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***DAMPING OF COLLECTIVE MODES AND  
QUASIPARTICLES IN  $d$ - WAVE SUPERCONDUCTORS***

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*These are preliminary lecture notes, intended only for distribution to participants.*



# DAMPING OF COLLECTIVE MODES AND QUASIPARTICLES IN $d$ -WAVE SUPERCONDUCTORS

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**Abstract.** The two-dimensional  $d$ -wave superconducting state of the high temperature superconductors has a number of different elementary excitations: the spin-singlet Cooper pairs, the spin  $S = 1/2$  fermionic quasiparticles, and a bosonic  $S = 1$  resonant collective mode,  $\phi_\alpha$ , at the antiferromagnetic wavevector. Although the  $\phi_\alpha$  quanta are strongly coupled to the gapped quasiparticles near the  $(\pi, 0)$ ,  $(0, \pi)$  wavevectors (the “hot spots”), they are essentially decoupled from the low energy quasiparticles near the nodes of the superconducting gap. Consequently, distinct and independent low energy quantum field theories can be constructed for the  $\phi_\alpha$  and nodal quasiparticle excitations. We review recent work introducing a 2+1 dimensional boundary conformal field theory for the damping of the  $\phi_\alpha$  excitations by non-magnetic impurities, which is built on the proximity to a magnetic ordering transition at which the  $\phi_\alpha$  condense; the results are compared with neutron scattering experiments. Photoemission and THz conductivity measurements indicate that the nodal quasiparticles undergo strong inelastic scattering at low temperatures; we propose that this is due to fluctuations near a quantum phase transition, and critically analyze candidate order parameters and field theories.

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## 1. Introduction

The description of high temperature superconductivity in the cuprate compounds has been a central problem at the frontier of quantum many body theory in the last decade. Although many anomalous properties have been observed in the normal state, both in the over-doped and under-doped regions, no theoretical consensus has emerged on their origin. Part of the difficulty is that there appear to be many competing instabilities and excitations as one cools down from high temperatures ( $T$ ), and they are all strongly coupled to each other at intermediate  $T$ .

However, simplifications do occur at temperatures  $T < T_c$ , the critical temperature below which there is an onset of  $d$ -wave superconductivity. In this review we shall argue, on the basis of recent experimental observations, that there is an important decoupling between different sectors of the excitation spectrum which carry a non-zero spin, and that this decoupling allows development of tractable quantum field theories of the low energy excitations [1, 2, 3, 4]. We will make quantitative predictions for the impurity-induced and intrinsic damping of these excitations and compare them to experimental results.

Let us list the elementary excitations of the  $d$ -wave superconductor and nearby phases:

(A) Cooper Pairs: The superconductivity is of course a consequence of the condensation of spin  $S = 0$ , charge  $2e$  Cooper pairs. Below  $T_c$ , the excitations of the phase of the condensate are responsible for the superflow, and for the plasmon excitations. In this paper, we will be primarily concerned with the damping of excitations which carry spin, and these couple only weakly to the phase excitations in a well-formed superconductor at low  $T$ : so we will neglect the phase excitations in the body of the paper. These phase excitations become more important near a  $T = 0$  superconducting-insulator transition, but we will not consider such a situation here. Above  $T_c$ , phase fluctuations [5, 6, 7, 8, 9] are surely important for the transport properties, and they also couple strongly to some of the fermionic quasiparticle excitations: we will briefly discuss this phenomenon further below.

(B)  $S = 1/2$  fermionic quasiparticles: These are the familiar Bogoliubov quasiparticles in a BCS theory of the superconducting state. Because of the  $d$ -wave symmetry of the order parameter, their energies vanishes at four nodal points in the Brillouin zone -  $(\pm K, \pm K)$ , with  $K = 0.391\pi$  for optimally doped  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$  [10]. We will denote the fermionic excitations in the vicinity of these points by the Nambu spinors  $\Psi_{1,2}$  (see Fig 1 and further details in Section 3). It is also interesting to consider

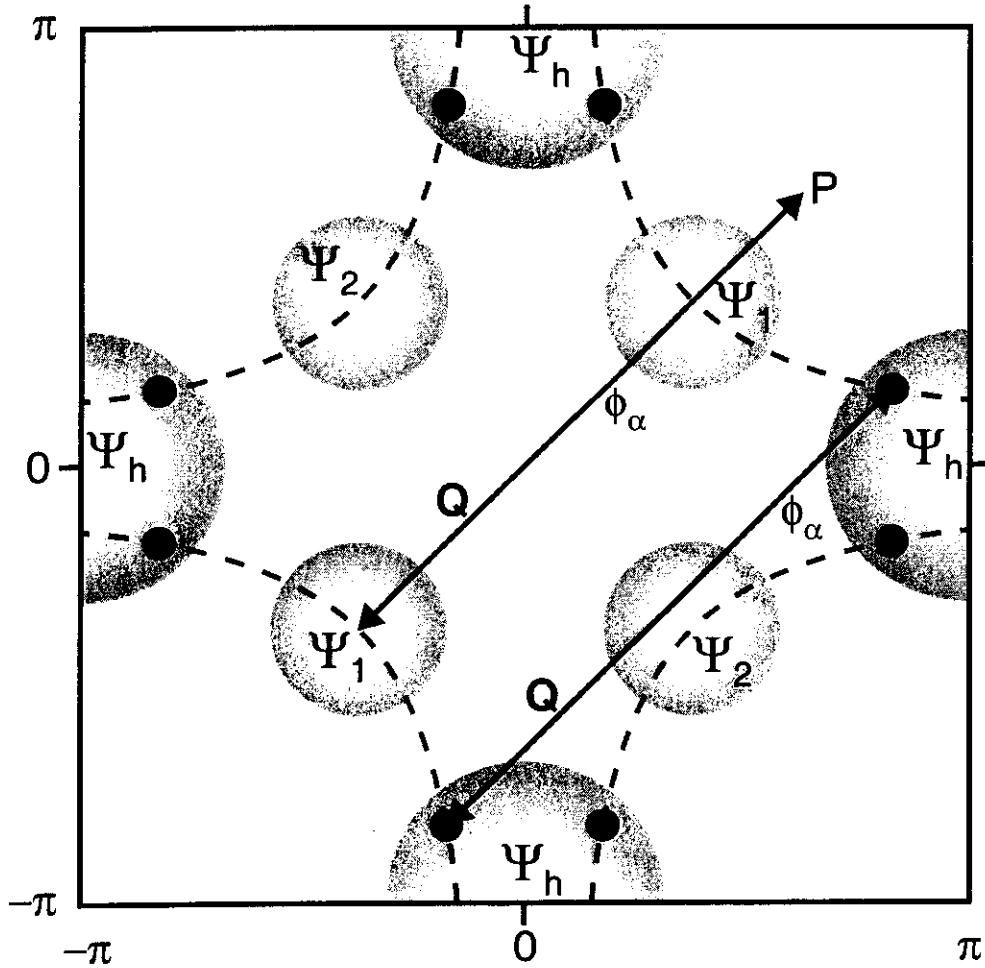


Figure 1. Brillouin zone of the high temperature superconductors at optimal doping (see *e.g.* Ref. [11]). The dashed line is the location of the incipient Fermi surface at intermediate temperatures: the ground state is a *d*-wave superconductor, and not a Fermi liquid, and so there is no sharply defined Fermi surface as  $T \rightarrow 0$ —the line is merely the location of smooth crossover in the momentum distribution function. The fermionic,  $S = 1/2$ , quasiparticles  $\Psi_{1,2}$  lie near the nodal points  $(\pm K, \pm K)$  (with  $K \approx 0.39\pi$  [10]) at which their excitation energy vanishes. The  $\Psi_h$  quasiparticles require an energy  $\approx \Delta$  for their excitation and lie in the vicinity of the “Fermi surface” points  $(\pm 0.18\pi, \pm\pi)$  and  $(\pm\pi, \pm 0.18\pi)$ . The double-headed arrow at wavevector  $\mathbf{Q} = (\pi, \pi)$  represents the bosonic,  $S = 1$ , resonant collective mode  $\phi_\alpha$  scattering fermions between two points in the Brillouin zone. Notice that the  $\Psi_{1,2}$  fermions are decoupled from the  $\phi_\alpha$  quanta. In contrast, the  $\Psi_h$  fermions couple strongly to the  $\phi_\alpha$ , especially in the vicinity of the “hot spots” denoted by the small filled circles.

the fermionic excitations near the  $(\pi, 0)$  and  $(0, \pi)$  points: here the pairing amplitude has its largest value and so there is a large energy gap,  $\Delta$ , towards exciting the quasiparticles. We will denote these high energy quasiparticles generically by  $\Psi_h$  (see Fig 1). Because of their large pairing amplitude, the  $\Psi_h$  quasiparticles couple efficiently to the phase fluctuations discussed above in (A), and are expected to have a rapidly decreasing lifetime once the phase fluctuations proliferate above  $T_c$  [12]. In contrast, the nodal quasiparticles,  $\Psi_{1,2}$ , are in a region of vanishing pairing, and are essentially decoupled from the phase fluctuations: as we will discuss below, other mechanisms will be required to damp the  $\Psi_{1,2}$  quasiparticles.

(C)  $S = 1$  resonant collective mode: Neutron scattering experiments observe a sharp resonance peak at an energy  $\Delta_{\text{res}}$  in the scattering cross section at the antiferromagnetic wavevector,  $\mathbf{Q}$  [13, 14, 15, 16, 17]. We will view this bosonic  $S = 1$  resonant collective mode,  $\phi_\alpha$  ( $\alpha = x, y, z$  are the spin components), as that expected in a paramagnetic phase across a magnetic disordering quantum phase transition [18]. (For the case where  $\mathbf{Q} = (\pi, \pi)$ , the  $\phi_\alpha$  are real, while for incommensurate  $\mathbf{Q}$ , the  $\phi_\alpha$  become complex; we will explicitly treat the commensurate case here, although the generalization to the incommensurate case is straightforward [3], and does not modify any of the scaling arguments, including the central result (6).) Our identification of  $\phi_\alpha$  is similar to the view that it is a  $S = 1$  particle-hole bound state in a  $d$ -wave superconductor [19, 20, 21]. However, as we shall discuss in more detail below, appealing to the proximity of a quantum phase transition allows a systematic treatment of the strongly relevant self-interactions of the  $\phi_\alpha$  field. Such a collective mode has also been discussed in models with a special  $\text{SO}(5)$  symmetry [22], but we shall not appeal to symmetries beyond the usual  $\text{SU}(2)$  spin symmetry in our treatment. The coupling of the  $\phi_\alpha$  to the fermionic quasiparticles in (B) is illustrated in Fig 1. Momentum conservation allows a strong coupling between the  $\Psi_h$  and the  $\phi_\alpha$  in the vicinity of the so-called “hot spots” [11, 23, 24, 25]. This coupling leads to strong mutual damping of  $\Psi_h$  and  $\phi_\alpha$  above  $T_c$ . Below  $T_c$ , the same coupling is surely an important ingredient in the pairing of the  $\Psi_h$  fermions [26]: this conclusion is supported by the “shake-off” satellite peaks, separated by the  $\Delta_{\text{res}} \sim 40$  meV, observed in the photoemission and optical spectra [24, 27, 28]. A key ingredient in our discussion is also clear from Fig 1: the coupling between the low energy  $\Psi_{1,2}$  fermions and the  $\phi_\alpha$  mode is strongly suppressed by momentum conservation—moving a distance  $\mathbf{Q}$  from a nodal point places one in a section of the Brillouin zone (e.g. near the point  $P$  in Fig 1) where the fermionic excitations cost over 100 meV. Remnants of the shake-off peaks just noted are also seen along the  $(1, \pm 1)$  directions [29] not too far from the point  $P$ , but this does

not qualitatively modify the low energy  $\Psi_{1,2}$  excitations within the shaded circles.

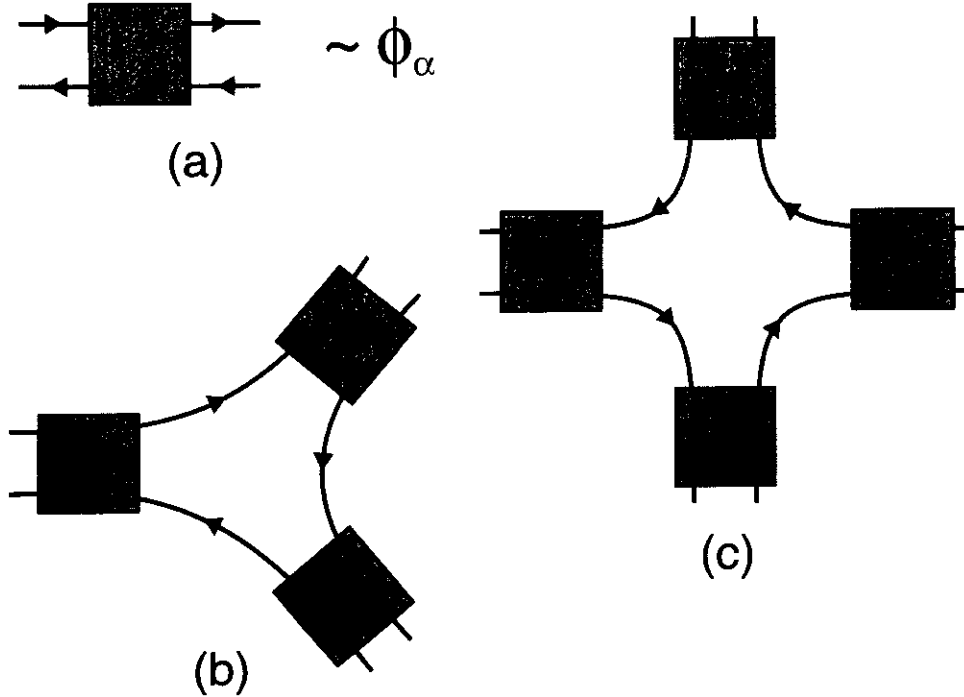
We have now collected all the ingredients necessary to motivate our recent computations.

In Section 2 we will discuss the  $T = 0$  broadening of the  $\phi_\alpha$  collective mode by the substitution of a small concentration of non-magnetic impurities like Zn or Li on the Cu sites, and compare our theoretical results [2, 3] to experimental observations [30]. It should be clear from the discussion above that, in the absence of such extrinsic broadening, it is possible for the  $\phi_\alpha$  resonance to be infinitely sharp at  $T = 0$  in a  $d$ -wave superconductor. The  $\phi_\alpha$  quanta couple strongly to the  $\Psi_h$  fermions, but these induce no damping as long as  $\Delta_{\text{res}} < 2\Delta$  ( $\Delta \approx 40$  meV near optimal doping [31]). Along the diagonals of the Brillouin zone, momentum conservation prohibits coupling to the gapless nodal fermions  $\Psi_{1,2}$ , and only allows couplings to quasiparticle excitations whose excitation energy exceeds  $\Delta_{\text{res}}$ .

Section 3 will consider *intrinsic*  $T$ -dependent damping of the nodal fermionic quasiparticles by inelastic scattering [1, 4]. We have already discussed the damping of the gapped  $\Psi_h$  above. Below  $T_c$ , photoemission experiments [32, 33, 34, 31, 35] observe negligible damping of the  $\Psi_h$  fermions, and this is consistent with our considerations: the phase fluctuations in (A) are suppressed, while the  $\phi_\alpha$  mode in (C) leads to coherent pairing of  $\Psi_h$  fermions. Above  $T_c$ , phase fluctuations proliferate, and their strong coupling to the  $\Psi_h$  is expected to lead to significant inelastic scattering of the  $\Psi_h$  [12]; additional scattering is also expected from the “hot-spot” coupling to the  $\phi_\alpha$ , and these expectations are consistent with experimental observations. However neither of the fluctuations in (A) or (C) couple to the nodal fermions  $\Psi_{1,2}$ : so, with our present considerations we would conclude that the nodal fermions should be very sharp both below and above  $T_c$ . The actual experimental situation is dramatically different [10, 36]—these nodal quasiparticles have a large inverse lifetime, which decreases roughly linearly with  $T$ , and an imaginary component of a self energy which is roughly linearly proportional to frequency,  $\omega$ , for  $\hbar\omega > k_B T$ . Moreover, these damping rates change smoothly through  $T_c$ , with little sign of the superconducting transition. We have to appeal to other inelastic damping mechanisms to explain these observations, and we will present a critical classification of candidates in Section 3.

## 2. Impurities and the $S = 1$ resonant collective mode

Before discussing the effect of impurities, we state our model for the spin collective mode,  $\phi_\alpha$ , in the clean  $d$ -wave superconductor. A popular approach in recent work [19, 20, 21] has been to compute the dynamic spin



*Figure 2.* (a) Particle-hole propagator which has a  $S = 1$  bound state at the antiferromagnetic wavevector representing the  $\phi_\alpha$  quanta. (b) and (c) Three- and four-point self interactions of the spin excitations.

susceptibility of the underlying electrons and to identify this mode as a bound state pole near the antiferromagnetic wavevector: this is schematically indicated in Fig 2a. However all such studies have so far neglected three- and four-point (and higher multi-point) self interactions of the spin excitations, which are schematically indicated in Figs 2b and 2c respectively. One of the key points of our work is that it is essential to include the interactions in Figs 2b and 2c in the low energy theory: these are strongly relevant perturbations, and in a sense, their effective strength is infinite (higher multi-point interactions can, however, be neglected). The simple results for impurity-induced damping we shall quote below rely on hyperscaling properties, and these are a direct consequence of the self interactions in Fig 2.

So how does one obtain a tractable theory which includes the interactions in Fig 2? Our strategy is to appeal to the proximity of a  $T = 0$  magnetic ordering transition at which the excitation energy of  $\phi_\alpha$ ,  $\Delta_{\text{res}}$ , vanishes. This is a quantum phase transition driven by the condensation of  $\phi_\alpha$  to a state with coexisting superconductivity and collinear spin den-

sity wave order. We will develop a theory for the quantum-critical point of this transition, and then use powerful field-theoretic methods to expand back into the region where  $\Delta_{\text{res}}$  is non-zero. The values of the interactions in Fig 2 will be universally determined by the underlying structure of the expansion in relevant perturbations of the critical field theory. Such an approach effectively reduces to an expansion in  $\Delta_{\text{res}}/J$ , where  $J$  is a microscopic exchange constant, and the smallness of this ratio is our primary assumption.

We shall use a theoretical model (which is supported by the ‘pseudo-gap’ phenomenology) in which the magnetic ordering transition occurs while the excitation energy of the  $\Psi_h$  remains non-zero, and so these fermions can be neglected in the critical theory of the transition. As discussed above, the  $\Psi_{1,2}$  fermions have little coupling to the magnetic excitations, and so an action for the transition can be expressed in terms of the  $\phi_\alpha$  alone. By analogy with theories developed for the magnetic transition in insulators, and using general symmetry arguments [37], we can write down the following effective action for the bulk  $\phi_\alpha$  fluctuations in the  $d$ -wave superconductor:

$$\mathcal{S}_b = \int d^2x \int d\tau \left[ \frac{1}{2} \left( (\partial_\tau \phi_\alpha)^2 + c^2 (\nabla_x \phi_\alpha)^2 + s \phi_\alpha^2 \right) + \frac{g_0}{4!} (\phi_\alpha^2)^2 \right]; \quad (1)$$

the fermionic excitations with spin,  $\Psi_{1,2}$ ,  $\Psi_h$  have been integrated out, and for reasons already discussed, serve only to renormalize the values of the couplings in (1). The parameter  $c$  is the velocity of spin-waves in the ordered phase, and  $s$  tunes the system between the two phases which lie on either side of a critical value  $s = s_c$ . The quartic non-linearity,  $g_0$ , corresponds to the interaction in Fig 2c. How about the cubic coupling in Fig 2b? This is implicitly accounted for in (1) in a manner we now describe. By momentum conservation, if two of the particle-hole propagators in Fig 2b carry momentum  $\mathbf{Q}$ , the third must carry momentum  $2\mathbf{Q} \approx 0$ , *i.e.*, it represents the ferromagnetic spin component  $L_\alpha$ . This has allowed three-point couplings with the  $\phi_\alpha$ , including the kinematic term [37] in the action  $\sim i\epsilon_{\alpha\beta\gamma} L_\alpha \phi_\beta \partial_\tau \phi_\gamma$ . The  $L_\alpha$  fluctuations are not critical, and after integrating them out, one obtains renormalizations of terms already present in (1) [37].

The magnetic properties of the bulk quantum phase transition described by  $\mathcal{S}_b$  have been worked out in some detail [38, 18], and many aspects are in agreement with trends in NMR and neutron scattering experiments on the high temperature superconductors [39, 40]. Here, we will only need a few well-known scaling properties of the critical point of (1). Upon interpreting  $\tau$  as a third spatial dimension,  $\mathcal{S}_b$  can also represent the partition function of a classical Heisenberg ferromagnet in dimension  $D = 3$  at finite temperature, and its Curie transition corresponds to the quantum-critical

point we are interested in. Both occur at a critical value  $r = r_c$ , where there the functional integral over  $\phi_\alpha$  is invariant under the scale transformation

$$\begin{aligned} x &\rightarrow x/b \\ \tau &\rightarrow \tau/b \\ \phi_\alpha &\rightarrow b^{(1+\eta_H)/2} \phi_\alpha \\ \phi_\alpha^2 &\rightarrow b^{3-1/\nu_H} \phi_\alpha^2, \end{aligned} \quad (2)$$

where  $b$  is a rescaling factor, and  $\nu_H$  and  $\eta_H$  are known critical exponents of the  $D = 3$  classical Heisenberg model. The last transformation in (2) represents the mapping of the composite operator  $\phi_\alpha^2$ , and its scaling dimension is not simply twice that of  $\phi_\alpha$  because of corrections due to the  $g_0$  interaction in (1).

We now turn to the effect of a dilute concentration of impurities. We will outline the central ingredients leading to our main result, and refer the reader to Ref. [3] for further details. Consider a single impurity at  $x = 0$ ; by “impurity” we mean an *arbitrary* localized deformation in the vicinity of  $x = 0$ . One consequence of any such deformation will be a change in the value of  $s$  near  $x = 0$ , and this will lead to the following term in the action

$$\zeta \int d\tau \phi_\alpha^2(x = 0, \tau). \quad (3)$$

Under the scale transformation (2), we see immediately that  $\zeta$  has scaling dimension  $1/\nu_H - 2 \approx -0.57$ , and is therefore *irrelevant* at the critical point of (1), and it has only weak effects on the bulk properties. To obtain a local, relevant, perturbation on the bulk fluctuations, we need to consider quantum mechanical effects associated with Berry phases. The Berry phases accumulated by the precession of spins in the host antiferromagnet cancel almost completely upon an average over the lattice sites [37]; however, in the presence of impurities it is entirely possible that this cancellation is disrupted, and a residual Berry phase of spin  $S$  ( $S$  must be an integer or half-odd-integer) survives [41, 42]. To account for this Berry phase we introduce a single unit vector  $n_\alpha(\tau)$  ( $n_\alpha^2(\tau) = 1$ ) representing the orientation of the net uncompensated spin, and the action

$$S_{\text{imp}} = iS \int d\tau A_\alpha(n) \frac{dn_\alpha(\tau)}{d\tau}, \quad (4)$$

where  $A_\alpha$  is a function of  $n_\alpha$  defined by  $\epsilon_{\alpha\beta\gamma} \partial A_\beta / \partial n_\gamma = n_\alpha$ . This Berry phase is intimately connected to the fact that an external magnetic field will lead to a Curie susceptibility  $= S(S+1)/3T$  from the impurity spin in the non-magnetic phase (in the absence of Kondo screening—see below);

this response is divergent as  $T \rightarrow 0$  and is a reflection of a  $(2S + 1)$ -fold degenerate level near the impurity. For the case of a non-magnetic Zn or Li ion replacing a magnetic  $S = 1/2$  Cu ion in the high temperature superconductors, the above arguments on Berry phases strongly suggest that each such impurity should contribute a term like (4) with  $S = 1/2$ . This conclusion is supported by NMR experiments [43, 44, 45], and we will assume its validity in our discussion below. Other authors have modeled the Zn ion solely as a non-magnetic scatterer in the unitarity limit [46, 47, 48, 49, 50]. In such a model, the fermionic quasiparticles form quasi-bound states at the impurity sites at the Fermi level; we believe that after accounting for the strong local Coulomb repulsion at the impurity site, each bound state will capture only a single electron, and the low energy physics will then be described by (4) with  $S = 1/2$  (see also Ref [48]).

We now need to couple the impurity degree of freedom,  $n_\alpha$ , to those of the host. The most important coupling is the simple linear term

$$S_c = \gamma \int d\tau n_\alpha(\tau) \phi_\alpha(x = 0, \tau). \quad (5)$$

To compute the scaling dimension of  $\gamma$  at the fixed point where the impurity and bulk degrees of freedom are decoupled, we note that if  $n_\alpha \rightarrow n_\alpha$  under the transformation (2), the impurity action (4) remains invariant. Under such a mapping,  $\gamma$  has dimension  $(1 - \eta_H)/2 \approx 0.48$ . Unlike  $\zeta$ , the coupling  $\gamma$  is therefore *relevant* at the  $\gamma = 0$  fixed point, and plays a central role in the main results presented below. We can also imagine a Kondo coupling,  $J_K$  between the spin,  $n_\alpha(\tau)$ , and the host fermions  $\Psi_{1,2}, \Psi_h$ . However, the fermionic, single particle density of states vanishes at the Fermi level, and this dramatically reduces the possibility of Kondo screening of the impurity spin: with particle-hole symmetry, there is no Kondo screening even upto  $J_K = \infty$ , while without particle-hole symmetry, the spin is screened only above an appreciable threshold value of  $J_K$  [51, 52, 53, 54]. We will assume that no Kondo screening has occurred over the experimentally relevant temperature range.

The above arguments suggest that a theory of the impurity spin dynamics will emerge from a complete renormalization group analysis of  $S_b + S_{\text{imp}} + S_c$ . This has been carried out in Ref [3], and the final results are quite simple: the bulk phase transition at  $s = s_c$  is the only critical point, and  $(s - s_c)$  remains the only relevant perturbation at this critical point; both  $g$  and  $\gamma$  approach fixed point values  $g^*$  and  $\gamma^*$  (related phenomena were noted earlier in simpler models [55, 56, 57]). There is no separate critical point associated with the impurity degrees of freedom, as is often the case in the theory critical phenomena on boundaries [58]. A remarkable consequence of this is that the single energy scale,  $\Delta_{\text{res}}$ , which characterized the dynamics of the paramagnet in the host system [37], is also all

that is needed to completely characterize the dynamics in the vicinity of the impurity. Of course, the quantized number  $S$  in (4) also influences the values of the various universal scaling functions.

Now consider a dilute concentration of impurities,  $n_{\text{imp}}$ , each described by the analog of (4) and (5), placed at random locations in the  $d$ -wave superconductor. We will answer the following key question, relevant to the neutron scattering experiments on Zn doped  $\text{YBa}_2\text{Cu}_3\text{O}_7$  by Fong *et al.* [30]. At what energy scale,  $\Gamma$ , does the  $S = 1$  resonant pole at energy  $\Delta_{\text{res}}$  get broadened by the impurities? The implication of the renormalization group arguments above is that, for small  $\Delta_{\text{res}}/J$ ,  $\Gamma$  is universally determined by the only dimensionful parameters available to us:  $n_{\text{imp}}$ ,  $\Delta_{\text{res}}$ , and the velocity  $c$ . Making the mild assumption that, for small  $n_{\text{imp}}$ ,  $\Gamma$  must be linearly proportional to  $n_{\text{imp}}$  (this is supported by explicit computations [3]), simple dimensional analysis of the length and time scales allows us to conclude

$$\Gamma = C_S \frac{(\hbar c)^2}{\Delta_{\text{res}}} n_{\text{imp}}, \quad (6)$$

where  $C_S$  is a universal number. All corrections to (6) will be suppressed by positive powers of  $\Delta_{\text{res}}/J$ . Notice also the inverse dependence on the small energy scale  $\Delta_{\text{res}}$ : this is an indication of the strong effect of the relevant coupling in (5). For an impurity with  $S = 0$ , the simplest allowed coupling is  $\zeta$ , and its irrelevance implies  $C_0 = 0$ , and the corrections just noted will be the leading contributions. We have estimated  $C_{1/2}$  in a self-consistent non-crossing approximation and obtained  $C_{1/2} \approx 1$ .

It is useful to rewrite our main result (6) in a different manner:

$$\frac{\Gamma}{\Delta_{\text{res}}} \sim n_{\text{imp}} \xi^2, \quad (7)$$

where  $\xi = \hbar c / \Delta_{\text{res}}$  is a correlation length. So if we imagine a “swiss cheese” model [59] where each impurity makes a hole of radius  $\xi$ , the inverse  $Q$  of the resonance is of order the fractional volume of holes in the swiss cheese.

The numerical predictions of (6) are in good agreement with the observations of Fong *et al.* [30]. We use  $n_{\text{imp}} = 0.005$ ,  $\Delta_{\text{res}} = 40$  meV, and the spin-wave velocity in the insulator  $\hbar c = 0.2$  eV, and obtain  $\Gamma = 5$  meV. This compares well with the observed value of 4.25 meV. We have also predicted detailed lineshapes for the impurity-induced broadening, and these will hopefully be tested in future, higher precision experiments.

### 3. Inelastic damping of the nodal quasiparticles

As we noted at the end of Section 1, recent experimental observations [10, 36] of a short lifetime  $\sim \hbar/k_B T$ , and a large frequency-dependent self

energy, for the nodal quasiparticles both above and below  $T_c$ , are puzzling in the light of the very weak coupling between the  $\Psi_{1,2}$  and both the  $\phi_\alpha$  and phase fluctuations. Consequently, we have to appeal to a separate decoupled sector of low energy fluctuations to explain this anomalous damping [1, 4].

A natural way of obtaining inverse lifetimes of order  $k_B T/\hbar$ , and self energies of order  $\omega$ , is to assume that the system is in the quantum-critical region of a  $T = 0$  quantum phase transition for which the  $\Psi_{1,2}$  fermions are central critical degrees of freedom [41, 37] (see Fig 3); the quantum-critical point should be described by an interacting quantum field theory below its upper critical dimension, so that universal low-energy fluctuations dominate the interactions. A further constraint is that the critical fluctuations should be decoupled from the  $\Psi_h$  fermions, as these remain undamped below  $T_c$ : so the new low energy mode associated with the onset of state  $X$  in Fig 3 couples strongly to the  $\Psi_{1,2}$  but not the  $\Psi_h$ . It will turn out that these constraints are rather difficult to satisfy, and lead to an essentially unique identification of the state  $X$  in Fig 3.

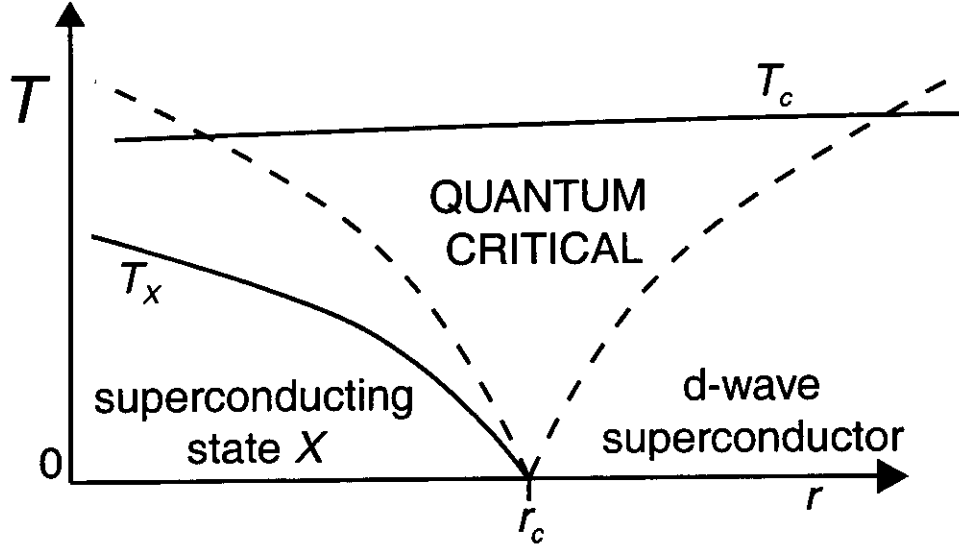
Clearly, the magnetic transition in Section 2 cannot be the required transition because the  $\Psi_{1,2}$  fermions are innocuous spectators of its critical field theory. We will list below many of the order parameters that have been considered in the literature in the last decade, and discuss whether they satisfy the requirements we have imposed on the state  $X$ .

Before we embark on this, let us recall the effective action for the nodal quasiparticles in the  $d$ -wave superconductor well away from the quantum-critical point,  $r \gg r_c$ . We denote the components of the electron annihilation operator,  $c_a$ , in the vicinity of the four nodal points  $(K, K)$ ,  $(-K, K)$ ,  $(-K, -K)$ ,  $(K, -K)$  by  $f_{1a}$ ,  $f_{2a}$ ,  $f_{3a}$ ,  $f_{4a}$  respectively, where  $a = \uparrow, \downarrow$  is the electron spin component. The 4-component Nambu spinors are  $\Psi_1 = (f_{1a}, \varepsilon_{ab} f_{3b}^\dagger)$  and  $\Psi_2 = (f_{2a}, \varepsilon_{ab} f_{4b}^\dagger)$  where  $\varepsilon_{ab}$  is an antisymmetric tensor with  $\varepsilon_{\uparrow\downarrow} = 1$ . The action is then

$$\begin{aligned} \mathcal{S}_\Psi = & \int \frac{d^d k}{(2\pi)^d} T \sum_{\omega_n} \Psi_1^\dagger (-i\omega_n + v_F k_x \tau^z + v_\Delta k_y \tau^x) \Psi_1 \\ & + \int \frac{d^d k}{(2\pi)^d} T \sum_{\omega_n} \Psi_2^\dagger (-i\omega_n + v_F k_y \tau^z + v_\Delta k_x \tau^x) \Psi_2. \end{aligned} \quad (8)$$

Here  $\tau^\alpha$  are Pauli matrices which act in the fermionic particle-hole space,  $k_{x,y}$  measure the wavevector from the nodal points and have been rotated by 45 degrees from the axes of the square lattice, and  $v_F$ ,  $v_\Delta$  are velocities. The action  $\mathcal{S}_\Psi$  has a scale invariance which will be useful in our considerations below

$$x \rightarrow x/b$$



*Figure 3.* Finite temperature ( $T$ ) phase diagram in the vicinity of a second order quantum phase transition from a  $d$ -wave superconductor as a function of some parameter in the Hamiltonian,  $r$  (which is possibly, but not necessarily, the hole concentration  $\delta$ ). Superconductivity is present at temperatures below  $T_c$ , and the superfluid density is non-zero on both sides of  $r_c$ . The state  $X$  is characterized by some other order parameter (in addition to superconductivity) which vanishes above a temperature  $T_X$ . We discuss a number of possibilities for the state  $X$  in the text—only one naturally satisfies the requirement of leading to an inverse lifetime  $\sim k_B T/\hbar$  for the nodal quasiparticles  $\Psi_{1,2}$  in the quantum-critical region, and negligible damping of the  $\Psi_h$  quasiparticles below  $T_c$ : the  $(d_{x^2-y^2} + id_{xy})$ -wave superconductor. Our computations for the quantum-critical point are carried out below  $T_c$ , but the results should also apply above  $T_c$  as long as the quantum-critical length  $\sim T^{-1/2}$  remains shorter than the phase coherence length. We emphasize that we are not requiring the high temperature superconductors to have  $(d_{x^2-y^2} + id_{xy})$  order in the ground state (although it is permitted): the coupling  $r$  could be larger than  $r_c$ , but it should be close enough that the system enters the quantum-critical region at some low  $T$ .

$$\begin{aligned} \tau &\rightarrow \tau/b \\ \Psi_{1,2}(x, \tau) &\rightarrow b\Psi_{1,2}(x/b, \tau/b), \end{aligned} \quad (9)$$

where  $b$  is a rescaling factor. We can illustrate the power of scaling arguments by showing how this transformation allows us to quickly deduce damping produced by ordinary screened Coulomb interactions (the screening is performed by the Cooper pairs in the condensate). These interactions lead to couplings like

$$v \int d^2x \int d\tau (\Psi_1^\dagger \tau^z \Psi_1)^2, \quad (10)$$

and it is easy to see that under (9),  $v$  has scaling dimension -1. Therefore  $v$  is *irrelevant*, and perturbation theory in  $v$  should be reliable. The fermion self energy,  $\Sigma_f$ , acquires an imaginary part at order  $v^2$ ; using the fact that under (9) the frequency and momentum dependent self energy transforms as  $\Sigma_f \rightarrow b\Sigma_f$ , the result (10) immediately leads to  $\text{Im}\Sigma_f \sim v^2 T^3$  (and similarly for the  $\omega$  dependence). This damping rate appears far too weak to explain the experimental observations.

We now turn to considerations of quantum critical points, and list various plausible candidates for the state  $X$  in Fig 3. We will only consider order parameters for  $X$  which break simple underlying symmetries of the Hamiltonian: the symmetry of the square lattice space group, time-reversal, and spin rotation (the last we have already discussed above). More complicated transitions with non-local order parameters and deconfinement transitions are also possible, but we will not consider them here [60].

Staggered flux phase: Many investigators [61, 62, 63, 64, 65, 66, 67] have considered the possibility of  $d$ -wave superconductivity coexisting with a staggered distribution of orbital currents (or an “orbital antiferromagnet”). This state  $X$  is characterized by the expectation value [66]

$$\langle c_{\mathbf{k}+\mathbf{Q},a}^\dagger c_{\mathbf{k},a} \rangle = i\phi(\cos k_x - \cos k_y), \quad (11)$$

where, as before,  $\mathbf{Q} = (\pi, \pi)$ , and  $\phi$  is a real order parameter. Momentarily neglecting the fermionic excitations, we can, just as for (1), write down an effective action for  $\phi$  fluctuations purely on symmetry grounds:

$$\mathcal{S}_\phi = \int d^2x \int d\tau \left[ \frac{1}{2} \left( (\partial_\tau \phi)^2 + c^2 (\nabla_x \phi)^2 + r\phi^2 \right) + \frac{\tilde{g}}{4!} \phi^4 \right]; \quad (12)$$

(without strict particle-hole symmetry, a first order time derivative term is potentially allowed, but the only possible relevant term,  $\phi \partial_\tau \phi$ , is a total derivative  $(1/2) \partial_\tau \phi^2$ ). As is well known,  $\mathcal{S}_\phi$  describes the phase transition in the Ising model in  $D = 3$  spacetime dimensions. At the critical point  $r = r_c$ ,  $\mathcal{S}_\phi$  is also invariant under the extension of (9) to the analog of the scale transformations in (2)

$$\begin{aligned} \phi &\rightarrow b^{(1+\eta_I)/2} \phi \\ \phi^2 &\rightarrow b^{3-1/\nu_I} \phi^2, \end{aligned} \quad (13)$$

where now  $\nu_I$  and  $\eta_I$  are the exponents of the  $D = 3$  Ising model. To decide if this purely Ising description of the transition is correct, we have to test its stability to a coupling between the  $\phi$  and the  $\Psi_{1,2}$ . From (11) we see that  $\phi$  carries momentum  $\mathbf{Q}$ , and so the momentum conservation

constraints upon its coupling to the fermions are identical to those of the spin mode  $\phi_\alpha$  in Fig 1: unless  $K = \pi/2$ , there is no linear coupling to the fermionic excitations which is linear in  $\phi$ . For general  $K$ , the simplest allowed coupling is

$$w \int d^2x \int d\tau \phi^2 \Psi_1^\dagger \tau^z \Psi_1, \quad (14)$$

and similarly for  $\Psi_2$ . Using (9), (13), we deduce that the scaling dimensions of  $w$  is  $1/\nu_I - 2 \approx -0.41$ : consequently  $w$  is *irrelevant*, and the critical theory for the transition is (12) alone. The fermions  $\Psi_{1,2}$  are not part of the critical theory, and their inverse lifetimes can be estimated by perturbation theory in  $w$ . Using the scaling dimension of  $w$  above, we deduce that the fermionic self energy has the following  $T$  dependence in the quantum-critical region of Fig 3:  $\text{Im}\Sigma_f \sim w^2 T^{5-2/\nu_I} \approx w^2 T^{1.83}$  (and  $\text{Im}\Sigma_f \sim w^2 \omega^{1.83}$  for  $\hbar\omega > k_B T$ ). This is a super-linear power, which does not appear compatible with experimental observations. Finally, we note that the special case  $K = \pi/2$  has also been considered in Ref [4]: then a coupling term linear in  $\phi$  is also allowed [66], but its contribution to  $\Sigma_f$  vanishes with an even higher power of  $T$ .

Charge Stripes: We consider the onset of a charge density wave in a  $d$ -wave superconductor, a transition to a state  $X$  defined by the order parameters

$$\langle c_{k+\mathbf{G}_x,a}^\dagger c_{k,a} \rangle = \Phi_x \quad ; \quad \langle c_{k+\mathbf{G}_y,a}^\dagger c_{k,a} \rangle = \Phi_y \quad (15)$$

where  $\mathbf{G}_x = (G, 0)$ ,  $\mathbf{G}_y = (0, G)$  are the ordering wavevectors, and  $\Phi_{x,y}$  are complex order parameters. Again, constraints from momentum conservation are rather severe. Unless  $G = 2K$ , there is no coupling between the  $\Phi_{x,y}$  and the  $\Psi_{1,2}$  fermions. Existing experimental observations of charge stripe formation easily satisfy  $G \neq 2K$ . Under these conditions, the damping of the  $\Psi_{1,2}$  from the critical charge fluctuations can be estimated as in the staggered-flux case above: the simplest allowed couplings, as in (14), are  $\sim |\Phi_x|^2 \Psi_1^\dagger \tau^z \Psi_1$  etc., and we obtain  $\text{Im}\Sigma_f \sim (\max(\omega, T))^{5-2/\nu}$ . Any reasonable model of the critical theory of the  $\Phi_{x,y}$  [4] has  $\nu > \nu_I$ , and so the fermion damping is rather weak. The special case  $G = 2K$  has also been analyzed in Refs [1, 4]: it does yield fermion damping compatible with experimental observations [10, 36], but, as we have already noted, this mode-locking of the charge stripe and fermionic nodal wavevectors is not supported by experiments.

$d + is$  superconductivity: Next consider a time-reversal symmetry breaking transition in which the Cooper pair wavefunction in state  $X$  acquires a

small  $s$ -wave component, but with a relative phase factor  $\pm\pi/2$  [68]:

$$\langle c_{k\uparrow}c_{-k\downarrow} \rangle = \Delta_0(\cos k_x - \cos k_y) + i\phi(\cos k_x + \cos k_y). \quad (16)$$

The order parameter is again a single real field  $\phi$ . On general symmetry grounds, we expect the effective action of the  $\phi$  fluctuations to also have the form (12). However, now there is an efficient coupling of  $\phi$  fluctuations to the nodal fermions, which is not preempted by momentum conservation. It is evident from (16) that  $\phi$  fluctuations Andreev scatter fermions with momenta  $k$  and  $-k$ . This scattering can occur between the nodal points  $(K, K)$  and  $(-K, -K)$  of the  $\Psi_1$  fermions, and similarly for  $\Psi_2$ ; it is represented by the allowed coupling:

$$\mathcal{S}_{\Psi\phi} = \int d^2x d\tau \left[ \lambda_0 \phi \left( \Psi_1^\dagger \tau^y \Psi_1 + \Psi_2^\dagger \tau^y \Psi_2 \right) \right]. \quad (17)$$

Now, computing the scaling dimension of the coupling  $\lambda_0$  under the transformations (9), (13), we observe a crucial difference from the two cases considered so far: the coupling  $\lambda_0$  has dimension  $(1 - \eta_I)/2 \approx 0.48$ , and is therefore *relevant*, and the  $\lambda_0 = 0$  fixed point is unstable. So the critical theory strongly couples the  $\phi$  and  $\Psi_{1,2}$  fluctuations, and a complete understanding requires a more detailed renormalization group analysis of  $\mathcal{S}_\Psi + \mathcal{S}_\phi + \mathcal{S}_{\Psi\phi}$ . This has been discussed in Ref. [4] (and for a similar model in a different physical context in Ref. [69]): we will not discuss this here apart from noting that both the non-linearities,  $\tilde{g}$  and  $\lambda_0$  approach fixed-point values, and there is only one relevant perturbation,  $(r - r_c)$ , at the interacting critical point. Under these conditions, strong scaling applies, and the lifetimes of excitations of the  $\phi$  and  $\Psi_{1,2}$  quanta are of order  $\hbar/k_B T$  in the quantum-critical region of Fig 3 [41, 37]. So for the case in which  $X$  is a  $d + is$  superconductor, the relaxation of the nodal quasiparticles  $\Psi_{1,2}$  appears to be in good accord with experimental observations. However, one significant discrepancy remains: the  $s$ -wave order parameter couples strongly to fermions in all directions, and so will also couple to the  $\Psi_h$  quasiparticles (see Fig 1). The gapped  $\Psi_h$  quasiparticles will easily radiate many of the low-energy  $\phi$  quanta, and acquire an appreciable width: this is not in accord with photo-emission experiments in which, as we noted earlier, the  $\Psi_h$  quasiparticles become sharp below  $T_c$ .

$d_{x^2-y^2} + id_{xy}$  superconductivity: Finally, we consider another case in which  $X$  breaks time-reversal, but now by acquiring a small  $d_{xy}$  component [70]. We will see that this case is very similar to the  $d + is$  case discussed above, but it also succeeds in very simply and naturally resolving the discrepancy with  $\Psi_h$  width we have just mentioned. The order parameter for this

transition, replacing (16), is

$$\langle c_{k\uparrow}c_{-k\downarrow} \rangle = \Delta_0(\cos k_x - \cos k_y) + i\phi \sin k_x \sin k_y. \quad (18)$$

The  $\phi$  fluctuations again Andreev scatter fermions between  $k$  and  $-k$ , and their coupling to the  $\Psi_{1,2}$  fields has the form (replacing (17)):

$$\bar{\mathcal{S}}_{\Psi\phi} = \int d^2x d\tau [\lambda_0 \phi (\Psi_1^\dagger \tau^y \Psi_1 - \Psi_2^\dagger \tau^y \Psi_2)]. \quad (19)$$

Notice that the only difference from (17) is the relative sign of the  $\Psi_1$  and  $\Psi_2$  terms: this is because the coefficient of  $\phi$  in (18) changes sign between the two pairs of nodal points, unlike the case in (17). The consequences of (19) are essentially identical to those of (17): the coupling  $\lambda_0$  approaches a fixed point value, and this leads immediately to a lifetime  $\sim \hbar/k_B T$  for the nodal quasiparticles in the quantum-critical region. Moreover, the  $\Psi_h$  fermions do *not* couple to the  $\phi$  fluctuations: the coefficient of  $\phi$  in (18) vanishes along the line between  $(\pi, \pi)$  and  $(\pi, 0)$  (and also between  $(0, 0)$  and  $(\pi, 0)$  and other symmetry-related lines), and so the fluctuating  $d_{xy}$  component of the pair wavefunction does not lead to appreciable broadening of the  $\Psi_h$  quasiparticles. Therefore, if the state  $X$  is a  $d_{x^2-y^2} + id_{xy}$  superconductor, the dynamics of the quantum-critical region are very naturally in accord with the constraints described at the beginning of Section 3.

Quite apart from the motivation provided by the above analysis, there are some appealing independent reasons [70] for suspecting that the  $d$ -wave superconductor may be on the verge of an instability to a  $d_{x^2-y^2} + id_{xy}$  state. The structure of the incipient Fermi surface in Fig 1 indicates that there is significant second-neighbor hopping on the square lattice. Accompanying this there should be a corresponding second-neighbor exchange,  $J_2$ . Just as the first neighbor exchange,  $J_1$ , prefers  $d_{x^2-y^2}$  pairing, the  $J_2$  exchange will induce  $d_{xy}$  pairing; the realistic case with both  $J_1$  and  $J_2$  non-zero should therefore prefer the intermediate  $d_{x^2-y^2} + id_{xy}$  state, with relative phase of  $\pm\pi/2$  ensuring that a gap opens over the entire fermion spectrum. Alternatively stated, the  $\Psi_h$  fermions are already strongly paired in the  $d_{x^2-y^2}$  state, while the  $\Psi_{1,2}$  fermions are essentially unpaired; the system will try to lower its energy by pairing the  $\Psi_{1,2}$  fermions, and this is most efficiently done by an additional  $d_{xy}$  component to the pair wave-function. An additional  $s$  component would also do the job, but has the disadvantage of also deforming the already optimal pairing of the  $\Psi_h$ , and so is not as efficient. Our mean-field calculations [3] on models with  $J_1, J_2$  both non-zero are consistent with these expectations. We can therefore identify the coupling  $r$  in Fig 3 as  $r \sim J_1/J_2$ .

It is also important to note that the instability from a  $d_{x^2-y^2}$  superconductor to a  $d_{x^2-y^2} + id_{xy}$  superconductor occurs below a finite value of

the coupling  $r = r_c$ . This is to be contrasted with pairing instabilities of a Fermi liquid, which occur at infinitesimal attraction, and so the analog of the effective action (12) has a logarithmic dependence upon the order parameter. However, when the parent state is a  $d_{x^2-y^2}$ -wave superconductor, the vanishing density of states at the Fermi level removes the usual BCS log divergence, and a finite attraction is required for further pairing in the  $d_{xy}$  channel. Only such a *finite-coupling* quantum phase transition can be described by an interacting quantum field theory with hyperscaling properties, and which leads to a  $T > 0$  quantum-critical region with lifetimes of order  $\hbar/k_B T$ .

Further tests of the above scenario will be provided by computations of transport properties, including the optical conductivities and the Hall coefficient, in the quantum-critical region of Fig 3 for the case where  $X$  is the  $d_{x^2-y^2} + id_{xy}$  superconductor: these are currently in progress.

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