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SPRING SCHOOL ON SUPERSTRINGS AND RELATED MATTERS

THE HOLOGRAPHIC PRINCIPLE

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1 Introduction

The Holographic Principle was to my knowledge first proposed by 't Hooft, around 1987. His argument went as follows. Take a volume in d spatial dimensions, say a box with sides L and volume

$$V = L^d$$

How many degrees of freedom does one need to describe all the physics that could possibly takes place inside this volume? Or in other words, in how many different quantum states can a system be that fits inside this volume? Naively, and without gravity, the number of degrees of freedom, or the number of quantum states grows like the volume, or more precisely, the exponent of the volume. Why?

Physics as we know it obeys the requirement of locality. Locality is directly linked with causality. A change in the system at one point can not instantaneously affect the system at another point. This requirement is implemented usually by introducing local degrees of freedom at (almost) every point in space. Suppose space is a lattice with lattice distance ℓ, with on each site some discrete spin with m degrees of freedom, or states. Then the total number of states inside the volume is

$$\#states = m^{\#sites} = e^S$$

where the entropy S is given by

$$S = c\left(\frac{L}{\ell}\right)^d$$

and $c = \log m$.

This counting does not take gravity in to account. In particular, it counts states with arbitrary energies. Energy is a source of gravity, and it significantly effects the geometry if the gravitational potential is large. 't Hooft argued that the maximal energy allowed for this system is equal to the mass of a black hole with Schwarschild radius equal to the size L. This implies that the maximal energy obeys

$$\frac{GE}{L^{d-2}} \sim 1$$

Suppose that the system is described by some free massless relativistic gass of particle with temperature T. Then we have

$$\begin{aligned} E &\sim L^d T^{d+1} \\ S &\sim L^d T^d \end{aligned}$$

and so

$$S \sim (EL)^{d/d+1}$$

Combined with the gravitational bound on the energy this gives a maximal entropy

$$S \sim \left(\frac{L^{d-1}}{G}\right)^{d/d+1}$$

In fact, Bekenstein has argued that there is in fact a universal maximal entropy for a given energy E and size L of the form

$$S \sim EL \sim \frac{L^{d-1}}{G}$$

His argument is roughly that the minimum energy quantum above the ground state is $\varepsilon \sim 1/L$ leading to EL different quanta. So based on these estimates of the maximal entropy for systems with a bounded energy, 't Hooft argued that the states of maximal entropy is given by a single black hole with mass $M = E$ and whose horizon, therefore coincides with the boundary of the volume.

According to the holographic principle the entropy, being defined as the logarithm of the number of states, should not grow like the volume, but as the area of of the volume. In its most strong form the principle states that

$$S = \frac{A}{4G}$$

where G denotes Newtons constant, which throughout this lectues I will try to keep explicitly present. I'll put only c and $\hbar$ equal to one. The holographic entropy has the same form in any dimension. Note that Newton's constant has dimension of [length]$^{d-1}$, and can be expressed in terms of the Plank length as

$$G = \ell_p^{d-1}$$

The holographic principle suggest that there may be a description of the number of states as a system that is defined on the boundary with a lattice spacing of one Plank length. This is of course a radical statement, since it implies that the whole notion of local degrees of freedom must be replaced by something else, and in such a way that the effective physics still looks local, and causal.

Note that the holographic principle relates microscopic physics, the number of states, with macroscopic physics, the geometry and dimensions of a region in space and time. One of the main points that I would like to get across in these lectures is that this is very analogous to the relationship between thermodynamics and statistical mechanics. There are indeed many indications that the Einstein equations that determine the macroscopic geometry in terms of the energy- momentum, are effective equations very much like the thermodynamic relations that express entropy in term of energy. The Einstein action

$$S_E = \frac{1}{16\pi G}\int \sqrt{g}(R-2\Lambda)$$

is just an approximation to a fundamental underlying theory, and is usually seen as just the (almost) leading term in an expansion

$$S_{eff} = \frac{a_0}{\ell_p^d}\int \sqrt{g} + \frac{a_1}{\ell_p^{d-2}}\int \sqrt{g}R + \frac{a_2}{\ell_p^{d-4}}\int \sqrt{g}R^2 + ...$$

the higher order terms are higher curvature terms with more derivative, and hence less important at low energies. They are suppressed by powers of the Planck length. The leading term in fact is the cosmological constant term. Strangely enough the same reasoning that would imply that the higher curvature terms are suppressed, would lead to the conclusion that the cosmological constant term is very large. Namely, one expects that all coefficients are of order one, but observations now indicate that a_0 is a finite number of order 10^{-120}.

2 Black Holes

Pure Einstein gravity has a unique spherically symmetric solution, the Schwarschild metric. It has the form

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{h(r)} + r^2 d\Omega_{d-1}^2$$

with

$$h(r) = 1 - \omega_d \frac{M}{r^{d-2}}$$

where

$$\omega_d = \frac{16\pi G}{(d-1)\mathrm{Vol}(S^{d-1})}$$

For a black hole in (anti-) de Sitter space the metric has a similar form with

$$h(r) = 1 - \omega_d \frac{M}{r^{d-2}} - \frac{2\Lambda}{d(d-1)} r^2$$

The horizon of the black hole is at

$$h(r_H) = 0$$

The Hawking temperature can be obtained as follows. Define

$$dr^* = \frac{dr}{h(r)}$$

so that the metric becomes

$$ds^2 = h(r)(-dt^2 + dr^{*2}) + ..$$

Now use the near horizon limit

$$h(r) = h'(r_H)(r - r_H)$$

so that

$$r^* = \frac{1}{h'(r_H)} \log(r - r_H)$$

and

$$.h(r) \sim h'(r_H) \exp\left(h'(r_H) r^*\right)$$

Now the local minkowski coordinates are

$$u, v = \frac{1}{\sqrt{h'(r_H)}} \exp \frac{1}{2} h'(r_H)(r^* \pm t)$$

and so after Euclidean continuation the Euclidean time has a periodicity

$$\beta_H = \frac{4\pi}{h'(r_H)}$$

corresponding to a Hawking temperature of

$$T_H = \frac{h'(r_H)}{4\pi}$$

The near-horizon metric takes the form

$$ds^2 = dudv + r_H^2 d\Omega^2$$

which is $R^2 \times S^{d-1}$.

2.1 BLACK HOLE ENTROPY

To find the entropy of the black hole one uses the second law of thermodynamics

$$TdS = dE$$

We have

$$h'(r_H)dr_H = \omega_d \frac{dM}{r_H^{d-2}}$$

and hence

$$dM = \frac{h'(r_H)}{4\pi} d\left(\frac{A}{4G}\right)$$

where

$$A = \mathrm{Vol}(S^{d-1})r_H^{d-1}$$

is the area of the horizon. The entropy may also be obtained from the free energy, which is identified with the value of the classical action for the euclidean black hole. We have

$$S_E = \frac{1}{16\pi G}\left[\int \sqrt{g}(R - 2\Lambda) + \int_{\partial M} \sqrt{h}K\right]$$

where K represents the extrinsic curvature. The evaluation of this action is somewhat tricky because of the inclusion of the extrinsic curvature, and because it has to be regularized. The result is

$$S_E = \beta(M - TS) = \beta\frac{\mathrm{Vol}(S^{d-1})r_H^{d-2}}{4G}\left(1 - \frac{2\Lambda}{d(d-1)}r_H^2\right)$$

Note that in this derivation we have not really made any distinction between zero or non-zero cosmological constant. Furthermore, we have not specified which zero of the function $h(r)$ we identified with the horizon. In fact, in de Sitter space there are two possible zeroes. One corresponding to the black hole horizon, the other to the cosmological event horizon. For both type of horizons one can define the temperature and entropy. I'll say more about the cosmological situation in one of the later lectures.

2.2 BLACK HOLES IN STRING THEORY

The metric of a D-brane is

$$ds^2 = f(r)\left(-dt^2 + dx_{\|}^2\right) + \frac{1}{f(r)}\left(dr^2 + r^2 d\Omega_{8-p}^2\right)$$

with

$$f(r) = \left(1 + \frac{\omega_p Q_p}{r^{7-p}}\right)^{-\frac{1}{2}}$$

where

$$\omega_p = g_s \ell_s^{7-p} \mathrm{Vol}(S^{8-p})$$

and

$$\phi = \frac{1}{2}(p-3)\log f(r)$$

The horizon is at $r = 0$. For $p = 3$ the near-horizon limit is

$$ds^2 = R^2\left[u^2\left(-dt^2 + dx_{\|}^2\right) + \frac{du^2}{u^2} + d\Omega_5^2\right]$$

with

$$u = \frac{r}{R^2}$$

and

$$R^2 = \ell_s^2\sqrt{g_s Q_3 \mathrm{Vol}(S^5)}$$

ENTROPY IN STRING THEORY I hope most of you have had a basic introduction in string theory. So you probably know that we can describe the string in terms of a set of string coordinates $X^\mu(\sigma,\tau)$, and a number of fermionic coordinates $\psi^\mu(\sigma,\tau)$, each of which define a set of oscillators. What may be less familiar is how one counts the number of states of a string, in a way that exhibits the maximal entropy. For simplicity, let us focus on the scalar coordinates. Eeach state is characterized by a oscilation numbers N_n of the n^{th} oscillator, satisfying

$$\sum nN_n = N$$

The partition sum

$$Z(q) = \sum_N d(N) q^N$$

for a single scalar coordinate is easily computed

$$Z(q) = \sum_{\{N_1,\ldots,N_n,\ldots\}} q^{\sum_n nN_n} = \prod_n \left(\sum_{N_n} q^{nN_n} \right) = \prod_n \frac{1}{1-q^n}$$

The degeneracies can be obtained from the partition sum via

$$d(N) = \frac{1}{2\pi i} \oint \frac{dq}{q^{N+1}} Z(q)$$

These degeneracies can be computed using the fact that Z satisfies the magical property

$$q^{-1/24} Z(q) = \tilde{q}^{-1/24} Z(\tilde{q})$$

where

$$q = e^{2\pi i \tau}, \quad \tilde{q} = e^{-2\pi i/\tau}$$

This property, known as modular invariance, is a consequence of the fact that the partition sum may be represented as a functional integral over the scalar coordinate defined on a torus defined by the latice $n+m\tau$. The integral representation for the degeneracy can now be computed using a saddle point approximation. One finds

$$\frac{1}{2\pi i} \oint \frac{dq}{q^{N+1}} \tilde{q}^{-c/24} = \int d\tau e^{-2\pi i N\tau + 2\pi i c/24\tau} = \exp 2\pi \sqrt{\frac{c}{6}N}$$

where we used that the saddle point value for τ is given by

$$\tau = i\sqrt{\frac{c}{24N}}$$

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On the Holographic Principle in a Radiation Dominated Universe

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Abstract

The holographic principle is studied in the context of a $n+1$ dimensional radiation dominated closed Friedman-Robertson-Walker (FRW) universe. The radiation is represented by a conformal field theory with a large central charge. Following recent ideas on holography, it is argued that the entropy density in the early universe is bounded by a multiple of the Hubble constant. The entropy of the CFT is expressed in terms of the energy and the Casimir energy via a universal Cardy formula that is valid for all dimensions. A new purely holographic bound is postulated which restricts the sub-extensive entropy associated with the Casimir energy. Unlike the Hubble bound, the new bound remains valid throughout the cosmological evolution. When the new bound is saturated the Friedman equation exactly coincides with the universal Cardy formula, and the temperature is uniquely fixed in terms of the Hubble parameter and its time-derivative.

1. Introduction

The holographic principle is based on the idea that for a given volume V the state of maximal entropy is given by the largest black hole that fits inside V. 't Hooft and Susskind [1] argued on this basis that the microscopic entropy S associated with the volume V should be less than the Bekenstein-Hawking entropy

$$S \leq \frac{A}{4G} \tag{1}$$

of a black hole with horizon area A equal to the surface area of the boundary of V. Here the dependence on Newton's constant G is made explicit, but as usual $\hbar$ and c are set to one.

To shed further light on the holographic principle and the entropy bounds derived from it, we study in this paper the standard cosmology of a closed radiation dominated Friedman-Robertson-Walker (FRW) universe with general space-time dimension

$$D = n + 1.$$

The metric takes the form

$$ds^2 = -dt^2 + R^2(t) d\Omega_n^2 \tag{2}$$

where $R(t)$ represents the radius of the universe at a given time t and $d\Omega_n^2$ is a short hand notation for the metric on the unit n-sphere S^n. Hence, the spatial section of a $(n{+}1)$d closed FRW universe is an n-sphere with a finite volume

$$V = \mathrm{Vol}(S^n) R^n.$$

The holographic bound is in its naive form (1) not really applicable to a closed universe, since space has no boundary. Furthermore, the argumentation leading to (1) assumes that it's possible to form a black hole that fills the entire volume. This is not true in a cosmological setting, because the expansion rate H of the universe as well as the given value of the total energy E restrict the maximal size of black hole. As will be discussed in this paper, this will lead to a modified version of the holographic bound.

The radiation in an FRW universe is usually described by free or weakly interacting mass-less particles. More generally, however, one can describe the radiation by an interacting conformal field theory (CFT). The number of species of mass-less particles translates into the value of the central charge c of the CFT. In this paper we will be particularly interested in radiation described by a CFT with a very large central charge. In a finite volume the energy E has a Casimir contribution proportional to c. Due to this Casimir effect, the entropy S is no longer a purely extensive function of E and V. The entropy of a $(1{+}1)$d CFT is given by the well-known Cardy formula [2]

$$S = 2\pi \sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}, \tag{3}$$

where L_0 represents the product ER of the energy and radius, and the shift of $\frac{c}{24}$ is caused by the Casimir effect. In this paper we show that, after making the appropriate identifications for L_0 and c, the same Cardy formula is also valid for CFTs in other dimensions. This is rather surprising, since the standard derivation of the Cardy formula based on modular invariance only appears to work for $n = 1$. By defining the central charge c in terms of the Casimir energy, we are able to argue that the Cardy formula is universally valid. Specifically, we will show that with the appropriate identifications, the entropy S for a $n+1$ dimensional CFT with an AdS-dual is exactly given by (3).

The main new result of this paper is the appearance of a deep and fundamental connection between the holographic principle, the entropy formulas for the CFT, and the FRW equations for a radiation dominated universe. In $n+1$ dimensions the FRW equations are given by

$$
\begin{aligned}
H^2 &= \frac{16\pi G}{n(n-1)}\frac{E}{V} - \frac{1}{R^2} && (4)\\
\dot{H} &= -\frac{8\pi G}{n-1}\left(\frac{E}{V} + p\right) + \frac{1}{R^2} && (5)
\end{aligned}
$$

where $H = \dot{R}/R$ is the Hubble parameter, and the dot denotes as usual differentiation with respect to the time t. The FRW equations are usually written in terms of the energy density $\rho = E/V$, but for the present study it is more convenient to work with the total energy E and entropy S instead of their respective densities ρ and $s = S/V$. Note that the cosmological constant has been put to zero; the case $\Lambda \neq 0$ will be described elsewhere [3].

Entropy and energy momentum conservation together with the equation of state $p = E/nV$ imply that E/V and p decrease in the usual way like $R^{-(n+1)}$. Hence, the cosmological evolution follows the standard scenario for a closed radiation dominated FRW universe. After the initial Big Bang, the universe expands until it reaches a maximum radius, the universe subsequently re-collapses and ends with a Big Crunch. No surprises happen in this respect.

The fun starts when one compares the holographic entropy bound with the entropy formulas for the CFT. We will show that when the bound is saturated the FRW equations and entropy formulas of the CFT merge together into one set of equation. One easily checks on the back of an envelope that via the substitutions

$$
\begin{aligned}
2\pi L_0 &\Rightarrow \frac{2\pi}{n}ER\\
2\pi\frac{c}{12} &\Rightarrow (n-1)\frac{V}{4GR}\\
S &\Rightarrow (n-1)\frac{HV}{4G}
\end{aligned}
\tag{6}
$$

the Cardy formula (3) *exactly* turns into the $n+1$ dimensional Friedman equation (4). This observation appears as a natural consequence of the holographic principle. In sections 2 and 3 we introduce three cosmological bounds each corresponding to one of the equations in (6) The Cardy formula is presented and derived in section 4. In section 5 we introduce a new cosmological bound, and show that the FRW equations and the entropy formulas are exactly matched when the bound is saturated. In section 6 we present a graphical picture of the entropy bounds and their time evolution.

2. Cosmological entropy bounds

This section is devoted to the description of three cosmological entropy bounds: the Bekenstein bound, the holographic Bekenstein-Hawking bound, and the Hubble bound. The relation with the holographic bound proposed by Fischler-Susskind and Bousso (FSB) will also be clarified.

2.1. The Bekenstein bound

Bekenstein [4] was the first to propose a bound on the entropy of a macroscopic system. He argued that for a system with limited self-gravity, the total entropy S is less or equal than a multiple of the product of the energy and the linear size of the system. In the present context, namely that of a closed radiation dominated FRW universe with radius R, the appropriately normalized Bekenstein bound is

$$S \leq S_B \tag{7}$$

where the Bekenstein entropy S_B is defined by

$$S_B \equiv \frac{2\pi}{n} E R. \tag{8}$$

The bound is most powerful for relatively low energy density or small volumes. This is due to the fact that S_B is super-extensive: under $V \to \lambda V$ and $E \to \lambda E$ it scales like $S_B \to \lambda^{1+1/n} S_B$.

For a radiation dominated universe the Bekenstein entropy is constant throughout the entire evolution, since $E \sim R^{-1}$. Therefore, once the Bekenstein bound is satisfied at one instance, it will remain satisfied at all times as long as the entropy S does not change. The Bekenstein entropy is the most natural generalization of the Virasoro operator $2\pi L_0$ to arbitrary dimensions, as is apparent from (6). Indeed, it is useful to think about S_B not really as an entropy but rather as the energy measured with respect to an appropriately chosen conformal time coordinate.

2.2. The Bekenstein-Hawking bound

The Bekenstein-bound is supposed to hold for systems with limited self-gravity, which means that the gravitational self-energy of the system is small compared to the total energy E. In the current situation this implies, concretely, that the Hubble radius H^{-1} is larger than the radius R of the universe. So the Bekenstein bound is only appropriate in the parameter range $HR \leq 1$. In a strongly self-gravitating universe, that is for $HR \geq 1$, the possibility of black hole formation has to be taken into account, and the entropy bound must be modified accordingly. Here the general philosophy of the holographic principle becomes important.

It follows directly from the Friedman equation (4) that

$$HR \lesseqgtr 1 \qquad \Leftrightarrow \qquad S_B \lesseqgtr (n-1)\frac{V}{4GR} \tag{9}$$

Therefore, to decide whether a system is strongly or weakly gravitating one should compare the Bekenstein entropy S_B with the quantity

$$S_{BH} \equiv (n-1)\frac{V}{4GR}. \tag{10}$$

When $S_B \leq S_{BH}$ the system is weakly gravitating, while for $S_B \geq S_{BH}$ the self-gravity is strong. We will identify S_{BH} with the holographic Bekenstein-Hawking entropy of a black hole with the size of the universe. S_{BH} indeed grows like an area instead of the volume, and for a closed universe it is the closest one can come to the usual expression $A/4G$.

As will become clear in this paper, the role of S_{BH} is not to serve as a bound on the total entropy, but rather on a sub-extensive component of the entropy that is association with the Casimir energy of the CFT. The relation (6) suggests that the Bekenstein-Hawking entropy is closely related to the central charge c. Indeed, it is well-known from $(1+1)d$ CFT that the central charge characterizes the number of degrees of freedom may be even better than the entropy. This fact will be further explained in sections 5 and 6, when we describe a new cosmological bound on the Casimir energy and its associated entropy.

2.3. The Hubble entropy bound

The Bekenstein entropy S_B is equal to the holographic Bekenstein-Hawking entropy S_{BH} precisely when $HR = 1$. For $HR > 1$ one has $S_B > S_{BH}$ and the Bekenstein bound has to be replaced by a holographic bound. A naive application of the holographic principle would imply that the total entropy S should be bounded by S_{BH}. This turns out to be incorrect, however, since a purely holographic bound assumes the existence of arbitrarily large black holes, and is irreconcilable with a finite homogeneous entropy density.

Following earlier work by Fischler and Susskind [5], it was argued by Easther and Lowe [6], Veneziano [7], Bak and Rey [8], Kaloper and Linde [9], that the maximal entropy inside the universe is produced by black holes of the size of the Hubble horizon, see also [10]. Following the usual holographic arguments one then finds that the total entropy should be less or equal than the Bekenstein-Hawking entropy of a Hubble size black hole times the number N_H of Hubble regions in the universe. The entropy of a Hubble size black hole is roughly $HV_H/4G$, where V_H is the volume of a single Hubble region. Combined with the fact that $N_H = V/V_H$ one obtains an upper bound on the total entropy S given by a multiple of $HV/4G$. The presented arguments of [6, 8, 9, 7] are not sufficient to determine the precise pre-factor, but in the following subsection we will fix the normalization of the bound by using a local version of the Fischler-Susskind-Bousso formulation of the holographic principle. The appropriately normalized entropy bound takes the form

$$S \leq S_H \qquad \text{for} \qquad HR \geq 1 \tag{11}$$

with

$$S_H \equiv (n-1)\frac{HV}{4G}. \tag{12}$$

The Hubble bound is only valid for $HR \geq 1$. In fact, it is easily seen that for $HR \leq 1$ the bound will at some point be violated. For example, when the universe reaches its maximum radius and starts to re-collapse the Hubble constant H vanishes, while the entropy is still non-zero.[1] This should not really come as a surprise, since the Hubble bound was based on the idea that the maximum size of a black hole is equal to the Hubble radius. Clearly, when the radius R of the universe is smaller than the Hubble radius H^{-1} one should reconsider the validity of the bound. In this situation, the self-gravity of the universe is less important, and the appropriate entropy bound is

$$S \leq S_B \qquad \text{for} \qquad HR \leq 1 \tag{13}$$

2.4. The Hubble bound and the FSB prescription.

Fischler, Susskind, and subsequently Bousso [12], have proposed an ingenious version of the holographic bound that restricts the entropy flow through contracting light sheets. The FSB-bound works well in many situations, but, so far, no microscopic derivation has been given. Wald and collaborators [13] have shown that the FSB bound follows from local inequalities on the entropy density and the stress energy. The analysis of [13] suggests the existence a local version of the FSB entropy bound, one that does not involve global information about the causal structure of the universe, see also [11]. The idea of to formulate the holographic principle via entropy flow through light sheets also occurred in the work of Jacobson [14], who used it to derive an intriguing relation between the Einstein equations and the first law of thermodynamics. In this subsection, a local FSB bound will be presented that leads to a precisely normalized upper limit on the entropy in terms of the Hubble constant.

According to the original FSB proposal, the entropy flow S through a contracting light sheet is less or equal to $A/4G$, where A is the area of the surface from which the light sheet originates. The following infinitesimal version of this FSB prescription will lead to the Hubble bound. For every $n-1$ dimensional surface at time $t+dt$ with area $A+dA$ one demands that

$$dS \leq \frac{dA}{4G}, \tag{14}$$

[1]To avoid this problem a different covariant version of the Hubble bound was proposed in [11].

where dS denotes the entropy flow through the infinitesimal light sheets originating at the surface at $t+dt$ and extending back to time t, and dA represents the increase in area between t and $t+dt$. For a surface that is kept fixed in co-moving coordinates the area A changes as a result of the Hubble expansion by an amount

$$dA = (n-1)HA\,dt, \tag{15}$$

where the factor $n-1$ simply follows from the fact that $A \sim R^{n-1}$. Now pick one of the two past light-sheets that originate at the surface: the inward or the outward going. The entropy flow through this light-sheet between t and $t+dt$ is given by the entropy density $s = S/V$ times the infinitesimal volume Adt swept out by the light-sheet. Hence,

$$dS = \frac{S}{V}A\,dt. \tag{16}$$

By inserting this result together with (15) into the infinitesimal FSB bound (14) one finds that the factor Adt cancels on both sides and one is left exactly with the Hubble bound $S \leq S_H$ with the Hubble entropy S_H given in (12). We stress that the relation with the FSB bound was merely used to fix the normalization of the Hubble bound, and should not be seen as a derivation.

3. Time-evolution of the entropy bounds.

Let us now return to the three cosmological entropy bounds discussed in section 2. The Friedman equation (4) can be re-written as an identity that relates the Bekenstein-, the Hubble-, and the Bekenstein-Hawking entropy. One easily verifies that the expressions given in (8), (10), and (12) satisfy the quadratic relation

$$S_H^2 + (S_B - S_{BH})^2 = S_B^2. \tag{17}$$

It is deliberately written in a Pythagorean form, since it suggests a useful graphical picture of the three entropy bounds. By representing each entropy by a line with length equal to its value one finds that due to the quadratic Friedman relation (17) all three fit nicely together in one diagram, see figure 1. The circular form of the diagram reflects the fact that S_B is constant during the cosmological evolution. Only S_H and S_{BH} depend on time.

Let us introduce a conformal time coordinate via

$$Rd\eta = (n-1)dt \tag{18}$$

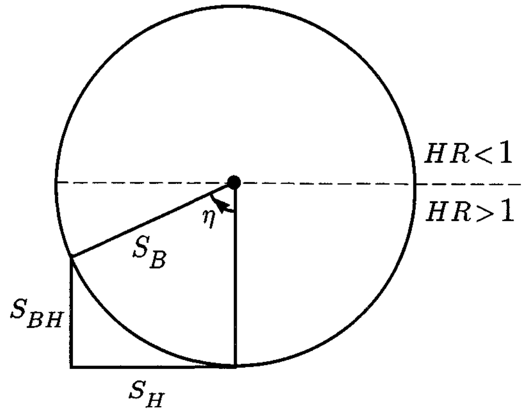

Fig.1. *A graphical representation of the Bekenstein entropy S_B, the Hubble entropy S_H and the Bekenstein-Hawking entropy S_{BH}. The angle η corresponds to the conformal time coordinate. The value of each entropy is represented by an actual distance: S_B is constant, while S_H and S_{BH} change with time.*

and let us compute the η-dependence of S_{BH} and S_H. For S_{BH} this easily follows from: $\dot{S}_{BH} = (n-1)HS_{BH} = (n-1)R^{-1}S_H$. For S_H the calculation is a bit more tedious, but with the help of the FRW equations, the result can eventually be put in the form

$$\begin{aligned} \frac{dS_H}{d\eta} &= S_B - S_{BH}, \\ \frac{dS_{BH}}{d\eta} &= -S_H. \end{aligned} \tag{19}$$

These equations show that the conformal time coordinate η can be identified with the angle η, as already indicated in figure 1. As time evolves the Hubble entropy S_H rotates into the combination $S_B - S_{BH}$ and visa versa. Equation (19) can be integrated to

$$\begin{aligned} S_H &= S_B \sin\eta \\ S_{BH} &= S_B(1-\cos\eta) \end{aligned} \tag{20}$$

The conformal time coordinate η plays the role of the time on a cosmological clock that only goes around once: at $\eta = 0$ time starts with a Big Bang and at $\eta = 2\pi$ it ends with a Big Crunch. Note that η is related to the parameter HR via

$$HR = \cot\frac{\eta}{2} \tag{21}$$

So far we have not yet included the CFT into our discussion. We will see that the entropy of the CFT will 'fill' part of the diagram, and in this way give rise to a special moment in time when the entropy bounds are saturated.

4. Casimir energy and the Cardy formula

We now turn to the discussion of the entropy of the CFT that lives inside the FRW universe. We begin with a study of the finite temperature Casimir energy with the aim to exhibit its relation with the entropy of the CFT. Subsequently a universal Cardy formula will be derived that expresses the entropy in terms of the energy and the Casimir energy, and is valid for all values of the spatial dimension n.

4.1. The Euler relation and Casimir energy.

In standard textbooks on cosmology [15, 16] it is usually assumed that the total entropy S and energy E are extensive quantities. This fact is used for example to relate the entropy density s to the energy density ρ and pressure p, via $Ts = \rho + p$. For a thermodynamic system in finite volume V the energy $E(S,V)$, regarded as a function of entropy and volume, is called extensive when it satisfies $E(\lambda S, \lambda V) = \lambda E(S,V)$. Differentiating with respect to λ and putting $\lambda = 1$ leads to the Euler relation[1]

$$E = V\left(\frac{\partial E}{\partial V}\right)_S + S\left(\frac{\partial E}{\partial S}\right)_V \tag{22}$$

The first law of thermodynamics $dE = TdS - pdV$ can now be used to re-express the derivatives via the thermodynamic relations

$$\left(\frac{\partial E}{\partial V}\right)_S = -p, \qquad \left(\frac{\partial E}{\partial S}\right)_V = T. \tag{23}$$

The resulting equation $TS = E + pV$ is equivalent to the previously mentioned relation for the entropy density s.

For a CFT with a large central charge the entropy and energy are not purely extensive. In a finite volume the energy E of a CFT contains a non-extensive Casimir contribution proportional to c. This is well known in (1+1) dimensions where it gives rise to the familiar shift of $c/24$ in the L_0 Virasoro operator. The Casimir energy is the result of finite size effects in the quantum fluctuations of the CFT, and disappears when the volume becomes infinitely large. It therefore leads to sub-extensive contributions to the total energy E. Usually the Casimir effect is discussed at zero temperature [17], but a similar effect occurs at finite temperature. The value of the Casimir energy will in that case generically depend on the temperature T.

We will now define the Casimir energy as the violation of the Euler identity (22)

$$E_C \equiv n(E + pV - TS) \tag{24}$$

[1]We assume here that there are no other thermodynamic functions like a chemical or electric potential. For a system with a 1st law like $TdS = dE + pdV + \mu dN + \Phi dQ$ the Euler relation reads $TS = E + pV + \mu N + \Phi Q$.

Here we inserted for convenience a factor equal to the spatial dimension n. From the previous discussion it is clear that E_C parameterizes the sub-extensive part of the total energy. The Casimir energy will just as the total energy be a function of the entropy S and the volume V. Under $S \to \lambda S$ and $V \to \lambda V$ it scales with a power of λ that is smaller than one. On general grounds one expects that the first subleading correction to the extensive part of the energy scales like

$$E_C(\lambda S, \lambda V) = \lambda^{1-2/n} E_C(S, V) \tag{25}$$

One possible way to see this is to write the energy as an integral over a local density expressed in the metric and its derivatives. Derivatives scale like $\lambda^{-1/n}$ and because derivatives come generally in pairs, the first subleading terms indeed has two additional factors of $\lambda^{-1/n}$. The total energy $E(S, V)$ may be written as a sum of two terms

$$E(S, V) = E_E(S, V) + \frac{1}{2} E_C(S, V) \tag{26}$$

where the first term E_E denotes the purely extensive part of the energy E and E_C represents the Casimir energy. Again the factor $1/2$ has been put in for later convenience. By repeating the steps that lead to the Euler relation one easily verifies the defining equation (24) for the Casimir energy E_C.

4.2. Universality of the Cardy formula and the Bekenstein bound

Conformal invariance implies that the product ER is independent of the volume V, and is only a function of the entropy S. This holds for both terms E_E and E_C in (26). Combined with the known (sub-)extensive behavior of E_E and E_C this leads to the following general expressions

$$E_E = \frac{a}{4\pi R} S^{1+1/n} \qquad\qquad E_C = \frac{b}{2\pi R} S^{1-1/n}$$

where a and b are a priori arbitrary positive coefficients, independent of R and S. The factors of 4π and 2π are put in for convenience. With these expressions, one now easily checks that the entropy S can be written as

$$S = \frac{2\pi R}{\sqrt{ab}} \sqrt{E_C(2E - E_C)}. \tag{27}$$

If we ignore for a moment the normalization, this is exactly the Cardy formula: insert $ER = L_0$ and $E_C R = c/12$, and one recovers (3). It is obviously an interesting question to compute the coefficients a and b for various known conformal invariant field theories. This should be particularly straightforward for free field theories, such as $d = 4$ Maxwell theory and the self-dual tensor theory in $d = 6$. This question is left for future study.

Given the energy E the expression (27) has a maximum value. For all values of E, E_C and R one has the inequality

$$S \leq \frac{2\pi}{\sqrt{ab}} ER$$

This looks exactly like the Bekenstein bound, except that the pre-factor is in general different from the factor $2\pi/n$ used in the previous section. In fact, in the following subsection we will show that for CFTs with an AdS-dual description, the value of the product ab is exactly equal to n^2, so the upper limit is indeed exactly given by the Bekenstein entropy. Although we have no proof of this fact, we believe that the Bekenstein bound is universal. This implies that the product ab for all CFTs in $n+1$ dimensions is larger or equal than n^2. Only then it is guaranteed that the upper limit on the entropy is less or equal than S_B.

The upper limit is reached when the Casimir energy E_C is equal to the total energy E. Formally, when E_C becomes larger that E the entropy S will again decrease. Although in principle this is possible, we believe that in actual examples the Casimir energy E_C is bounded by the total energy E. So, from now on we assume that

$$E_C \leq E \tag{28}$$

In the next subsection we provide further evidence for this inequality.

From now on we will assume that we are dealing with a CFT for which $ab = n^2$. In the next section I will show that this includes all CFTs that have an AdS-dual description.

4.3. The Cardy formula derived from AdS/CFT

Soon after Maldacena's AdS/CFT-correspondence [18] was properly understood [19, 20] it was convincingly argued by Witten [21] that the entropy, energy and temperature of CFT at high temperatures can be identified with the entropy, mass, and Hawking temperature of the AdS black hole previously considered by Hawking and Page [22]. Using this duality relation the following expressions can be derived for the energy and entropy[2] for a $D = n+1$ dimensional CFT on $R \times S^n$:

$$\begin{aligned} S &= \frac{c}{12}\frac{V}{L^n} \\ E &= \frac{c}{12}\frac{n}{4\pi L}\left(1 + \frac{L^2}{R^2}\right)\frac{V}{L^n} \end{aligned} \tag{29}$$

[2]These expressions differ somewhat from the presented formulas in [21] due to the fact that (i) the $D+1$ dimensional Newton constant has been eliminated using its relation with the central charge, (ii) the coordinates have been re-scaled so that the CFT lives on a sphere with radius equal to the black hole horizon. We will not discuss the AdS perspective in this paper, since the essential physics can be understood without introducing an extra dimension. The discussion of the CFT/FRW cosmology from an AdS perspective will be described elsewhere [3].

The temperature again follows from the first law of thermodynamics. One finds

$$T = \frac{1}{4\pi L}\left((n+1) + (n-1)\frac{L^2}{R^2}\right). \tag{30}$$

The length scale L of the thermal CFT arises in the AdS/CFT correspondence as the curvature radius of the AdS black hole geometry. The expression for the energy clearly exhibits a non-extensive contribution, while also the temperature T contains a corresponding non-intensive term. Inserting the equations (29,30) into (24) yields the following result for the Casimir energy

$$E_C = \frac{c}{12}\frac{n}{2\pi R}\frac{V}{L^{n-1}R}. \tag{31}$$

Now let us come to the Cardy formula. The entropy S, energy E and Casimir energy E_C are expressed in c, L and R. Eliminating c and L leads to a unique expression for S in terms of E, E_C and R. One easily checks that it takes the form of the Cardy formula

$$S = \frac{2\pi R}{n}\sqrt{E_C\,(2E - E_C)} \tag{32}$$

In the derivation of these formulas it was assumed that $R \gg L$. One may worry therefore that these formulas are not applicable in the early universe. Fortunately this is not a problem because during an adiabatic expansion both L and R scale in the same way so that R/L is fixed. Hence the formulas are valid provided the (fixed) ratio of the thermal wave-length and the radius R is much smaller than one. Effectively this means, as far as the CFT is concerned, we are in a high temperature regime. We note further that with in this parameter range, the Casimir energy E_C is indeed smaller than the total energy E.

Henceforth, we will assume that the CFT that describes the radiation in the FRW universe will have an entropy given by (32) with the specific normalization of $2\pi/n$. Note that if we take $n = 1$ and make the previously mentioned identifications $ER = L_0$ and $E_C R = c/12$ that this equation exactly coincides with the usual Cardy formula. We will therefore in the following refer to (32) simply as the Cardy formula. To check the precise coefficient of the Cardy formula for a CFT we have made use of the AdS/CFT correspondence. The rest of our discussions in the preceding and in the following sections do not depend on this correspondence. So, in this paper we will not make use of any additional dimensions other than the ones present in the FRW-universe.

5. A new cosmological bound

In this section a new cosmological bound will be presented, which is equivalent to the Hubble bound in the strongly gravitating phase, but which unlike the Hubble bound remains valid in the phase of weak self-gravity. When the bound is saturated the FRW equations and the CFT formulas for the entropy and Casimir energy completely coincide.

5.1. A cosmological bound on the Casimir energy

Let us begin by presenting another criterion for distinguishing between a weakly or strongly self-gravitating universe. When the universe goes from the strongly to the weakly self-gravitating phase, or vice-versa, the Bekenstein entropy S_B and the Bekenstein-Hawking entropy S_{BH} are equal in value. Given the radius R, we now define the 'Bekenstein-Hawking' energy E_{BH} as the value of the energy E for which S_B and S_{BH} are exactly equal. This leads to the condition

$$\frac{2\pi}{n} E_{BH} R \equiv (n-1)\frac{V}{4GR}. \tag{33}$$

One may interpret E_{BH} as the energy required to form a black hole with the size of the entire universe. Now, one easily verifies that

$$\begin{array}{lllll} E & \leq & E_{BH} & \text{for} & HR \leq 1 \\ E & \geq & E_{BH} & \text{for} & HR \geq 1. \end{array} \tag{34}$$

Hence, the universe is weakly self-gravitating when the total energy E is less than E_{BH} and strongly gravitating for $E > E_{BH}$.

We are now ready to present a proposal for a new cosmological bound. It is not formulated as a bound on the entropy S, but as a restriction on the Casimir energy E_C. The physical content of the bound is the Casimir energy E_C by itself can not be sufficient to form a universe-size black hole. Concretely, this implies that the Casimir energy E_C is less or equal to the Bekenstein-Hawking energy E_{BH}. Hence, we postulate

$$E_C \leq E_{BH} \tag{35}$$

To put the bound in a more conventional notation one may insert the definition (24) of the Casimir energy together with the defining relation (33) of the Bekenstein-Hawking energy. We leave this to the reader.

The virtues of the new cosmological bound are: (i) it is universally valid and does not break down for a weakly gravitating universe, (ii) in a strongly gravitating universe it is equivalent to the Hubble bound, (iii) it is purely holographic and can be formulated in terms of the Bekenstein-Hawking entropy S_{BH} of a universe-size black hole, (iv) when the bound is saturated the laws of general relativity and quantum field theory converge in a miraculous way, giving a strong indication that they have a common origin in a more fundamental unified theory.

The first point on the list is easily checked because E_C decays like R^{-1} while E_{BH} goes like R^{-n}. Only when the universe re-collapses and returns to the strongly gravitating phase the bound may again become saturated. To be able to proof the other points on the list of advertised virtues, we have to take a closer look to the FRW equations and the CFT formulas for the entropy an entropy.

5.2. A cosmological Cardy formula

To show the equivalence of the new bound with the Hubble bound let us write the Friedman equation as an expression for the Hubble entropy S_H in terms of the energy E, the radius R and the Bekenstein-Hawking energy E_{BH}. Here, the latter is used to remove the explicit dependence on Newton's constant G. The resulting expression is unique and takes the form

$$S_H = \frac{2\pi}{n} R \sqrt{E_{BH} \left(2E - E_{BH}\right)} \tag{36}$$

This is exactly the Cardy formula (32), except that the role of the Casimir energy E_C in CFT formula is now replaced by the Bekenstein-Hawking energy E_{BH}. Somehow, miraculously, the Friedman equation knows about the Cardy formula for the entropy of a CFT!

With the help of (36) is now a straightforward matter to proof that when $HR \geq 1$ the new bound $E_C \leq E_{BH}$ is equivalent to the Hubble bound $S \leq S_H$. First, let us remind that for $HR \geq 1$ the energy E satisfies $E \geq E_{BH}$. Furthermore, we always assume that the Casimir energy E_C is smaller than the total energy E. The entropy S is a monotonically increasing function of E_C as long as $E_C \leq E$. Therefore in the range

$$E_C \leq E_{BH} \leq E \tag{37}$$

the maximum entropy is reached when $E_C = E_{BH}$. In that case the Cardy formula (32) for S exactly turns into the cosmological Cardy formula (36) for S_H. Therefore, we conclude that S_H is indeed the maximum entropy that can be reached when $HR \geq 1$. Note that in the weakly self-gravitating phase, when $E \leq E_{BH}$, the maximum is reached earlier, namely for $E_C = E$. The maximum entropy is in that case given by the bekenstein entropy S_B. The bifurcation of the new bound in two entropy bounds is a direct consequence of the fact that the Hubble bound is written as the square-root of a quadratic expression.

5.3. A limiting temperature

So far we have focussed on the entropy and energy of the CFT and on the first of the two FRW equations, usually referred to as the Friedman equation. We will now show that also the second FRW equation has a counterpart in the CFT, and will lead to a constraint on the temperature T. Specifically, we will find that the bound on E_C implies that the temperature

T in the early universe is bounded from below by

$$T_H \equiv -\frac{\dot{H}}{2\pi H} \tag{38}$$

The minus sign is necessary to get a positive result, since in a radiation dominated universe the expansion always slows down. Further, we assume that we are in the strongly self-gravitating phase with $HR \geq 1$, so that there is no danger of dividing by zero.

The second FRW equation in (5) can now be written as a relation between E_{BH}, S_H and T_H that takes the familiar form

$$E_{BH} = n(E + pV - T_H S_H) \tag{39}$$

This equation has exactly the same form as the defining relation $E_C = n(E + pV - TS)$ for the Casimir energy. In the strongly gravitating phase we have just argued that the bound $E_C \leq E_{BH}$ is equivalent to the Hubble bound $S \leq S_H$. It follows immediately that the temperature T in this phase is bounded from below by T_H. One has

$$T \geq T_H \qquad \text{for} \quad HR \geq 1 \tag{40}$$

When the cosmological bound is saturated all inequalities turn into equalities. The Cardy formula and the defining Euler relation for the Casimir energy in that case exactly match the Friedman equation for the Hubble constant and the FRW equation for its time derivative.

6. The entropy bounds revisited.

We now return to the cosmological entropy bounds introduced in sections 2 and 3. In particular, we are interested in the way that the entropy of the CFT may be incorporated in the entropy diagram described in section 3. For this purpose it will be useful to introduce a non-extensive component of the entropy that is associated with the Casimir energy.

The cosmological bound $E_C \leq E_{BH}$ can also be formulated as an entropy bound, not on the total entropy, but on a non-extensive part of the entropy that is associated with the Casimir energy. In analogy with the definition of the Bekenstein entropy (8) one can introduce a 'Casimir' entropy defined by

$$S_C \equiv \frac{2\pi}{n} E_C R. \tag{41}$$

For $d = (1+1)$ the Casimir entropy is directly related to the central charge c. One has $S_C = 2\pi\, c/12$. In fact, it is more appropriate to interpret the Casimir entropy S_C as a

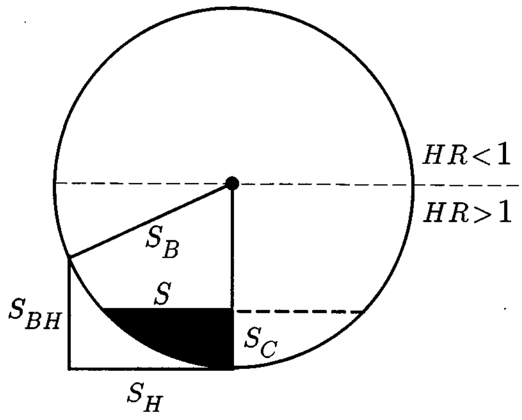

Fig.2. *The entropy S and Casimir entropy S_C fill part of the cosmological entropy diagram. The diagram shows: (i) the Bekenstein bound $S \le S_B$ is valid at all times (ii) the Hubble bound $S \le S_H$ restricts the allowed range of η in the range $HR > 1$, but is violated for $HR < 1$, (iii) the new bound $S_C \le S_{BH}$ is equivalent to the Hubble bound for $HR > 1$, and remains valid for $HR < 1$.*

generalization of the central charge to $n{+}1$ dimensions than what is usually called the central charge c. Indeed, if one introduces a dimensionless 'Virasoro operator' $\widetilde{L}_0 \equiv \frac{1}{2\pi} S_B$ and a new central charge $\frac{\widetilde{c}}{12} \equiv \frac{1}{2\pi} S_C$, the $n{+}1$ dimensional entropy formula (32) is exactly identical to (3).

The Casimir entropy S_C is sub-extensive because under $V \to \lambda V$ and $E \to \lambda E$ it goes like $S_C \to \lambda^{1-1/n} S_C$. In fact, it scales like an area! This is a clear indication that the Casimir entropy has something to do with holography. The total entropy S contains extensive as well as sub-extensive contributions. One can show that for $E_C \le E$ the entropy S satisfies the following inequalities

$$S_C \le S \le S_B \tag{42}$$

where both equal signs can only hold simultaneously. The precise relation between S and its super- and sub-extensive counterparts S_B and S_C is determined by the Cardy formula, which can be expressed as

$$S^2 + (S_B - S_C)^2 = S_B^2. \tag{43}$$

This identity has exactly the same form as the relation (17) between the cosmological entropy bounds, except that in (17) the role of the entropy and Casimir entropy are taken over by the Hubble entropy S_H and Bekenstein-Hawking entropy S_{BH}. This fact will be used to incorporate the entropy S and the Casimir entropy S_C in the entropy diagram introduce in section 3.

The cosmological bound on the Casimir energy presented in the section 4 can be formulated as an upper limit on the Casimir entropy S_C. From the definitions of S_C and E_{BH} it follows directly that the bound $E_C \leq E_{BH}$ is equivalent to

$$S_C \leq S_{BH} \tag{44}$$

where we made use of the relation (33) to re-write E_{BH} again in terms of the Bekenstein-Hawking entropy S_{BH}. Thus the bound puts a holographic upper limit on the d.o.f. of the CFT as measured by the Casimir entropy S_C.

In figure 2 we have graphically depicted the quadratic relation between the total entropy S and the Casimir entropy S_C in the same diagram we used to related the cosmological entropy bounds. From this diagram it easy to determine the relation between the new bound and the Hubble bound. One clearly sees that when $HR > 1$ that the two bounds are in fact equivalent. When the new bound is saturated, which means $S_C = S_{BH}$, then the Hubble bound is also saturated, *ie.* $S = S_H$. The converse is not true: there are two moments in the region $HR < 1$ when the $S = S_H$, but $S_C \neq S_{BH}$. In our opinion, this is an indication that the bound on the Casimir energy has a good chance of being a truly fundamental bound.

7. Summary and conclusion

In this paper we have used the holographic principle to study the bounds on the entropy in a radiation dominated universe. The radiation has been described by a continuum CFT in the bulk. Surprisingly the CFT appears to know about the holographic entropy bounds, and equally surprising the FRW-equations know about the entropy formulas for the CFT. Our main results are summarized in the following two tables. Table 1. contains an overview of the bounds that hold in the early universe on the temperature, entropy and Casimir energy. In table 2. the Cardy formula for the CFT and the Euler relation for the Casimir energy are matched with the Friedman equations written in terms of the quantities listed in table 1.

CFT-bound	*FRW-definition*
$T \geq T_H$	$T_H \equiv -\dot{H}/2\pi H$
$S \leq S_H$	$S_H \equiv (n-1)HV/4G$
$E_C \leq E_{BH}$	$E_{BH} \equiv n(n-1)V/8\pi G R^2$

Table 1: summary of cosmological bounds

CFT-formula	*FRW-equation*
$S = \frac{2\pi R}{n}\sqrt{E_C(2E - E_C)}$	$S_H = \frac{2\pi R}{n}\sqrt{E_{BH}(2E - E_{BH})}$
$E_C \equiv n(E + pV - TS)$	$E_{BH} = n(E + pV - T_H S_H)$

Table 2: Matching of the CFT-formulas with the FRW-equations

The presented relation between the FRW equations and the entropy formulas precisely holds at this transition point, when the holographic bound is saturated or threatens to be violated. The miraculous merging of the CFT and FRW equations strongly indicates that both sets of these equations arise from a single underlying fundamental theory.

The discovered relation between the entropy, Casimir energy and temperature of the CFT and their cosmological counterparts has a very natural explanation from a RS-type braneworld scenario [23] along the lines of [24]. The radiation dominated FRW equations can be obtained by studying a brane with fixed tension in the background of a AdS-black hole. In this description the radius of the universe is identified with the distance of the brane to the center of the black hole. At the Big Bang the brane originates from the past singularity. At some finite radius determined by the energy of the black hole, the brane crosses the horizon. It keeps moving away from the black hole, until it reaches a maximum distance, and then it falls back into the AdS-black hole. The special moment when the brane crosses the horizon precisely corresponds to the moment when the cosmological entropy bounds are saturated. This world-brane perspective on the cosmological bounds for a radiation dominated universe will be described in detail in [3].

We have restricted our attention to matter described by a CFT in order to make our discussion as concrete and coherent as possible. Many of the used concepts, however, such as the entropy bounds, the notion of a non-extensive entropy, the matching of the FRW equations, and possibly even the Cardy formula are quite independent of the equation of state of the matter. One point at which the conformal invariance was used is in the diagrammatic representation of the bounds. The diàgram is only circular when the energy E goes like R^{-1}. But it is possible that a similar non-circular diagram exists for other kinds of matter. It would be interesting to study other examples in more detail.

Finally, the cosmological constant has been put to zero, since only in that case all of the formulas work so nicely. It is possible to modify the formalism to incorporate a cosmological constant, but the analysis becomes less transparent. In particular, one finds that the Hubble entropy bound needs to be modified by replacing H with the square root of $H^2 - \Lambda/n$. At this moment we have no complete understanding of the case $\Lambda \neq 0$, and postpone its discussion to future work.

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CFT and Entropy on the Brane

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Abstract

We consider a brane-universe in the background of an Anti-de Sitter/Schwarschild geometry. We show that the induced geometry of the brane is exactly given by that of a standard radiation dominated FRW-universe. The radiation is represented by a strongly coupled CFT with an *AdS*-dual description. We show that when the brane crosses the horizon of the *AdS*-black hole the entropy and temperature are simply expressed in the Hubble constant and its time derivative. We present formulas for the entropy of the CFT which are generally valid, and which at the horizon coincide with the FRW equations. These results shed new light on recently proposed entropy bounds in the context of cosmology.

1. Introduction

Recently the holographic principle was studied in a Friedmann-Robertson-Walker (FRW) universe filled with a conformal field theory (CFT) with a dual anti-de Sitter (AdS) description [1], see also [2–7]. An interesting and surprising relationship was found between the FRW equations controlling the cosmological expansion and the formulas that relate the energy and entropy of the CFT. The aim of the present paper will be to shed further light on this coincidence by studying the CFT/FRW-cosmology from a Randall-Sundrum type brane-world perspective [8,9].

Brane cosmology has previously been studied from an AdS/CFT perspective in [10,11]. Following these papers we describe the CFT dominated universe as a co-dimension one brane, with fixed tension, in the background of an AdS-black hole. In this description the movement of the brane turns out to be exactly described by the standard Friedmann equation in which the size of the universe directly corresponds to the distance of the brane to the center of the black hole. The brane starts out inside the black hole, it passes through the horizon and keeps expanding until it reaches a maximal radius, after which it re-contracts and falls back into the black hole. From the AdS-perspective there are two special moments, one in the early and one in the late universe, when the brane crosses the horizon. The main goal of this paper is to show that at those moments the entropy density on the brane takes a special value given in terms of the Hubble constant and Newtons constant. Furthermore, at these times the Friedmann equation turns into an equation that expresses the entropy density in terms of the energy density and exactly coincides with a generalized form of the Cardy formula for the entropy of the CFT.

We begin by presenting the brane description of a CFT-dominated cosmology in section 2. The dimension $d = n+1$ of the brane-universe will be taken to be arbitrary, but its relation with the dimension $D = d+1$ of the AdS space is of course fixed. In section 3 we argue that the radiation on the brane can be identified with the CFT dual to the AdS-space and use this fact to fix the normalization of Newtons constant and derive the FRW equations. The entropy density and temperature of the CFT at the moment that the brane crosses the horizon are calculated in section 4. We find that these quantities have a simple expression in terms of the Hubble constant and its time-derivative. In section 5 we derive the entropy formulas for the CFT and show the correspondence with the FRW equations. Finally, sections 6 and 7 contain some concluding remarks.

2. Brane cosmology

We consider an $(n+1)$-dimensional brane with a constant tension in the background of an $(n+2)$-dimensional AdS-Schwarzschild black hole. Following the AdS/CFT prescription [12,13] we regard the brane as the boundary of the AdS-geometry. An important difference is, however, that now the location and the metric on the boundary are, at least partly, dynamical. The movement of the brane is described by the boundary action

$$\mathcal{L}_b = \frac{-1}{8\pi \mathbf{G}_N} \int_{\partial\mathcal{M}} \sqrt{g}\,\mathcal{K} + \frac{\kappa}{8\pi \mathbf{G}_N} \int_{\partial\mathcal{M}} \sqrt{g}\,. \tag{1}$$

Here $\mathcal{K} \equiv \mathcal{K}_i^{\ i}$ is the trace of the extrinsic curvature, κ is a parameter related to the tension of the brane, $\mathbf{G}_N$ is the $(n+2)$-dimensional bulk Newton constant, g is the determinant of the induced metric and $\partial\mathcal{M}$ denotes the surface of the brane. The equation of motion of the brane that follows from this Lagrangian is

$$\mathcal{K}_{ij} = \frac{\kappa}{n} g_{ij}^{\text{induced}}. \tag{2}$$

This equation implies that $\partial\mathcal{M}$ is a surface of constant extrinsic curvature.

The bulk action is given by the $(n+2)$-dimensional Einstein action with cosmological term. The AdS-Schwarzschild metric provides a solution of the bulk equations of motion and can be written in the following form,

$$ds_{n+2}^2 = \frac{1}{h(a)}da^2 - h(a)dt^2 + a^2 d\Omega_n^2\,, \tag{3}$$

$$h(a) = \frac{a^2}{L^2} + 1 - \frac{\omega_{n+1} M}{a^{n-1}}, \tag{4}$$

where

$$\omega_{n+1} = \frac{16\pi \mathbf{G}_N}{n\text{Vol}(\mathbf{S}^n)}. \tag{5}$$

In these equations, L is the curvature radius of AdS. The pre-factor ω_{n+1} is chosen such that M is the mass of the black hole as measured by an observer who uses t as his time coordinate.

Our aim is to find the spherically symmetric solutions corresponding to a homogeneous and isotropic induced metric on the brane. Let us parameterize the location of the brane by giving a as a function of the AdS-time t. Equivalently, we may introduce a new time parameter τ and specify the functions

$$a = a(\tau),\ t = t(\tau). \tag{6}$$

We will choose the time parameter τ such that the following relation is satisfied,

$$\frac{1}{h(a)}\left(\frac{da}{d\tau}\right)^2 - h(a)\left(\frac{dt}{d\tau}\right)^2 = -1. \tag{7}$$

This condition ensures that the induced metric on the brane takes the standard Robertson-Walker form,

$$ds_{n+1}^2 = -d\tau^2 + a^2(\tau)d\Omega_n^2\,. \tag{8}$$

We note that the size of the $(n+1)$-dimensional universe is determined by the radial distance, a, from the center of the black hole.

The extrinsic curvature, $\mathcal{K}_{ij}$, of the brane can be straightforwardly calculated and expressed in term of the functions $a(\tau)$ and $t(\tau)$. One then finds that the equation of motion (2) translates into

$$\frac{dt}{d\tau} = \frac{\kappa a}{h(a)}. \tag{9}$$

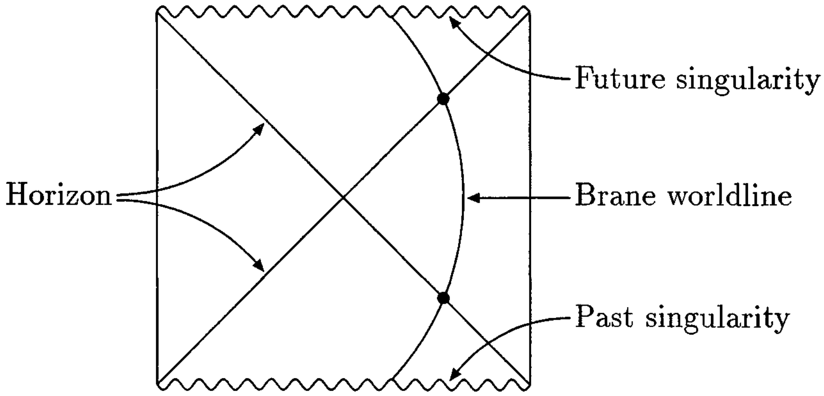

Figure 1: Penrose diagram of an AdS_{n+2}-Schwarzschild black hole with the trajectory of the brane. The brane originates in the past singularity, expands to a certain size and subsequently falls into the future singularity as it re-collapses. The dots indicate the moments when the brane crosses the black hole horizon.

In the following we will tune the $(n+1)$-dimensional cosmological constant to zero by setting $\kappa = 1/L$. Combining (9) with (7) leads to an equation that looks suspiciously like the Friedmann equation for a radiation dominated universe,

$$H^2 = -\frac{1}{a^2} + \frac{\omega_{n+1} M}{a^{n+1}}. \tag{10}$$

In this equation, $H \equiv \dot{a}/a$ is the Hubble 'constant' and the dot denotes differentiation with respect to the cosmological time τ. For future purpose, we also give the equation for the time derivative of H,

$$\dot{H} = \frac{1}{a^2} - \frac{(n+1)}{2} \frac{\omega_{n+1} M}{a^{n+1}}, \tag{11}$$

which is simply obtained by differentiating (10).

3. CFT on the brane

We now want to identify the equation of motion (10) with the $(n+1)$-dimensional Friedmann equation. In particular, we will argue that the radiation can be identified with the finite temperature CFT that is dual to the AdS-geometry. To do so, we interpret the last term on the r.h.s. as the contribution of the energy density ρ of the CFT times the $(n+1)$-dimensional Newton constant G_N. In the brane-world scenario the relation between the Newton constant $\mathbf{G}_N$ in the bulk and the Newton constant G_N on the brane is given by

$$\mathbf{G}_N = \frac{G_N L}{(n-1)}. \tag{12}$$

One possible way to derive this fact is to add a small amount of stress energy on the brane and determine how it effects the equation of motion. This same relation is, as we will discuss, also consistent with the identification of the radiation with the dual CFT.

In [14] it was argued that the energy, entropy and temperature of a CFT at high temperatures can be identified with the mass, entropy and Hawking temperature of the AdS black hole [15]. The CFT lives on a space-time which, after Euclidean continuation, has the topology of $\mathbf{S}^1 \times \mathbf{S}^n$ and whose geometry is identified with the asymptotic boundary of the Euclidean AdS-black hole. We remind the reader that the standard GKPW prescription [12,13] of the AdS/CFT correspondence [16] only fixes the conformal class of the CFT metric. It thus specifies only the ratio of the radius of the n-sphere to the Hawking temperature but does not fix the overall scale of the boundary metric. One is therefore free to re-scale the metric as one wishes. It is important to note, however, that such a rescaling does also affect the energy and temperature of the CFT.

To make this more precise, let us consider the asymptotic form of the AdS-Schwarschild metric. We have

$$\lim_{a\to\infty}\left[\frac{L^2}{a^2}\,ds^2_{n+2}\right] = -dt^2 + L^2 d\Omega_n^2\,, \tag{13}$$

from which we see that the CFT time is equal to the AdS time t only when the radius of the spatial sphere is set equal to L. Therefore, if we want the sphere to have a radius equal to say a, the CFT time will be equal to at/L. The same factor a/L then appears in the relation between the energy E and the black hole mass M. One thus finds that the energy for a CFT on a sphere with radius a, of volume

$$V = a^n \mathrm{Vol}(\mathbf{S}^n),$$

is given by

$$E = M\frac{L}{a}. \tag{14}$$

Note that the total energy E is not constant during the cosmological expansion, but decreases like a^{-1}. This is consistent with the fact that for a CFT the energy density,

$$\rho = \frac{E}{V},$$

scales like $a^{-(n+1)}$. Inserting the relation (14) combined with (12) into the equation of motion (10) leads to

$$H^2 = -\frac{1}{a^2} + \frac{16\pi G_N}{n(n-1)}\rho\,. \tag{15}$$

This is the standard Friedmann equation with the appropriate normalization for both terms. By differentiating once with respect to τ and using the fact that $\dot\rho = nH(\rho+p)$, one derives the second FRW equation,

$$\dot H = \frac{1}{a^2} - \frac{8\pi G_N}{(n-1)}(\rho + p)\,, \tag{16}$$

which is equivalent to (11). An observer on the brane, who knows nothing about the AdS-bulk gravity, just notices the normal cosmological expansion. The brane description contains more information, since it also knows about the size of the AdS-black hole.

The movement of the brane in the AdS-black hole background is depicted in the Penrose diagram in figure 1. The diagram represents the full geodesically complete black hole geometry including the asymptotic region $a \to \infty$. If one wants to take the brane as the real boundary, one has to cut away the part to the right of the brane. We see that the brane indeed starts inside the black hole at the past singularity and then, as it expands, it moves away from $a = 0$. At late times it does the opposite. The points where the brane crosses the black hole horizon will play a central role in the following discussion and have been marked in the figure. These moments are clearly distinguished from the AdS-perspective, even though nothing special happens to the induced geometry on the brane. So what do these moments mean for an observer on the brane?

4. Entropy and temperature at the horizon

Let us now consider the points at which the brane crosses the horizon. The horizon of the AdS-black hole is located at radius $a = a_H$, where a_H is the largest solution to the equation $h(a) = 0$, i.e.

$$\frac{a_H^2}{L^2} + 1 - \frac{\omega_{n+1} M}{a_H^{n-1}} = 0 . \tag{17}$$

From this equation and the equation of motion (10), one immediately concludes that the Hubble constant at the horizon obeys

$$H^2 = \frac{1}{L^2},$$

and hence $H = \pm 1/L$ depending on whether the brane is expanding or contracting.

Next, let us consider the entropy density. According to [14], the entropy of the CFT is equal to the Bekenstein-Hawking entropy of the AdS-black hole, which is given by the area of the horizon measured in bulk planckian units. The total entropy may thus be expressed as

$$S = \frac{V_H}{4\mathbf{G}_N}, \tag{18}$$

where V_H is the area of the horizon,

$$V_H \equiv a_H^n \mathrm{Vol}(\mathbf{S}^n).$$

Note that the area of an n-sphere in AdS equals the volume of the corresponding spatial section for an observer on the brane. The total entropy S is constant during the cosmological evolution but the entropy density,

$$s = \frac{S}{V},$$

of course varies with time. It equals

$$s = (n-1)\frac{a_H^n}{4G_N L a^n}, \tag{19}$$

where we made use of the relation (12). What makes the moments that the brane crosses the horizon special is that the entropy density is given by a simple multiple of the Hubble constant H. At the horizon $V = V_H$ and hence the entropy density on the brane is $s = 1/4\mathbf{G}_N$. Now, using the relation (12) and the fact that $H = 1/L$ one finds that the entropy density equals

$$s = (n-1)\frac{H}{4G_N}, \qquad \text{at } a = a_H\,. \tag{20}$$

The significance of this relation will be further discussed below.

Also the temperature turns out to have a special value at the horizon. The Hawking temperature measured by an observer who uses t as his time coordinate is [10,14]

$$T_H = \frac{h'(a_H)}{4\pi}, \tag{21}$$

where the prime denotes differentiation with respect to a. Since the CFT time differs from t by a factor a/L the CFT-temperature T will differ from the Hawking temperature T_H by the same a-dependent factor,

$$T = T_H \frac{L}{a}. \tag{22}$$

Using the explicit form of $h'(a_H)$ and using the fact that $h(a_H) = 0$, we eventually find

$$T = \frac{1}{4\pi a}\left((n+1)\frac{a_H}{L} + (n-1)\frac{L}{a_H}\right). \tag{23}$$

Now, from the derivation of the brane equation of motion, it follows that the quantities H^2 and $-h(a)/a^2$ only differ by a constant and therefore, at the horizon where $h(a_H) = 0$, we have that $\dot{H} = -h'(a_H)/2a_H$. This can be used to show that the temperature at the horizon may be expressed in the Hubble constant H and its time derivative $\dot{H}$ as

$$T = -\frac{\dot{H}}{2\pi H}, \qquad \text{at } a = a_H\,. \tag{24}$$

5. Entropy formulas and FRW equations

The above relations between the entropy density and temperature on the one hand, and the Hubble constant, its time derivative and Newtons constant on the other are valid only when the brane crosses the horizon. However, since the entropy density, temperature and energy density all vary in a precisely prescribed manner as a function of the radius a, these relations imply a set of entropy formulas that remain valid at all times.

Before making this point clear, let us first briefly discuss some basic thermodynamics. The first law of thermodynamics,

$$TdS = dE + pdV,$$

can after some straightforward manipulations be rewritten in terms of the entropy and energy densities s and ρ as

$$Tds = d\rho + n(\rho + p - Ts)\frac{da}{a}, \tag{25}$$

where we used $dV = nVda/a$. The combination $(\rho + p - Ts)$ is in most standard textbooks on cosmology [17, 18] assumed to vanish, which is equivalent to saying that the entropy and energy are purely extensive. But let us now compute it for the CFT. The energy density is given by

$$\rho = \frac{ML}{a^{n+1}\mathrm{Vol}(\mathbf{S}^n)}. \tag{26}$$

For our purpose, it is convenient to rewrite ρ in terms of the horizon radius a_H using $h(a_H) = 0$. This gives

$$\rho = \frac{n a_H^n}{16\pi \mathrm{G}_N a^{n+1}}\left(\frac{L}{a_H} + \frac{a_H}{L}\right). \tag{27}$$

The pressure follows from ρ through the equation of state $p = \rho/n$. Combined with (19) and (23), one gets

$$\frac{n}{2}(\rho + p - Ts) = \frac{\gamma}{a^2}, \tag{28}$$

where the quantity γ is given by

$$\gamma = \frac{n(n-1)a_H^{n-1}}{16\pi G_N a^{n-1}}. \tag{29}$$

Equation (28) may be regarded as the definition of γ. Physically one can think of γ as describing the response of the energy density under variations of the radius a or, more precisely, the spatial curvature $1/a^2$. It thus represents the geometrical Casimir part of the energy density.

We are now ready to present the main entropy formula for CFT's with an AdS dual. In [1] an entropy formula was already derived and expressed in terms of the total energy and entropy. Here we will give the local version in terms of densities. From the given expressions for the entropy density s, energy density ρ and γ, one finds that s may be expressed as

$$s^2 = \left(\frac{4\pi}{n}\right)^2 \gamma\left(\rho - \frac{\gamma}{a^2}\right). \tag{30}$$

As noted in [1], this formula resembles the Cardy formula of a $(1+1)$-dimensional CFT but is valid for all spatial dimensions n.

The formulas (28) and (30) are valid at all times. It will be interesting, however, to study these formulas at the special time when the brane crosses the horizon. First we note that at that time the Casimir quantity γ equals

$$\gamma = \frac{n(n-1)}{16\pi G_N}, \qquad \text{at } a = a_H \,. \tag{31}$$

Let us now consider the entropy formula (30). By making the identifications (20) and (31) one sees that this formula exactly reproduces the Friedmann equation! Similarly, one finds that equation (28) reduces to the second FRW equation for $\dot{H}$ by making the same substitutions for s and γ and replacing the temperature T by the r.h.s. of (24). In fact, the equations (28) and (30) are equations of state of the CFT and in principle have an interpretation that is independent of gravity or cosmology. It seems therefore rather surprising that the Friedmann equation knows about the thermodynamic properties of the CFT.

6. Euclidean brane cosmology

In principle one can use the present setup to calculate the correlation functions of operators in the CFT/FRW cosmology, in particular the stress energy tensor, using the same methods as in the standard AdS/CFT setup. This would for example give information about fluctuations in the energy density in the early universe. As described above, the brane starts out as a point in the past singularity of the black hole. The presence of this singularity may lead to problems in performing these calculations in Minkowski signature. On the gravity side a singularity is associated with the UV properties of the theory, i.e. to very high energies. However, through the UV/IR-connection [19] known from AdS/CFT, on the field theory side this in fact corresponds to the IR, i.e. to very low energies. As it is the CFT that describes the matter in the universe, this seems strange since conventionally one associates the UV with the early universe.

To calculate correlation functions one can circumvent this problem by analytically continuing to the Euclidean setup. So let us briefly discuss how to describe the Euclidean FRW universe as a brane in an Euclidean AdS-Schwarzschild background. Going through the calculation in a similiar way as performed above, one arrives at the following Friedmann equation

$$H_{\mathrm{E}}^2 = \frac{1}{a^2} - \frac{16\pi G_N}{n(n-1)}\rho. \tag{32}$$

From this one easily deduces that the universe, when regarded in Euclidean time, undergoes a reverse evolution, starting out very big, collapsing to a minimal size and subsequently re-expanding. This is depicted in figure 2. From the CFT point of view, this means that the universe starts in the far UV, then cools down to a certain minimum temperature after which it re-heats. Note that in this case, the brane does not cross the horizon at all.

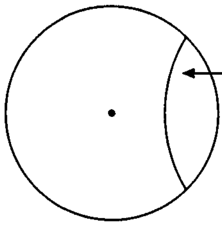

Figure 2: Diagram of Euclidean AdS_{n+2}-Schwarzschild with the trajectory of the brane. The horizon is represented by the dot in the middle of the diagram; only the region $a \geq a_H$ is drawn. The brane originates at spatial infinity, collapses to a certain miminal size and subsequently re-expands. It remains outside of the black hole horizon during the entire evolution.

7. Conclusion

In [1] it was argued that the discovered relation between the FRW equations and the entropy formulas sheds light on the meaning of the holographic principle in a cosmological setting [20]. Indeed, it was suggested that the values for s and T on the horizon should be regarded as bounds on these respective quantities. Although we still have no proof of this fact, we would like to present some further arguments in favor of this. At the moment when the brane crosses the horizon, the quantity γ is essentially equal to the inverse Newton constant. This means that the response of the energy density to a variation of the curvature is comparable to that of the Einstein action itself. Namely, from (28) and (31) one finds

$$a\left(\frac{\partial\rho}{\partial a}\right)_s = \frac{-n(n-1)}{8\pi G_N a^2}, \qquad \text{at } a = a_H. \tag{33}$$

The right hand side also gives the contribution of the spatial curvature in the equation of motion. Clearly, when this is the case one should reconsider the validity of the usual formulation of gravity, since quantum effects (the Casimir energy density) are of the same order as the spatial curvature. This suggests that a classical description of the geometry of the universe may no longer be well defined and one has to go over to a different, more fundamental formulation of the theory. We have indeed noticed that, at the transition points, the laws that govern the gravitational evolution and the entropy and energy expressions for the CFT, that describes the radiation, merge in a surprising way. This indicates that both sets of equations have a common origin in a single underlying fundamental theory.

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