (5.12) for $M_W >> m_s$ allows one to determine the spectrum in a wide range of energies $0 < E << M_W$, and to compute various off-shell correlation functions of the observables discussed in section 3. Interactions can be turned on gradually by increasing g_{cigar} (5.13). Two and three point functions as well as the resulting spectrum were analyzed in [36]. We will next illustrate the resulting structure by discussing an example.

Consider the operator $\text{Tr}\, B^N(x)$. The dual vertex operator (5.7) can be written in terms of the background (5.12) as

$$\text{Tr}\, B^N(x) \leftrightarrow e^{-\varphi-\bar{\varphi}} e^{ik_\mu x^\mu} V_{j;m,m} \tag{5.14}$$

where φ, $\bar{\varphi}$ are the standard bosonized ghosts needed for the -1 picture, $V_{j;m,\bar{m}}$ is a Virasoro primary on the cigar carrying p units of momentum and w units of winding, with

$$m = \frac{1}{2}(p + wN)\ ;\ \ \bar{m} = -\frac{1}{2}(p - wN)\ . \tag{5.15}$$

In the case (5.14), $p=0$ while $w=1$ (*i.e.* $m=\bar{m}=N/2$). The worldsheet scaling dimension of $V_{j;m,m}$ is

$$\Delta = \bar{\Delta} = \frac{m^2 - j(j+1)}{N} \tag{5.16}$$

Requiring that (5.14) be physical gives rise to the mass-shell condition

$$\alpha' k_\mu k^\mu = \frac{4}{N}(j-m+1)(j+m) \tag{5.17}$$

To compute the two point function of Tr $B^N(k_\mu)$ we use the correspondence (3.9):

$$\langle \mathrm{Tr}\, B^N(k_\mu) \mathrm{Tr}\, \bar{B}^N(-k_\mu)\rangle = \langle e^{-\varphi-\bar{\varphi}} e^{ik_\mu x^\mu} V_{j;m,m} e^{-\varphi-\bar{\varphi}} e^{-ik_\mu x^\mu} V_{j;-m,-m}\rangle \tag{5.18}$$

The only non-trivial part of the correlator on the r.h.s. is $\langle VV\rangle$. It was computed in [37]:

$$\langle V_{j;m,\bar{m}} V_{j;-m,-\bar{m}}\rangle = N[\nu(N)]^{2j+1} \frac{\Gamma(1-\frac{2j+1}{N})\Gamma(-2j-1)\Gamma(j-m+1)\Gamma(1+j+\bar{m})}{\Gamma(\frac{2j+1}{N})\Gamma(2j+2)\Gamma(-j-m)\Gamma(\bar{m}-j)} . \tag{5.19}$$

where

$$\nu(N) \equiv \frac{1}{\pi}\frac{\Gamma(1+\frac{1}{N})}{\Gamma(1-\frac{1}{N})} . \tag{5.20}$$

The two point function (5.19) has a series of poles; these can be interpreted as contributions of on-shell states in DSLST, which are created from the vacuum by the operator (5.14). The masses of these states can be computed by using the relation (5.17) between j and $M^2 = -k_\mu k^\mu$. The locations of the poles are given by

$$|m| = j+n; \quad n = 1,2,3,\cdots \tag{5.21}$$

These values of m and j belong to the principal discrete series representations of $SL(2)$. The corresponding states can be thought of as bound states that live near the tip of the cigar [38]. Such bound states are to be expected since winding modes around the cigar feel an effective attractive potential towards the tip – their energy decreases as they approach the tip and shrink.

For the particular case (5.14), $m = \bar{m} = N/2$, and the masses of these states are given by

$$\begin{aligned} &\frac{\alpha'}{2} M_n^2 = \frac{2}{N}(n-1)(N-n) , \\ &N+1 > 2n > 1 , \end{aligned} \tag{5.22}$$

The second line in (5.22) comes from a unitarity constraint on j which must be imposed, $-1/2 < j < (N-1)/2$. Note that all the masses in (5.22) are non-negative; For $n = 1$ one finds massless states, which correspond to the eigenvalues of the scalar matrix B.

A few comments are in order here:

(1) By analyzing the behavior of the two point function (5.18), (5.19) one can check that the residues of the poles corresponding to the states (5.22) are positive, in agreement with the unitarity of the theory.

(2) In addition to the discrete spectrum given by (5.22), one also has the continuum discussed in section 3 (3.18). One can show that the continuum starts right above the heaviest state (5.22). Thus the spectrum of states that can be created from the vacuum by the operator (5.14) is a finite discrete set, followed by a continuum (similar to the spectrum of bound states and scattering states in quantum mechanics).

(3) One can repeat the above discussion for other observables as well. The resulting picture is similar; one always finds a finite set of discrete states which live near the tip of the cigar, followed by a continuum of states which propagate in the semi-infinite throat [36].

(4) Since there is a Hagedorn growth in the number of observables (coming from oscillator states on (5.12)), one finds a Hagedorn density of states in LST. But the exponent β_H (4.1) does not grow like $\sqrt{N}$ as expected from (4.10). Instead one gets $\beta_H \sim 1/m_s$. This is not particularly surprising since (4.10) is the expected behavior for high energies $E >> M_W$, whereas the present analysis is only valid in the intermediate regime $m_s << E << M_W$.

(5) Three point functions of the off-shell observables discussed above can be computed as well using the results of [37]. One finds a similar analytic structure to that exhibited by the two point functions. There are poles associated with external legs going on-shell; their locations correspond again to the spectrum (5.22). The residues of these poles describe the scattering amplitudes of the physical states; they seem to have sensible physical properties. See [36] for details.

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