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A Basic Introduction to Surgery Theory

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A Basic Introduction to Surgery Theory

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II

Preface

This manuscript contains extended notes of the lectures presented by the author at the summer school "High-dimensional Manifold Theory" in Trieste in May/June 2001. It is written not for experts but for talented and well educated graduate students or Ph.D. students. Sugery theory has been and is a very successful and well established theory. It was initiated and developed by Browder, Kervaire, Milnor, Novikov, Sullivan, Wall and others and is still a very active research area. The idea of these notes is to give young mathematicians the possibility to get access to the field and to see at least a small part of the results which have grown out of surgery theory. Of course there are other good text books and survey articles about surgery theory, some of them are listed in the references.

We remark that these notes are not yet finished.

Münster, May 14, 2001 *Wolfgang Lück*

Contents

Chapter 1

The s-Cobordism Theorem

Introduction

In this chapter we want to discuss and prove the following result

Theorem 1.1 (s-cobordism theorem) *Let M_0 be a closed connected oriented manifold of dimension $n \geq 5$ with fundamental group $\pi = \pi_1(M_0)$. Then*

1. *Let $(W; M_0, f_0, M_1, f_1)$ be an h-cobordism over M_0. Then W is trivial over M_0 if and only if its Whitehead torsion $\tau(W, M_0) \in \mathrm{Wh}(\pi)$ vanishes;*
2. *For any $x \in \mathrm{Wh}(\pi)$ there is an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0 with $\tau(W, M_0) = x \in \mathrm{Wh}(\pi)$;*
3. *The function assigning to an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0 its Whitehead torsion yields a bijection from the diffeomorphism classes relative M_0 of h-cobordism over M_0 to the Whitehead group $\mathrm{Wh}(\pi)$.*

Here are some explanations. An *n-dimensional cobordism* (sometimes also called just bordism) $(W; M_0, f_0, M_1, f_1)$ consists of a compact oriented n-dimensional manifold W, closed $(n-1)$-dimensional manifolds M_0 and M_1, a disjoint decomposition $\partial W = \partial_0 W \coprod \partial_1 W$ of the boundary ∂W of W and orientation preserving diffeomorphisms $f_0 : M_0 \to \partial W_0$ and $f_1 : M_1^- \to \partial W_1$. Here and in the sequel we denote by M_1^- the manifold M_1 with the reversed orientation and we use on ∂W the orientation with respect to the decomposition $T_x W = T_x \partial W \oplus \mathbb{R}$ coming from an inward normal field for the boundary. If we equip D^2 with the standard orientation coming from the standard orientation on $\mathbb{R}^2$, the induced orientation on $S^1 = \partial D^2$ corresponds to the anti-clockwise orientation on S^1. If we want to specify M_0, we say that W is a *cobordism over M_0*. If $\partial_0 W = M_0$, $\partial_1 W = M_1^-$ and f_0 and f_1 are given by the identity, we briefly write $(W; \partial_0 W, \partial_1 W)$. Two cobordisms (W, M_0, f_0, M_1, f_1) and $(W', M_0, f_0', M_1', f_1')$ over M_0 are *diffeomorphic relative M_0* if there is an orientation preserving diffeomorphism $F : W \to W'$ with $F \circ f_0 = f_0'$. We call an h-cobordism over M_0 *trivial*, if it is diffeomorphic relative M_0 to the trivial h-cobordism $(M_0 \times [0,1]; M_0 \times \{0\}, (M_0 \times \{1\})^-)$. Notice that the choice of the diffeomorphism f_i do play a role although they are often surpressed in the notation. We call a cobordism $(W; M_0, f_0, M_1, f_1)$ an *h-cobordism*, if the inclusions $\partial_i W \to W$ for $i = 0, 1$ are homotopy equivalences.

We will later see that the Whitehead group of the trivial group vanishes. Thus the s-Cobordism Theorem 1.1 implies

Theorem 1.2 (h-Cobordism Theorem) *Any h-cobordism* $(W; M_0, f_0, M_1, f_1)$ *over a simply connected closed n-dimensional manifold* M_0 *with* $\dim(W) \geq 6$ *is trivial.*

Theorem 1.3 (Poincaré conjecture) *The Poincaré Conjecture is true for a closed n-dimensional manifold M with* $\dim(M) \geq 5$*, namely, if M is simply connected and its homology* $H_p(M)$ *is isomorphic to* $H_p(S^n)$ *for all* $p \in \mathbb{Z}$*, then M is homeomorphic to* S^n*.*

Proof : We only give the proof for $\dim(M) \geq 6$. Since M is simply connected and $H_*(M) \cong H_*(S^n)$, one can conclude from the Hurewicz Theorem and Whitehead Theorem [65, Theorem IV.7.13 on page 181 and Theorem IV.7.17 on page 188] that there is a homotopy equivalence $f : M \to S^n$. Let $D_i^n \subset M$ for $i = 0, 1$ be two embedded disjoint disks. Put $W = M - (\operatorname{int}(D_0^n) \coprod \operatorname{int}(D_1^n))$. Then W turns out to be a simply connected h-cobordism. Hence we can find a diffeomorphism $F : (\partial D_0^n \times [0,1], \partial D_0^n \times \{0\}, \partial D_0^n \times \{1\}) \to (W, \partial D_0^n, \partial D_1^n)$ which is the identity on $\partial D_0^n = \partial D_0^n \times \{0\}$ and induces some (unknown) diffeomorphism $f_1 : \partial D_0^n \times \{1\} \to \partial D_1^n$. By the *Alexander trick* one can extend $f_1 : \partial D_0^n = \partial D_0^n \times \{1\} \to \partial D_1^n$ to a homeomorphism $\overline{f_1} : D_0^n \to D_1^n$. Namely, any homeomorphism $f : S^{n-1} \to S^{n-1}$ extends to a homeomorphism $\overline{f} : D^n \to D^n$ by sending $t \cdot x$ for $t \in [0,1]$ and $x \in S^{n-1}$ to $t \cdot f(x)$. Now define a homeomorphism $h : D_0^n \times \{0\} \cup_{i_0} \partial D_0^n \times [0,1] \cup_{i_1} D_0^n \times \{1\} \to M$ for the canonical inclusions $i_k : \partial D_0^n \times \{k\} \to \partial D_0^n \times [0,1]$ for $k = 0, 1$ by $h|_{D_0^n \times \{0\}} = \mathrm{id}$, $h|_{\partial D_0^n \times [0,1]} = F$ and $h|_{D_0^n \times \{1\}} = \overline{f_1}$. Since the source of h is obviously homeomorphic to S^n, Theorem 1.3 follows.

In the case $\dim(M) = 5$ one uses the fact that M is the boundary of a contractible 6-dimensional manifold W and applies the s-cobordism theorem to W with an embedded disc removed. ■

Remark 1.4 Notice that the proof of the Poincaré Conjecture in Theorem 1.3 works only in the topological category but not in the smooth category. In other words, we cannot conclude the existence of a diffeomorphism $h : S^n \to W$. The proof in the smooth case breaks down when we apply the Alexander trick. The construction of $\overline{f}$ given by coning f yields only a homeomorphism $\overline{f}$ and not a diffeomorphism even if we start with a diffeomorphism f. The map $\overline{f}$ is smooth outside the origin of D^n but not necessarily at the origin. We will see that not any diffeomorphism $f : S^{n-1} \to S^{n-1}$ can be extended to a diffeomorphism $D^n \to D^n$ and that there exist so called *exotic spheres*, i.e. closed manifolds which are homeomorphic to S^n but not diffeomorphic to S^n. The classification of these exotic spheres is one of the early very important achievements of surgery theory and one motivation for its further development.

Remark 1.5 In some sense the s-Cobordism Theorem 1.1 is one of the first theorems, where diffeomorphism classes of certain manifolds are determined by an algebraic invariant, namely the Whitehead torsion. Moreover, the Whitehead group $\mathrm{Wh}(\pi)$ depends only on the fundamental group $\pi = \pi_1(M_0)$, whereas the diffeomorphism classes of h-cobordisms over M_0 a priori depends on M_0 itself. The s-Cobordism Theorem 1.1 is one step in a program to decide whether two closed manifolds M and N are diffeomorphic what is in general a very hard question. The idea is to construct an h-cobordism $(W; M, f, N, g)$ with vanishing Whitehead torsion. Then W is diffeomorphic to the trivial h-cobordism over M what implies that M and N are diffeomorphic. So the *surgery program* would be:

1. Construct a homotopy equivalence $f : M \to N$;
2. Construct a cobordism $(W; M, N)$ and a map $(F, f, \mathrm{id}) : (W; M, N) \to (N \times [0,1], N \times \{0\}, N \times \{1\})$;
3. Modify W and F relative boundary by so called surgery such that F becomes a homotopy equivalence and thus W becomes an h-cobordism. During these processes one should make certain that the Whitehead torsion of the resulting h-cobordism is trivial.

The advantage of this approach will be that it can be reduced to problems in homotopy theory and algebra which can sometimes be handled by well-known techniques. In particular one will get sometimes computable obstructions for two homotopy equivalent manifolds to be diffeomorphic. Often surgery theory has proven to be very useful when one wants to distinguish two closed manifolds which have very similar properties. The classification of homotopy spheres (see Chapter 6) is one example. Moreover, surgery techniques can be applied to problems which are of different nature than of diffeomorphism or homeomorphism classifications.

In this chapter we want to present the proof of the s-Cobordism Theorem and to explain why the notion of Whitehead torsion comes in. We will encounter a typical situation in mathematics. We will consider an h-cobordism and try to prove that it is trivial. We will introduce modifications which we can apply to a handlebody decomposition without changing the diffeomorphism type and which are designed to reduce the number of handles. If we could get rid of all handles, the h-cobordism would be trivial. When attempting to cancel all handles, we run into an algebraic difficulty. A priori this difficulty could be a lack of a good idea or technique. But it will turn out to be the principal obstruction and lead us to the definition of the Whitehead torsion and Whitehead group.

The rest of this Chapter is devoted to the proof of the s-cobordism Theorem 1.1. Its proof is interesting and illuminating and it motivates the definition of

Whitehead torsion. But we mention that it is not necessary to go through it in order to understand the following chapters.

1.1 Handlebody decompositions

In this section we explain basic facts about handles and handlebody decompositions.

Definition 1.6 *The* n-dimensional handle of index q *or briefly* q*-handle is* $D^q \times D^{n-q}$. *Its* core *is* $D^q \times \{0\}$. *The* boundary of the core *is* $S^{q-1} \times \{0\}$. *Its* cocore *is* $\{0\} \times D^{n-q}$ *and its* transverse sphere *is* $\{0\} \times S^{n-q-1}$.

Let $(M, \partial M)$ *be an* n*-dimensional manifold with boundary* ∂M. *If* $\phi^q : S^{q-1} \times D^{n-q-1} \to \partial M$ *is an embedding, then we say that the manifold* $M + (\phi^q)$ *defined by* $M \cup_{\phi^q} D^q \times D^{n-q}$ *is obtained from* M *by attaching a handle of index* q *by* ϕ^q.

Obviously $M + (\phi^q)$ carries the structure of a topological manifold. To get a smooth structure, one has to use the technique of straigthening the angle to get rid of the corners at the place, where the handle is glued to M. The boundary $\partial(M + (\phi^q))$ can be described as follows. Delete from ∂M the interior of the image of ϕ^q. We obtain a manifold with boundary together with a diffeomorphism from its boundary to $S^{q-1} \times S^{n-q-2}$ induced by $\phi^q|_{S^{q-1}\times S^{n-q-2}}$. If we use this diffeomorphism to glue $D^q \times S^{n-q-2}$ to it, we obtain a closed manifold, namely, $\partial(M + (\phi^q))$.

Let W be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Then we want to construct W from $\partial_0 W \times [0,1]$ by attaching handles as follows. Notice that the following construction will not change $\partial_0 W = \partial_0 W \times \{0\}$. If $\phi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$ is an embedding, we get by attaching a handle the compact manifold $W_1 = \partial_0 W \times [0,1] + (\phi^q)$ which is given by $W \cup_{\phi^q} D^q \times D^{n-q}$. Its boundary is a disjoint sum $\partial_0 W_1 \coprod \partial_1 W_1$, where $\partial_0 W_1$ is the same as $\partial_0 W$. Now we can iterate this process, where we attach a handle to $\partial_1 W_1$. Thus we obtain a compact manifold with boundary

$$W = \partial_0 W \times [0,1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \ldots + (\phi_r^{q_r}),$$

whose boundary is the disjoint union $\partial_0 W \coprod \partial_1 W$, where $\partial_0 W$ is just $\partial_0 W \times \{0\}$. We call such a description of W as above a *handlebody decomposition* of W relative $\partial_0 W$. We get from Morse theory [31, Chapter 6], [45, part I].

Lemma 1.7 *Let* W *be a compact manifold whose boundary* ∂W *is the disjoint sum* $\partial_0 W \coprod \partial_1 W$. *Then* W *possesses a handlebody decomposition relative* $\partial_0 W$, *i.e.* W *is up to diffeomorphism relative* $\partial_0 W = \partial_0 W \times \{0\}$ *of the form*

$$W = \partial_0 W \times [0,1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \ldots + (\phi_r^{q_r}).$$

If we want to show that W is diffeomorphic to $\partial_0 W \times [0,1]$ relative $\partial_0 W = \partial_0 W \times \{0\}$, we must get rid of the handles. For this purpose we have to find possible modifications of the handlebody decomposition which reduce the number of handles without changing the diffeomorphism type of W relative $\partial_0 W$.

Lemma 1.8 (Isotopy lemma) *Let W be an n-dimensional compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. If $\phi^q, \psi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$ are isotopic embeddings, then there is a diffeomorphism $W + (\phi^p) \to W + (\psi^q)$ relative $\partial_0 W$.*

Proof : Let $i : S^{q-1} \times D^{n-q} \times [0,1] \to \partial_1 W$ be an isotopy from ϕ^q to ψ^q. Then one can find a diffeotopy $H : W \times [0,1] \to W$ with $H_0 = \mathrm{id}_W$ such that the composition of H with $\phi^q \times \mathrm{id}_{[0,1]}$ is i and H is stationary on $\partial_0 W$ [31, Theorem 1.3 in Chapter 8 on page 184]. Thus $H_1 : W \to W$ is a diffeomorphism relative $\partial_0 W$ and satisfies $H_1 \circ \phi^q = \psi^q$. It induces a diffeomorphism $W + (\phi^p) \to W + (\psi^q)$ relative $\partial_0 W$. ■

Lemma 1.9 (Associativity lemma) *Let W resp. W' be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$ resp. $\partial_0 W' \coprod \partial_1 W'$. Let $F : W \to W'$ be a diffeomorphism which induces a diffeomorphism $f_0 : \partial_0 W \to \partial_0 W'$. Let $\phi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$ be an embedding. Then there is an embedding $\overline{\phi}^q : S^{q-1} \times D^{n-q} \to \partial_1 W'$ and a diffeomorphism $F' : W + (\phi^q) \to W' + (\overline{\phi}^q)$ which induces f_0 on $\partial_0 W$.*

Proof : Put $\overline{\phi}^q = F \circ \phi^q$. ■

Lemma 1.10 *Let W be an n-dimensional compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Suppose that $V = W + (\psi^r) + (\phi^q)$ for $q \leq r$. Then V is diffeomorphic relative $\partial_0 W$ to $V' = W + (\overline{\phi}^q) + (\psi^r)$ for an appropriate $\overline{\phi}^q$.*

Proof : By transversality and the assumption $(q-1) + (n-1-r) < n-1$ we can show that the embedding $\phi^q|_{S^{q-1} \times \{0\}} : S^{q-1} \times \{0\} \to \partial_1(W + (\psi^r))$ is isotopic to an embedding which does not meet the transverse sphere of the handle (ψ^r) attached by ψ^r [31, Theorem 2.3 in Chapter 3 on page 78]. This isotopy can be embedded in a diffeotopy on $\partial_1(W + (\psi^r))$. Thus the embedding $\phi^q : S^{q-1} \times D^{n-q} \to \partial_1(W + (\psi^r))$ is isotopic to an embedding whose restriction to $S^{q-1} \times \{0\}$ does not meet the transverse sphere of the handle (ψ^r). Since we can isotope an embedding $S^{q-1} \times D^{n-q} \to W + (\psi^r)$ such that its image becomes arbitrary close to the image of $S^{q-1} \times \{0\}$, we can isotope $\phi^q : S^{q-1} \times D^{n-q} \to \partial_1(W + (\psi^r))$ to an embedding which does not meet a closed neighborhood $U \subset \partial_1(W + (\psi^r))$ of the transverse sphere of the handle (ψ^r). There is an obvious diffeotopy on $\partial_1(W + (\psi^r))$ which is stationary on the transverse sphere of (ψ^r) and moves any point on $\partial_1(W + (\psi^r))$ which belongs to the handle (ψ^r) but not to U to a point outside the handle (ψ^r). Thus we can find an isotopy of ϕ^q to an embedding $\overline{\phi}^p$ which does not meet the handle (ψ^r) at all. Obviously $W + (\psi^r) + (\overline{\phi}^q)$ and $W + (\overline{\phi}^q) + (\psi^r)$ agree. By the Isotopy Lemma 1.8 there is a diffeomorphism relative $\partial_0 W$ from $W + (\psi^r) + (\overline{\phi}^q)$ to $W + (\psi^r) + (\phi^q)$. ■

Example 1.11 Here is a standard situation, where attaching first a q-handle and then a $(q+1)$-handle does not change the diffeomorphism type of an n-dimensional compact manifold W whose boundary is the disjoint union $\partial_0 W \coprod \partial_1 W$.

Consider an embedding $\mu : S^{q-1} \times D^{n-q} \cup_{S^{q-1}\times S_+^{n-1-q}} D^q \times S_+^{n-1-q} \to \partial_1 W$, where S_+^{n-1-q} is the upper hemisphere in $S^{n-1-q} = \partial D^{n-q}$. Let $\phi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$ be its restriction to $S^{q-1} \times D^{n-q}$. Let $\phi_+^{q+1} : S_+^q \times S_+^{n-q-1} \to \partial_1(W + (\phi^q))$ be the embedding which is given by

$$S_+^q \times S_+^{n-q-1} = D^q \times S_+^{n-q-1} \subset D^q \times S^{n-q-1} = \partial(\phi^q) \subset \partial_1(W + (\phi^q)).$$

It does not meet the interior of W. Let $\phi_-^{q+1} : S_-^q \times S_+^{n-1-q} \to \partial_1(W \cup (\phi^q))$ be the embedding obtained from μ by restriction to $S_-^q \times S_+^{n-1-q} = D^q \times S_+^{n-1-q}$. Then ϕ_-^{q+1} and ϕ_+^{q+1} fit together to yield an embedding $\psi^{q+1} : S^q \times D^{n-q-1} = S_-^q \times S_+^{n-q-1} \cup_{S^{q-1}\times S_+^{n-q-1}} S_+^q \times S_+^{n-q-1} \to \partial_1(W + (\phi^q))$. Then it is not difficult to check that $W + (\phi^q) + (\psi^{q+1})$ is diffeomorphic relative $\partial_0 W$ to W.

This cancellation of two handles of consecutive index can be generalized as follows.

Lemma 1.12 (Cancellation lemma) *Let W be an n-dimensional compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Let $\phi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$ be an embedding. Let $\psi^{q+1} : S^q \times D^{n-1-q} \to \partial_1(W + (\phi^q))$ be an embedding. Suppose that $\psi^{q+1}(S^q \times \{0\})$ is transversal to the transverse sphere of the handle (ϕ^q) and meets the transverse sphere in exactly one point. Then there is a diffeomorphism relative $\partial_0 W$ from W to $W + (\phi^q) + (\psi^{q+1})$.*

Proof : Given any neighborhood $U \subset \partial(\phi^q)$ of the tranverse sphere of (ϕ^q), there is an obvious diffeotopy on $\partial_1(W + (\phi^q))$ which is stationary on the transverse sphere of (ϕ^q) and moves any point on $\partial_1(W + (\phi^q))$ which belongs to the handle (ϕ^q) but not to U to a point outside the handle (ϕ^q). Thus we can achieve that ψ^{q+1} maps the lower hemisphere $S_-^q \times \{0\}$ to points outside (ϕ^q) and is on the upper hemisphere $S_+^q \times \{0\}$ given by the obvious inclusion $D^q \times \{x\} \to D^p \times D^{n-q} = (\phi^q)$ for some $x \in S^{n-q-1}$ and the obvious identification of $S_+^q \times \{0\}$ with $D^q \times \{x\}$. Now it is not hard to construct an diffeomorphism relative $\partial_0 W$ from $W + (\phi^q) + (\psi^{q+1})$ to W modelling the standard situation of Example 1.11. ■

The Cancellation Lemma 1.12 will be our only tool to reduce the number of handles. Notice that one can never get rid of one handle alone, there must always be involved at least two handles simultaneously. The reason is that the Euler characteristic $\chi(W, \partial_0 W)$ is independent of the handle decomposition and can be computed by $\sum_{q\geq 0}(-1)^q \cdot p_q$, where p_q is the number of q-handles (see Section 1.2).

We conclude from the Cancellation Lemma 1.12

Lemma 1.13 *Let $\phi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$ be a trivial embedding. Then there is an embedding $\psi^{q+1} : S^q \times D^{n-1-q} \to \partial_1(W + (\phi^q))$ such that W and $W + (\phi^q) + (\phi^{q+1})$ are diffeomorphic relativ $\partial_0 W$.*

Consider a compact n-dimensional manifold W whose boundary is the disjoint union $\partial_0 W \coprod \partial_1 W$. In view of Lemma 1.7 and Lemma 1.10 we can write it

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_0}(\phi_i^0) + \sum_{i=1}^{p_1}(\phi_i^1) + \ldots + \sum_{i=1}^{p_n}(\phi_i^n), \qquad (1.14)$$

where $\cong$ means diffeomorphic relative $\partial_0 W$.

Notation 1.15 *Put for $-1 \leq q \leq n$*

$$\begin{aligned} W_q &:= \partial_0 W \times [0,1] + \sum_{i=1}^{p_0}(\phi_i^0) + \sum_{i=1}^{p_1}(\phi_i^1) + \ldots + \sum_{i=1}^{p_q}(\phi_i^q); \\ \partial_1 W_q &:= \partial W_q - \partial_0 W \times \{0\}; \\ \partial_1^\circ W_q &:= \partial_1 W_q - \coprod_{i=1}^{p_{q+1}} \phi_i^{q+1}(S^q \times \operatorname{int}(D^{n-1-q})). \end{aligned}$$

Notice for the sequel that $\partial_1^\circ W_q \subset \partial_1 W_{q+1}$.

Lemma 1.16 (Elimination Lemma) *Fix an integer q with $1 \leq q \leq n-3$. Suppose that $p_j = 0$ for $j < q$, i.e. W looks like*

$$W = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q}(\phi_i^q) + \sum_{i=1}^{p_{q+1}}(\phi_i^{q+1}) + \ldots + \sum_{i=1}^{p_n}(\phi_i^n).$$

Fix an integer i_0 with $1 \leq i_0 \leq p_q$. Suppose that there is an embedding $\psi^{q+1} : S^q \times D^{n-1-q} \to \partial_1^\circ W_q$ with the following properties:

1. *$\psi^{q+1}|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_q$ to an embedding $\psi_1^{q+1} : S^q \times \{0\} \to \partial_1 W_q$ which meets the transverse sphere of the handle $(\phi_{i_0}^q)$ transversally and in exactly one point and is disjoint from to the transverse sphere of ϕ_i^q for $i \neq i_0$;*
2. *$\psi^{q+1}|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_{q+1}$ to a trivial embedding $\psi_2^{q+1} : S^q \times \{0\} \to \partial_1 W_{q+1}$.*

Then W is diffeomorphic relative $\partial_0 W$ to a manifold of the shape

$$\partial_0 W \times [0,1] + \sum_{i=1,2,\ldots p_q, i \neq i_0}(\phi_i^q) + \sum_{i=1}^{p_{q+1}}(\overline{\phi}_i^{q+1}) + (\psi^{q+2}) + \sum_{i=1}^{p_{q+2}}(\overline{\phi}_i^{q+2}) + \ldots + \sum_{i=1}^{p_n}(\overline{\phi}_i^n).$$

Proof : Since $\psi^{q+1}|_{S^q \times \{0\}}$ is isotopic to ψ_1^{q+1} and ψ_2^{q+1} is trivial, we can extend ψ_1^{q+1} and ψ_2^{q+1} to embeddings denoted in the same way $\psi_1^{q+1} : S^q \times D^{n-q-1} \to \partial_1 W_q$ and $\psi_2^{q+1} : S^q \times D^{n-1-q} \to \partial_1^\circ W_{q+1}$ with the following properties [31, Theorem 1.5 in Chapter 8 on page 180]: ψ^{q+1} is isotopic to ψ_1^{q+1} in $\partial_1 W_q$, ψ_1^{q+1} does not meet the transverse spheres of the handles (ϕ_i^q) for $i \neq i_0$, $\psi_1^{q+1}|_{S^q \times \{0\}}$ meets the transverse sphere of the handle $(\phi_{i_0}^q)$ transversally and in exactly one point, ψ^{q+1} is isotopic to ψ_2^{q+1} within $\partial_1 W_{q+1}$ and ψ_2^{q+1} is trivial. Because of the Associativity Lemma 1.9 we can assume without loss of generality that there are no handles of index $\geq q+2$, i.e. $p_{q+2} = p_{q+3} = \ldots = p_n = 0$. It suffices to show for appropriate embeddings $\overline{\phi}_i^{q+1}$ and ψ^{q+2} that

$$\begin{aligned} &\partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) \\ &\quad\cong \quad \partial_0 W \times [0,1] + \sum_{i=1,2,\ldots p_q, i \neq i_0}^{} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\overline{\phi}_i^{q+1}) + (\psi^{q+2}), \end{aligned}$$

where $\cong$ means diffeomorphic relative $\partial_0 W$. Because of Lemma 1.13 there is an embedding (ψ^{q+2}) satisfying

$$\begin{aligned} &\partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) \\ &\quad\cong \quad \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+1}) + (\psi^{q+2}). \end{aligned}$$

We conclude from the Isotopy Lemma 1.8 and the Associativity Lemma 1.9 for appropriate embeddings ψ_k^{q+1} for $k = 1, 2$

$$\begin{aligned} &\partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+1}) + (\psi^{q+2}) \\ \cong \quad &\partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi^{q+1}) + (\psi_1^{q+2}) \\ \cong \quad &\partial_0 W \times [0,1] + \sum_{i=1,2,\ldots p_q, i \neq i_0}^{p_q} (\phi_i^q) + (\phi_{i_0}^q) + (\psi^{q+1}) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+2}). \end{aligned}$$

We get from the Associativity Lemma 1.9 and the Cancellation Lemma 1.12 for appropriate embeddings $\overline{\phi}_i^{q+2}$ and ψ_3^{q+1}

$$\begin{aligned} &\partial_0 W \times [0,1] + \sum_{i=1,2,\ldots p_q, i \neq i_0}^{p_q} (\phi_i^q) + (\phi_{i_0}^q) + (\psi^{q+1}) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+2}) \\ &\quad\cong \quad \partial_0 W \times [0,1] + \sum_{i=1,2,\ldots p_q, i \neq i_0}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\overline{\phi}_i^{q+1}) + (\psi_3^{q+2}). \end{aligned}$$

This finishes the proof of the Elimination Lemma 1.16. ∎

1.2 Handlebody decompositions and CW-structures

Next we explain how we can associate to a handlebody decomposition (1.14) a CW-pair $(X, \partial_0 W)$ such that there is a bijective correspondence between the

q-handles of the handlebody decomposition and the q-cells of $(X, \partial_0 W)$. The key ingredient is the elementary fact that the projection $(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (D^q, S^{q-1})$ is a homotopy equivalence and actually – as we will explain later – a simple homotopy equivalence.

Recall that a (relative) CW-complex (X, A) consists of a pair of topological spaces (X, A) together with a filtration

$$X_{-1} = A \subset X_0 \subset X_1 \subset \ldots X_q \subset X_{q+1} \subset \ldots \cup_{q \geq 0} X_q = X$$

such that X carries the colimit topology with respect to this filtration and for any $q \geq 0$ there exists a pushout of spaces

$$\begin{array}{ccc} \coprod_{i \in I_q} S^{q-1} & \xrightarrow{\coprod_{i \in I_q} \phi_i^q} & X_{q-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_q} D^q & \xrightarrow[\coprod_{i \in I_q} \Phi_i^q]{} & X_q \end{array}$$

The map ϕ_i^q is called the *attaching map* and the map (Φ_i^q, ϕ_i^q) is called the *characteristic map* of the q-cell belonging to $i \in I_q$. The pushouts above are not part of the structure, only their existence is required. Only the filtration $\{X_q \mid q \geq -1\}$ is part of the structure. The path components of $X_q - X_{q-1}$ are called the *open cells*. The open cells coincide with the sets $\Phi_i^q(D^q - S^{q-1})$. The closure of an open cell $\Phi_i^q(D^q - S^{q-1})$ is called *closed cell* and turns out to be $\Phi_i^q(D^q)$.

Suppose that X is connected with fundamental group π. Let $p : \widetilde{X} \to X$ be the universal covering of X. Put $\widetilde{X_q} = p^{-1}(X_q)$ and $\widetilde{A} = p^{-1}(A)$. Then $(\widetilde{X}, \widetilde{A})$ inherits a CW-structure from (X, A) by the filtration $\{\widetilde{X_q} \mid q \geq -1\}$. The cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{X}, \widetilde{A})$ has as q-th $\mathbb{Z}\pi$-chain module the singular homology $H_q(\widetilde{X_q}, \widetilde{X_{q-1}})$ with $\mathbb{Z}$-coefficients and the π-action coming from the deck transformations. The q-th differential d_q is given by the composition

$$H_q(\widetilde{X_q}, \widetilde{X_{q-1}}) \xrightarrow{\partial_q} H_{q-1}(\widetilde{X_{q-1}}) \xrightarrow{i_q} H_{q-1}(\widetilde{X_{q-1}}, \widetilde{X_{q-2}}),$$

where ∂_q is the boundary operator of the long exact sequence of the pair $(\widetilde{X_q}, \widetilde{X_{q-1}})$ and i_q is induced by the inclusion. If we choose for each $i \in I_q$ a lift $(\widetilde{\Phi_i^q}, \widetilde{\phi_i^q}) : (D^q, S^{q-1}) \to (\widetilde{X_q}, \widetilde{X_{q-1}})$ of the characteristic map (Φ_i^q, ϕ_i^q), we obtain a $\mathbb{Z}\pi$-basis $\{b_i \mid i \in I_n\}$ for $C_n(\widetilde{X}, \widetilde{A})$, if we define b_i as the image of a generator in $H_q(D^q, S^{q-1}) \cong \mathbb{Z}$ under the map $H_q(\widetilde{Q_i^q}, \widetilde{q_i^q}) : H_q(D^q, S^{q-1}) \to H_q(\widetilde{X_q}, \widetilde{X_{q-1}}) = C_q(\widetilde{X}, \widetilde{A})$. We call $\{b_i \mid i \in I_n\}$ *the cellular basis.* Notice that we have made several choices in defining the cellular basis. We call two $\mathbb{Z}\pi$-basis $\{\alpha_j \mid j \in J\}$ and $\{\beta_k \mid k \in K\}$ for $C_q(\widetilde{X}, \widetilde{A})$ *equivalent* if there is a bijection $\phi : J \to K$ and elements $\epsilon_j \in \{\pm 1\}$ and $\gamma_j \in \pi$ for $j \in J$ such that $\epsilon \cdot \gamma_j \cdot \alpha_j = \beta_{\phi(j)}$. The equivalence class of the basis $\{b_i \mid i \in I_n\}$ constructed above does only depend on the CW-structure on (X, A) and is independent of all further choices such as (Φ_i^q, ϕ_i^q), its lift $(\widetilde{\Phi_i^q}, \widetilde{\phi_i^q})$ and the generator of $H_n(D^n, S^{n-1})$.

Now suppose we are given a handlebody decomposition (1.14). Next we construct by induction over $q = -1, 0, 1, \ldots n$ a sequence of spaces $X_{-1} = \partial_0 W \subset X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n$ together with homotopy equivalences $f_q : W_q \to X_q$ such that $f_q|_{W_{q-1}} = f_{q-1}$ and $(X, \partial_0 W)$ is a CW-complex with respect to the filtration $\{X_q \mid q = -1, 0, 1, \ldots n\}$. The induction beginning $f_1 : W_{-1} = \partial_0 W \times [0,1] \to X_1 = \partial_0 W$ is given by the projection. The induction step from $(q-1)$ to q is done as follows. We attach for each handle (ϕ_i^q) for $i = 1, 2, \ldots p_q$ a cell D^q to X_{q-1} by the attaching map $f_{q-1} \circ \phi_i^q|_{S^{q-1} \times \{0\}}$. In other words, we define X_q by the pushout

$$\begin{array}{ccc}
\coprod_{i=1}^{p_q} S^{q-1} & \xrightarrow{\coprod_{i=1}^{p_q} f_{q-1} \circ \phi_i^q|_{S^{q-1} \times \{0\}}} & X_{q-1} \\
\downarrow & & \downarrow \\
\coprod_{i=1}^{p_q} D^q & \longrightarrow & X_q
\end{array}$$

Recall that W_q is the pushout

$$\begin{array}{ccc}
\coprod_{i=1}^{p_q} S^{q-1} \times D^{n-q} & \xrightarrow{\coprod_{i=1}^{p_q} \phi_i^q} & W_{q-1} \\
\downarrow & & \downarrow \\
\coprod_{i=1}^{p_q} D^q \times D^{n-q} & \longrightarrow & W_q
\end{array}$$

Define a space Y_q by the pushout

$$\begin{array}{ccc}
\coprod_{i=1}^{p_q} S^{q-1} & \xrightarrow{\coprod_{i=1}^{p_q} \phi_i^q|_{S^{q-1} \times \{0\}}} & W_{q-1} \\
\downarrow & & \downarrow \\
\coprod_{i=1}^{p_q} D^q & \longrightarrow & Y_q
\end{array}$$

Define $(g_q, f_{q-1}) : (Y_q, W_{q-1}) \to (X_q, X_{q-1})$ by the pushout property applied to homotopy equivalences given by $f_{q-1} : W_{q-1} \to X_{q-1}$ and the identity maps on S^{q-1} and D^q. Define $(h_q, \mathrm{id}) : (Y_q, W_{q-1}) \to (W_q, W_{q-1})$ by the pushout property applied to homotopy equivalences given by the obvious inclusions $S^{q-1} \to S^{q-1} \times D^{n-q}$ and $D^q \to D^q \times D^{n-q}$ and the identity on W_{q-1}. The resulting maps are homotopy equivalences of pairs since the upper horizontal arrows in the three pushouts above are cofibrations (see [10, page 249]). Choose a homotopy inverse $(h_q^{-1}, \mathrm{id}) : (W_q, W_{q-1}) \to (Y_q, W_{q-1})$. Define f_q by the composition $g_q \circ h_q^{-1}$.

In particular we see that the inclusions $W_q \to W$ are q-connected since the inclusion of the q-skeleton $X_q \to X$ is always q-connected for a CW-complex X.

Denote by $p : \widetilde{W} \to W$ the universal covering with $\pi = \pi_1(W)$ as group of deck transformations. Let $\widetilde{W_q}$ be the preimage of W_q under p. Notice that this is the universal covering for $q \geq 2$ since each inclusion $W_q \to W$ induces an isomorphism on the fundamental groups. Let $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ be the $\mathbb{Z}\pi$-chain complex whose q-th chain group is $H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$ and whose q-th differential is given by the composition

$$H_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \xrightarrow{\partial_p} H_q(\widetilde{W_{q-1}}) \xrightarrow{i_q} H_{q-1}(\widetilde{W_{q-1}}, \widetilde{W_{q-2}}),$$

where ∂_q is the boundary operator of the long homology sequence associated to the pair $(\widetilde{W_p}, \widetilde{W_{p-1}})$ and i_q is induced by the inclusion. The map $f : W \to X$ induces an isomorphism of $\mathbb{Z}\pi$-chain complex

$$C_*(\widetilde{f}) : C_*(\widetilde{W}, \widetilde{\partial_0 W}) \xrightarrow{\cong} C_*(\widetilde{X}, \widetilde{\partial_0 W}). \tag{1.17}$$

Each handle (ϕ_i^q) determines an element

$$[\phi_i^q] \in C_q(\widetilde{W}, \widetilde{\partial_0 W}) \tag{1.18}$$

after choosing a lift $(\widetilde{\Phi_i^q}, \widetilde{\phi_i^q}) : (D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (\widetilde{W_q}, \widetilde{W_{q-1}})$ of its characteristic map $(\Phi_i^q, \phi_i^q) : (D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (W_q, W_{q-1})$, namely, the image of the preferred generator in $H_q(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \cong H_0(\{*\} = \mathbb{Z}$ under the map $H_q(\widetilde{\Phi_i^q}, \widetilde{\phi_i^q})$. This element is only well-defined up to multiplication with an element $\gamma \in \pi$. The elements $\{[\phi_i^q] \mid i = 1, 2, \ldots p_q\}$ form a $\mathbb{Z}\pi$-basis for $C_q(\widetilde{W}, \widetilde{\partial_0 W})$. Its image under the isomorphism (1.17) is a cellular $\mathbb{Z}\pi$-basis.

If W has no handles of index ≤ 2, i.e. $p_0 = p_1 = 0$, one can express $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ also in terms of homotopy groups as follows. Fix a base point $z \in \partial_0 W$ and a lift $\widetilde{z} \in \widetilde{\partial_0 W}$. All homotopy groups are taken with respect to these base points. Let $\pi_*(W_*, W_{*-1})$ be the $\mathbb{Z}\pi$-chain complex, whose q-th $\mathbb{Z}\pi$-module is $\pi_q(W_q, W_{q-1})$ for $q \geq 2$ and zero for $q \leq 1$ and whose q-th differential is given by the composition

$$\pi_q(W_q, W_{q-1}) \xrightarrow{\partial_q} \pi_{q-1}(W_{q-1}) \xrightarrow{\pi_{q-1}(i)} \pi_{q-1}(W_{q-1}, W_{q-2}).$$

The $\mathbb{Z}\pi$-action comes from the canonical $\pi_1(Y)$-action on the group $\pi_q(Y, A)$ [65, Theorem I.3.1 on page 164]. Notice that $\pi_q(Y, A)$ is abelian for any pair of spaces (Y, A) for $q \geq 3$ and is abelian also for $q = 2$ if A is simply connected or empty. For $q \geq 2$ the Hurewicz homomorphism is an isomorphism [65, Corollary IV.7.11 on page 181] $\pi_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \to H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$ and the projection $p : \widetilde{W} \to W$ induces isomorphisms $\pi_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \to \pi_q(W_q, W_{q-1})$. Thus we obtain an isomorphism of $\mathbb{Z}\pi$-chain complexes

$$C_*(\widetilde{W}, \widetilde{\partial_0 W}) \xrightarrow{\cong} \pi_*(W_*, W_{*-1}). \tag{1.19}$$

Fix a path w_i in W from a point in the transverse sphere of (ϕ_i^q) to the base point z. Then the handle (ϕ_i^q) determines an element

$$[\phi_i^q] \in \pi_q(W_q, W_{q-1}). \tag{1.20}$$

It is represented by the obvious map $(D^q \times \{0\}, \times S^{q-1} \times \{0\}) \to (W_q, W_{q-1})$ together with w_i. It agrees with the element $[\phi_i^q] \in C_q(\widetilde{W}, \widetilde{\partial_0 W})$ defined in (1.18) under the isomorphism (1.19) if we use the lift of the characteristic map determined by the path w_i.

1.3 Reducing the handlebody decomposition of an h-cobordism

In the next step we want to get rid of the handles of index zero and one in the handlebody decomposition (1.14).

Lemma 1.21 *Let W be an n-dimensional manifold for $n \geq 6$ whose boundary is the disjoint union $\partial W = \partial_0 W \coprod \partial_1 W$. Then the following statements are equivalent*

1. *The inclusion $\partial_0 W \to W$ is 1-connected;*
2. *We can find a diffeomorphism relative $\partial_0 W$*

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_2}(\phi_i^2) + \sum_{i=1}^{p_3}(\overline{\phi}_i^3) + \sum_{i=1}^{p_n}(\overline{\phi}_i^n).$$

Proof : (2) $\Rightarrow$ (1) has already been proven in Section 1.2. It remains to conclude (2) provided that (1) holds.

We first get rid of all 0-handles in the handlebody decomposition (1.14). It suffices to give a procedure to reduce the number of handles of index 0 by one. Since the inclusion $\partial_0 W \to W$ is 1-connected, the inclusion $\partial_0 W \to W_1$ induces a bijection on the set of path components. Given any index i_0, there must be an index i_1 such that the core of the handle $\phi_{i_1}^1$ is a path connecting a point in $\partial_0 W \times \{1\}$ with a point in $(\phi_{i_0}^0)$. We conclude from the Associativity Lemma 1.9 and the Cancellation Lemma 1.12 that (ϕ_{i_0}) and $(\phi_{i_1}^1)$ cancel another, i.e. we have

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1,2,\ldots p_0, i \neq i_0} (\phi_i^0) + \sum_{i=1,2,\ldots p_1, i \neq i_1} (\phi_i^1) + \ldots + \sum_{i=1}^{p_n}(\phi_i^n).$$

Hence we can assume $p_0 = 0$ in (1.14).

Next we want to get rid of the 1-handles assuming that the inclusion $\partial_0 W \to W$ is 1-connected. It suffices to give a procedure to reduce the number of handles of index 1 by one. We want to do this by constructing an embedding $\psi^2 : S^1 \times D^{n-2} \to \partial_1^\circ W_1$ which satisfies the two conditions of the Elimination Lemma 1.16 and then applying the Elimination Lemma 1.16. Consider the embedding $\psi_+^2 : S_+^1 = D^1 = D^1 \times \{x\} \subset D^1 \times D^{n-1} = (\phi_1^1)$ for some fixed $x \in S^{n-2} = \partial D^{n-1}$. The inclusion $\partial_1^\circ W_0 \to \partial_1 W_0 = \partial_0 W \times \{1\}$ induces an isomorphism on the fundamental group since $\partial_1^\circ W_0$ is obtained from $\partial_1 W_0 = \partial_0 W \times \{1\}$ by removing the interior of a finite number of embedded $(n-1)$-dimensional disks. Since by assumption the inclusion $\partial_0 W \to W$ is 1-connected, the inclusion $\partial_1^\circ W_0 \to W$ induces an epimorphism on the fundamental groups. Therefore we can find an embedding $\psi_-^2 : S_-^1 \to \partial_1^\circ W_0$ with $\psi_-^2|_{S^0} = \psi_+^2|_{S^0}$ such that the map $\psi_0^2 : S^1 = S_+^1 \cup_{S^0} S_-^1 \to \partial_1 W_1$ given by $\psi_+^2 \cup \psi_-^2$ is nullhomotopic in W. One can isotop the attaching maps $\phi_i^2 : S^1 \times D^{n-2} \to \partial_1 W_1$ of the 2-handles (ϕ_i^2) such that they do not meet the image of ψ_0^2 because the sum of the dimension of the source of ψ_0^2 and of $S^1 \times \{0\} \subset S^1 \times D^{n-2}$ is less than the dimension $(n-1)$ of $\partial_1 W_1$ and one can always shrink inside D^{n-2}. Thus we can assume without loss of generality by the Isotopy Lemma 1.8 and the Associativity Lemma 1.9 that the image of ψ_0^2 lies in $\partial_1^\circ W_1$. The inclusion $\partial_1 W_2 \to W$ is 2-connected. Hence ψ_0^2 is nullhomotopic in $\partial_1 W_2$. Let $h : D^2 \to \partial_1 W_2$ be a nullhomotopy for ψ_0^2. Since $2 \cdot \dim(D^2) < \dim(\partial_1 W_2)$, we can change h relative to S^1 into an embedding. (Here we need for the first time the assumption $n \geq 6$.) Since D^2 is contractible the normal bundle of h and thus of $\psi_1^2 \cup \psi_2^2$ are trivial. Therefore we can extend ψ_0^2 to an embedding $\psi^2 : S^1 \times D^{n-1} \to \partial_1^\circ W_1$ which is isotopic to a trivial embedding in $\partial_1 W_2$ and meets the transverse sphere of the handle (ϕ_1^1) transversally and in exactly one point and does not meet the transverse spheres of the handles (ϕ_i^1) for $2 \leq i \leq p_1$. Now Lemma 1.21 follows from the Elimination Lemma 1.16. ■

Now consider an h-cobordism $(W; \partial_0 W, \partial_1 W)$. Because of Lemma 1.21 we can write it as

$$W \quad \cong \quad \partial_0 W \times [0,1] + \sum_{i=1}^{p_2} (\phi_i^2) + \sum_{i=1}^{p_3} (\overline{\phi}_i^3) + \ldots .$$

Lemma 1.22 (Homology lemma) *Suppose $n \geq 6$. Fix $2 \leq q \leq n-3$ and $i_0 \in \{1, 2, \ldots p_q\}$. Let $S^q \to \partial_1 W_q$ be an embedding. Then the following statements are equivalent*

1. *f is isotopic to an embedding $g : S^q \to \partial_1 W_q$ such that g meets the transverse sphere of $(\phi_{i_0}^q)$ transversally and in exactly one point and is disjoint from transverse spheres of the handles (ϕ_i^q) for $i \neq i_0$;*
2. *Let $\widetilde{f} : S^q \to \widetilde{W_q}$ be a lift of f under $p|_{\widetilde{W_q}} : \widetilde{W_q} \to W_q$. Let $[\widetilde{f}]$ be the image of the class represented by $\widetilde{f}$ under the obvious composition*

$$\pi_q(\widetilde{W_q}) \to \pi_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \to H_q(\widetilde{W_q}, \widetilde{W_{q-1}}) = C_q(\widetilde{W}).$$

Then there is $\gamma \in \pi$ with

$$[\widetilde{f}] = \pm\gamma \cdot [\phi_{i_0}^q].$$

Proof : (1) ⇒ (2) We can isotop f such that $f|_{S_+^q} : S_+^q \to \partial_1 W_q$ looks like the canonical embedding $S_+^q = D^q \times \{x\} \subset D^q \times S^{n-1-q} = \partial(\phi_{i_0}^q)$ for some

$x \in S^{n-1-q}$ and $f(S^q_-)$ does not meet any of the handles (ϕ^q_i) for $i = 1, 2, \ldots p_q$. One easily checks that then (2) is true.

(2) $\Rightarrow$ (1) We can isotop f such that it is transversal to the transverse spheres of the handles (ϕ^q_i) for $i = 1, 2, \ldots p_q$. Because the sum of the dimension of the source of f and of the dimension of the transverse spheres is the dimension of $\partial_1 W_q$, the intersection of the image of f with the transverse sphere of the handle (ϕ^q_i) consists of finitely many points $x_{i,1}, x_{i,2}, \ldots, x_{i,r_i}$ for $i = 1, 2, \ldots p_q$. Fix a base point $y \in S^q$. It yields a base point $z = f(y) \in W$. Fix for each handle (ϕ^q_i) a path w_i in W from a point in its transverse sphere to z. Let $u_{i,j}$ be a path in S^q with the property that $u_{i,j}(0) = y$ and $f(u_{i,j}(1)) = x_{i,j}$ for $1 \leq j \leq r_i$ and $1 \leq i \leq p_q$. Let $v_{i,j}$ be any path in the transverse sphere from $x_{i,j}$ to $w_i(0)$. Then the composition $f(u_{i,j}) * v_{i,j} * w_i$ is a loop in W with base point z and thus represents an element denoted by $\gamma_{i,j}$ in $\pi = \pi_1(W, z)$. It is independent of the choice of $u_{i,j}$ and $v_{i,j}$ since S^q and the transverse sphere of each handle (ϕ^q_i) are simply connected. The tangent space $T_{x_{i,j}} \partial_1 W_q$ is the direct sum of $T_{f^{-1}(x_{i,j})} S^p$ and the tangent space of the transverse sphere $\{0\} \times S^{n-1-q}$ of the handle (ϕ^q_i) at $x_{i,j}$. All these three tangent spaces come with preferred orientations. We define elements $\epsilon_{i,j} \in \{\pm 1\}$ by requiring that it is 1 if these orientations fit together and -1 otherwise. Now one easily checks that

$$[\widetilde{f}] = \sum_{i=1}^{q} \sum_{j=1}^{r_i} \epsilon_{i,j} \cdot \gamma_{i,j} \cdot [\phi^q_i],$$

where $[\phi^q_i]$ is the element associated to the handle (ϕ^q_i) after the choice of the path w_i (see (1.18) and (1.20)). We have by assumption $[\widetilde{f}] = \pm \cdot \gamma \cdot [\phi^q_{i_0}]$ for some $\gamma \in \pi$. We want to isotop f such that f does not meet the transverse spheres of the handles (ϕ^q_i) for $i \neq i_0$ and the transverse sphere of $(\phi^q_{i_0})$ transversally and in exactly one point. Therefore it suffices to show that in the case that the number $\sum_{i=1}^{p_q} r_i$ of all intersection points of f with the transverse spheres of the handles (ϕ^q_i) for $i = 1, 2, \ldots p_i$ is bigger than one that we can change f up to isotopy such that this number becomes smaller. We have

$$\pm \cdot \gamma \cdot [\phi^q_{i_0}] = \sum_{i=1}^{p_q} \sum_{j=1}^{r_i} \epsilon_{i,j} \cdot \gamma_{i,j} \cdot [\phi^q_i].$$

Recall that the elements $[\phi^q_i]$ for $i = 1, 2, \ldots p_q$ form a $\mathbb{Z}\pi$-basis. Hence we can find an index $i \in \{1, 2, \ldots p_q\}$ and two different indices $j_1, j_2 \in \{1, 2, \ldots r_i\}$ such that the composition of the paths $f(u_{i,j_1}) * v_{i,j_1} * v^-_{i,j_2} * f(u^-_{i,j_2})$ is nullhomotopic in W and hence in $\partial_1 W_q$ and the signs ϵ_{i,j_1} and ϵ_{i,j_2} are different. Now by the Whitney trick (see [46, Theorem 6.6 on page 71], [67]) we can change f up to isotopy such that the two intersection points x_{i,j_1} and x_{i,j_2} disappear, the other intersection points of f with transverse spheres of the handles (ϕ^q_i) for $i \in \{1, 2, \ldots p_q\}$ remain and no further intersection points are introduced. For the application of the Whitney trick we need the assumption $n - 1 \geq 5$. This finishes the proof of the Homology Lemma 1.22. ∎

Lemma 1.23 (Modification Lemma) *Let $f : S^q \to \partial_1^\circ W_q$ be an embedding and let $x_j \in \mathbb{Z}\pi$ be elements for $j = 1, 2 \ldots p_{q+1}$. Then there is an embedding $g : S^q \to \partial_1^\circ W_q$ with the following properties:*

1. *f and g are isotopic in $\partial_1 W_{q+1}$;*
2. *For a given lifting $\widetilde{f} : S^q \to \widetilde{W_q}$ of f one can find a lifting $\widetilde{g} : S^q \to \widetilde{W_q}$ of g such that we get in $C_q(\widetilde{W})$*

$$[\widetilde{g}] \;=\; [\widetilde{f}] + \sum_{j=1}^{p_{q+1}} x_j \cdot d_{q+1}[\phi_j^{q+1}],$$

where d_{q+1} is the $(q+1)$-th differential in $C_*(\widetilde{W}, \widetilde{\partial_0 W})$.

Proof : Any element in $\mathbb{Z}\pi$ can be written as a sum of elements of the shape $\pm\gamma$ for $\gamma \in \pi$. Hence it suffices to prove for a fixed number $j \in \{1, 2 \ldots p_q\}$, fixed element $\gamma \in \pi$ and fixed sign $\epsilon \in \{\pm 1\}$ that one can find an embedding $g : S^q \to \partial_1^\circ W_q$ which is isotopic to f in $\partial_1 W_{q+1}$ and satisfies for an appropriate lifting $\widetilde{g}$

$$[\widetilde{g}] \;=\; [\widetilde{f}] + \epsilon \cdot \gamma \cdot d_{p+1}[\phi_j^{q+1}].$$

Consider the embedding $t_j : S^q = S^q \times \{z\} \subset S^q \times S^{n-2-q} \subset \partial(\phi_j^{q+1}) \subset \partial_1 W_q$ for some point $z \in S^{n-2-q} = \partial D^{n-1-q}$. It is in $\partial_1 W_{q+1}$ isotopic to a trivial embedding. Choose a path w in $\partial_1^\circ W_q$ connecting a point in the image of f with a point in the image of t_j. Without loss of generality we can arrange w to be an embedding. Moreover, we can thicken $w : [0,1] \to \partial_1^\circ W_q$ to an embedding $\overline{w} : [0,1] \times D^q \to \partial_1^\circ W_q$ such that $\overline{w}(\{0\} \times D^q)$ and $\overline{w}(\{1\} \times D^q)$ are embedded q-dimensional disks in the images of f and t_j and $\overline{w}((0,1) \times D^q)$ does not meet the images of f and t_j. Now one can form a new embedding, the connected sum $g := f\sharp_w t_j : S^q \to \partial_1^\circ W_q$. It is essentially given by restriction of f and t_j to the part of S^q, which is not mapped under f and t_j to the interior of the disks $\overline{w}(\{0\} \times D^q)$, $\overline{w}(\{1\} \times D^q)$, and $\overline{w}|_{[0,1] \times S^{q-1}}$. Since t_j is isotopic to a trivial embedding in $\partial_1 W_{q+1}$, the embedding g is isotopic in $\partial_1 W_{q+1}$ to f. Recall that we have fixed a lifting $\widetilde{f}$ of f. This determines a unique lifting of $\widetilde{g}$, namely, we require that $\widetilde{f}$ and $\widetilde{g}$ coincide on those points, where f and g already coincide. For an appropriate element $\gamma' \in \pi$ one gets $[\widetilde{g}] = [\widetilde{f}] + \gamma' \cdot d_{q+1}([\phi_j^{q+1}])$, since $t_j : S^q \to \partial_1 W_q \subset W_q$ is homotopic to $\phi_j^{q+1}|_{S^q \times \{0\}} : S^q \times \{0\} = S^q \to W_q$ in W_q. We can change the path w by composing it with a loop representing $\gamma \cdot (\gamma')^{-1} \in \pi$. Then we get for the new embedding g that

$$[\widetilde{g}] = [\widetilde{f}] + \gamma \cdot d_{q+1}([\phi_j^{q+1}]).$$

If we compose t_j with a diffeomorphism $S^q \to S^q$ of degree -1, we still get an embedding g which is isotopic to f in $\partial_1 W_{q+1}$ and satisfies

$$[\widetilde{g}] = [\widetilde{f}] - \gamma \cdot d_{q+1}([\phi_j^{q+1}]).$$

This finishes the proof of the Modification Lemma 1.23. ■

Lemma 1.24 (Normal form lemma) *Let $(W;\partial_0 W, \partial_1 W)$ be an n-dimensional oriented compact h-cobordism for $n \geq 6$. Let q be an integer with $2 \leq q \leq n-3$. Then there is a handlebody decomposition which has only handles of index q and $(q+1)$, i.e. there is a diffeomorphism relative $\partial_0 W$*

$$W \quad \cong \quad \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^r) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}).$$

Proof : In the first step we show that we can arrange $W = W_q$, i.e. $p_r = 0$ for $r \leq q-1$. We do this by induction over q. The induction begin $q = 2$ has already been carried out in Lemma 1.21. In the induction step from q to $(q+1)$ we must explain how we can decrease the number of q-handles provided that there are no handles of index $< q$. In order to get rid of the handle (ϕ_1^q) we want to attach a new $(q+1)$-handle and a new $(q+2)$-handle such that (ϕ_1^q) and the new $(q+1)$-handle cancel and the new $(q+1)$-handle and the new $(q+2)$-handle cancel each other. The effect will be that the number of q-handles is decreased by one at the cost of increasing the number of $(q+2)$-handles by one.

Fix a trivial embedding $\overline{\psi}^{q+1} : S^q \times D^{n-1-q} \to \partial_1^\circ W_q$. Since the inclusion $\partial_0 W \to W$ is a homotopy equivalence, $H_p(\widetilde{W}, \widetilde{\partial_0 W}) = 0$ for all $p \geq 0$. Since the p-th homology of $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is $H_p(\widetilde{W}, \widetilde{\partial_0 W}) = 0$, the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is acyclic. Since $C_{q-1}(\widetilde{W}, \widetilde{\partial_0 W})$ is trivial, the q-th differential of $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is zero and hence the $(q+1)$-th differential d_{q+1} is surjective. We can choose elements $x_j \in \mathbb{Z}\pi$ such that

$$[\phi_1^q] \quad = \quad \sum_{i=1}^{p_{q+1}} x_j \cdot d_{q+1}([\phi_i^{q+1}]).$$

Since $\alpha := \overline{\psi}^{q+1}|_{S^q \times \{0\}} \to \partial_1^\circ W_q$ is nullhomotopic, $[\widetilde{\alpha}] = 0$ in $H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$. Because of the Modification Lemma 1.23 we can find an embedding ψ^{q+1} : $S^q \times D^{n-1-q} \to \partial_1^\circ W_q$ such that $\beta := \psi^q|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_{q+1}$ to α and we get

$$[\widetilde{\beta}] \quad = \quad [\widetilde{\alpha}] + \sum_{i=1}^{p_{q+1}} x_j \cdot d_{q+1}([\phi_i^{q+1}]) \quad = \quad [\phi_1^q].$$

Because of the Homology Lemma 1.22 the embedding $\beta = \psi^q|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_q$ to an embedding $\gamma : S^q \to \partial_1 W_q$ which meets the transverse sphere of (ϕ_1^q) transversally and in exactly one point and is disjoint from the transverse spheres of all other handles of index q. By construction ψ^{q+1} is isotopic in $\partial_1 W_{q+1}$ to the trivial embedding $\overline{\psi}^{q+1}$. Now we can apply the Elimination Lemma 1.16. This finishes the proof that we can arrange $W = W_q$.

Next we explain the *dual handlebody decomposition*. Suppose that W is obtained from $\partial_0 W \times [0,1]$ by attaching one q-handle (ϕ^q), i.e. $W = \partial_0 W \times [0,1] + (\phi^q)$. Then we can interchange the role of $\partial_0 W$ and $\partial_1 W$ and try to built W from $\partial_1 W$ by handles. It turns out that W can be written as

$$W \quad = \quad \partial_1 W \times [0,1] + (\psi^{n-q}) \tag{1.25}$$

by the following argument.

Let M be the manifold with boundary $S^{q-1} \times S^{n-1-q}$ obtained from $\partial_0 W$ by removing the interior of $\phi^q(S^{q-1} \times D^{n-q})$. We get

$$\begin{aligned} W &\cong \partial_0 W \times [0,1] \cup_{S^{q-1}\times D^{n-q}} D^q \times D^{n-q} \\ &= M \times [0,1] \cup_{S^{q-1}\times S^{n-2-q}\times[0,1]} \\ &\qquad \left(S^{q-1} \times D^{n-1-q} \times [0,1] \cup_{S^{q-1}\times D^{n-q}\times\{1\}} D^q \times D^{n-q}\right). \end{aligned}$$

Inside $S^{q-1} \times D^{n-1-q} \times [0,1] \cup_{S^{q-1}\times D^{n-q}\times\{1\}} D^q \times D^{n-q}$ we have the following submanifolds

$$\begin{aligned} X &:= S^{q-1} \times 1/2 \cdot D^{n-1-q} \times [0,1] \cup_{S^{q-1}\times 1/2\cdot D^{n-q}\times\{1\}} D^q \times 1/2 \cdot D^{n-q}; \\ Y &:= S^{q-1} \times 1/2 \cdot S^{n-1-q} \times [0,1] \cup_{S^{q-1}\times 1/2\cdot S^{n-q}\times\{1\}} D^q \times 1/2 \cdot S^{n-q}. \end{aligned}$$

The pair (X,Y) is diffeomorphic to $(D^q \times D^{n-q}, D^q \times S^{n-1-q})$, i.e. it is a handle of index $(n-q)$. Let N be obtained from W by removing the interior of X. Then W is obtained from N by adding a $(n-q)$-handle, the so called *dual handle.* One easily checks that N is diffeomorphic to $\partial_1 W \times [0,1]$ relative $\partial_1 W \times \{1\}$. Thus (1.25) follows.

Suppose that W is relatively $\partial_0 W$ of the shape

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_0}(\phi_i^0) + \sum_{i=1}^{p_1}(\phi_i^1) + \ldots + \sum_{i=1}^{p_n}(\phi_i^n),$$

Then we can conclude inductively using the Associativity Lemma 1.9 and (1.25) that W is diffeomorphic relative to $\partial_1 W$ to

$$W \cong \partial_1 W \times [0,1] + \sum_{i=1}^{p_n}(\overline{\phi}_i^0) + \sum_{i=1}^{p_{n-1}}(\overline{\phi}_i^1) + \ldots + \sum_{i=1}^{p_0}(\overline{\phi}_i^n). \qquad (1.26)$$

This corresponds to replacing a Morse function f by $-f$. The effect is that the number of q-handles becomes now the number of $(n-q)$-handles.

Now applying the first step to the dual handlebody decomposition for q replaced by $(n-q-1)$ and then considering the dual handlebody decomposition of the result finishes the proof of the Normal Form Lemma 1.24. ■

1.4 Handlebody decomposition of an h-cobordism and Whitehead groups

Let $(W, \partial_0 W, \partial_1 W)$ be an n-dimensional compact oriented h-cobordism for $n \geq 6$. By the Normal Form Lemma 1.24 we can fix a handlebody decomposition for some fixed number $2 \leq q \leq n-3$

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q}(\phi_i^q) + \sum_{i=1}^{p_{q+1}}(\phi_i^{q+1}).$$

Recall that the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is acyclic. Hence the only non-trivial differential $d_{q+1} : H_{q+1}(\widetilde{W_{q+1}}, \widetilde{W_q}) \to H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$ is bijective. Recall that $\{[\phi_i^{q+1}] \mid i = 1, 2 \ldots p_{q+1}\}$ is a $\mathbb{Z}\pi$-basis for $H_{q+1}(\widetilde{W_{q+1}}, \widetilde{W_q})$ and $\{[\phi_i^q] \mid i = 1, 2 \ldots p_q\}$ is a $\mathbb{Z}\pi$-basis for $H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$. In particular $p_q = p_{q+1}$. The matrix A, which describes the differential d_{q+1} with respect to these basis, is an invertible (p_q, p_q)-matrix over $\mathbb{Z}\pi$. Since we are working with left modules, d_{q+1} sends an element $x \in (\mathbb{Z}G)^n$ to $x \cdot A \in \mathbb{Z}G^n$, or equivalently, $d_{q+1}([\phi_i^{q+1}]) = \sum_{j=1}^n a_{i,j}[\phi_j^q]$.

Next we define an abelian group $\mathrm{Wh}(\pi)$ as follows. It is the set of equivalence classes of invertible matrices of arbitrary size with entries in $\mathbb{Z}\pi$, where we call an invertible (m, m)-matrix A and an invertible (n, n)-matrix B over $\mathbb{Z}\pi$ equivalent, if we can pass from A to B by a sequence of the following operations:

1. B is obtained from A by adding the k-th row multiplied with x from the left to the l-th row for $x \in \mathbb{Z}\pi$ and $k \neq l$;
2. B is obtained by taking the direct sum of A and the $(1,1)$-matrix $I_1 = (1)$, i.e. B looks like the block matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;
3. A is the direct sum of B and I_1. This is the inverse operation to (2);
4. B is obtained from A by multiplying the i-th row from the left with a trivial unit , i.e. with an element of the shape $\pm\gamma$ for $\gamma \in \pi$;
5. B is obtained from A by interchanging two rows or two columns.

The group structure is given on representatives A and B as follows. By taking the direct sum $A \oplus I_m$ and $B \oplus I_n$ with the identity matrices I_m and I_n of size m and n for appropriate m and n one can arrange that $A \oplus I_m$ and $B \oplus I_n$ are invertible matrices of the same size and can be multiplied. Define $[A] \cdot [B]$ by $[(A \oplus I_m) \cdot (B \oplus I_n)]$. The zero element $0 \in \mathrm{Wh}(\pi)$ is represented by I_n for any positive integer n. The inverse of $[A]$ is given by $[A^{-1}]$. We will show later in Lemma 2.4 that the multiplication is well-defined and yields an abelian group $\mathrm{Wh}(\pi)$.

Lemma 1.27 *1. Let $(W, \partial_0 W, \partial_1 W)$ be an n-dimensional compact oriented h-cobordism for $n \geq 6$ and A be the matrix defined above. If $[A] = 0$ in $\mathrm{Wh}(\pi)$, then the h-cobordism W is trivial relative $\partial_0 W$;*

2. Consider an element $u \in \mathrm{Wh}(\pi)$, a closed oriented manifold M of dimension $n - 1 \geq 5$ with fundamental group π and an integer q with $2 \leq q \leq n - 3$. Then we can find an h-cobordism of the shape

$$W = M \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^r) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1})$$

such that $[A] = u$.

Proof : (1) It suffices to show that we can modify the given handlebody decomposition in normal form of W with associated matrix A such that we get a new handlebody decomposition in normal form whose associated matrix B is obtained from A by applying one of the operations (1), (2), (3), (4) and (5).

We begin with (1). Consider $W' = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q}(\phi_i^q) + \sum_{j=1,j\neq l}^{p_{q+1}}(\phi_j^{q+1})$. Notice that we get from W' our h-cobordism W if we attach the handle (ϕ_l^{q+1}). By the Modification Lemma 1.23 we can find an embedding $\overline{\phi}_l^{q+1} : S^q \times D^{n-1-q} \to \partial_1 W'$ such that $\overline{\phi}_l^{q+1}$ is isotopic to ϕ_l^{q+1} and we get

$$\left[\widetilde{\overline{\phi}_l^{q+1}|_{S^q\times\{0\}}}\right] = \left[\widetilde{\phi_l^{q+1}|_{S^q\times\{0\}}}\right] + x \cdot d_{q+1}([\phi_k^{q+1}]).$$

If we attach to W' the handle $(\overline{\phi}_l^{q+1})$, the result is diffeomorphic to W relative $\partial_0 W$ by the Isotopy Lemma 1.8. One easily checks that the associated invertible matrix B is obtained from A by adding the k-th row multiplied with x from the left to the the l-th row.

The claim for the operations (2) and (3) follow from the Cancellation Lemma 1.12 and the Homology Lemma 1.22. The claim for the operation (4) follows from the observation that we can replace the attaching map of a handle $\phi^q : S^q \times D^{n-1-q} \to \partial_1 W_q$ by its composition with $f \times \mathrm{id}$ for some diffeomorphism $f : S^q \to S^q$ of degree -1 and that the base element $[\phi_i^q]$ can also be changed to $\gamma \cdot [\phi_i^q]$ by choosing a different lift along $\widetilde{W_q} \to W_q$. Operation (5) can be realized by interchanging the numeration of the q-handles and $(q+1)$-handles.

(2) Fix an invertible matrix $A = (a_{i,j}) \in GL(n, \mathbb{Z}\pi)$. Choose trivial pairwise disjoint embeddings $\phi_i^2 : S^1 \times D^{n-2} \to M_0 \times \{1\}$. Consider

$$W_2 = M_0 \times [0,1] + (\phi_1^2) + (\phi_2^2) + \ldots + (\phi_n^2).$$

Since the embeddings ϕ_i^2 are trivial, we can construct embeddings $\phi_i^3 : S^2 \times D^{n-3} \to \partial_1 W_2$ and lifts $\widetilde{\phi_i^3} : S^2 \times D^{n-3} \to \widetilde{\partial_1 W_2}$ such that in $\pi_2(\widetilde{W_2}, \widetilde{\partial_0 W})$

$$[\widetilde{\phi_i^3|_{S^2\times\{0\}}}] = \sum_{j=1}^{n} a_{i,j} \cdot [\phi_j^2].$$

Put $W = W_2 + (\phi_1^3) + (\phi_2^3) + \ldots + (\phi_n^3)$. One easily checks that W is an h-cobordism over M_0 with a handlebody decompostion which realizes the matrix A. This finishes the proof Lemma 1.27. ■

Remark 1.28 If π is trivial, then $\mathrm{Wh}(\pi)$ is trivial. This follows from the fact that any invertible matrix over the integers can be reduced by elementary column operations, permutations of columns and rows and multiplication of a row with ± 1 to the identity matrix. This is essentially a consequence of the existence of an Euclidean algorithm for $\mathbb{Z}$. Hence Lemma 1.27 (2) implies already the h-Cobordism Theorem 1.2. As soon as we have shown that $[A] \in \mathrm{Wh}(\pi)$ agrees with the Whitehead torsion $\tau(W, M_0)$ of the h-cobordism W over M_0

and that this invariant depends only on the diffeomorphism type of W relative M_0, the s-Cobordism Theorem 1.1 (1) will follow.

Obviously Lemma 1.27 (2) implies the s-Cobordism Theorem Theorem 1.1 (2). We will later see that assertion (3) of the s-Cobordism Theorem 1.1 follows from assertions (1) and (2) if we have more information about the Whitehead torsion, namely the sum and the composition formulas.

1.5 Miscellaneous

The s-Cobordism Theorem 1.1 is due to Barden, Mazur, Stallings. Its topological version was proven by Kirby and Siebenmann [36, Essay II]. More information about the s-cobordism theorem can be found for instance in [33], [46] [55, page 87-90]. The s-cobordism theorem is known to be false for $n = \dim(M_0) = 4$ in general, by the work of Donaldson [22], but it is true for $n = \dim(M_0) = 4$ for so called "good" fundamental groups in the topological category by results of Freedman [26], [27]. The trivial group is an example of a "good" fundamental groups. Counterexamples in the case $n = \dim(M_0) = 3$ $n = \dim(M_0) = 4$ are constructed by Cappell and Shaneson [17]. The Poincaré Conjecture (see Theorem 1.3) is at the time of writing known in all dimensions except dimension 3.

Chapter 2

Whitehead torsion

Introduction

In this section we will assign to a homotopy equivalence $f : X \to Y$ of finite CW-complexes its Whitehead torsion $\tau(f)$ in the Whitehead group $\mathrm{Wh}(\pi(Y))$ associated to Y. The main properties of this invariant are summarized in the following

Theorem 2.1 *1. Sum formula*

Let the following two diagrams be cellular pushouts of finite CW-complexes

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & X \end{array} \qquad\qquad \begin{array}{ccc} Y_0 & \xrightarrow{k_1} & Y_1 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle l_1} \\ Y_2 & \xrightarrow{l_2} & Y \end{array}$$

Put $l_0 = l_1 \circ k_1 = l_2 \circ k_2 : Y_0 \to Y$. Let $f_i : X_i \to Y_i$ be homotopy equivalences for $i = 0, 1, 2$ satisfying $f_1 \circ i_1 = k_1 \circ f_0$ and $f_2 \circ i_2 = k_2 \circ f_0$. Denote by $f : X \to Y$ the map induced by f_0, f_1 and f_2 and the pushout property. Then f is a homotopy equivalence and

$$\tau(f) \;=\; (l_1)_*\tau(f_1) + (l_2)_*\tau(f_2) - (l_0)_*\tau(f_0);$$

2. Homotopy invariance

Let $f \simeq g : X \to Y$ be homotopic maps of finite CW-complexes. Then the homomorphisms $f_*, g_* : \mathrm{Wh}(\pi(X)) \to \mathrm{Wh}(\pi(Y))$ agree. If additionally f and g are homotopy equivalences, then

$$\tau(g) \;=\; \tau(f);$$

3. *Composition formula*

   Let $f : X \to Y$ and $g : Y \to Z$ be homotopy equivalences of finite CW-complexes. Then
$$\tau(g \circ f) \;=\; g_*\tau(f) + \tau(g);$$

4. *Product formula*

   Let $f : X' \to X$ and $g : Y' \to Y$ be homotopy equivalences of connected finite CW-complexes. Then
$$\tau(f \times g) \;=\; \chi(X) \cdot j_*\tau(g) + \chi(Y) \cdot i_*\tau(f),$$
   where $\chi(X), \chi(Y) \in \mathbb{Z}$ denote the Euler characteristics, $j_* : \mathrm{Wh}(\pi(Y)) \to \mathrm{Wh}(\pi(X \times Y))$ is the homomorphism induced by $j : Y \to X \times Y, y \mapsto (y, x_0)$ for some base point $x_0 \in X$ and i_* is defined analogously;

5. *Topological invariance*

   Let $f : X \to Y$ be a homeomorphism of finite CW-complexes. Then
$$\tau(f) \;=\; 0.$$

Given an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0, we define its Whitehead torsion $\tau(W, M_0)$ by the Whitehead torsion of the inclusion $\partial_0 W \to W$ (see (2.14)). This is the invariant appearing in the s-Cobordism Theorem 1.1. We will give some information about the Whitehead group $\mathrm{Wh}(\pi(Y))$ in Section 2.1. We will present the algebraic definition of Whitehead torsion and the proof of Theorem 2.1 in Section 2.2. A geometric approach to the Whitehead torsion is summarized in Section 2.3. A similar invariant, the Reidemeister torsion, will be treated in Section 2.4. It will be used to classify lens spaces. In order to understand the following chapters, it suffices to comprehend the statements in the s-Cobordism Theorem 1.1 and Theorem 2.1

2.1 Whitehead groups

In this section we define $K_1(R)$ for an associative ring R with unit and the Whitehead group $\mathrm{Wh}(G)$ of a group G and relate the definitions of this section with the one of Section 1.4. Furthermore we give some basic information about its computation.

Let R be an associative ring with unit. Denote by $GL(n, R)$ the group of invertible (n, n)-matrices with entries in R. Define the group $GL(R)$ by the colimit of the system indexed by the natural numbers $\ldots \subset GL(n, R) \subset GL(n+1, R) \subset \ldots$, where the inclusion $GL(n, R)$ to $GL(n+1, R)$ is given by stabilization
$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$
Define $K_1(R)$ by the abelianization $GL(R)/[GL(R), GL(R)]$ of $GL(R)$. Let $\widetilde{K}_1(R)$ be the cokernel of the map $K_1(\mathbb{Z}) \to K_1(R)$ induced by the canonical

ring homomorphism $\mathbb{Z} \to R$. The homomorphism $\det : K_1(\mathbb{Z}) \to \{\pm 1\}$, $[A] \mapsto \det(A)$ is a bijection, because $\mathbb{Z}$ is a ring with Euclidian algorithm. Hence $\widetilde{K}_1(R)$ is the same as the quotient of $K_1(R)$ by the cyclic subgroup of at most order two generated by the class of the $(1,1)$-matrix (-1). Define the *Whitehead group* $\mathrm{Wh}(G)$ of a group G to be the cokernel of the map $G \times \{\pm 1\} \to K_1(\mathbb{Z}G)$ which sends $(g, \pm 1)$ to the class of the invertible $(1,1)$-matrix $(\pm g)$. This will be the group, where Whitehead torsion will take its values in.

The Whitehead group $\mathrm{Wh}(G)$ is known to be trivial if G is the free abelian group $\mathbb{Z}^n$ of rank n [5] or the free group $*_{i=1}^n \mathbb{Z}$ of rank n [60]. There is the conjecture that it vanishes for any torsionfree group. This has been proven by Farrell and Jones [23], [24], [25] for a large class of groups. This class contains any subgroup $G \subset G'$, where G' is a discrete cocompact subgroup of a Lie group with finitely many path components, and any group G which is the fundamental group of a non-positively curved closed Riemannian manifold or of a complete pinched negatively curved Riemannian manifold. The Whitehead group satisfies $\mathrm{Wh}(G * H) = \mathrm{Wh}(G) \oplus \mathrm{Wh}(H)$ [60].

If G is finite, then $\mathrm{Wh}(G)$ is very well understood (see [50]). Namely, $\mathrm{Wh}(G)$ is finitely generated, its rank as abelian group is the number of conjugacy classes of unordered pairs $\{g, g^{-1}\}$ in G minus the number of conjugacy classes of cyclic subgroups, and its torsion subgroup is isomorphic to the kernel $SK_1(G)$ of the change of coefficient homomorphism $K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G)$. For a finite cyclic group G the Whitehead group $\mathrm{Wh}(G)$ is torsionfree. For instance the Whitehead group $\mathrm{Wh}(\mathbb{Z}/p)$ of a cyclic group of order p for an odd prime p is the free abelian group of rank $(p-3)/2$ and $\mathrm{Wh}(\mathbb{Z}/2) = 0$. The Whitehead group of the symmetric group S_n is trivial. The Whitehead group of $\mathbb{Z}^2 \times \mathbb{Z}/4$ is not finitely generated as abelian group.

Next we want to relate the definitions above to the one of Section 1.4. Denote by $E_n(i,j)$ for $n \geq 1$ and $1 \leq i,j \leq n$ the (n,n)-matrix whose entry at (i,j) is one and is zero elsewhere. Denote by I_n the identity matrix of size n. An elementary (n,n)-matrix is a matrix of the form $I_n + r \cdot E_n(i,j)$ for $n \geq 1$, $1 \leq i,j \leq n$, $i \neq j$ and $r \in R$. Let A be a (n,n)-matrix. The matrix $B = A \cdot (I_n + r \cdot E_n(i,j))$ is obtained from A by adding the i-th column multiplied with r from the right to the j-th column. The matrix $C = (I_n + r \cdot E_n(i,j)) \cdot A$ is obtained from A by adding the j-th row multiplied with r from the left to the i-th row. Let $E(R) \subset GL(R)$ be the subgroup generated by all elements in $GL(R)$ which are represented by elementary matrices.

Lemma 2.2 *We have $E(R) = [GL(R), GL(R)]$. In particular $E(R) \subset GL(R)$ is a normal subgroup and $K_1(R) = GL(R)/E(R)$.*

Proof : For $n \geq 3$, pairwise distinct numbers $1 \leq i,j,k \leq n$ and $r \in R$ we can write $I_n + r \cdot E_n(i,k)$ as a commutator in $GL(n,R)$, namely

$$\begin{aligned} I_n + r \cdot E_n(i,k) &= (I_n + r \cdot E_n(i,j)) \cdot (I_n + E_n(j,k)) \cdot \\ &\qquad (I_n + r \cdot E_n(i,j))^{-1} \cdot (I_n + E_n(j,k))^{-1}. \end{aligned}$$

This implies $E(R) \supset [GL(R), GL(R)]$.

Let A and B be two elements in $GL(n,R)$. Let $[A]$ and $[B]$ be the elements in $GL(R)$ represented by A and B. Given two elements x and y in $GL(R)$, we write $x \sim y$ if there are elements e_1 and e_2 in $E(R)$ with $x = e_1 y e_2$, in other words, if the classes of x and y in $E(R)\backslash GL(R)/E(R)$ agree. One easily checks

$$[AB] \sim \left[\begin{pmatrix} AB & 0 \\ 0 & I_n \end{pmatrix}\right] \sim \left[\begin{pmatrix} AB & A \\ 0 & I_n \end{pmatrix}\right] \sim \left[\begin{pmatrix} 0 & A \\ -B & I_n \end{pmatrix}\right] \sim \left[\begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix}\right],$$

since each step is given by multiplication from the right or left with a block matrix of the form $\begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix}$ or $\begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix}$ and such a block matrix is obviously obtained from I_{2n} by a sequence of column and row operations and hence its class in $GL(R)$ belongs to $E(R)$. Analogously we get

$$[BA] \sim \left[\begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix}\right].$$

Since the element in $GL(R)$ represented by $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ belongs to $E(R)$, we conclude

$$\left[\begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix}\right] \sim \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right] \sim \left[\begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix}\right].$$

This shows

$$[AB] \quad \sim [BA]. \tag{2.3}$$

This implies for any element $x \in GL(R)$ and $e \in E(R)$ that $xex^{-1} \sim ex^{-1}x = e$ and hence $xex^{-1} \in E(R)$. Therefore $E(R)$ is normal. Given a commutator $xyx^{-1}y^{-1}$ for $x, y \in GL(R)$, we conclude for appropriate elements e_1, e_2, e_3 in $E(R)$

$$xyx^{-1}y^{-1} = e_1 yxe_2 x^{-1}y^{-1} = e_1 yxx^{-1}y^{-1}(yx)e_2(yx)^{-1} = e_1e_3 \in E(R).$$

This finishes the proof of Lemma 2.2. ■

Lemma 2.4 *The definition of* Wh(G) *of Section 1.4 makes sense and yields an abelian group which can be identified with the definition of* Wh(G) *given in this section above.*

Proof : Notice that the operation (4) appearing in the definition of Wh(G) in Section 1.4 corresponds to multiplication with an elementary matrix from the left. Since $E(R)$ is normal by Lemma 2.2, two invertible matrices A and B over $\mathbb{Z}G$ are equivalent under the equivalence relation appearing in the definition of Wh(G) as explained in Section 1.4 if and only their classes $[A]$ and $[B]$ in Wh(G) as defined in this section agree. Now the claim follows from Lemma 2.2. ■

Remark 2.5 Often $K_1(R)$ is defined in a little bit more conceptual way in terms of automorphisms as follows. Namely, $K_1(R)$ is defined as the abelian group whose generators $[f]$ are conjugacy classes of automorphisms $f : P \to P$ of finitely generated projective R-modules P and which satisfies the following relations. For any commutative diagram of finitely generated projective R-modules with exact rows and automorphisms as vertical arrows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0 & \xrightarrow{i} & P_1 & \xrightarrow{p} & P_2 & \longrightarrow & 0 \\ & & f_0 \downarrow \cong & & f_1 \downarrow \cong & & f_2 \downarrow \cong & & \\ 0 & \longrightarrow & P_0 & \xrightarrow{i} & P_1 & \xrightarrow{p} & P_2 & \longrightarrow & 0 \end{array}$$

we get the relation $[f_0] - [f_1] + [f_2] = 0$. If $f, g : P \to P$ are automorphisms of a finitely generated projective R-module P, then $[g \circ f] = [g] + [f]$. Using Lemma 2.2 one easily checks that sending the class $[A]$ of an invertible (n,n)-matrix A to the class of the automorphism $R_A : R^n \to R^n,\ x \mapsto xA$ defines an isomorphism from $GL(R)/[GL(R), GL(R)]$ to the abelian group defined above.

2.2 Algebraic approach to Whitehead torsion

In this section we give the definition and prove the basic properties of the Whitehead torsion using an algebraic approach via chain complexes. This will enable us to finish the proof of the s-Cobordism Theorem 1.1. The idea which underlies the notion of Whitehead torsion will become more transparent in Section 2.3, where we will develop a geometric approach to Whitehead torsion and link it to the strategy of proof of the s-Cobordism Theorem 1.1.

We begin with some input about chain complexes. Let $f_* : C_* \to D_*$ be a chain map of R-chain complexes for some ring R. Define $\operatorname{cyl}_*(f_*)$ to be the chain complex with p-th differential

$$C_{p-1} \oplus C_p \oplus D_p \xrightarrow{\begin{pmatrix} -c_{p-1} & 0 & 0 \\ -\operatorname{id} & c_p & 0 \\ f_{p-1} & 0 & d_p \end{pmatrix}} C_{p-2} \oplus C_{p-1} \oplus D_{p-1}.$$

Define $\operatorname{cone}_*(f_*)$ to be the quotient of $\operatorname{cyl}_*(f_*)$ by the obvious copy of C_*. Hence the p-th differential of $\operatorname{cone}_*(f_*)$ is

$$C_{p-1} \oplus D_p \xrightarrow{\begin{pmatrix} -c_{p-1} & 0 \\ f_{p-1} & d_p \end{pmatrix}} C_{p-2} \oplus D_{p-1}.$$

Given a chain complex C_*, define ΣC_* to be the quotient of $\operatorname{cone}_*(\operatorname{id}_{C_*})$ by the obvious copy of C_*, i.e. the chain complex with p-th differential

$$C_{p-1} \xrightarrow{-c_{p-1}} C_{p-2}.$$

Definition 2.6 *We call* $\mathrm{cyl}_*(f_*)$ *the* mapping cylinder, $\mathrm{cone}_*(f_*)$ *the* mapping cone *of the chain map* f_* *and* ΣC_* *the* suspension *of the chain complex* C_*.

These algebraic notions of mapping cylinder, mapping cone and suspension are modelled on their geometric counterparts. Namely, the cellular chain complex of a mapping cylinder of a cellular map of CW-complexes is the mapping cylinder of the chain map induced by f. From the geometry it is also clear why one obtains obvious exact sequences such as $0 \to C_* \to \mathrm{cyl}(f_*) \to \mathrm{cone}(f_*) \to 0$ and $0 \to D_* \to \mathrm{cone}_*(f_*) \to \Sigma C_* \to 0$.

A *chain contraction* γ_* for a R-chain complex C_* is a collection of R-homomorphisms $\gamma_p : C_p \to C_{p+1}$ for $p \in \mathbb{Z}$ such that $c_{p+1} \circ \gamma_p + \gamma_{p-1} \circ c_p = \mathrm{id}_{C_p}$ holds for all $p \in \mathbb{Z}$. We call a R-chain complex C_* *finite based free* if there is a number N with $C_p = 0$ for $|p| > N$ and each R-chain module C_p is a finitely generated free R-module with a preferred basis. Suppose that C_* is a finite based free R-chain complex which is *contractible*, i.e. which possesses a chain contraction. Put $C_{\mathrm{odd}} = \oplus_{p\in\mathbb{Z}} C_{2p+1}$ and $C_{\mathrm{ev}} = \oplus_{p\in\mathbb{Z}} C_{2p}$. Let γ_* and δ_* be two chain contractions. Define R-homomorphisms

$$\begin{aligned} (c_* + \gamma_*)_{\mathrm{odd}} : C_{\mathrm{odd}} &\to C_{\mathrm{ev}}; \\ (c_* + \delta_*)_{\mathrm{ev}} : C_{\mathrm{ev}} &\to C_{\mathrm{odd}}. \end{aligned}$$

Let A be the matrix of $(c_* + \gamma_*)_{\mathrm{odd}}$ with respect to the given bases. Let B be the matrix of $(c_* + \delta_*)_{\mathrm{ev}}$ with respect to the given bases. Put $\mu_n := (\gamma_{n+1} - \delta_{n+1}) \circ \delta_n$ and $\nu_n := (\delta_{n+1} - \gamma_{n+1}) \circ \gamma_n$. One easily checks that $(\mathrm{id} + \mu_*)_{\mathrm{odd}}$, $(\mathrm{id} + \nu_*)_{\mathrm{ev}}$ and both compositions $(c_* + \gamma_*)_{\mathrm{odd}} \circ (\mathrm{id} + \mu_*)_{\mathrm{odd}} \circ (c_* + \delta_*)_{\mathrm{ev}}$ and $(c_* + \delta_*)_{\mathrm{ev}} \circ (\mathrm{id} + \nu_*)_{\mathrm{ev}} \circ (c_* + \gamma_*)_{\mathrm{odd}}$ are given by upper triangular matrices whose diagonal entries are identity maps. Hence A and B are invertible and their class $[A], [B] \in \widetilde{K}_1(R)$ satisfy $[A] = -[B]$. Since $[B]$ is independent of the choice of γ_*, the same is true for $[A]$. Thus we can associate to a finite based free contractible R-chain complex C_* an element

$$\tau(C_*) \quad = \quad [A] \quad \in \widetilde{K}_1(R). \tag{2.7}$$

Let $f_* : C_* \to D_*$ be a homotopy equivalence of finite based free R-chain complexes. Its mapping cone $\mathrm{cone}(f_*)$ is a contractible finite based free R-chain complex. Define the *Whitehead torsion* of f_* by

$$\tau(f_*) \quad := \quad \tau(\mathrm{cone}_*(f_*)) \quad \in \widetilde{K}_1(R). \tag{2.8}$$

We call a sequence of finite based free R-chain complexes $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{q_*} E_* \to 0$ *based exact* if for any $p \in \mathbb{Z}$ the basis B for D_p can be written as a disjoint union $B' \coprod B''$ such that the image of the basis of C_p under i_p is B' and the image of B'' under q_p is the basis for E_p.

Lemma 2.9 *1. Consider a commutative diagram of finite based free R-chain complexes whose rows are based exact.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_* & \longrightarrow & D'_* & \longrightarrow & E'_* & \longrightarrow & 0 \\ & & {\scriptstyle f_*}\downarrow & & {\scriptstyle g_*}\downarrow & & {\scriptstyle h_*}\downarrow & & \\ 0 & \longrightarrow & C_* & \longrightarrow & D_* & \longrightarrow & E_* & \longrightarrow & 0 \end{array}$$

Suppose that two of the chain maps f_*, g_* and h_* are R-chain homotopy equivalences. Then all three are R-chain homotopy equivalences and

$$\tau(f_*) - \tau(g_*) + \tau(h_*) \;=\; 0;$$

2. Let $f_* \simeq g_* : C_* \to D_*$ be homotopic R-chain homotopy equivalences of finite based free R-chain complexes. Then

$$\tau(f_*) \;=\; \tau(g_*);$$

3. Let $f_* : C_* \to D_*$ and $g_* : D_* \to E_*$ be R-chain homotopy equivalences of based free R-chain complexes. Then

$$\tau(g_* \circ f_*) = \tau(g_*) + \tau(f_*).$$

Proof : (1) A chain map of projective chain complexes is a homotopy equivalence if and only if it induces an isomorphism on homology. The five-lemma and the long homology sequence of a short exact sequence of chain complexes imply that all three chain maps f_*, h_* and g_* are chain homotopy equivalences if two of them are.

To prove the sum formula, it suffices to show for a based free exact sequence $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{q_*} E_* \to 0$ of contractible finite based free R-chain complexes that

$$\tau(C_*) - \tau(D_*) + \tau(E_*) \;=\; 0. \tag{2.10}$$

Let $u_* : F_* \to G_*$ be an isomorphism of contractible finite based free R-chain complexes. Since the choice of chain contraction does not affect the values of the Whitehead torsion, we can compute $\tau(F_*)$ and $\tau(G_*)$ with respect to chain contractions which are compatible with u_*. Then one easily checks in $\widetilde{K}_1(R)$

$$\tau(G_*) - \tau(F_*) \;=\; \sum_{p \in \mathbb{Z}} (-1)^p \cdot [u_p], \tag{2.11}$$

where $[u_p]$ is the element represented by the matrix of u_p with respect to the given bases.

Let ϵ_* be a chain contraction for E_*. Choose for any $p \in \mathbb{Z}$ an R-homomorphism $\sigma_p : E_p \to D_p$ satisfying $p_q \circ \sigma_q = \mathrm{id}$. Define $s_p : E_p \to D_p$ by $d_{p+1} \circ \sigma_{p+1} \circ \epsilon_p + \sigma_p \circ \epsilon_{p-1} \circ e_p$. One easily checks that the collection of the s_p-s defines a chain map $s_* : E_* \to D_*$ with $q_* \circ s_* = \mathrm{id}$. Thus we obtain an isomorphism of contractible based free R-chain complexes

$$i_* \oplus q_* : C_* \oplus E_* \to D_*.$$

Since the matrix of $i_p \oplus s_p$ with respect to the given basis is a block matrix of the shape $\begin{pmatrix} I_m & * \\ 0 & I_n \end{pmatrix}$ we get $[i_p \oplus s_p] = 0$ in $\widetilde{K}_1(R)$. Now (2.11) implies $\tau(C_* \oplus D_*) = \tau(E_*)$. Since obviously $\tau(C_* \oplus D_*) = \tau(C_*) + \tau(D_*)$, (2.10) and thus assertion (1) follows.

(2) If $h_* : f_* \simeq g_*$ is a chain homotopy, we obtain an isomorphism of based free R-chain complexes

$$\begin{pmatrix} \mathrm{id} & 0 \\ h_{*-1} & \mathrm{id} \end{pmatrix} : \mathrm{cone}_*(f_*) = C_{*-1} \oplus D_* \to \mathrm{cone}_*(g_*) = C_{*-1} \oplus D_*.$$

We conclude from (2.11)

$$\tau(g_*) - \tau(f_*) = \sum_{p \in \mathbb{Z}} (-1)^p \cdot \left[\begin{pmatrix} \mathrm{id} & 0 \\ h_{*-1} & \mathrm{id} \end{pmatrix}\right] = 0.$$

(3) Define a chain map $h_* : \Sigma^{-1} \mathrm{cone}_*(g_*) \to \mathrm{cone}_*(f_*)$ by

$$\begin{pmatrix} 0 & 0 \\ -\mathrm{id} & 0 \end{pmatrix} : D_p \oplus E_{p+1} \to C_{p-1} \oplus D_p.$$

There is an obvious based exact sequence of contractible finite based free R-chain complexes $0 \to \mathrm{cone}_*(f_*) \to \mathrm{cone}(h_*) \to \mathrm{cone}(g_*) \to 0$. There is also a based exact sequence of contractible finite based free R-chain complexes $0 \to \mathrm{cone}_*(g_* \circ f_*) \xrightarrow{i_*} \mathrm{cone}_*(h_*) \to \mathrm{cone}_*(\mathrm{id} : D_* \to D_*) \to 0$, where i_p is given by

$$\begin{pmatrix} f_{p-1} & 0 \\ 0 & \mathrm{id} \\ \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix} : C_{p-1} \oplus E_p \to D_{p-1} \oplus E_p \oplus C_{p-1} \oplus D_p.$$

We conclude from assertion (1)

$$\begin{aligned} \tau(h_*) &= \tau(f_*) + \tau(g_*); \\ \tau(h_*) &= \tau(g_* \circ f_*) + \tau(\mathrm{id}_* : D_* \to D_*); \\ \tau(\mathrm{id}_* : D_* \to D_*) &= 0. \end{aligned}$$

This finishes the proof of Lemma 2.9. ■

Now we can pass to CW-complexes. Let $f : X \to Y$ be a homotopy equivalence of connected finite CW-complexes. Let $p_X : \widetilde{X} \to X$ and $p_Y : \widetilde{Y} \to Y$ be the universal coverings. Fix base points $\widetilde{x} \in \widetilde{X}$ and $\widetilde{y} \in \widetilde{Y}$ such that f maps $x = p_X(\widetilde{x})$ to $y = p_Y(\widetilde{y})$. Let $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ be the unique lift of f satisfying

$\widetilde{f}(\widetilde{x}) = \widetilde{y}$. We abbreviate $\pi = \pi_1(Y, y)$ and identify $\pi_1(X, x)$ in the sequel with π by $\pi_1(f, x)$. After the choice of the base points $\widetilde{x}$ and $\widetilde{y}$ we get unique operations of π on $\widetilde{X}$ and $\widetilde{Y}$. The lift $\widetilde{f}$ is π-equivariant. It induces a $\mathbb{Z}\pi$-chain homotopy equivalence $C_*(\widetilde{f}) : C_*(\widetilde{X}) \to C_*(\widetilde{Y})$. We can apply (2.8) to it and thus obtain an element

$$\tau(f) \quad \in \quad \mathrm{Wh}(\pi_1(Y, y)). \tag{2.12}$$

So far this definition depends on the various choices of base points. We can get rid of these choices as follows. If y' is a second base point, we can choose path w from y to y' in Y. Conjugation with w yields a homomorphism $c_w : \pi_1(Y, y) \to \pi_1(Y, y')$ which induces $(c_w)_* : \mathrm{Wh}(\pi_1(Y, y)) \to \mathrm{Wh}(\pi_1(Y, y'))$. If v is a different path from y to y', then c_w and c_v differ by an inner automorphism of $\pi_1(Y, y)$. Since an inner automorphism of $\pi_1(Y, y)$ induces the identity on $\mathrm{Wh}(\pi_1(Y, y))$, we conclude that $(c_w)_*$ and $(c_v)_*$ agree. Hence we get a unique isomorphism $t(y, y') : \mathrm{Wh}(\pi_1(Y, y)) \to \mathrm{Wh}(\pi_1(Y, y'))$ depending only on y and y'. Moreover $t(y, y) = \mathrm{id}$ and $t(y, y'') = t(y', y'') \circ t(y, y')$. Therefore we can define $\mathrm{Wh}(\pi(Y))$ independently of a choice of a base point by $\coprod_{y \in Y} \mathrm{Wh}(\pi_1(Y, y)) / \sim$, where $\sim$ is the obvious equivalence relation generated by $a \sim b \Leftrightarrow t(y, y')(a) = b$ for $a \in \mathrm{Wh}(\pi_1(Y, y))$ and $b \in \mathrm{Wh}(\pi_1(Y, y'))$. Define $\tau(f) \in \mathrm{Wh}(\pi_1(Y))$ by the element represented by the element introduced in (2.12). Notice that $\mathrm{Wh}(\pi(Y))$ is isomorphic to $\mathrm{Wh}(\pi_1(Y, y))$ for any base point $y \in Y$. It is not hard to check using Lemma 2.9 that $\tau(f)$ depends only on $f : X \to Y$ and not the choice of the universal coverings and base points. Finally we want to drop the assumption that Y is connected. Notice that f induces a bijection $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$.

Definition 2.13 *Let $f : X \to Y$ be a homotopy equivalence of finite CW-complexes. Define the Whitehead group $\mathrm{Wh}(\pi(Y))$ of Y and the Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi(Y))$ by*

$$\begin{aligned} \mathrm{Wh}(\pi(Y)) &= \oplus_{C \in \pi_0(Y)} \mathrm{Wh}(\pi_1(C)); \\ \tau(f) &= \oplus_{C \in \pi_0(Y)} \tau\left(f|_{\pi_0(f)^{-1}(C)} : \pi_0(f)^{-1}(C) \to C\right). \end{aligned}$$

In the notation $\mathrm{Wh}(\pi(Y))$ one should think of $\pi(Y)$ as the fundamental groupoid of Y. Notice that a map $f : X \to Y$ induces a homomorphism $f_* : \mathrm{Wh}(\pi(X)) \to \mathrm{Wh}(\pi(Y)$ such that $\mathrm{id}_* = \mathrm{id}$, $(g \circ f)_* = g_* \circ f_*$ and $f \simeq g \Rightarrow f_* = g_*$.

Suppose that the following diagram is a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[g]{} & Y \end{array}$$

the map i is an inclusion of CW-complexes and f is a *cellular* map of CW-complexes, i.e. respects the filtration given by the CW-structures. Then Y

inherits a CW-structure by defining Y_n as the union of $j(B_n)$ and $g(X_n)$. If we equip Y with this CW-structure, we call the pushout above a *cellular pushout*.

Next we give the proof of Theorem 2.1.

(1), (2) and (3) follow from Lemma 2.9.

(4) Because of assertion (3) we have

$$\tau(f \times g) \;=\; \tau(f \times \mathrm{id}_Y) + (f \times \mathrm{id}_Y)_* \tau(\mathrm{id}_X \times g).$$

Hence it suffices to treat the case $g = \mathrm{id}_Y$. Now one proceeds by induction over the cells of Y using assertions (1), (2) and (3).

(5) This (in comparision with the other assertions a much deeper result) is due to Chapman [18], [19]. This finishes the proof of Theorem 2.1. ■

We define the Whitehead torsion of an h-cobordism $(W; M_0, M_1, f_0, f_1)$

$$\tau(W, M_0) \;\in\; \mathrm{Wh}(\pi(M_0)). \tag{2.14}$$

by $\tau(W, M_0) := (i_0 \circ f_0)_*^{-1} \left(\tau(i_0 \circ f_0 : M_0 \to W)\right)$, where we equip W and M_0 with some CW-structure, for instance one coming from a smooth triangulation. This is independent of the choice of CW-structure by Theorem 2.1 (5). Because of Theorem 2.1 (5) two h-cobordism over M_0 which are diffeomorphic relative M_0 have the same Whitehead torsion.

Let R be a *ring with involution* $i : R \to R$,$r \to \overline{r}$, i.e. a map satisfying $\overline{r+s} = \overline{r} + \overline{s}$, $\overline{r \cdot s} = \overline{s} \cdot \overline{r}$ and $\overline{1} = 1$. Given a (m,n)-matrix $A = (a_{i,j})$ define the (n,m)-matrix A^* by $(\overline{a_{j,i}})$. We obtain an involution

$$* : K_1(R) \to K_1(R) \quad , [A] \mapsto [A^*]. \tag{2.15}$$

Let P be a left R-module. Define the *dual R-module* P^* to be the left R-module whose underlying abelian group is $P^* = \hom(P, R)$ and whose left R-module structure is given by $(rf)(x) := f(x)\overline{r}$ for $f \in P^*$ and $x \in P$. Then the involution on $K_1(R)$ corresponds to $[f : P \to P] \mapsto [f^* : P^* \to P^*]$ if one defines $K_1(R)$ as in Remark 2.5. We equip $\mathbb{Z}G$ with the involution $\overline{\sum_{g \in G} \lambda_g \cdot g} = \sum_{g \in G} \lambda_g \cdot g^{-1}$. Thus we get an involution on $K_1(\mathbb{Z}G)$ which induces an involution $* : \mathrm{Wh}(G) \to \mathrm{Wh}(G)$.

Lemma 2.16 *1. Let $(W; M_0, f_0, M_1, f_1)$ and $(W'; M_0', f_0', M_1', f_1')$ be h-cobordisms over M_0 and M_0' and let $g : M_1 \to M_0'$ be a diffeomorphism. Let $W \cup W'$ be the h-cobordism over M_0 obtained from W and W' by glueing with the diffeomorphism $f_0' \circ g \circ f_1^{-1} : \partial_1 W \to \partial_0 W'$. Let $u : \mathrm{Wh}(M_0') \to \mathrm{Wh}(M_0)$ be the isomorphism given by the composition $(f_0)_*^{-1} \circ (i_0)_*^{-1} \circ (i_1)_* \circ (f_1)_* \circ (g_*)^{-1}$, where $i_k : \partial_k W \to W$ is the inclusion for $k = 0, 1$. Then*

$$\tau(W \cup W', M_0) = \tau(W, M_0) + u\left(\tau(W', M_0')\right).$$

2. Let $(W; M_0, f_0, M_1, f_1)$ be an h-cobordism over M_0. Let $v : \mathrm{Wh}(M_1) \to \mathrm{Wh}(M_0)$ be the isomorphism given by the composition $(f_0)_*^{-1} \circ (i_0)_*^{-1} \circ (i_1)_* \circ (f_1)_*$. Then

$$*(\tau(W, M_0)) = (-1)^{\dim(M_0)} \cdot v(\tau(W, M_1)).$$

Proof : (1) follows from Theorem 2.1.

(2) Let $C_*(\widetilde{W}, \widetilde{\partial_k W})$ be the cellular $\mathbb{Z}\pi$-chain complex with respect to a triangulation of W of the universal covering $\widetilde{W} \to W$ for $\pi = \pi_1(W) = \pi_1(\partial_0 W) = \pi_1(\partial_1 W)$. Put $n = \dim(W)$. The dual $\mathbb{Z}\pi$-chain complex $C^{n-*}(\widetilde{W}, \widetilde{\partial_1 W})$ has as p-th chain module the dual module of $C_{n-p}(\widetilde{W}, \widetilde{\partial_1 W})$ and its p-th differential is the dual of $c_{n-p+1} : C_{n-p+1}(\widetilde{W}, \widetilde{\partial_1 W}) \to C_{n-p}(\widetilde{W}, \widetilde{\partial_1 W})$. Poincaŕe duality yields a $\mathbb{Z}\pi$-chain homotopy equivalence for $n = \dim(W)$ and

$$\cap[W, \partial W] : C^{n-*}(\widetilde{W}, \widetilde{\partial_1 W}) \to C_*(\widetilde{W}, \widetilde{\partial_0 W}),$$

where $C^{n-*}(\widetilde{W}, \widetilde{\partial_1 W})$ is the dual chain complex of $C_*(\widetilde{W}, \widetilde{\partial_1 W})$. Inspecting the proof of Poincaré duality using dual cells [37] shows that this is a base preserving chain map if one passes to subdivisions. This implies that this $\mathbb{Z}\pi$-chain homotopy equivalence has trivial Whitehead torsion. We conclude from Lemma 2.9

$$\tau(C_*(\widetilde{W}, \widetilde{\partial_0 W})) = \tau(C^{n-*}(\widetilde{W}, \widetilde{\partial_1 W})) = (-1)^{n-1} \cdot * \left(\tau(C_*(\widetilde{W}, \widetilde{\partial_1 W}))\right).$$

This finishes the proof of Lemma 2.16 ■

Next we can finish the proof of the s-Cobordism Theorem 1.1. As mentioned already in Remark 1.28, Theorem 1.1 (1) would follow if we can show that the class of the matrix $[A] \in \mathrm{Wh}(\pi)$ agrees with $\tau(W, M_0)$ and that $\tau(W, M_0)$ depends only on the diffeomorphism type of W relative M_0 (see Lemma 1.27 (1)). Using Theorem 2.1 one can show that the homotopy equivalence $f : W \to X$ of Section 1.2 satisfies $\tau(f) = 0$ and hence $\tau(W, M_0) = \tau(C_*(\widetilde{X}, \widetilde{\partial_0 W}))$. We conclude $[A] = \tau(C_*(\widetilde{X}, \widetilde{\partial_0 W}))$ from the existence of the base preserving isomorphism (1.17). In view of Remark 1.28 and Theorem 1.1 assertions (1) and (2) of Theorem 1.1 follow. In order to prove Theorem 1.1 (3), we must show for two h-cobordisms $(W; M_0, M_1, f_0, f_1)$ and $(W'; M_0, M_1', f_0', f_1')$ over M_0 with $\tau(W, M_0) = \tau(W', M_0)$ that they are diffeomorphic relative M_0. Choose an h-cobordism $(W''; M_1, M_2', f_1', f_2')$ over M_1 such that $\tau(W''; M_1, M_2, f_1', f_2) \in \mathrm{Wh}(\pi(M_1))$ is the image of $\tau(W, M_0)$ under the isomorphism $(f_1 \circ i_1)_*^{-1} \circ (f_0 \circ i_0)_* : \mathrm{Wh}(\pi(M_0)) \to \mathrm{Wh}(\pi(M_1))$, where $i_k : \partial_k W \to W$ for $k = 0, 1$ is the inclusion. We can glue W and W'' along M_1 to get an h-cobordism $W \cup_{M_1} W''$ over M_0. From Lemma 2.16 (1) we get $\tau(W \cup_{M_1} W'', M_0) = 0$. Hence there is a diffeomorphism $G : W \cup_{M_1} W'' \to M \times [0, 1]$ which induces the identity on $M_0 = M_0 \times \{0\}$ and a diffeomorphism $g_1 : M_2 \to M_0 \times \{1\} = M_0$. Now we can form the h-cobordism $W \cup_{M_1} W'' \cup_{g_1} W'$. Using G we can construct a diffeomorphism relative M_0 from $W \cup_{M_1} W'' \cup_{g_1} W'$ to W'. Similarly one can show that $W'' \cup_{g_1} W'$ is diffeomorphic relative M_1 to the trivial h-cobordism over M_1. Hence there is also a diffeomorphism relative M_0 from $W \cup_{M_1} W'' \cup_{g_1} W'$ to W. Hence W and W' are diffeomorphic relative M_0. This finishes the proof of the s-Cobordism Theorem 1.1. ■

2.3 The geometric approach to Whitehead torsion

In this section we introduce the concept of a simple homotopy equivalence $f : X \to Y$ of finite CW-complexes geometrically. We will show that the obstruction for a homotopy equivalence $f : X \to Y$ of finite CW-complexes to be simple is the Whitehead torsion. This proof is a CW-version or homotopy version of the proof of the s-Cobordism Theorem 1.1.

We have the inclusion of spaces $S^{n-2} \subset S^{n-1}_+ \subset S^{n-1} \subset D^n$, where $S^{n-1}_+ \subset S^{n-1}$ is the upper hemisphere. The pair (D^n, S^{n-1}_+) carries an obvious relative CW-structure. Namely, attach a $(n-1)$-cell to S^{n-1}_+ by the attaching map $\mathrm{id} : S^{n-2} \to S^{n-2}$ to obtain S^{n-1}. Then we attach to S^{n-1} an n-cell by the attaching map $\mathrm{id} : S^{n-1} \to S^{n-1}$ to obtain D^n. Let X be a CW-complex. Let $q : S^{n-1}_+ \to X$ be a map satisfying $q(S^{n-2}) \subset X_{n-2}$ and $q(S^{n-1}_+) \subset X_{n-1}$. Let Y be the space $D^n \cup_q X$, i.e. the push out

$$\begin{array}{ccc} S^{n-1}_+ & \xrightarrow{q} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ D^n & \xrightarrow[g]{} & Y \end{array}$$

where i is the inclusion. Then Y inherits a CW-structure by putting $Y_k = j(X_k)$ for $k \le n-2$, $Y_{n-1} = j(X_{n-1}) \cup g(S^{n-1})$ and $Y_k = j(X_k) \cup g(D^n)$ for $k \ge n$. Notice that Y is obtained from X by attaching one $(n-1)$-cell and one n-cell. Since the map $i : S^{n-1}_+ \to D^n$ is a homotopy equivalence and cofibration, the map $j : X \to Y$ is a homotopy equivalence and cofibration. We call j an *elementary expansion* and say that Y is obtained from X by an elementary expansion. There is a map $r : Y \to X$ with $r \circ j = \mathrm{id}_X$. This map is unique up to homotopy relative $j(X)$. We call any such map an *elementary collaps* and say that X is obtained from Y by an elementary collaps.

An elementary expansion is the CW-version or homotopy version of the construction in Example 1.11, where we have added a q-handle and a $(q+1)$-handle to W without changing the diffeomorphism type of W. This corresponds to an elementary expansion for the CW-complex X which we have assigned to W in Section 1.2.

Definition 2.17 *Let $f : X \to Y$ be a map of finite CW-complexes. We call it a simple* homotopy equivalence *if there is a sequence of maps*

$$X = X[0] \xrightarrow{f_0} X[1] \xrightarrow{f_1} X[2] \ldots \xrightarrow{f_{n-1}} X[n] = Y$$

such that each f_i is an elementary expansion or elementary collaps and f is homotopic to the composition of the maps f_i.

The idea of the definition of a simple homotopy equivalence is that such a map can be written as a composition of elementary maps which are obviously homotopy equivalences. This is similar to the idea in knot theory that two knots are equivalent if one can pass from one knot to the other by a sequence of elementary moves, the so called Reidemeister moves. A Reidemeister move obviously does not change the equivalence class of a knot and, indeed, it turns out that one can pass from one knot to a second knot by a sequence of Reidemeister moves if the two knots are equivalent. The analogous statement is not true for homotopy equivalences $f : X \to Y$ of finite CW-complexes because there is an obstruction for f to be simple, namely its Whitehead torsion.

Lemma 2.18 *1. Let $f : X \to Y$ be a simple homotopy equivalence. Then its Whitehead torsion $\tau(f) \in \mathrm{Wh}(Y)$ vanishes;*

2. Let X be a finite CW-complex. Then for any element $x \in \mathrm{Wh}(\pi(X))$ there is an inclusion $i : X \to Y$ of finite CW-complexes such that i is a homotopy equivalence and $i_*^{-1}(\tau(i)) = x$.

Proof : (1) Because of Theorem 2.1 it suffices to prove for an elementary expansion $j : X \to Y$ that its Whitehead torsion $\tau(j) \in \mathrm{Wh}(Y)$ vanishes. We can assume without loss of generality that Y is connected. In the sequel we write $\pi = \pi_1(Y)$ and identify $\pi = \pi_1(X)$ by $\pi_1(f)$. The following diagram of based free finite $\mathbb{Z}\pi$-chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(\widetilde{X}) & \xrightarrow{C_*(\widetilde{j})} & C_*(\widetilde{Y}) & \xrightarrow{\mathrm{pr}_*} & C_*(\widetilde{Y},\widetilde{X}) & \longrightarrow & 0 \\
 & & \mathrm{id}_* \big\uparrow & & C_*(\widetilde{j}) \big\uparrow & & 0_* \big\uparrow & & \\
0 & \longrightarrow & C_*(\widetilde{X}) & \xrightarrow{\mathrm{id}_*} & C_*(\widetilde{X}) & \xrightarrow{\mathrm{pr}_*} & 0 & \longrightarrow & 0
\end{array}$$

has based exact rows and $\mathbb{Z}\pi$-chain homotopy equivalences as vertical arrows. We conclude from Lemma 2.9 (1)

$$\tau(C_*(\widetilde{j})) = \tau(\mathrm{id}_* : C_*(\widetilde{X}) \to C_*(\widetilde{X})) + \tau(0_* : 0 \to C_*(\widetilde{Y},\widetilde{X})) = \tau(C_*(\widetilde{Y},\widetilde{X})).$$

The $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{Y},\widetilde{X})$ is concentrated in two consecutive dimensions and its only non-trivial differential is $\mathrm{id} : \mathbb{Z}\pi \to \mathbb{Z}\pi$ if we identify the two non-trivial $\mathbb{Z}\pi$-chain modules with $\mathbb{Z}\pi$ using the cellular basis. This implies $\tau(C_*(\widetilde{Y},\widetilde{X})) = 0$ and hence $\tau(j) := \tau(C_*(\widetilde{j})) = 0$.

(2) We can assume without loss of generality that X is connected. Put $\pi = \pi_1(X)$. Choose an element $A \in GL(n, \mathbb{Z}\pi)$ representing $x \in \mathrm{Wh}(\pi)$. Choose $n \geq 2$. In the sequel we fix a zero-cell in X as base point. Put $X' = X \vee \bigvee_{j=1}^n S^n$. Let $b_j \in \pi_n(X')$ be the element represented by the inclusion of the j-th copy of S^n into X for $j = 1, 2 \ldots n$. Recall that $\pi_n(X')$ is a $\mathbb{Z}\pi$-module. Choose for $i = 1, 2 \ldots n$ a map $f_i : S^n \to X'_n$ such that $[f_i] = \sum_{j=1}^n a_{i,j} \cdot b_j$ holds in $\pi_n(X')$. Attach to X' for each $i \in \{1, 2 \ldots n\}$ an $(n+1)$-cell by $f_i : S^n \to X'_n$. Let Y be the resulting CW-complex and $i : X \to Y$ be the inclusion. Then i is an inclusion of finite CW-complexes and induces an isomorphism on the fundamental groups. In the sequel we identify π and $\pi_1(Y)$ by $\pi_1(i)$. The

cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{Y},\widetilde{X})$ is concentrated in dimensions n and $(n+1)$ and its $(n+1)$-differential is given by the matrix A with respect to the cellular basis. Hence $C_*(\widetilde{Y},\widetilde{X})$ is a contractible finite based free $\mathbb{Z}\pi$-chain complex with $\tau(C_*(\widetilde{Y},\widetilde{X})) = [A]$ in $\mathrm{Wh}(\pi)$. This implies that $i : X \to Y$ is a homotopy equivalence with $i_*^{-1}(\tau(i)) = x$. This finishes the proof of Lemma 2.18. ■

Notice that Lemma 2.18 (2) is the CW-analogue of Theorem 1.1 (2).

Recall that the *mapping cylinder* $\mathrm{cyl}(f)$ of a map $f : X \to Y$ is defined by the pushout

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X \times [0,1] & \longrightarrow & \mathrm{cyl}(f) \end{array}$$

There are natural inclusions $i_X : X = X \times \{1\} \to \mathrm{cyl}(f)$ and $i_Y : Y \to \mathrm{cyl}(f)$ and a natural projection $p : \mathrm{cyl}(f) \to Y$. Notice that i_X is a cofibration and $p \circ i_X = f$ and $p_Y \circ Y = \mathrm{id}_Y$. Define *the mapping cone* $\mathrm{cone}(f)$ by the quotient $\mathrm{cyl}(f)/i_X(X)$.

Lemma 2.19 *Let $f : X \to Y$ be a cellular map of finite CW-complexes and $A \subset X$ be a CW-subcomplex. Then the inclusion $\mathrm{cyl}(f|_A) \to \mathrm{cyl}(f)$ and in particular $i_Y : Y \to \mathrm{cyl}(f)$ is a composition of elementary expansions and hence a simple homotopy equivalence.*

It suffices to treat the case, where X is obtained from A by attaching an n-cell by an attaching map $q : S^{n-1} \to X$. Then there is an obvious pushout

$$\begin{array}{ccc} S^{n-1} \times [0,1] \cup_{S^{n-1}\times\{0\}} D^n \times \{0\} & \longrightarrow & \mathrm{cyl}(f|_A) \\ \downarrow & & \downarrow \\ D^n \times [0,1] & \longrightarrow & \mathrm{cyl}(f) \end{array}$$

and an obvious homeomorphism $(D^n \times [0,1], S^{n-1} \times [0,1] \cup_{S^{n-1}\times\{0\}} D^n \times \{0\}) \to (D^{n+1}, S^n_+)$. ■

Lemma 2.20 *A map $f : X \to Y$ of finite CW-complexes is a simple homotopy equivalence if and only if $i_X : X \to \mathrm{cyl}(f)$ is a simple homotopy equivalence.*

Proof : follows from Lemma 2.19 since a composition of simple homotopy equivalence and a homotopy inverse of a simple homotopy equivalence is again a simple homotopy equivalence. ■

Fix a finite CW-complex X. Consider to pairs of finite CW-complexes (Y,X) and (Z,X) such that the inclusions of X into Y and Z are homotopy equivalent. We call them equivalent, if there is a chain of pairs of finite CW-complexes

$$(Y,X) = (Y[0],X), (Y[1],X), (Y[2],X), \ldots, (Y[n],X) = (Z,X),$$

such that for each $k \in \{1, 2, \ldots n\}$ either $Y[k]$ is obtained from $Y[k-1]$ by an elementary expansion or $Y[k-1]$ is obtained from $Y[k]$ by an elementary expansion. Denote by $\mathrm{Wh}^{\mathrm{geo}}(X)$ the equivalence classes $[Y, X]$ of such pairs (Y, X). This becomes an abelian group under the addition $[Y, X] + [Z, X] := [Y \cup_X Z, X]$. The zero element is given by $[X, X]$. The inverse of $[Y, X]$ is constructed as follows. Choose a map $r : Y \to X$ with $r_X = \mathrm{id}$. Let $p : X \times [0,1] \to X$ be the projection. Then $[(\mathrm{cyl}(r) \cup_p X) \cup_r X, X] + [Y, X] = 0$. A map $g : X \to X'$ induces a homomorphism $g_* : \mathrm{Wh}^{\mathrm{geo}}(X) \to \mathrm{Wh}^{\mathrm{geo}}(X')$ by sending $[Y, X]$ to $[Y \cup_g X', X']$. We obviously have $\mathrm{id}_* = \mathrm{id}$ and $(g \circ h)_* = g_* \circ h_*$. In other words, we obtain a covariant functor on the category of finite CW-complexes with values in abelian groups.

The next result may be viewed as the homotopy theoretic analogon of the s-Cobordism Theorem 1.1 (3), where $\mathrm{Wh}^{\mathrm{geo}}(X)$ plays the role of the set of the diffeomorphism classes relative M_0 of h-cobordism over M_0

Theorem 2.21 *1. Let X be a finite CW-complex. The map*

$$\tau : \mathrm{Wh}^{\mathrm{geo}}(X) \to \mathrm{Wh}(X)$$

sending $[Y, X]$ to $i_*^{-1}\tau(i)$ for the inclusion $i : X \to Y$ is a natural isomorphism;

2. A homotopy equivalence $f : X \to Y$ is a simple homotopy equivalence if and only if $\tau(f) \in \mathrm{Wh}(Y)$ vanishes.

Proof : (1) The map τ is a well-defined homomorphism by Theorem 2.1 and Lemma 2.18 (1). It is surjective by Lemma 2.18 (2).

We give only a sketch of the proof of injectivity which is similar but much easier than the proof of s-Cobordism Theorem 1.1 (1). Consider an element $[Y, X]$ in $\mathrm{Wh}^{\mathrm{geo}}(X)$ with $i_*^{-1}\tau(i) = 0$ for the inclusion $i : X \to Y$. We want to show that $[Y, X] = [X, X]$ by reducing the number of cells, which must be attached to X to obtain Y, to zero without changing the class $[Y, X] \in \mathrm{Wh}^{\mathrm{geo}}(X)$. This corresponds in the proof of the s-Cobordism Theorem 1.1 (1) to reducing the number of handles in the handlebody decomposition to zero without changing the diffeomorphism type of the s-cobordism.

In the first step one arranges that Y is obtained from X by attaching only cells in two dimensions r and $(r+1)$ for some integer r. This is analogous to, but much easier to achieve than in the case of the s-Cobordims Theorem 1.1 (1) (see Normal Form Lemma 1.24). Details of this construction for CW-complexes can be found in [21, page 25-26].

Let $A \in GL(n, \mathbb{Z}\pi)$ be the matrix describing the $(r+1)$-differential in the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{Y}, \widetilde{X})$. As in the proof of the s-Cobordims Theorem 1.1 (1) (see also [21, chapter II, Section §8]) one shows that we can modify (Y, X) without changing its class $[Y, X] \in \mathrm{Wh}^{\mathrm{geo}}(X)$ such that the new matrix B is obtained from A by applying one of the operations (1), (2), (3), (4) and (5) introduced in Section 1.4. Since one can reduce A by a sequence of these operation to the trivial matrix if and only if its class $[A] \in \mathrm{Wh}(X)$ vanishes and this class $[A]$ is $i_*^{-1}\tau(i)$, the map τ is injective. Hence τ is a natural isomorphism of abelian groups.

(2) follows from Lemma 2.18 (1), Lemma 2.20 and the obvious fact that $i : X \to Y$ is a simple homotopy equivalence if $[Y, X] = 0$ in $\mathrm{Wh}^{\mathrm{geo}}(X)$. This finishes the proof of Theorem 2.21

2.4 Reidemeister torsion and lens spaces

In this section we deal with Reidemeister torsion which was defined earlier than Whitehead torsion and motivated the definition of Whitehead torsion. Reidemeister torsion was the first invariant in algebraic topology which could distinguish between spaces which are homotopy equivalent but not homeomorphic. Namely, it can be used to classify lens spaces up to homeomorphism.

Let X be a finite CW-complex with fundamental group π. Let U be an orthogonal finite-dimensional π-representation. Denote by $H_p(X;U)$ the homology of X with coefficients in U, i.e. the homology of the $\mathbb{C}$-chain complex $U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$. Suppose that X is *U-acyclic*, i.e. $H_p(X;U) = 0$ for all $p \geq 0$. If we fix a cellular basis for $C_*(\widetilde{X})$ and some orthogonal $\mathbb{R}$-basis for U, then $U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$ is a contractible based free finite $\mathbb{R}$-chain complex and defines an element $\tau(U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})) \in \widetilde{K}_1(\mathbb{R})$ (see (2.7)). Define the Reidemeister torsion

$$\rho(X;U) \in \mathbb{R}^{>0} \tag{2.22}$$

to be the image of $\tau(U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})) \in \widetilde{K}_1(\mathbb{R})$ under the homomorphism $\widetilde{K}_1(\mathbb{R}) \to \mathbb{R}^{>0}$ sending the class $[A]$ of $A \in GL(n, \mathbb{R})$ to $\det(A)^2$. Notice that for any trivial unit $\pm\gamma$ the automorphism of U given by multiplication with $\pm\gamma$ is orthogonal and that the square of the determinant of any orthogonal automorphism of U is 1. Therefore $\rho(X;U) \in \mathbb{R}^{>0}$ is independent of the choice of cellular basis for $C_*(\widetilde{X})$ and the orthogonal basis for U and hence is an invariant of the CW-complex X and U.

Lemma 2.23 *Let $f : X \to Y$ be a homotopy equivalence of connected finite CW-complexes and let U be an orthogonal finite-dimensional $\pi = \pi_1(Y)$-representation. Suppose that Y is U-acyclic. Let f^*U be the orthogonal $\pi_1(X)$-representation obtained from U by restriction with the isomorphism $\pi_1(f)$. Let $\det_U : \mathrm{Wh}(\pi(Y)) \to \mathbb{R}^{>0}$ be the map sending the class $[A]$ of $A \in GL(n, \mathbb{Z}\pi_1(Y))$ to $\det(\mathrm{id}_U \otimes_{\mathbb{Z}\pi} R_A : U \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^n \to U \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^n)^2$, where $R_A : \mathbb{Z}\pi^n \to \mathbb{Z}\pi^n$ is the $\mathbb{Z}\pi$-automorphism induced by A. Then*

$$\frac{\rho(Y, U)}{\rho(X, f^*U)} = \det_U(\tau(f)).$$

Proof : This follows from Lemma 2.9 (1) applied to the based exact sequence of contractible based free finite $\mathbb{R}$-chain complexes $0 \to U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{Y}) \to U \otimes_{\mathbb{Z}\pi} \mathrm{cone}_*(C_*(\widetilde{f})) \to \Sigma\left(U \otimes_{\mathbb{Z}\pi_1(f)} C_*(\widetilde{X})\right) \to 0$. ∎

Next we introduce the family of spaces which we want to classify completely using Reidemeister torsion. Let G be a cyclic group of finite order $|G|$. Let V be a unitary finite-dimensional G-representation. Define its *unit sphere* SV and its *unit disk* DV to be the G-subspaces $SV = \{v \in V \mid ||v|| = 1\}$ and $DV = \{v \in V \mid ||u|| \leq 1\}$ of V. Notice that a complex finite-dimensional vector space has a preferred orientation as real vector space, namely the one given by the $\mathbb{R}$-basis $\{b_1, ib_1, b_2, ib_2, \ldots b_n, ib_n\}$ for any $\mathbb{C}$-basis $\{b_1, b_2, \ldots b_n\}$. Any $\mathbb{C}$-linear automorphism of a complex finite-dimensional vector space preserves this orientation. Thus SV and DV are compact oriented Riemannian manifolds with isometric orientation preserving G-action. We call a unitary G-representation V *free* if the induced G-action on its unit sphere $SV = \{v \in V \mid ||u|| = 1\}$ is free. Then $SV \to G\backslash SV$ is a covering and the quotient space $L(V) := G\backslash SV$ inherits from SV the structure of an oriented closed Riemannian manifold.

Definition 2.24 *We call the closed oriented Riemannian manifold $L(V)$ the* lens space *associated to the finite-dimensional unitary representation V of the finite cyclic group G.*

One can specify these lens spaces also by numbers as follows.

Notation 2.25 *Let $\mathbb{Z}/t$ be the cyclic group of order $t \geq 2$. The 1-dimensional unitary representation V_k for $k \in \mathbb{Z}/t$ has as underlying vector space $\mathbb{C}$ and $l \in \mathbb{Z}/t$ acts on it by multiplication with $\exp(2\pi ikl/t)$. Notice that V_k is free if and only if $k \in \mathbb{Z}/t^*$, and is trivial if and only if $k = 0$ in $\mathbb{Z}/t$. Define the lens space $L(t; k_1, \ldots k_c)$ for an integer $c \geq 1$ and elements $k_1, \ldots k_c$ in $\mathbb{Z}/t^*$ by $L(\oplus_{i=1}^c V_{k_i})$.*

These lens spaces form a very interesting family of manifolds which can be completely classified as we will see. Two lens spaces $L(V)$ and $L(W)$ of the same dimension $n \geq 3$ have the same homotopy groups, namely their fundamental group is G and their p-th homotopy group is isomorphic to $\pi_p(S^n)$. They also have the same homology with integral coefficients, namely $H_p(L(V)) \cong \mathbb{Z}$ for $p = 0, 2n-1$, $H_p(L(V)) \cong G$ for p odd and $1 \leq p < n$ and $H_p(L(V)) = 0$ for all other values of p. Also their cohomology groups agree. Nevertheless not of all them are homotopic. Moreover, there are homotopic lens spaces which are not diffeomorphic (see Example 2.41).

Suppose that $\dim_{\mathbb{C}}(V) \geq 2$. We want to give an explicit identification

$$\pi_1(L(V), x) \quad = \quad G. \tag{2.26}$$

Given a point $x \in L(V)$, we obtain an isomorphism $s(x) : \pi_1(L(V), x) \xrightarrow{\cong} G$ by sending the class of a loop w in $L(V)$ with base point x to the element $g \in G$ for which there is a lift $\widetilde{w}$ in SV of w with $\widetilde{w}(1) = g \cdot \widetilde{w}(0)$. One easily checks using elementary covering theory that this is a well-defined isomorphism. If y is another base point, we obtain a homomorphism $t(x, y) : \pi_1(L(V), x) \to \pi_1(L(V), y)$ by conjugation with any path v in $L(V)$ from x to y. Since $\pi_1(L(V), x)$ is abelian, $t(x, y)$ is independent of the choice of v. One easily checks $t(x, x) = \mathrm{id}$,

$t(y,z) \circ t(x,y) = t(x,z)$ and $s(y) \circ t(x,y) = s(x)$. Hence we can in the sequel identify $\pi_1(L(V),x)$ with G and ignore the choice of the base point $x \in L(V)$.

Let $p : EG \to BG$ be a model for the *universal principal G-bundle.* It has the property that for any principal G-bundle $q : E \to B$ there is a map $f : B \to BG$ called *classifying map of q* which is up to homotopy uniquely determined by the property that the pull back of p with f is isomorphic over B to q. Equivalently p can be characterized by the property that BG is a CW-complex and EG is contractible. The space BG is called the *classifying space for G.* Let $f(V) : L(V) \to BG$ be the classifying map of the principal G-bundle $SV \to L(V)$. Put $n = \dim(L(V)) = 2 \cdot \dim_{\mathbb{C}}(V) - 1$. Define the element

$$l(V) \quad \in \quad H_n(BG) \tag{2.27}$$

by the image of the fundamental class $[L(V)] \in H_n(L(V))$ associated to the preferred orientation of $L(V)$ under the map $H_n(f(V)) : H_n(L(V)) \to H_n(BG)$ induced by $f(V)$ on homology with integer coefficients. The map $f(V) : L(V) \to BG$ is n-connected since its lift $SV \to EG$ is n-connected. Hence $H_n(f(V))$ is surjective. As $H_n(L(V))$ is infinite cyclic with $[L(V)]$ as generator, $l(V)$ generates $H_n(BG)$. Notice that n is odd and that $H_n(BG)$ is isomorphic to $\mathbb{Z}/|G|$ for a cyclic group G of finite order $|G|$. A map $f : L(V) \to L(W)$ of lens spaces of the same dimension $n = \dim(L(V)) = \dim(L(W))$ for two free unitary G-representations V and W induces a homomorphism $\pi_1(f,x) : \pi_1(L(V)) \to \pi_1(L(W))$. Under the identification (2.26) this is an endomorphism $\pi_1(f)$ of G. Define

$$e(f) \quad \in \quad \mathbb{Z}/|G| \tag{2.28}$$

to be the element for which $\pi_1(f)$ sends $g \in G$ to $g^{e(f)}$. Notice that $e(f)$ depends only on the homotopy class of f and satisfies $e(g \circ f) = e(g) \cdot e(f)$ and $e(\mathrm{id}) = 1$. In particular $e(f) \in \mathbb{Z}/|G|^*$ for a homotopy equivalence $f : L(V) \to L(W)$. Define the *degree*

$$\deg(f) \quad \in \quad \mathbb{Z} \tag{2.29}$$

to be the integer for which $H_n(f)$ sends $[L(V)] \in H_n(L(V))$ to $\deg(f) \cdot [L(W)] \in H_n(L(W))$.

Given two spaces X and Y, define their *join* $X * Y$ by the push out

$$\begin{array}{ccc} X \times Y & \longrightarrow & X \times \operatorname{cone}(Y) \\ \downarrow & & \downarrow \\ \operatorname{cone}(X) \times Y & \longrightarrow & X * Y \end{array}$$

If X and Y are G-spaces, $X * Y$ inherits a G-operation by the diagonal operation. Given two free finite-dimensional unitary G-representations, there is a G-homeomorphism

$$S(V \oplus W) \quad \cong \quad SV * SW. \tag{2.30}$$

Theorem 2.31 (Homotopy classification of lens spaces) *Let $L(V)$ and $L(W)$ be two lens spaces of the same dimension $n \geq 3$. Then*

1. *The map*

$$e \times \deg : [L(V), L(W)] \to \mathbb{Z}/|G| \times \mathbb{Z}, \qquad [f] \mapsto e(f), \deg(f)$$

is injective, where $[L(V), L(W)]$ is the set of homotopy classes of maps from $L(V)$ to $L(W)$;

2. *An element $(u, d) \in \mathbb{Z}/|G| \times \mathbb{Z}$ is in the image of $e \times \deg$ if and only if we get in $H_n(BG)$*

$$d \cdot l(W) = e(f)^{\dim_{\mathbb{C}}(V)} \cdot l(V);$$

3. *$L(V)$ and $L(W)$ are homotopy equivalent if and only if there is an element $e \in \mathbb{Z}/|G|^*$ satisfying in $H_n(BG)$*

$$\pm l(W) = e^{\dim_{\mathbb{C}}(V)} \cdot l(V).$$

$L(V)$ and $L(W)$ are oriented homotopy equivalent if and only if there is an element $e \in \mathbb{Z}/|G|^*$ satisfying in $H_n(BG)$

$$l(W) = e^{\dim_{\mathbb{C}}(V)} \cdot l(V);$$

4. *The lens spaces $L(t; k_1, \ldots k_c)$ and $L(t; l_1, \ldots, l_c)$ are homotopy equivalent if and only if there is $e \in \mathbb{Z}/t^*$ such that we get $\prod_{i=1}^c k_i = \pm e^c \cdot \prod_{i=1}^c l_i$ in $\mathbb{Z}/t^*$.*

The lens spaces $L(t; k_1, \ldots k_c)$ and $L(t; l_1, \ldots, l_c)$ are oriented homotopy equivalent if and only if there is $e \in \mathbb{Z}/t^*$ such that we get $\prod_{i=1}^c k_i = e^c \cdot \prod_{i=1}^c l_i$ in $\mathbb{Z}/t^*$.

Proof : (1) Obviously $e(f)$ and $\deg(f)$ depend only on the homotopy type of f. Consider two maps $f_0, f_1 : L(V) \to L(W)$ with $e(f_0) = e(f_1)$ and $\deg(f_0) = \deg(f_1)$. Choose lifts $\widetilde{f_0}, \widetilde{f_1} : SV \to SW$. Let $\alpha : G \to G$ be the automorphism sending g to $g^{e(f_0)} = g^{e(f_1)}$. Then both lifts $\widetilde{f_k}$ are α-equivariant. Since G acts orientation preserving and freely on SV and SW, the projection induces a map $H_n(SV) \to H_n(L(V))$ resp. $H_n(SW) \to H_n(L(W))$ which send the fundamental class $[SV]$ resp. $[SW]$ to $|G| \cdot [L(V)]$ resp. $|G| \cdot [L(W)]$. Hence $\deg(f_k) = \deg(\widetilde{f_k})$ for $k = 0, 1$. Thus $\deg(f_0) = \deg(f_1)$ implies $\deg(\widetilde{f_0}) = \deg(\widetilde{f_1})$. Let $\alpha^* SW$ be the G-space obtained from SW by restricting the group action with α. Then $\widetilde{f_0}$ and $\widetilde{f_1} : SV \to \alpha^* SW$ are G-maps with $\deg(\widetilde{f_0}) = \deg(\widetilde{f_1})$. It suffices to show that they are G-homotopic.

We outline an elementary proof of this fact. There is a G-CW-structure on SV such that there is exactly one equivariant cell $G \times D^k$ in each dimension. It induces a G-CW-structure on $SV \times [0, 1]$ using the standard CW-structure on $[0, 1]$. We want to define inductively for $k = -1, 0, \ldots, n$ G-maps h_k :

$SV \times \{0,1\} \cup SV_k \times [0,1] \to \alpha^*SW$ such that h_k extends h_{k-1} and $h_{-1} = \widetilde{f_0} \coprod \widetilde{f_1} : SV \times \{0,1\} \to \alpha^*SW$. Then h_n will be the desired G-homotopy between f_0 and f_1. Notice that $SV \times \{0,1\} \cup SV_k \times [0,1]$ is obtained from $SV \times \{0,1\} \cup SV_{k-1} \times [0,1]$ by attaching one equivariant cell $G \times D^{k+1}$ with some attaching G-map $q_k : G \times S^k \to SV$. In the induction step we must show that the composition $h_{k-1} \circ q_k : G \times S^k \to SW$ can be extended to a G-map $G \times D^{k+1} \to SW$. This is possible if and only if its restriction to $S^k = \{1\} \times S^k$ can be extended to a (non-equivariant) map $D^{k+1} \to SW$. Since SW is n-connected, this can be done for any map $S^k \to SW$ for $k < n$. In the final step we run into the obstruction that a map $S^n \to SW$ can be extended to a map $D^{n+1} \to SW$ if and only if its degree is zero. Now one has to check that the degree of the map $(h_{n-1} \circ q_n)|_{\{1\}\times S^n} : \{1\} \times S^n = S^n \to SW$ is exactly the difference $\deg(f_0) - \deg(f_1)$ and hence zero.

(2) Given a map $f : LV \to LW$, let $\pi_1(f) : G \to G$ be the map induced on the fundamental groups under the identification (2.26). Then the following diagram commutes

$$\begin{array}{ccc} H_n(LV) & \xrightarrow{H_n(f)} & H_n(LW) \\ {\scriptstyle H_n(f(V))}\downarrow & & \downarrow{\scriptstyle H_n(f(W))} \\ H_n(BG) & \xrightarrow[H_n(B\pi_1(f))]{} & H_n(BG) \end{array}$$

This implies

$$\deg(f) \cdot l(W) \quad = \quad H_n(B\pi_1(f))(l(V)). \tag{2.32}$$

Given $e \in \mathbb{Z}$, let $m_e : \mathbb{Z}/t \to \mathbb{Z}/t$ be multiplication with e. Let e, $k_1, \ldots k_c$, $l_1, \ldots l_c$ be integers which are prime to t. Fix integers $k_1', \ldots k_c'$ such that we get in $\mathbb{Z}/t$ the equation $k_i \cdot k_i' = 1$. Define a m_e-equivariant map $d_i : V_{k_i} \to V_{l_i}$ by $z \mapsto z^{k_i' l_i e}$. It has degree $k_i' l_i e$. The m_e-equivariant map $*_{i=1}^c d_i : *_{i=1}^c SV_{k_i} \to *_{i=1}^c SV_{l_i}$ yields under the identification (2.30) a m_e-equivariant map $\widetilde{f} : S(\oplus_{i=1}^c V_{k_i}) \to S(\oplus_{i=1}^c V_{l_i})$ of degree $e^c \cdot \prod_{i=1}^c k_i' l_i$. By taking the quotient under the G-action yields a map $f : L(t; k_1, \ldots k_c) \to L(t; l_1, \ldots l_c)$ of $n = (2c-1)$-dimensional lens spaces with $\deg(f) = e^c \cdot \prod_{i=1}^c k_i' l_i$ and $e(f) = e$. We conclude from (2.32) in the special case $k_i = l_i$ for $i = 1, \ldots, c$ that $H_n(Bm_e) : H_n(B\mathbb{Z}/t) \to H_n(B\mathbb{Z}/t)$ is multiplication with e^c since $l(t; k_1 \ldots k_c) := l(\oplus_{i=1}^c V_{k_i})$ is always a generator of $H_n(BG)$. Thus (2.32) becomes

$$\deg(f) \cdot l(W) \quad = \quad e(f)^{\dim_{\mathbb{C}}(V)} \cdot l(V). \tag{2.33}$$

We conclude from (2.33) in the special case $e = 1$ that we get in $H_n(B(\mathbb{Z}/t))$ the equation

$$\prod_{i=1}^{c} l_i \cdot l(t; l_1, \ldots l_c) \quad = \quad \prod_{i=1}^{c} k_i \cdot l(t; k_1, \ldots k_c). \tag{2.34}$$

Next we show for a map $f : L(V) \to L(W)$ and $m \in \mathbb{Z}$ that we can find another map $f' : LV \to LW$ with $e(f) = e(f')$ and $\deg(f') = \deg(f) + m \cdot |G|$.

Fix some embedded disk $D^n \subset SV$ such that $g \cdot D^n \cap D^n \neq \emptyset$ implies $g = 1$. Let $\widetilde{f} : SV \to SW$ be a lift of f. Recall that f is m_e-equivariant for $m_e : G \to G,\ g \mapsto g^e$. Define a new m_e-equivariant map $\widetilde{f'} : SV \to SW$ as follows. Outside $G \cdot D^n$ the maps $\widetilde{f}$ and $\widetilde{f'}$ agree. Let $\frac{1}{2}D^n$ be $\{x \in D^n \mid ||x|| \leq 1/2\}$. Define $\widetilde{f'}$ on $G \cdot \left(D^n - \frac{1}{2}D^n\right)$ by sending $(g, t \cdot x)$ for $g \in G$, $t \in [1/2, 1]$ and $x \in S^{n-1} = \partial D^n$ to $g^e \cdot \widetilde{f}((2t-1)x)$. Let $c : (\frac{1}{2}D^n, \partial\frac{1}{2}D^n) \to (SW, \widetilde{f}(0))$ for $0 \in D^n$ the origin be a map such that the induced map $(\frac{1}{2}D^n)/\partial(\frac{1}{2}D^n) \to SW$ has degree m. Define $f'|_{G\cdot\frac{1}{2}D^n} : G \cdot \frac{1}{2}D^n \to SW$ by sending (g, x) to $g^e \cdot c(x)$. One easily checks that $\widetilde{f'}$ has degree $\deg(\widetilde{f}) + m \cdot |G|$. Then $G\backslash\widetilde{f'}$ is the desired map f'.

Notice that there is at least one map $f : L(V) \to L(W)$ with $e(f) = e$ for any given $e \in \mathbb{Z}/|G|$. This follows by an argument similar to the one above, since G acts freely on SV and SW is $(\dim(SV) - 1)$-connected. Now assertion (2) follows.

(3) Let $f : SV \to SW$ be a map. Then f is a homotopy equivalence if and only if it induces isomorphisms on all homotopy groups. This is the case if and only if $\pi_1(f)$ is an automorphism and $\widetilde{f} : SV \to SW$ induces an isomorphism on all homotopy groups. Hence f is a homotopy equivalence if and only if $e(f) \in \mathbb{Z}/|G|^*$ and $\deg(f) = \deg(\widetilde{f}) = \pm 1$. Recall that f is an oriented homotopy equivalence if and only if f is a homotopy equivalence and $\deg(f) = 1$. Now the claim follows from assertion (2).

(4) follows from (2.34) and assertion (3). This finishes the proof of Theorem 2.31. ■

Lemma 2.35 *Let G be a cyclic group of finite order $|G|$.*

1. *Let V be a free unitary finite-dimensional G-representation. Let U be an orthogonal finite-dimensional G-representation with $U^G = 0$. Then $L(V)$ is U-acyclic and the Reidemeister torsion $\rho(L(V); U) \in \mathbb{R}^{>0}$ is defined;*

2. *Let V, V_1, V_2 be free unitary finite-dimensional G-representations. Let U, U_1 and U_2 be orthogonal finite-dimensional G-representations with $U^G = U_1^G = U_2^G = 0$. Then*

$$\begin{aligned} \rho(L(V_1 \oplus V_2), U) &= \rho(L(V_1); U) \cdot \rho(L(V_2); U)); \\ \rho(L(V), U_1 \oplus U_2) &= \rho(L(V); U_1) \cdot \rho(L(V); U_2)). \end{aligned}$$

Proof : (1) Let X be a finite CW-complex with fundamental group G. We show that X is U-acyclic if G acts trivial on $H_p(\widetilde{X})$ and U is an orthogonal finite-dimensional G-representation with $U^G = 0$. We have to show that $H_p(C_*(\widetilde{X}) \otimes_{\mathbb{Z}G} U) \cong H_p(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{R} \otimes_{\mathbb{R}G} U$ vanishes. By assumption $H_p(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{R}$ is a direct sum of copies of the trivial G-representation $\mathbb{R}$ and U is a direct sum of non-trivial irreducible representations. Since for any non-trivial irreducible G-representation W the tensor product $\mathbb{R} \otimes_{\mathbb{R}G} W$ is trivial, the claim follows.

(2) The claim for the second equation is obvious, it remains to prove the first. Since G-acts trivially on the homology of SV and SW and hence on on the homology of $SV \times SW$, $SV \times DV$ and $DV \times SW$, we conclude from Theorem 2.1 (1), Lemma 2.9 (1), Lemma 2.23 and (2.30)

$$\begin{aligned} &\rho(L(V \oplus W); U) \\ &= \rho(G\backslash(DV \times SW; U) \cdot \rho(G\backslash(SV \times DW; U) \cdot (\rho(G\backslash(SV \times SW; U))^{-1}. \end{aligned}$$

Hence it remains to show

$$\begin{aligned} \rho(G\backslash(DV \times SW; U) &= \rho(L(W); U); \\ \rho(G\backslash(SV \times DW; U) &= \rho(L(V); U); \\ \rho(G\backslash(SV \times SW; U) &= 1. \end{aligned}$$

These equations will follow from the following slightly more general formula (2.36) below. Let D_* be a finite $\mathbb{R}G$-chain complexes such that G-acts trivially on its homology. Assume that D_* comes with a $\mathbb{R}$-basis. Then $C_*(SV) \otimes_{\mathbb{Z}} D_*$ with the diagonal G-action is a finite $\mathbb{R}G$-chain complex such that G acts trivially on its homology and there is a preferred $\mathbb{R}G$-basis. Let $\chi_{\mathbb{R}}(D_*)$ be the Euler characteristic of D_*, i.e.

$$\chi_{\mathbb{R}}(D_*) := \sum_{p \in \mathbb{Z}} (-1)^p \cdot \dim_{\mathbb{R}}(D_p) = \sum_{p \in \mathbb{Z}} (-1)^p \cdot \dim_{\mathbb{R}}(H_p(D_*)).$$

Then

$$\rho(C_*(SV) \otimes_{\mathbb{Z}} D_* \otimes_{\mathbb{R}G} U) = \rho(L(V); U)^{\chi_{\mathbb{R}}(D_*)}, \tag{2.36}$$

where $\rho(C_*(SV) \otimes_{\mathbb{Z}} D_* \otimes_{\mathbb{R}G} U) \in \mathbb{R}^{>0}$ is the Reidemeister torsion of the acyclic based free finite $\mathbb{R}$-chain complex $C_*(SV) \otimes_{\mathbb{Z}} D_* \otimes_{\mathbb{R}G} U$ which is defined to be the image of the Whitehead torsion $\tau(C_*(SV) \otimes_{\mathbb{Z}} D_* \otimes_{\mathbb{R}G} U) \in \widetilde{K}_1(\mathbb{R})$ (see (2.7)) under the homomorphism $\widetilde{K}_1(\mathbb{R}) \to \mathbb{R}^{>0}$, $[A] \mapsto \det(A)^2$.

It remains to prove (2.36). Since $\mathbb{R}G$ is semi-simple, there is a $\mathbb{R}G$-chain homotopy equivalence $p_* : D_* \to H_*(D_*)$, where we consider $H_*(D_*)$ as $\mathbb{R}G$-chain complex with the trivial differential.

Equip $H_*(D_*)$ with an $\mathbb{R}$-basis. Then we get in $\widetilde{K_1}(\mathbb{R}G)$ from Lemma 2.9 (1)

$$\begin{aligned} &\tau\left(\mathrm{id}_{C_*(SV)} \otimes p_* : C_*(SV) \otimes_{\mathbb{Z}} D_* \to C_*(SV) \otimes_{\mathbb{Z}} H_*(D_*)\right) \\ &= \sum_{q \in \mathbb{Z}} (-1)^p \cdot \tau\left(\mathrm{id}_{C_q(SV)} \otimes p_* : C_q(SV) \otimes_{\mathbb{Z}} D_* \to C_q(SV) \otimes_{\mathbb{Z}} H_*(D_*)\right) \\ &= \sum_{q \in \mathbb{Z}} (-1)^p \cdot \dim_{\mathbb{Z}G}(C_p(SV)) \cdot \tau\left(\mathrm{id}_{\mathbb{Z}G} \otimes p_* : \mathbb{Z}G \otimes_{\mathbb{Z}} D_* \to \mathbb{Z}G \otimes_{\mathbb{Z}} H_*(D_*)\right) \\ &= \chi(L(V)) \cdot \tau\left(\mathrm{id}_{\mathbb{Z}G} \otimes p_* : \mathbb{Z}G \otimes_{\mathbb{Z}} D_* \to \mathbb{Z}G \otimes_{\mathbb{Z}} H_*(D_*)\right) \\ &= 0. \end{aligned}$$

The chain complex version of Lemma 2.23 shows

$$\rho(C_*(SV) \otimes_{\mathbb{Z}} D_* \otimes_{\mathbb{R}G} U) \quad = \quad \rho(C_*(SV) \otimes_{\mathbb{Z}} H_*(D_*) \otimes_{\mathbb{R}G} U).$$

Obviously

$$\rho(C_*(SV) \otimes_{\mathbb{Z}} H_*(D_*) \otimes_{\mathbb{R}G} U) \; = \; \rho(C_*(SV) \otimes_{\mathbb{Z}G} U)^{\chi_{\mathbb{R}}(D_*)} \; = \; \rho(L(V); U)^{\chi_{\mathbb{R}}(D_*)}.$$

This finishes the proof of Lemma 2.35. ■

Theorem 2.37 (Diffeomorphism classification of lens spaces) *Let $L(V)$ and $L(W)$ be two lens spaces of the same dimension $n \geq 3$. Then the following statements are equivalent.*

1. *There is an automorphism $\alpha : G \to G$ such that V and $\alpha^* W$ are isomorphic as orthogonal G-representations;*
2. *There is an isometric diffeomorphism $L(V) \to L(W)$;*
3. *There is a diffeomorphism $L(V) \to L(W)$;*
4. *There is homeomorphism $L(V) \to L(W)$;*
5. *There is simple homotopy equivalence $L(V) \to L(W)$;*
6. *There is an automorphism $\alpha : G \to G$ such that for any orthogonal finite-dimensional-dimensional representation U with $U^G = 0$*
$$\rho(L(W), U) = \rho(L(V), \alpha^* U)$$
holds.
7. *There is an automorphism $\alpha : G \to G$ such that for any non-trivial 1-dimensional unitary G-representation U*
$$\rho(L(W), \operatorname{res} U) = \rho(L(V), \alpha^* \operatorname{res} U)$$
holds, where the orthogonal representation $\operatorname{res} U$ is obtained from U by restricting the scalar multiplication from $\mathbb{C}$ to $\mathbb{R}$.

Proof : The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (7) are obvious or follow directly from Theorem 2.1 (5) and Lemma 2.23. Hence it remains to prove the implication (7) ⇒ (1).

Fix a generator $g \in G$, or equivalently, an identification $G = \mathbb{Z}/|G|$. Choose $e \in \mathbb{Z}/|G|^*$ such that $\alpha : G \to G$ sends g to g^e. Recall that V_k denotes the 1-dimensional unitary G-representation for which g acts by multiplication with $\exp(2\pi i k/|G|)$. The based $\mathbb{Z}G$-chain complex of SV_k is concentrated in dimension 0 and 1 and its first differential is $g^k - 1 : \mathbb{Z}G \to \mathbb{Z}G$. If g acts on U by multiplication with the $|G|$-th root of unity ζ, then we conclude

$$\begin{aligned} \rho(LV_k; U) &= ||\zeta^k - 1||^2; \\ \rho(LV_k; \alpha^* U) &= ||\zeta^{ek} - 1||^2. \end{aligned}$$

We can write for appropriate numbers $c \in \mathbb{Z}$, $c \geq 1$, $k_i, l_j \in \mathbb{Z}/G^*$

$$\begin{aligned} V &= \oplus_{i=1}^c V_{k_i}; \\ W &= \oplus_{j=1}^c V_{l_j}. \end{aligned}$$

We conclude from Lemma 2.35

$$\begin{aligned} \rho(LV;U) &= \prod_{i=1}^c (\zeta^{k_i} - 1) \cdot (\zeta^{-k_i} - 1); \\ \rho(LW;\alpha^* U) &= \prod_{i=1}^c (\zeta^{el_j} - 1) \cdot (\zeta^{-el_j} - 1). \end{aligned}$$

This implies that for any $|G|$-th root ζ of unity with $\zeta \neq 1$ the following equation holds

$$\prod_{i=1}^c (\zeta^{k_i} - 1) \cdot (\zeta^{-k_i} - 1) \quad = \quad \prod_{i=1}^c (\zeta^{el_j} - 1) \cdot (\zeta^{-el_j} - 1). \qquad (2.38)$$

We will need the following number theoretic result due to Franz whose proof can be found for instance in [34].

Lemma 2.39 (Franz' Independence Lemma) *Let $t \geq 2$ be an integer and $S = \{j \in \mathbb{Z} \mid 0 < j < t, (j,t) = 1\}$. Let $(a_j)_{j \in S}$ be a sequence of integers indexed by S such that $\sum_{j \in S} a_j = 0$, $a_j = a_{t-j}$ for $j \in S$ and $\prod_{j \in S} (\zeta^j - 1)^{a_j} = 1$ holds for every t-th root of unity $\zeta \neq 1$. Then $a_j = 0$ for $j \in S$.*

Put $t = |G|$. For $j \in S = \{j \in \mathbb{Z} \mid 0 < j < |G|, (j, |G|) = 1\}$ let x_j be the number of elements in the sequence k_1, $-k_1$, k_2, $-k_2$, ..., k_c, $-k_c$, which are congruent j modulo $|G|$. Each of the elements k_i is prime to $|G|$ and hence $\pm x_j$ is congruent mod $|G|$ to some $j \in S$. This implies $\sum_{j \in S} x_j = 2c$. Obviously $x_j = x_{|G|-j}$ for $j \in S$. Define analogously a sequence $(y_j)_{j \in S}$ for the sequence $el_1, -el_1, el_2, -el_2, \ldots, el_c, -el_c$. Put $a_j = x_j - y_j$ for $j \in S$. Then $\sum_{\in S} a_j = 0$, $a_j = a_{|G|-j}$ for $j \in S$ and we conclude from (2.38) that $\prod_{j \in S} (\zeta^j - 1)^{a_j} = 1$ for any $|G|$-th root of unity $\zeta \neq 1$. We conclude from Franz Independence Lemma 2.39 that $a_j = 0$ and hence $x_j = y_j$ holds for $j \in S$. This implies that there is a permutation $\sigma \in \Sigma_c$ together with signs $\epsilon_i \in \{\pm 1\}$ for $i = 1, 2, \ldots c$ such that k_i and $\epsilon_i \cdot l_{\sigma(i)}$ are congruent modulo $|G|$. But this implies that V_i and $\alpha^* W_{\sigma(i)}$ are isomorphic as orthogonal G-representations. Hence V and $\alpha^* W$ are isomorphic as orthogonal representations. This finishes the proof of Theorem 2.37. ■

Corollary 2.40 *Two lens spaces $L(t; k_1, \ldots, k_c)$ and $L(t; l_1, \ldots, l_c)$ are homeomorphic if and only there are $e \in \mathbb{Z}/t^*$, signs $\epsilon_i \in \{\pm 1\}$ and a permutation $\sigma \in \Sigma_c$ such that $k_i = \epsilon_i \cdot e \cdot l_{\sigma(i)}$ holds in $\mathbb{Z}/t^*$ for $i = 1, 2, \ldots, c$.*

Notice that $\oplus_{i=1}^c V_{k_i}$ and $\oplus_{i=1}^c V_{l_i}$ are isomorphic as orthogonal representations if and only if there are signs $\epsilon_i \in \{\pm 1\}$ and a permutation $\sigma \in \Sigma_c$ such that $k_i =$

$\epsilon_i \cdot l_{\sigma(i)}$ holds in $\mathbb{Z}/t^*$ for $i = 1, 2, \ldots, c$. If $m_e : \mathbb{Z}/t \to \mathbb{Z}/t$ is multiplication with $e \in \mathbb{Z}/t^*$, then the restriction $m_e^* (\oplus_{i=1}^c V_{l_i})$ is $\oplus_{i=1}^c V_{el_i}$. Now apply Theorem 2.37. ■

Example 2.41 We conclude from Theorem 2.31 and Corollary 2.40 the following facts:

1. Any homotopy equivalence $L(7; k_1, k_2) \to L(7; k_1, k_2)$ has degree 1. Thus $L(7; k_1, k_2)$ possesses no orientation reversing selfdiffeomorphism;
2. $L(5; 1, 1)$ and $L(5; 2, 1)$ have the same homotopy groups, homology groups and cohomology groups, but they are not homotopy equivalent;
3. $L(7; 1, 1)$ and $L(7; 2, 1)$ are homotopy equivalent, but not homeomorphic.

2.5 Miscellaneous

We mention that lens spaces are the only closed manifolds M which carry a Riemannian metric with sectional curvature which is constant 1, provided $\pi_1(M)$ is cyclic. Reidemeister torsion can be used to classify all such manifolds without any assumption on $\pi_1(M)$ and to show that two finite-dimensional (not necessarily free) orthogonal representations V and W have G-diffeomorphic unit spheres SV and SW if and only if they are isomorphic as orthogonal representations (see [54]). The corresponding statement is false if one replaces G-diffeomorphic by G-homeomorphic. [16],[15], [29].

Let (W, L, L') is an h-cobordism of lens spaces which is compatible with the orientations and the identifications of $\pi_1(L)$ abd $\pi_1/L')$ with G. Then W is diffeomorphic relative L to $L \times [0, 1]$ and L and L' are diffeomorphic [47, Corollary 12.13 on page 410].

We refer to [21] and [47] for more information about Whitehead torsion and lens spaces.

Chapter 3

Normal maps and the surgery problem

Introduction

In this chapter we want to take the first step to the following problem

Problem 3.1 *Let X be a topological space. When is X homotopy equivalent to a closed manifold?*

We will begin with discussing Poincaré duality, which is the first obstruction from algebraic topology for a positive answer to Problem 3.1 above, in Section 3.1. In Section 3.2 we will deal with the Spivak normal fibration of a finite Poincaré complex X which is a reminiscense of the normal bundle of the embedding of a closed manifold into some high-dimensional Euclidean space. We will explain and motivate the notion of a normal map of degree one in Section 3.3. This is a map of degree one $f : M \to X$ from a closed manifold M to a finite Poincaré complex X covered by some bundle data. The surgery problem is to change it to a homotopy equivalence leaving the target fixed and changing the source without loosing the structure of a closed manifold. In Section 3.4 we will introduce the surgery step. This is the manifold analogue of the process in the world of CW-complexes which is given by adding a cell in order to kill an element in the homotopy group. We will show that we can make a normal map $f : M \to X$ highly connected by carrying out a finite number of surgery steps. The surgery obstructions will later arise as obstructions to make f connected in the middle dimension. If f is also connected in the middle dimension, then Poincaré duality implies that f is a homotopy equivalence from a closed manifold to X.

We mention that Problem 3.1 is an important problem but the main success of surgery comes from its contribution to the question whether two closed manifolds are diffeomorphic (see the surgery program in Remark 1.5) and the construction of exotic structures, i.e. different smooth structures on the same topological manifold. We will restrict our attention for some time to Problem 3.1 because it is a good motivation and guide line for certain techniques like the surgery step, certain notions like normal maps of degree one and obstructions and obstruction groups like the signature and the L-groups. After we have developed the machinery which allows us to solve Problem 3.1 in principle, we will develop it further in order to carry out the surgery program.

3.1 Poincaré Duality

In order to state Poincaré duality on the level we will need we have to introduce some algebra. Recall that a *ring with involution*R is an associative ring R with unit together with an *involution of rings* $\bar{\ } : R \to R,\ r \mapsto \overline{r}$ such that $\overline{r+s} = \overline{r} + \overline{s}$, $\overline{rs} = \overline{s}\overline{r}$ and $\overline{1} = 1$ holds for $r, s \in R$. Our main example will be the group ring AG for some commutative associative ring A with unit and a group G together with a homomorphism $w : G \to \{\pm 1\}$. The so called *w-twisted involution* sends $\sum_{g\in G} a_g \cdot g$ to $\sum_{g \in G} w(g) \cdot a_g \cdot g^{-1}$. Let M be a left R-module. Then $M^* := \hom_R(M, R)$ carries a canonical right R-module structure given by $(fr)(m) = f(m) \cdot r$ for a homomorphism of left R-modules $f : M \to R$ and $m \in M$. The involution allows us to view $M^* = \hom_R(M; R)$ as a left R-module, namely define rf for $r \in R$ and $f \in M^*$ by $(rf)(m) := f(m) \cdot \overline{r}$ for $m \in M$. Given an R-chain complex of left R-modules C_* and $n \in \mathbb{Z}$, we define its dual chain complex C^{n-*} to be the chain complex of left R-modules whose p-th chain module is $\hom_R(C_{n-p}, R)$ and whose p-th differential is given by

$$\hom_R(c_{n-p+1}, \mathrm{id}) : (C^{n-*})_p = \hom_R(C_{n-p}, R) \to (C^{n-*})_{p-1} = \hom_R(C_{n-p+1}, R).$$

Consider a connected finite CW-complex X with fundamental group π and a group homomorphism $w : \pi \to \{\pm 1\}$. In the sequel we use the w-twisted involution. Denote by $C_*(\widetilde{X})$ the cellular $\mathbb{Z}\pi$-chain complex of the universal covering. Recall that this is a free $\mathbb{Z}\pi$-chain complex and the cellular structure on X determines a cellular $\mathbb{Z}\pi$-basis on it such that each basis element corresponds to a cell in X. This basis is not quite unique but its equivalence class depends only on the CW-structure of X (see Section 1.2). The product $\widetilde{X} \times \widetilde{X}$ equipped with the diagonal π-action is again a π-CW-complex. The diagonal map $D : \widetilde{X} \to \widetilde{X} \times \widetilde{X}$ sending $\widetilde{x}$ to $(\widetilde{x}, \widetilde{x})$ is π-equivariant but not cellular. By the equivariant cellular Approximation Theorem (see for instance [39, Theorem 2.1 on page 32]) there is up to cellular π-homotopy precisely one cellular π-map $\overline{D} : \widetilde{X} \to \widetilde{X} \times \widetilde{X}$ which is π-homotopic to D. It induces a $\mathbb{Z}\pi$-chain map unique up to $\mathbb{Z}\pi$-chain homotopy

$$C_*(\overline{D}) : C_*(\widetilde{X}) \to C_*(\widetilde{X} \times \widetilde{X}). \tag{3.2}$$

There is a natural isomorphism of based free $\mathbb{Z}\pi$-chain complexes

$$i_* : C_*(\widetilde{X}) \otimes_{\mathbb{Z}} C_*(\widetilde{X}) \xrightarrow{\cong} C_*(\widetilde{X} \times \widetilde{X}). \tag{3.3}$$

Denote by $\mathbb{Z}^w$ the $\mathbb{Z}\pi$-module whose underlying abelian group is $\mathbb{Z}$ and on which $g \in G$ acts by $w(g) \cdot \mathrm{id}$. Given two projective $\mathbb{Z}\pi$-chain complexes C_* and D_* we obtain a natural $\mathbb{Z}$-chain map unique up to $\mathbb{Z}$-chain homotopy

$$s : \mathbb{Z}^w \otimes_{\mathbb{Z}\pi} (C_* \otimes_{\mathbb{Z}} D_*) \to \hom_{\mathbb{Z}\pi}(C^{-*}, D_*) \tag{3.4}$$

by sending $1 \otimes x \otimes y \in \mathbb{Z} \otimes C_p \otimes D_q$ to

$$s(1 \otimes x \otimes y) : \hom_{\mathbb{Z}\pi}(C_p, \mathbb{Z}\pi) \to D_q, \quad (\phi : C_p \to \mathbb{Z}\pi) \mapsto \overline{\phi(x)} \cdot y.$$

The composite of the chain map (3.4) for $C_* = D_* = C_*(\widetilde{X})$, the inverse of the chain map (3.3) and the chain map (3.2) yields a $\mathbb{Z}$-chain map

$$\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) \to \hom_{\mathbb{Z}\pi}(C^{-*}(\widetilde{X}), C_*(\widetilde{X})).$$

Notice that the n-homology of $\hom_{\mathbb{Z}\pi}(C^{-*}(\widetilde{X}), C_*(\widetilde{X}))$ is the set of $\mathbb{Z}\pi$-chain homotopy classes $[C^{n-*}(\widetilde{X}), C_*(\widetilde{X})]_{\mathbb{Z}\pi}$ of $\mathbb{Z}\pi$-chain maps from $C^{n-*}(\widetilde{X})$ to $C_*(\widetilde{X})$. Define $H_n(X; \mathbb{Z}^w) := H_n(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}))$. Taking the n-th homology group yields a well-defined $\mathbb{Z}$-homomorphism

$$\cap : H_n(X; \mathbb{Z}^w) \to [C^{n-*}(\widetilde{X}), C_*(\widetilde{X})]_{\mathbb{Z}\pi} \tag{3.5}$$

which sends a class $x \in H_n(X; \mathbb{Z}^w) = H_n(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}))$ to the $\mathbb{Z}\pi$-chain homotopy class of a $\mathbb{Z}\pi$-chain map denoted by $? \cap x : C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$.

Definition 3.6 *A* connected finite n-dimensional Poincaré complex *is a connected finite CW-complex of dimension n together with a group homomorphism* $w = w_1(X) : \pi_1(X) \to \{\pm 1\}$ *called* orientation homomorphism *and an element* $[X] \in H_n(X; \mathbb{Z}^w)$ *called* fundamental class *such that the* $\mathbb{Z}\pi$*-chain map* $? \cap [X] : C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$ *is a* $\mathbb{Z}\pi$*-chain homotopy equivalence. We will call it the* Poincaré $\mathbb{Z}\pi$-chain homotopy equivalence.

The orientation homomorphism $w : \pi_1(X) \to \{\pm 1\}$ is uniquely determined by the homotopy type of X by the following argument. Denote by $C^{n-*}(\widetilde{X})_{\mathrm{untw}}$ the $\mathbb{Z}\pi$-chain complex which is analogously defined as $C^{n-*}(\widetilde{X})$, but now with respect to the untwisted involution. Its n-th homology $H_n(C^{n-*}(\widetilde{X})_{\mathrm{untw}})$ depends only on the homotopy type of X. If X carries the structure of a Poincaré complex with respect to $w : \pi_1(X) \to \{\pm 1\}$, then the Poincaré $\mathbb{Z}\pi$-chain homotopy equivalence induces a $\mathbb{Z}\pi$-isomorphism $H_n(C^{n-*}(\widetilde{X})_{\mathrm{untw}}) \cong \mathbb{Z}^w$. Thus we rediscover w from $H_n(C^{n-*}(\widetilde{X})_{\mathrm{untw}})$. Obviously there are two possible choice for $[X]$, since it has to be a generator of the infinite cyclic group $H_n(X, \mathbb{Z}^w) \cong H^0(X; \mathbb{Z}) \cong \mathbb{Z}$. A choice of $[X]$ will be part of the Poincaré structure.

The connected finite n-dimensional Poincaré complex X is called *simple* if the Whitehead torsion (see (2.8)) of the $\mathbb{Z}\pi$-chain homotopy equivalence of finite based free $\mathbb{Z}\pi$-chain complexes $? \cap [X] : C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$ vanishes.

Theorem 3.7 *Let M be a connected closed manifold of dimension n. Then M carries the structure of a simple connected finite n-dimensional Poincaré complex.*

For a proof we refer for instance to [64, Theorem 2.1 on page 23]. We explain at least the idea and give some evidence for it.

Any closed manifold admits a smooth triangulation $h : K \to M$, i.e. a finite simplicial complex together with a homeomorphism $h : K \to M$ such that h restricted to a simplex is a smooth C^∞-embedding. Two such smooth triangulations have common subdivisions. In particular M is homeomorphic to a finite CW-complex. Fix such a triangulation K. Denote by K' its barycentric subdivision. The vertices of K' are the barycenters $\widehat{\sigma^r}$ of simplices σ^r in K. A p-simplex in K' is given by a sequence $\widehat{\sigma^{i_0}}\widehat{\sigma^{i_1}}\ldots\widehat{\sigma^{i_p}}$, where σ^{i_j} is a face of $\sigma^{i_{j+1}}$. Next we define the dual CW-complex K^* as follows. It is not a simplicial complex but shares the property of a simplicial complex that all attaching maps are embeddings. Each p-simplex σ in K determines a $(n-p)$-dimensional cell σ^* of K^* which is the union of all simplices in K' which begin with $\widehat{\sigma^p}$. So K has as many p-simplices as K^* has $(n-p)$-cells. The cap product with the fundamental cycle, which is given by the sum of the n-dimensional simplices, yields an isomorphism of $\mathbb{Z}\pi$-chain complexes $C^{n-*}(\widetilde{K^*}) \to C_*(\widetilde{K})$. It preserves the cellular $\mathbb{Z}\pi$-bases and in particular its Whitehead torsion is trivial. Since K' is a common subdivision of K and K^*, there are canonical $\mathbb{Z}\pi$-chain homotopy equivalences $C_*(\widetilde{K'}) \to C_*(\widetilde{K})$ and $C_*(\widetilde{K'}) \to C_*(\widetilde{K^*})$ which have trivial Whitehead torsion. Thus we can write the $\mathbb{Z}\pi$-chain map $?\cap[M] : C^{n-*}(\widetilde{K'}) \to C_*(\widetilde{K'})$ as a composite of three simple $\mathbb{Z}\pi$-chain homotopy equivalences. Hence it is a simple $\mathbb{Z}\pi$-chain homotopy equivalence.

Remark 3.8 Theorem 3.7 gives us the first obstruction for a topological space X to be homotopy equivalent to a connected closed n-dimensional manifold (see Problem 3.1). Namely, X must be homotopy equivalent to a connected finite simple n-dimensional Poincaré complex.

Remark 3.9 Suppose that X is a Poincaré complex with respect to the trivial orientation homomorphism. Definition 3.6 of Poincaré duality implies that Poincaré duality holds for any G-covering $\overline{X} \to X$ and yields Poincaré duality for all possible coefficient systems. In particular we get a $\mathbb{Z}$-chain homotopy equivalence

$$\mathbb{Z} \otimes_{\mathbb{Z}\pi} (? \cap [X]) : \mathbb{Z} \otimes_{\mathbb{Z}\pi} C^{n-*}(\widetilde{X}) = C^{n-*}(X) \ \to\ \mathbb{Z} \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) = C_*(X),$$

which induces for any commutative ring R an R-isomorphism on (co-)homology with R-coefficients

$$? \cap [X] : H^{n-*}(X;R) \xrightarrow{\cong} H_*(X;R).$$

Remark 3.10 From a Morse theoretic point of view Poincaré duality corresp-nds to the dual handlebody decomposition of a manifold which we have already described in (1.26).

Remark 3.11 From the analytic point of view Poincaré duality can be explained as follows. Let M be a connected closed oriented Riemannian manifold. Let $(\Omega^*(M), d^*)$ be the de Rham complex of smooth p-forms on M. The p-th Laplacian is defined by $\Delta_p = (d^p)^* d^p + d^{p-1}(d^{p-1})^* : \Omega^p(M) \to \Omega^p(M)$, where $(d^p)^*$ is the adjoint of the p-differential d^p. The kernel of the p-th Laplacian is the space $\mathcal{H}^p(M)$ of harmonic p-forms on M. The Hodge-de Rham Theorem yields an isomorphism

$$A^p : \mathcal{H}^p(M) \xrightarrow{\cong} H^p(M;\mathbb{R}) \tag{3.12}$$

from the space of harmonic p-forms to the singular cohomology of M with coefficients in $\mathbb{R}$. Let $[M]_{\mathbb{R}} \in H^n(M;\mathbb{R})$ be the fundamental cohomology class with $\mathbb{R}$-coefficients which is characterized by the property $\langle [M]_{\mathbb{R}}, i_*([M])\rangle = 1$ for $\langle\ ,\ \rangle$ the Kronecker product and $i_* : H_n(M;\mathbb{Z}) \to H_n(M;\mathbb{R})$ the change of rings homomorphism. Then A^n sends the volume form $d\mathrm{vol}$ to the class $\frac{1}{\mathrm{vol}(M)} \cdot [M]_{\mathbb{R}}$. The Hodge-star operator $* : \Omega^{n-p}(M) \to \Omega^p(M)$ induces isomorphisms

$$* : \mathcal{H}^{n-p}(M) \xrightarrow{\cong} \mathcal{H}^p(M). \tag{3.13}$$

We obtain from (3.12) and (3.13) isomorphisms

$$H^{n-p}(M;\mathbb{R}) \xrightarrow{\cong} H^p(M;\mathbb{R}).$$

This is the analytic version of Poincaré duality. It is equivalent to the claim that the bilinear pairing

$$P^p : \mathcal{H}^p(M) \otimes_{\mathbb{R}} \mathcal{H}^{n-p}(M) \to \mathbb{R} \qquad (\omega, \eta) \mapsto \int_M \omega \wedge \eta. \tag{3.14}$$

is non-degenerate. Recall that for any commutative ring R with unit we have the intersection pairing

$$I^p : H^p(M;R) \otimes_R H^{n-p}(M;R) \to R, \quad (x,y) \mapsto \langle x \cup y, i_*[M]\rangle, \tag{3.15}$$

where i_* is the change of coefficients map associated to $\mathbb{Z} \to R$. The fact that the intersection pairing is non-degenerate is for R a field equivalent to the bijectivity of the homomorphism $? \cap [X] : H^{n-*}(X;R) \to H_*(X;R)$ appearing in Remark 3.9. If we take $R = \mathbb{R}$, then the pairings (3.14) and (3.15) agree under the Hodge-de Rham isomorphism (3.12).

One basic invariant of a finite CW-complex X is its *Euler characteristic* $\chi(X)$. It is defined by $\chi(X) := \sum_p (-1)^p \cdot n_p$, where n_p is the number of p-cells. Equivalently it can be defined in terms of its homology by $\chi(X) =$

$\sum_p (-1)^p \cdot \dim_{\mathbb{Q}} H_p(X;\mathbb{Q})$. A basic invariant for Poincaré complexes is defined next.

Consider a symmetric bilinear non-degenerate pairing $s : V \otimes_{\mathbb{R}} V \to \mathbb{R}$ for a finite-dimensional real vector space V. Choose a basis for V and let A be the square matrix describing s with respect to this basis. Since s is symmetric and non-degenerate, A is symmetric and invertible. Hence A can be diagonalized by an orthogonal matrix U to a diagonal matrix whose entries on the diagonal are non-zero real numbers. Let n_+ be the number of positive entries and n_- be the number of negative entries on the diagonal. These two numbers are independent of the choice of the basis and the orthogonal matrix U. Namely n_+ is the maximum of the dimensions of subvector spaces $W \subset V$ on which s is positive-definite, and analogous for n_-. Obviously $n_+ + n_- = \dim_{\mathbb{R}}(V)$. Define the *signature* of s to be the integer $n_+ - n_-$.

Definition 3.16 *Let X be a finite connected Poincaré complex. Suppose that X is orientable, i.e. $w_1(X) : \pi_1(X) \to \{\pm 1\}$ is trivial and that its dimension $n = 4k$ is divisible by four. Define its* intersection pairing *to be the symmetric bilinear non-degenerate pairing*

$$I : H^{2k}(X;\mathbb{R}) \otimes_{\mathbb{R}} H^{2k}(X;\mathbb{R}) \xrightarrow{\cup} H^n(X;\mathbb{R}) \xrightarrow{\langle -,[X]_{\mathbb{R}}\rangle} \mathbb{R}.$$

Define the signature $\operatorname{sign}(X)$ *to be the signature of the intersection pairing.*

Remark 3.17 The notion of a Poincaré complex can be extended to pairs as follows. Let X be a connected finite n-dimensional CW-complex with fundamental group π together with a subcomplex $A \subset X$ of dimension $(n-1)$. Denote by $\widetilde{A} \subset \widetilde{X}$ the preimage of A under the universal covering $\widetilde{X} \to X$. We call (X, A) a finite n-dimensional Poincaré pair with respect to the orientation homomorphism $w : \pi_1(X) \to \{\pm 1\}$ if there is a fundamental class $[X, A] \in H_n(X, A; \mathbb{Z}^w)$ such that the $\mathbb{Z}\pi$-chain maps $? \cap [X, A] : C^{n-*}(\widetilde{X}, \widetilde{A}) \to C_*(\widetilde{X})$ and $? \cap [X, A] : C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X}, \widetilde{A})$ are $\mathbb{Z}\pi$-chain equivalences. We call (X, A) *simple* if the Whitehead torsion of these $\mathbb{Z}\pi$-chain maps vanish.

Each component C of the space A inherits from (X, A) the structure of a finite $(n-1)$-dimensional Poincaré complex. Its orientation homomorphisms $w_1(C)$ is obtained from $w_1(X)$ by restriction with the homomorphism $\pi_1(C) \to \pi_1(X)$ induced by the inclusion. The various fundamental classes $[C]$ of the components $C \in \pi_0(A)$ are given by the image of the fundamental class $[X, A]$ under the boundary map $H_n(X, A, \mathbb{Z}^w) \to H_{n-1}(A, \mathbb{Z}^{w_1(A)}) \cong \oplus_{C \in \pi_0(A)} H_{n-1}(C, \mathbb{Z}^{w_1(C)})$. If M is a compact connected manifold of dimension n with boundary ∂M, then $(M, \partial M)$ is a simple finite n-dimensional Poincaré pair.

The signature will be the first and the most elementary surgery obstruction which we will encounter. This is due to the following lemma.

Lemma 3.18 *Let (X, A) be a $(4k+1)$-dimensional oriented finite Poincaré pair. Then*

$$\sum_{C \in \pi_0(A)} \operatorname{sign}(C) = 0.$$

For its proof and later purposes we need the following lemma.

Lemma 3.19 *Let $s : V \otimes_{\mathbb{R}} V \to \mathbb{R}$ be a symmetric bilinear non-degenerate pairing for a finite-dimensional real vector space V. Then* $\operatorname{sign}(s) = 0$ *if and only if there exists a subvector space $L \subset V$ such that* $\dim_{\mathbb{R}}(V) = 2 \cdot \dim_{\mathbb{R}}(L)$ *and $s(a, b) = 0$ for $a, b \in L$.*

Proof : Suppose such an $L \subset V$ exists. Choose subvector spaces V_+ and V_- of V such that s is positive-definite on V_+ and negative-definite on V_- and V_+ and V_- are maximal with respect to this property. Then $V_+ \cap V_- = \{0\}$ and $V = V_+ \oplus V_-$. Obviously $V_+ \cap L = V_- \cap L = \{0\}$. From

$$\dim_{\mathbb{R}}(V_\pm) + \dim_{\mathbb{R}}(L) - \dim_{\mathbb{R}}(V_\pm \cap L) \leq \dim_{\mathbb{R}}(V)$$

we conclude $\dim_{\mathbb{R}}(V_\pm) \leq \dim_{\mathbb{R}}(V) - \dim_{\mathbb{R}}(L)$. Since $2 \cdot \dim_{\mathbb{R}}(L) = \dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(V_+) + \dim_{\mathbb{R}}(V_-)$ holds, we get $\dim_{\mathbb{R}}(V_\pm) = \dim_{\mathbb{R}}(L)$. This implies

$$\operatorname{sign}(s) \;=\; \dim_{\mathbb{R}}(V_+) - \dim_{\mathbb{R}}(V_-) = \dim_{\mathbb{R}}(L) - \dim_{\mathbb{R}}(L) = 0.$$

Now suppose that $\operatorname{sign}(s) = 0$. Then one can find a orthonormal (with respect to s) basis $\{b_1, b_2, \ldots, b_{n_+}, c_1, c_2, \ldots, c_{n_-}\}$ such that $s(b_i, b_i) = 1$ and $s(c_j, c_j) = -1$ holds. Since $0 = \operatorname{sign}(s) = n_+ - n_-$, we can define L to be the subvector space generated by $\{b_i + c_i \mid i = 1, 2, \ldots n_+\}$. One easily checks that L has the desired properties. ■

Now we can give the proof of Lemma 3.18.

Proof : Let $i : A \to M$ be the inclusion. Then the following diagram commutes for $n = 4k$.

$$\begin{array}{ccccc}
H^{2k}(X;\mathbb{R}) & \xrightarrow{H^{2k}(i)} & H^{2k}(A;\mathbb{R}) & \xrightarrow{\delta^{2k}} & H^{2k+1}(X,A;\mathbb{R}) \\
{\scriptstyle ?\cap[X,A]}\downarrow\cong & & {\scriptstyle ?\cap\partial_{4k+1}([X,A])}\downarrow\cong & & {\scriptstyle ?\cap[X,A]}\downarrow\cong \\
H_{2k+1}(X,A;\mathbb{R}) & \xrightarrow{\partial_{2k+1}} & H_{2k}(A;\mathbb{R}) & \xrightarrow{H_{2k}(i)} & H_{2k}(X;\mathbb{R})
\end{array}$$

This implies $\dim_{\mathbb{R}}(\ker(H_{2k}(i))) = \dim_{\mathbb{R}}(\operatorname{im}(H^{2k}(i)))$. Since $\mathbb{R}$ is a field, we get from the Kronecker pairing an isomorphism $H^{2k}(X;\mathbb{R}) \cong (H_{2k}(X;\mathbb{R}))^*$ and analogously for A. Under these identifications $H^{2k}(i)$ becomes $(H_{2k}(i))^*$. Hence $\dim_{\mathbb{R}}(\operatorname{im}(H_{2k}(i))) = \dim_{\mathbb{R}}(\operatorname{im}(H^{2k}(i)))$. From

$$\dim_{\mathbb{R}}(H_{2k}(A;\mathbb{R})) = \dim_{\mathbb{R}}(\ker(H_{2k}(i))) + \dim_{\mathbb{R}}(\operatorname{im}(H_{2k}(i)))$$

we conclude

$$\dim_{\mathbb{R}}(H^{2k}(A;\mathbb{R})) = 2 \cdot \dim_{\mathbb{R}}(\operatorname{im}(H^{2k}(i))).$$

We have for $x, y \in H^{2k}(M;\mathbb{R})$

$$\begin{aligned}\langle H^{2k}(i)(x) \cup H^{2k}(i)(y), \partial_{4k+1}([X,A])\rangle &= \langle H^{2k}(i)(x \cup y), \partial_{4k+1}([X,A])\rangle \\ &= \langle x \cup y, H_{2k}(i) \circ \partial_{4k+1}([X,A])\rangle \\ &= \langle x \cup y, 0\rangle \ = \ 0.\end{aligned}$$

If we apply Lemma 3.19 to the non-degenerate symmetric bilinear pairing

$$H^{2k}(A;\mathbb{R}) \otimes_{\mathbb{R}} H^{2k}(A;\mathbb{R}) \xrightarrow{\cup} H^{4k}(A;\mathbb{R}) \xrightarrow{\langle ?, \partial_{4k+1}([X,A])\rangle} H^0(A;\mathbb{R}) \cong \oplus_{\pi_0(A)}\mathbb{R} \xrightarrow{\Sigma} \mathbb{R}$$

with L the image of $H^{2k}(i) : H^{2k}(X;\mathbb{R}) \to H^{2k}(A;\mathbb{R})$, we see that the signature of this pairing is zero. One easily checks that its signature is the sum of the signatures of the components of A. ■

The signature has the following further properties.

Lemma 3.20 *1. Let M and N be compact oriented manifolds and $f : \partial M \to \partial N$ be an orientation reversing diffemorphism. Then $M \cup_f N$ inherits an orientation from M and N and*

$$\operatorname{sign}(M \cup_f N) \ = \ \operatorname{sign}(M) + \operatorname{sign}(N);$$

2. Let $p : \overline{M} \to M$ be a finite covering with d sheets of closed oriented manifolds. Then

$$\operatorname{sign}(\overline{M}) \ = \ d \cdot \operatorname{sign}(N).$$

Proof : (1) is due to Novikov. For a proof see for instance [3, Proposition 7.1 on page 588].

(2) For a smooth manifold M this follows from Atiyah's L^2-index theorem [2, (1.1)]. Topological closed manifolds are treated in [57, Theorem 8]. ■

Example 3.21 Wall has constructed a finite connected Poincaré space X together with a finite covering with d sheets $\overline{X} \to X$ such that the signature does not satisfy $\operatorname{sign}(\overline{X}) = d \cdot \operatorname{sign}(X)$ (see [53, Example 22.28], [63, Corollary 5.4.1])). Hence X cannot be homotopy equivalent to a closed manifold by Lemma 3.20.

3.2 The Spivak normal fibration

In this section we introduce the Spivak normal fibration which is the homotopy theoretic analogue of the normal fiber bundle of an embedding of a manifold into some Euclidean space. In order to motivate the construction we briefly recall the Pontrjagin-Thom construction which we will need later anyway.

3.2.1 Pontrjagin-Thom construction

Let (M, i) be an embedding $i : M^n \to \mathbb{R}^{n+k}$ of a closed n-dimensional manifold M into $\mathbb{R}^{n+k}$. Notice that $T\mathbb{R}^{n+k}$ comes with an explicit trivialization $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \xrightarrow{\cong} T\mathbb{R}^{n+k}$ and the standard Euclidean inner product induces a Riemannian metric on $T\mathbb{R}^{n+k}$. Denote by $\nu(M) = \nu(M, i)$ the *normal bundle* of i which is the orthogonal complement of TM in $i^*T\mathbb{R}^{n+k}$. For a vector bundle $\xi : E \to X$ with Riemannian metric define its *disk bundle* $p_{DE} : DE \to X$ by $DE = \{v \in E \mid ||v|| \leq 1\}$ and its *sphere bundle* $p_{SE} : SE \to X$ by $SE = \{v \in E \mid ||v|| = 1\}$, where p_{DE} and p_{SE} are the restrictions of p. Its *Thom space* $\mathrm{Th}(\xi)$ is defined by DE/SE. It has a preferred base point $\infty = SE/SE$. The Thom space can be defined without of choice of a Riemannian metric as follows. Put $\mathrm{Th}(\xi) = E \cup \{\infty\}$ for some extra point ∞. Equip $\mathrm{Th}(\xi)$ with the topology for which $E \subset \mathrm{Th}(E)$ is an open subset and a basis of open neighborhoods for ∞ is given by the complements of closed subsets $A \subset E$ for which $A \cap E_x$ is compact for each fiber E_x. If X is compact, E is locally compact and $\mathrm{Th}(\xi)$ is the one-point-compactification of E. The advantage of this definition is that any bundle map $(\overline{f}, f) : \xi_0 \to \xi_1$ of vector bundles $\xi_0 : E_0 \to X_0$ and $\xi_1 : E_1 \to X_1$ induces canonically a map $\mathrm{Th}(\overline{f}) : \mathrm{Th}(\xi_0) \to \mathrm{Th}(\xi_1)$. Notice that we require that $\overline{f}$ induces a bijective map on each fiber. Denote by $\underline{\mathbb{R}^k}$ the trivial vector bundle with fiber $\mathbb{R}^k$. We mention that there are homeomorphisms

$$\mathrm{Th}(\xi \oplus \eta) \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta); \tag{3.22}$$

$$\mathrm{Th}(\xi \oplus \underline{\mathbb{R}^k}) \cong \Sigma^k \,\mathrm{Th}(\xi), \tag{3.23}$$

where $\wedge$ stands for the *smash product* of pointed spaces

$$X \wedge Y := X \times Y / X \times \{y\} \cup \{x\} \times Y \tag{3.24}$$

and $\Sigma^k Y = S^k \wedge Y$ is the *(reduced) suspension.* Let $(N(M), \partial N(M))$ be a tubular neighborhood of M. Recall that there is a diffeomorphism $u : (D\nu(M), S\nu(M)) \to (N(M), \partial N(M))$ which is up to isotopy relative M uniquely determined by the property that its restriction to M is i and its differential at M is $\epsilon \cdot \mathrm{id}$ for small $\epsilon > 0$ under the canonical identification $T(D\nu(M))|_M = TM \oplus \nu(M) = i^*T\mathbb{R}^{n+k}$. The *collapse map*

$$c : S^{n+k} = \mathbb{R}^{n+k} \coprod \{\infty\} \to \mathrm{Th}(\nu(M)) \tag{3.25}$$

is the pointed map which is given by the diffeomorphism u^{-1} on the interior of $N(M)$ and sends the complement of the interior of $N(M)$ to the preferred base point ∞. The homology group $H_{n+k}(\mathrm{Th}(TM)) \cong H_{n+k}(N(M), \partial N(M))$ is infinite cyclic, since $N(M)$ is a compact orientable (n+k)-dimensional manifold with boundary $\partial N(M)$. The Hurewicz homomorphism $h : \pi_{n+k}(Th(TM)) \to H_{n+k}(\mathrm{Th}(TM))$ sends the class $[c]$ of c to a generator. This follows from the fact that any point in the interior of $N(M)$ is a regular value of c and has precisely one point in his preimage.

Before we deal with the Spivak normal fibration, we apply this construction to bordism. Fix a space X together with a k-dimensional vector bundle $\xi : E \to$

X. Let us recall the definition of the bordism set $\Omega_n(X,\xi)$. An element in it is represented by a quadruple $(M,i,f,\overline{f})$ which consists of a closed n-dimensional manifold M, an embedding $i : M \to \mathbb{R}^{n+k}$, a map $f : M \to X$ and a bundle map $\overline{f} : \nu(M) \to \xi$ covering f. We briefly explain what a bordism $(W,I,F,\overline{F})$ from one such quadruple $(M_0,i_0,f_0,\overline{f_0})$ to another quadruple $(M_1,i_1,f_1,\overline{f_1})$ is. We need a compact $(n+1)$-dimensional manifold W together with a map $F : W \to X \times [0,1]$. Its boundary ∂W is written as a disjoint sum $\partial_0 W \coprod \partial_1 W$ such that F maps $\partial_0 W$ to $X \times \{0\}$ and $\partial_1 W$ to $X \times \{1\}$. There is an embedding $I : W \to \mathbb{R}^{n+k} \times [0,1]$ such that $I^{-1}(\mathbb{R}^{n+k} \times \{j\}) = \partial_j W$ holds for $j = 0,1$ and W meets $\mathbb{R}^{n+k} \times \{j\}$ for $j = 0,1$ transversally. We require a bundle map $(\overline{F},F) : \nu(W) \to \xi \times [0,1]$. Moreover for $j = 0,1$ there is a diffeomorphism $U_j : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k} \times \{j\}$ which maps M_j to $\partial_j W$. It satisfies $F \circ U_j|_{M_j \times \{j\}} = f_j$. Notice that U_j induces a bundle map $\nu(U_j) : \nu(M_j) \to \nu(W)$ covering $U_j|_{M_j}$. The composition of $\overline{F}$ with $\nu(U_j)$ is required to be $\overline{f_j}$.

Theorem 3.26 (Pontrjagin-Thom construction) *Let $\xi : E \to X$ be a k-dimensional vector bundle over a CW-complex X. Then the map*

$$P_n(\xi) : \Omega_n(X,\xi) \xrightarrow{\cong} \pi_{n+k}(\mathrm{Th}(\xi)),$$

which sends the class of $(M,i,f,\overline{f})$ to the class of the composite $S^{n+k} \xrightarrow{c} \mathrm{Th}(\nu(M)) \xrightarrow{\mathrm{Th}(\overline{f})} \mathrm{Th}(\xi)$ is a well-defined bijection and natural in ξ.

Proof : The details can be found in [7, Satz 3.1 on page 28, Satz 4.9 on page 35,]. The basic idea becomes clear after we have explained the construction of the inverse for a finite CW-complex X. Consider a pointed map $(S^{n+k},\infty) \to (\mathrm{Th}(\xi),\infty)$. We can change f up to homotopy relative $\{\infty\}$ such that f becomes transverse to X. Notice that transversality makes sense although X is not a manifold, one needs only the fact that X is the zero-section in a vector bundle. Put $M = f^{-1}(X)$. The transversality construction yields a bundle map $\overline{f}$: $\nu(M) \to \xi$ covering $f|_M$. Let $i : M \to \mathbb{R}^{n+k} = S^{n+k} - \{\infty\}$ be the inclusion. Then the inverse of $P_n(\xi)$ sends the class of f to the class of $(M,i,f|_M,\overline{f})$. ■

Let $\Omega_n(X)$ be the bordism group of pairs (M,f) of oriented closed n-dimensional manifolds M together with reference maps $f : M \to X$. Let $\xi_k : E_k \to BSO(k)$ be the universal oriented k-dimensional vector bundle. In the sequel we will denote for a finite-dimensional vector space V by $\underline{V}$ the trivial bundle with fiber V. Let $\overline{j_k} : \xi_k \oplus \underline{\mathbb{R}} \to \xi_{k+1}$ be a bundle map covering a map $j_k : BSO(k) \to BSO(k+1)$. Up to homotopy of bundle maps this map is unique. Denote by γ_k the bundle $X \times E_k \to X \times BSO(k)$ and by $(\overline{i_k}, i_k) : \gamma_k \oplus \underline{\mathbb{R}} \to \gamma_{k+1}$ the bundle map $\mathrm{id}_X \times (\overline{j_k}, j_k)$. The bundle map $(\overline{i_k}, i_k)$ is unique up to homotopy of bundle maps and hence induces a well-defined map $\Omega_n(\overline{i_k}) : \Omega_n(\gamma_k) \to \Omega_n(\gamma_{k+1})$, which sends the class of $(M,i,f,\overline{f})$ to the class of the quadruple which comes from the embedding $j : M \xrightarrow{i} \mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1}$ and the canonical isomorphism $\nu(i) \oplus \underline{\mathbb{R}} = \nu(j)$. Consider the homomorphism

$$V_k : \Omega_n(\gamma_k) \to \Omega_n(X)$$

which sends the class of $(M, i, f, \overline{f})$ to $(M, \mathrm{pr}_X \circ f)$, where we equip M with the orientation determined by $\overline{f}$. Let $\mathrm{colim}_{k \to \infty} \Omega_n(\gamma_k)$ be the colimit of the directed system indexed by $k \geq 0$

$$\ldots \xrightarrow{\Omega_n(\overline{i_{k-1}})} \Omega_n(\gamma_k) \xrightarrow{\Omega_n(\overline{i_k})} \Omega_n(\gamma_{k+1}) \xrightarrow{\Omega_n(\overline{i_{k+1}})} \ldots$$

Since $V_{k+1} \circ i_k = V_k$ holds for all $k \geq 0$, we obtain a map

$$V : \mathrm{colim}_{k \to \infty} \Omega_n(\gamma_k) \xrightarrow{\cong} \Omega_n(X). \tag{3.27}$$

This map is bijective because of the classifying property of γ_k and the facts that for $k > n+1$ any closed manifold M of dimension n can be embedded into $\mathbb{R}^{n+k}$ and two such embeddings are isotopic.

We see a sequences of spaces $\mathrm{Th}(\gamma_k)$ together with maps

$$\mathrm{Th}(\overline{i_k}) : \Sigma\,\mathrm{Th}(\gamma_k) = \mathrm{Th}(\gamma_k \oplus \underline{\mathbb{R}}) \to \mathrm{Th}(\gamma_{k+1}).$$

They induce homomorphisms

$$s_k : \pi_{n+k}(\mathrm{Th}(\gamma_k)) \to \pi_{n+k+1}(\Sigma\,\mathrm{Th}(\gamma_k)) \xrightarrow{\pi_{n+k}(\mathrm{Th}(i_k))} \pi_{n+k+1}(\mathrm{Th}(\gamma_{k+1})),$$

where the first map is the suspension homomorphism. Define $\mathrm{colim}_{k \to \infty} \pi_{n+k}(\mathrm{Th}(\gamma_k))$ to be the colimit of the directed system

$$\ldots \xrightarrow{s_{k-1}} \pi_{n+k}(\mathrm{Th}(\gamma_k)) \xrightarrow{s_k} \pi_{n+k+1}(\mathrm{Th}(\gamma_{k+1})) \xrightarrow{s_{k+1}} \ldots.$$

We get from Theorem 3.26 a bijection

$$P : \mathrm{colim}_{k \to \infty} \Omega_n(\gamma_k) \xrightarrow{\cong} \mathrm{colim}_{k \to \infty} \pi_{n+k}(\mathrm{Th}(\gamma_k)).$$

This implies

Theorem 3.28 (Pontrjagin Thom construction and oriented bordism) *There is an isomorphism of abelian groups natural in X*

$$P : \Omega_n(X) \xrightarrow{\cong} \mathrm{colim}_{k \to \infty} \pi_{n+k}(\mathrm{Th}(\gamma_k)).$$

Remark 3.29 Notice that this is the beginning of the theory of spectra and stable homotopy theory. A *spectrum* $\mathbf{E}$ consists of a sequence of spaces $(E_k)_{k \in \mathbb{Z}}$ together with so called structure maps $\sigma_k : \Sigma E_k \to E_{k+1}$. The n-th *stable homotopy group* $\pi_n(\mathbf{E})$ is the defined as the colimit $\mathrm{colim}_{k \to \infty} \pi_{n+k}(E_k)$ with respect to the directed system given by the composites $\pi_{n+k}(E_k) \to \pi_{n+k+1}(\Sigma E_k) \xrightarrow{\pi_{n+k}(\sigma_k)} \pi_{n+k+1}(E_{k+1})$.

Theorem 3.28 is a kind of mile stone in homotopy theory since it is the prototype of a result, where the computation of geometrically defined objects are translated into a computation of (stable) homotopy groups. It applies to all other kind of bordism groups, where one puts additional structures on the manifolds, for instance a Spin-structre. The bijection is always of the same type, but the sequence of bundles ξ_k depends on the additional structure. If we want to deal with the unoriented bordism ring we have to replace the bundle $\xi_k \to BSO(k)$ by the universal k-dimensional vector bundle over $BO(k)$.

3.2.2 Spherical fibrations

A *spherical* $(k-1)$*-fibration* $p : E \to X$ is a fibration, i.e. a map having the homotopy lifting property, whose typical fiber is homotopy equivalent to S^{k-1}. Define its associated disk fibration $Dp : DE \to X$ by $DE = \text{cyl}(p)$, where $\text{cyl}(p)$ is the mapping cylinder of p and Dp the obvious map. Define its *Thom space* $\text{Th}(p)$ to be the pointed space $\text{cone}(p)$, where $\text{cone}(p)$ is the mapping cone of p with its canonical base point. Notice that $\text{Th}(p) = DE/E$.

If $\xi : E \to X$ is a k-dimensional vector bundle with Riemannian metric, its sphere bundle $p : SE \to X$ is an example of a $(k-1)$-spherical fibration. The role of the disk bundle $D\xi$ is now played by $DE = \text{cyl}(p)$. Notice that the canonical inclusion of X in $\text{cyl}(p)$ is a homotopy equivalence analogous to the fact that the inclusion of the zero-section of ξ into E is a homotopy equivalence. The canonical inclusion of E into $\text{cyl}(p)$ corresponds to the inclusion of $SE \subset DE$. Hence it is clear that $\text{Th}(p) = DE/E = \text{cone}(p)$ for a $(k-1)$-spherical fibration corresponds to $\text{Th}(\xi) = DE/SE$ for a k-dimensional vector bundle $\xi : E \to X$.

If one has two vector bundles $\xi_0 : E_0 \to X$ and $\xi_1 : E_1 \to X$, one can form the Whitney sum $\xi_0 \oplus \xi_1$. Notice that $S(E_0 \oplus E_1)_x$ is homeomorphic to the join $S(E_0)_x * S(E_1)_x$. This allows us to rediscover $S(\xi_1 \oplus \xi_2)$ as a spherical fibration from $S\xi_0$ and $S\xi_1$ by the fiberwise join construction. Namely, given a $(k-1)$-spherical fibration $p_0 : E_0 \to X$ and an $(l-1)$-spherical fibration $p_1 : E_1 \to X$, define the $(k+l-1)$-spherical fibration $p_0 * p_1 : E_0 * E_1 \to X$ called *fiberwise join* as follows. The total space $E_0 * E_1$ is the quotient of the space $\{(e_0, e_1, t) \in E_0 \times E_1 \times [0,1] \mid p_0(e_0) = p_1(e_1)\}$ under the equivalence relation generated by $(e_0, e_1, 1) \sim (e_0, e_1', 1)$ and $(e_0, e_1, 0) \sim (e_0', e_1, 0)$. The projection $p_0 * p_1$ sends the class of (e_0, e_1, t) to $p_0(e_0) = p_1(e_1)$. Given a spherical $(k-1)$-fibration $p : E \to X$, its l-fold suspension is the spherical $(k+l-1)$-fibration given by the fiberwise join of p and the trivial $(l-1)$-spherical fibration $\text{pr} : X \times S^{l-1} \to X$.

There are canonical homeomorphisms for spherical fibrations ξ and η

$$\text{Th}(\xi * \eta) \cong \text{Th}(\xi) \wedge \text{Th}(\eta); \tag{3.30}$$

$$\text{Th}(\xi * \underline{\mathbb{R}^{k-1}}) \cong \Sigma^{k-1}\, \text{Th}(\xi). \tag{3.31}$$

Given two fibrations $p_0 : E_0 \to X$ and $p_1 : E_1 \to X$, a fiber map $(\overline{f}, f) : p_0 \to p_1$ consists of maps $\overline{f} : E_0 \to E_1$ and $f : X_0 \to X_1$ satisfying $p_1 \circ \overline{f} = f \circ p_0$. There is an obvious notion of fiber homotopy $(\overline{h}, h)$. A fiber homotopy is called a strong fiber homotopy if $h_t : X_0 \to X_1$ is stationary for $t \in [0,1]$. Two fibrations p_0 and p_1 over the same base space are called *strongly fiber homotopy equivalent* if there are fiber maps $(\overline{f}, \text{id}) : p_0 \to p_1$ and $(\overline{g}, \text{id}) : p_1 \to p_0$ such that both composition are strongly fiber homotopy equivalent to the identity. Given a topological space F, let $G(F)$ be the monoid of selfhomotopy equivalences of F. One can associate to such a monoid a classifying space $BG(F)$ together with a fibration $p_F : EG(F) \to BG(F)$ with typical fiber F such that the pullback construction yields a bijection between the homotopy classes of maps from X to $BG(F)$ and the set of strong fiber homotopy classes of fibrations over X with typical fiber F [61]. We will abbreviate in the sequel $G(k) := G(S^{k-1})$.

Given a fibration $p : E \to X$, and $x \in X$, the fiber transport along loops at x defines a map of monoids $t_x : \pi_1(X, x) \to [p^{-1}(x), p^{-1}(x)]$. We call p *orientable* if t_x is trivial for any base point $x \in X$. In the case of $(k-1)$-spherical fibration this fiber transport is the same as a homomorphism $w : \pi_1(X) \to \{\pm 1\}$ called *the orientation homomorphism* since the degree defines a bijection $[S^{k-1}, S^{k-1}] \to \mathbb{Z}$ for $k \geq 2$. An *orientation* for an orientable spherical $(k-1)$-fibration p is the choice of fundamental class $[p^{-1}(x)] \in H_k(p^{-1}(x)) \cong \mathbb{Z}$ for any fiber $p^{-1}(x)$ for $x \in X$ such that for any path w in X the fiber transport along w yields a homotopy equivalence $p^{-1}(w(0)) \to p^{-1}(w(1))$ for which the induced map on H_{k-1} sends $[p^{-1}(w(0))]$ to $[p^{-1}(w(1))]$.

Theorem 3.32 (Thom isomorphism) *Let $p : E \to X$ be a $(k-1)$-spherical fibration over a connected CW-complex X. Then there exists a group homomorphism $w : \pi_1(X) \to \{\pm 1\}$ and a so called* Thom class $U_p \in H^k(DE, E; \mathbb{Z}^w)$ *such that the composites*

$$H_p(DE, E; \mathbb{Z}) \xrightarrow{U_p \cap ?} H_{p-k}(DE; \mathbb{Z}^w) \xrightarrow{H_{p-k}(p)} H_p(X; \mathbb{Z}^w); \qquad (3.33)$$

$$H_p(DE, E; \mathbb{Z}^w) \xrightarrow{U_p \cap ?} H_{p-k}(DE; \mathbb{Z}) \xrightarrow{H_{p-k}(p)} H_p(X; \mathbb{Z}); \qquad (3.34)$$

$$H^{p+k}(X; \mathbb{Z}) \xrightarrow{H^{p+k}(p)} H^p(DE; \mathbb{Z}) \xrightarrow{? \cup U_p} H^{p+k}(DE, SE; \mathbb{Z}^w); \qquad (3.35)$$

$$H^{p+k}(X; \mathbb{Z}^w) \xrightarrow{H^{p+k}(p)} H^p(DE; \mathbb{Z}^w) \xrightarrow{? \cup U_p} H^{p+k}(DE, SE; \mathbb{Z}); \qquad (3.36)$$

are bijective. These maps are called Thom isomorphisms.

There are precisely two possible choices for U_p because $H^k(DE, E; \mathbb{Z}^w) \cong H^0(X; \mathbb{Z}) \cong \mathbb{Z}$ is infinite cyclic. Moreover, w is uniquely determined by the spherical $(k-1)$-fibration. The proof is analogous to the one for Poincaré complexes.

If the spherical fibration p is orientable, then U_p is uniquely determined by the property that the composition $H^k(DE, E) \to H^k(p_{DE}^{-1}(x), p^{-1}(x); \mathbb{Z}) \xrightarrow{(\delta^{k-1})^{-1}} H^{k-1}(p^{-1}(x))$ sends U_p to $[p^{-1}(x)]$. Moreover, a choice of orientation for p is the same as a choice of Thom class.

3.2.3 The existence and uniqueness of the Spivak normal fibration

Definition 3.37 *A* Spivak normal fibration *for an n-dimensional connected finite Poincaré complex X is a $(k-1)$-spherical fibration $p = p_X : E \to X$ together with a pointed map $c = c_X : S^{n+k} \to \mathrm{Th}(p)$ such that X and p have the same orientation homomorphism $w : \pi_1(X) \to \{\pm 1\}$ and for some choice of Thom class $U_p \in H^k(DE, E; \mathbb{Z}^w)$ the fundamental class $[X] \in H_n(X; \mathbb{Z}^w)$ and the image $h(c) \in H_{n+k}(\mathrm{Th}(p)) \cong H_{n+k}(DE, E; \mathbb{Z})$ of $[c]$ under the Hurewicz homomorphism $h : \pi_{n+k}(\mathrm{Th}(p)) \to H_{n+k}(\mathrm{Th}(p), \mathbb{Z})$ are related by the formula*

$$[X] = H_n(p)(U_p \cap h(c)).$$

Remark 3.38 A closed manifold M of dimension n admits a Spivak normal fibration. Namely, choose an embedding $i : M \to \mathbb{R}^{n+k}$ and take p to be the sphere bundle $S\nu(i) \to M$ and c to be the collaps map defined in (3.25).

Theorem 3.39 (Existence and uniqueness of the Spivak normal fibration) *Let X be a connected finite n-dimensional Poincaré complex. Then:*

1. *For $k > n$ there exists a Spivak normal $(k-1)$-fibration for X;*

2. *Let $p_i : E_i \to X$ together with $c_i : S^{n+k_i} \to \mathrm{Th}(p_i)$ be a Spivak normal $(k_i - 1)$-fibration of X for $i = 0, 1$. Let k be an integer satisfying $k \geq k_0, k_1$. Then there is up to strong fiber homotopy precisely one strong fiber homotopy equivalence $\overline{f} : \Sigma^{k-k_0}p_0 \to \Sigma^{k-k_1}p_1$ for which $\pi_{n+k}(\mathrm{Th}(\overline{f})) : \pi_{n+k}(\mathrm{Th}(p_0)) \to \pi_{n+k}(\mathrm{Th}(p_0))$ sends $[c_0]$ to $[c_1]$.*

At least we want to give the idea of the proof of assertion (1) of Theorem 3.39 provided that the orientation homomorphism is trivial. For a detailed proof we refer for instance to [9, I.4], [59].

Consider a connected finite n-dimensional CW-complex X. One can always find an embedding $X \subset \mathbb{R}^{n+k}$ for $k > n$ together with a regular neighborhood $(N(X), \partial N(X))$. The regular neighboorhood is a compact manifold $N(K)$ with boundary $\partial N(K)$ such that $X \subset N(K)$ is a strong deformation retraction [55, Chapter 3]. This regular neighborhood corresponds in the case, where X is a closed manifold M, to a tubular neighborhood $(N(M), \partial N(M)) \cong (D\nu(M), S\nu(M))$. Let $[N(X), \partial N(X)] \in H_{n+k}(N(X), \partial N(X))$ be a fundamental class of the compact manifold $N(X)$ with boundary $\partial N(X)$ which corresponds to the orientation inherited from the standard orientation on $\mathbb{R}^{n+k}$. Let $i : X \to N(X)$ be the inclusion which is a homotopy equivalence. Then the following diagram commutes for any class $u \in H^k(N(K), \partial N(X))$

$$\begin{array}{ccc} H^{n-p}(X) & \xrightarrow{?\cap(H_n(i)^{-1}(u\cap[N(X),\partial N(X)]))} & H_p(X) \\ {\scriptstyle (u\cup ?)\circ H^{n-p}(i)^{-1}}\downarrow & & \downarrow{\scriptstyle H_p(i)} \\ H^{n+k-p}(N(X),\partial N(X)) & \xrightarrow[?\cap[N(X),\partial N(X)]]{} & H_p(N(X)) \end{array}$$

The lower horizontal arrow and the right vertical arrow are bijective by Poincaré duality and homotopy invariance. Hence the upper horizontal arrow is bijective if and only if the left vertical arrow is bijective. Notice that bijectivity of the upper horizontal arrow corresponds to Poincaré duality for X with $H_n(i)^{-1}(u \cap [N(X), \partial N(X)])$ as fundamental class and the bijectivity of the left vertical arrow corresponds to the Thom isomorphism with u as Thom class.

Suppose that X is a connected finite Poincaré complex with fundamental class $[X] \in H_n(X)$. Let $u \in H^k(N(X), \partial N(X))$ be the class uniquely determined by the property that $H_n(i)^{-1}(u \cap [N(X), \partial N(X)]) = [X]$. Then the map

$$H^{n-p}(X) \to H^{n+k-p}(N(X), \partial N(X), \mathbb{Z}) \qquad v \mapsto u \cup H^{n-p}(i)^{-1}(v)$$

is bijective. Our first approximation of the Spivak normal $(k-1)$-fibration is the composition $f : \partial N(X) \xrightarrow{j} N(X) \xrightarrow{i^{-1}} X$, where j is the inclusion and i^{-1} a homotopy inverse of i. Of course f is not a fibration but possess a candidate for a Thom class, namely u. By a general construction we can turn f into a fibration. More precisely, for any map $g : X \to Y$ there is a functorial construction which yields a fibration $p_g : E_g \to Y$ together with a homotopy equivalence $i_g : X \to E_g$ satisfying $p_g \circ i_g = g$. Namely, define $E_g = \{(x,w) \in X \times \mathrm{map}([0,1],Y) \mid f(x) = w(0)\}$, $p_g(x,w) = w(1)$ and $i_g(x) = (x, c_{f(x)})$ for $c_{f(x)}$ the constant path at $f(x)$ (see [65, Theorem 7.30 on page 42]). We apply this to $f : \partial N(X) \to X$ and obtain a fibration $p_f : E_f \to X$ together with a homotopy equivalence $i_f : \partial N(X) \to E_f$ satisfying $p_f \circ i_f = f$. Since $j : \partial N(X) \to N(X)$ is a cofibration and $i^{-1} : N(X) \to X$ a homotopy equivalence, we can find an extension of i_f to a homotopy equivalence of pairs $(I_f, i_f) : (N(X), \partial N(X)) \to (DE_f, E_f)$ for $DE_f = \mathrm{cyl}(p_f)$. This extension is unique up to homotopy relative $\partial N(X)$. Let $U_{p_f} \in H^k(DE_f, E_f)$ be the preimage of $u \in H^k(N(X), \partial N(X))$ under the isomorphism induces by (I_f, i_f). One easily checks that the map

$$H^p(DE_f, E) \xrightarrow{? \cup U_{p_f}} H^{p+k}(DE_f) \xrightarrow{H^{p+k}(p_f)^{-1}} H^{p+q}(X)$$

is bijective. So U_f looks like a Thom class. We have already seen that a spherical fibration has a Thom class and it turns out that this does characterize the homotopy fiber of a fibration. Hence $p_f : \partial N(X) \to X$ is a spherical $(k-1)$-fibration. The collaps map $c : S^{n+k} \to N(X)/\partial N(X)$ can be composed with the map $N(X)/\partial N(X) \to \mathrm{Th}(p_f)$ induced by (I_g, i_g) and yields a pointed map $c_{p_f} : S^{n+k} \to \mathrm{Th}(p_f)$. It has the desired property $[X] = H_n(p_f)(U_{p_f} \cap h(c_{p_f}))$. Hence p_f and U_{p_f} yield a normal Spivak fibration for X.

Remark 3.40 Recall that we want to address Problem 3.1, whether a space X is homotopy equivalent to a connected closed n-dimensional manifold. We have already seen in Remark 3.8 that we only have to consider connected finite n-dimensional Poincaré complexes X. From Theorem 3.39 and Remark 3.38 we get the following new necessary condition. Namely, we must be able to find for $k > n$ a k-dimensional vector bundle $\xi : E \to X$ such that the associated sphere bundle $SE \to X$ is strongly fiber homotopy equivalent to the Spivak normal $(k-1)$-fibration of X.

Lemma 3.41 *Let $p_i : E_i \to X_i$ be a spherical $(k-1)$-fibration over a connected finite n-dimensional Poincaré complexe X_i for $i = 0, 1$. Let $(\overline{f}, f) : p_0 \to p_1$ be a fiber map which is fiberwise a homotopy equivalence. Then we get for the orientation homomorphisms $w(p_1) \circ \pi_1(f) = w(p_0)$. Consider a pointed map $c_0 : S^{n+k} \to \mathrm{Th}(p_0)$. Let $c_1 : S^{n+k} \to \mathrm{Th}(p_0)$ be the composition $\mathrm{Th}(\overline{f}) \circ c_0$. Then*

1. *Suppose that the degree of f is ± -1. Then (p_0, c_0) is the Spivak normal fibration of X_0 if and only if (p_1, c_1) is the Spivak normal fibration for X_1;*

2. *Suppose that* (p_i, c_i) *is the Spivak normal* $(k-1)$*-fibration of* X_i *for* $i = 0, 1$. *Then the degree of* f *is* ± 1.

Proof : The claim about the orientation homomorphisms follows from the fact that the fiber transport of p_0 and p_1 are compatible with $\pi_1(f)$. We can choose the Thom classes of p_0 and p_1 such that the map $H^k(\mathrm{Th}(\overline{f}))$: $H^{n+k}(DE_1, E_1; \mathbb{Z}^{w(p_1)}) \to H^k(DE_0, E_0; \mathbb{Z}^{w(p_0)})$ sends U_{p_1} to U_{p_0}. Thus we get

$$H_n(f)\,(H_n(p_0)\,(U_{p_0} \cap h(c_0))) \;=\; H_n(p_1)\,(U_{p_1} \cap h(c_1))\,.$$

Now the claim follows from the definitions. ■

3.3 Normal maps

Motivated by Remark 3.40 we define

Definition 3.42 *Let* X *be a connected finite* n*-dimensional Poincaré complex. A* normal k-invariant (ξ, c) *consists of a* k*-dimensional vector bundle* $\xi : E \to X$ *together with an element* $c \in \pi_{n+k}(\mathrm{Th}(\xi))$ *such that for some choice of Thom class* $U_p \in H^k(DE, SE; w^{\mathbb{Z}})$ *the equation* $[X] = H_n(p)(U_p \cap h(c))$ *holds. We call a normal* k*-invariant* (ξ_0, c_0) *and a normal* k*-invariant* (ξ_1, c_1) *equivalent if there is a bundle isomorphism* $(\overline{f}, \mathrm{id}) : \xi_0 \xrightarrow{\cong} \xi_1$ *such that* $\pi_{n+k}(\mathrm{Th}(\overline{f}))$: $\pi_{n+k}(\mathrm{Th}(E_0)) \xrightarrow{\cong} \pi_{n+k}(\mathrm{Th}(\xi_1))$ *maps* c_0 *to* c_1. *The* set of normal k-invariants $\mathcal{T}_n(X, k)$ *is the set of equivalence classes of normal* k*-invariant of* X.

Given a normal k-invariant (ξ, c), we obtain a normal $(k + 1)$-invariant $(\xi \oplus \underline{\mathbb{R}}, \Sigma c)$, where $\Sigma : \pi_k(\mathrm{Th}(\xi)) \to \pi_{k+1}(\Sigma\,\mathrm{Th}(\xi)) = \pi_{k+1}(\mathrm{Th}(\xi \oplus \underline{\mathbb{R}}))$ is the suspension homomorphism. Define

Definition 3.43 *Let* X *be a connected finite* n*-dimensional Poincaré complex* X. *Define the* set of normal invariants $\mathcal{T}_n(X)$ *of* X *to be the colimit* $\mathrm{colim}_{k\to\infty}\, \mathcal{T}_n(X, k)$.

Let $J_k : BO(k) \to BG(k)$ be the classifying map for the universal k-dimensional vector bundle $\xi_k : E_k \to BO(k)$ viewed as a spherical fibration. Taking the Whitney sum with $\underline{\mathbb{R}}$ or the fiberwise join with $S\underline{\mathbb{R}}$ yields stabilization maps $BO(k) \to BO(k + 1)$ and $BG(k) \to BG(k + 1)$. This corresponds to the obvious stabilization maps $O(k) \to O(k + 1)$ and $G(k) \to G(k + 1)$ given by direct sum with the identity map $\mathbb{R} \to \mathbb{R}$ and by suspending a selfhomotopy equivalence of S^k. One can arrange that these stabilization maps are cofibrations by a mapping cone construction and are compatible with the various maps J_k by a cofibration argument. Define $O = \mathrm{colim}_{k\to\infty}\, O(k)$ and $G = \mathrm{colim}_{k\to\infty}\, G(k)$ and $J : BO \to BG$ by $\mathrm{colim}_{k\to\infty}\, J_k$. $BO = \mathrm{colim}_{k\to\infty}\, BO(k)$ and $BG = \mathrm{colim}_{k\to\infty}\, BG(k)$ and $J : BO \to BG$ by $\mathrm{colim}_{k\to\infty}\, J_k$.

From Theorem 3.39 we get for a connected finite n-dimensional Poincaré complex X a map $s_X : X \to BG$ which is given by the classifying map of the Spivak normal $(k - 1)$-bundle for large k. It is unique up to homotopy. We obtain from the universal properties of the classifying spaces

Theorem 3.44 *Let X be a connected finite n-dimensional Poincaré complex. Then $\mathcal{T}(X)$ is non-empty if and only there is a map $S : X \to BO$ such that $J \circ S$ is homotopic to s_X.*

Remark 3.45 In view of Remark 3.41 we see that a necessary condition for a connected finite n-dimensional Poincaré complex to be homotopy equivalent to a closed manifold is that the classifying map $s : X \xrightarrow{s_X} BG(k)$ lifts along $J : BO \to BG$, where BG is defined by $BG = \operatorname{colim}_{k\to\infty} BG(k)$ analogously to $BO = \operatorname{colim}_{k\to\infty} BO(k)$. There is a fibration $BO \to BG \to BG/O$. Hence this condition is equivalent to the statement that the composition $X \xrightarrow{s_X} BG \to BG/O$ is homotopic to the constant map. There exists a finite Poincaré complex X for which composition $X \xrightarrow{s_X} BG \to BG/O$ is not nullhomotopic (see [42, page 32 f]). In particular X cannot be homotopy equivalent to closed manifold.

Let G/O be the homotopy fiber of $J : BO \to BG$. This is the fiber of the fibration $\widehat{J} : E_J \to BG$ associated to J. Then the following holds

Theorem 3.46 *Let X be a connected finite n-dimensional Poincaré complex. Suppose that $\mathcal{T}_n(X)$ is non-empty. Then there is a canonical group structure on the set $[X, G/O]$ of homotopy classes of maps from X to G/O and a transitive free operation of this group on $\mathcal{T}_n(X)$.*

Proof : Define an abelian group $\mathcal{G}/\mathcal{O}(X)$ as follows. We consider pairs (ξ, t) consisting of a k-dimensional vector bundle ξ for some $k \geq 1$ and a strong fiber homotopy equivalence $t : S\xi \to \underline{S^{k-1}}$ from the associated spherical fibration given by the sphere bundle $S\xi$ and the trivial spherical $(k-1)$-fibration. We call two such pairs (ξ_0, t_0) and (ξ_1, t_1) equivalent, if for some k which is greater or equal to both $k_0 = \dim(\xi_0)$ and $k_1 = \dim(\xi_1)$ there is a bundle isomorphism $(\overline{f}, \mathrm{id}) : \xi_0 \oplus \underline{\mathbb{R}^{k-k_0}} \xrightarrow{\cong} \xi_1 \oplus \underline{\mathbb{R}^{k-k_1}}$ such that $\Sigma^{k-k_1} t_1 \circ S\overline{f}$ is fiber homotopic to $\Sigma^{k-k_0} t_0$. Let $\mathcal{G}/\mathcal{O}(X)$ be the set of equivalence classes $[\xi, t]$ of such pairs (ξ, t). Notice that $[\xi, t]$ for a pair (ξ, t) depends only on the fiber homotopy class of t. Addition is given by the Whitney sum. The neutral element is represented by $(\underline{\mathbb{R}^k}, \mathrm{id})$ for some $k \geq 1$. The existence of inverses follows from the fact that for a vector bundle ξ over X we can find another vector bundle η such that $\xi \oplus \eta$ is trivial and for a map $f : X \to G(k)$ we can find l and a map $f' : X \to G(l)$ such that the map given by the join $f * f' : X \to G(k+l)$ is homotopic to the identity.

Next we describe an action $\rho : \mathcal{G}/\mathcal{O}(X) \times \mathcal{T}_n(X) \to \mathcal{T}_n(X)$. Consider a pair (ξ, t) representing an element $[\xi, t] \in \mathcal{G}/\mathcal{O}(X)$ and a normal k_1-invariant (η, c) representing an element $[\eta, c] \in \mathcal{T}_n(X)$. If k_0 is the dimension of ξ, then $\xi \oplus \eta$ is a $(k_0 + k_1)$-dimensional vector bundle. Consider the composition

$$\mathrm{Th}(\xi \oplus \eta) \xrightarrow{\cong} \mathrm{Th}(S\xi * S\eta) \xrightarrow{\cong} \mathrm{Th}(S\xi) \wedge \mathrm{Th}(S\eta)$$

$$\xrightarrow{t \wedge \mathrm{id}} \mathrm{Th}(\underline{S^{k_0-1}}) \wedge \mathrm{Th}(S\eta) \xrightarrow{\cong} \Sigma^{k_0} \mathrm{Th}(\eta).$$

This together with the suspension $\Sigma^{k_0}c$ yields an element $d \in \pi_{n+k_0+k_1}(\mathrm{Th}(\xi \oplus \eta))$. One easily checks that $(\xi \oplus \eta, d)$ is a normal $(k_0 + k_1)$-invariant for X. Define

$$\rho([\xi, t], [\eta, c]) := [\xi \oplus \eta, d].$$

We leave it to the reader to check that this is a well-defined group action. It follows from Theorem 3.39 that this operation is transitive and free. Hence it remains to construct a bijection

$$\mu : [X, G/O] \xrightarrow{\cong} \mathcal{G}/\mathcal{O}(X).$$

Recall that G/O is the fiber of the fibration $\widehat{J} : E_J \to BG$ associated to $J : BO \to BG$ over some point $z \in BG$. By definition this is the space $\{(y, w) \in BO \times \mathrm{map}([0,1], BG) \mid J(y) = w(0), w(1) = z\}$. Hence a map $F : X \to G/O$ is the same as a pair (f, h) consisting of a map $f : X \to BO$ and a homotopy $h : X \times [0,1] \to BG$ such that $h_0 = J \circ f$ and h_1 is the constant map c_z. Since X is compact, the can find k such that the image of f and h lie in $BO(k)$ and $BG(k)$. Recall that $J_k : BO(k) \to BG(k)$ is covered by a fiber map $\overline{J_k} : S\xi_k \to \eta_k$ which is fiberwise a homotopy equivalence, where ξ_k is the universal k-dimensional bundle over $BO(k)$ and η_k the universal spherical $(k-1)$-fibration. By the pullback construction applied to f and ξ_k and h and η_k, we get a vector bundle ξ over X and a spherical $(k-1)$-fibration η over $X \times [0,1]$ together with fiber homotopy equivalences $(u, \mathrm{id}) : S\xi \to \eta|_{X \times \{0\}}$ and $(v, \mathrm{id}) : \eta|_{X \times \{1\}} \to X \times S^{k-1}$. Up to fiber homotopy there is precisely one fiber homotopy equivalence $(\overline{g}, \mathrm{pr}) : \eta \to X \times [0,1] \times S^{k-1}$ whose restriction to $X \times \{1\} = X$ is v [62, Proposition 15.11 on page 342]. Thus we obtain a fiber homotopy equivalence $v|_{X \times \{0\}} : \eta_{X \times \{0\}} \to X \times S^{k-1}$ covering the identity $X \times \{0\} \to X$ which is unique up to fiber homotopy. Composing $v|_{X \times \{0\}}$ and u yields a fiber homotopy equivalence unique up to fiber homotopy $(w, \mathrm{id}) : S\xi \to \underline{S^{k-1}}$. Thus we can assign to map $F : X \to G/O$ an element $[\xi, w] \in \mathcal{G}/\mathcal{O}(X)$. We leave it to the reader to check that this induces the desired bijection $\mu : [X, G/O] \xrightarrow{\cong} \mathcal{G}/\mathcal{O}(X)$. This finishes the proof of Theorem 3.46. ■

Notice that Theorem 3.46 yields after a choice of an element in $\mathcal{T}_n(X)$ a bijection of sets $[X, G/O] \xrightarrow{\cong} \mathcal{T}_n(X)$.

Definition 3.47 *Let X be a connected finite n-dimensional Poincare complex together with a k-dimensional vector bundle $\xi : E \to X$. A* normal k-map *$(M, i, f, \overline{f})$ consists of a closed manifold M of dimension n together with an embedding $i : M \to \mathbb{R}^{n+k}$ and a bundle map $(\overline{f}, f) : \nu(M) \to \xi$. A* normal map of degree one *is a normal map such that the degree of $f : M \to X$ is one.*

Notice that this definition is the same as the definition of an element representing a class in $\Omega_n(X; \xi)$ except that we additionally require the map of degree 1. Analogously one requires in the definition of a bordism that the map $F : (W, \partial W) \to (X \times [0,1], X \times \partial[0,1])$ has degree one. Denote by

$\mathcal{N}_n(X,k)$ the set of normal bordism classes of normal k-maps to X. Define $\mathcal{N}_n(X,k) \to \mathcal{N}_n(X,k+1)$ by sending the class of $(M,i,f,\overline{f})$ to the class of $(M,i',f,\overline{f}')$, where $i' : M \to \mathbb{R}^{n+k+1}$ is the composition of i with the standard inclusion $\mathbb{R}^{n+k} \to \mathbb{R}^{n+k+1}$ and $\overline{f}'$ is $\nu(i') = \nu(i) \oplus \underline{\mathbb{R}} \xrightarrow{\nu(f)\oplus \mathrm{id}} \xi \oplus \underline{\mathbb{R}}$.

Definition 3.48 *Let X be a connected finite n-dimensional Poincaré complex. Define the* set of normal maps to X

$$\mathcal{N}_n(X) \;:=\; \operatorname{colim}_{k\to\infty} \mathcal{N}_n(X,k).$$

The proof of the next result is similar to the one of Theorem 3.26.

Theorem 3.49 *Let X be a connected finite CW-complex. Then the Pontrjagin-Thom construction yields for each $k \geq 1$ a bijection*

$$P_k(X) : \mathcal{N}_n(X,k) \xrightarrow{\cong} \mathcal{T}_n(X,k).$$

This induces a bijection

$$P(X) : \mathcal{N}_n(X) \xrightarrow{\cong} \mathcal{T}_n(X).$$

Remark 3.50 In view of the Pontrjagin Thom construction it is convenient to work with the normal bundle. On the other hand one always needs an embedding and one would prefer an intrinsic definition. This is possible if one defines the normal map in terms of the tangent bundle which we will do below. Both approaches are equivalent. We will use in the sequel the one which is adequate for the concrete purpose. We mention that for a generalization to the equivariant setting the approach using the tangent bundle is more useful [40], [41].

Definition 3.51 *Let X be a connected finite n-dimensional Poincaré complex together with a vector bundle $\xi : E \to X$. A* normal map with respect to the tangent bundle *$(\overline{f},f) : M \to X$ consists of an oriented closed manifold M of dimension n together with a bundle map $(\overline{f},f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ for some integer $a \geq 0$. If f has degree one, we call $(\overline{f},f)$ a* normal map of degree one with respect to the tangent bundle.

We define also a bordism relation as follows. Consider two normal maps of degree one with respect to the tangent bundles $(\overline{f}_i,f_i) : TM \oplus \underline{\mathbb{R}^{a_i}} \to \xi_i$ covering $f_i : M_i \to X$. A normal bordism is a normal map of degree one $(\overline{F},F) : TW \oplus \underline{\mathbb{R}}^b \to \eta$ covering $F : W \to X$ such that ∂W is a disjoint union $\partial_0 W \coprod \partial_1 W$ and we have the following data for $i = 0,1$. We require a diffeomorphisms $u_i : M_i \to \partial_i W$ with $F \circ u_i = f_i$. Moreover, we require bundle isomorphisms $v_i : \xi_i \oplus \underline{\mathbb{R}^{b-a_i+1}} \to \eta$ covering the identity on X such that following diagram commutes

$$\begin{array}{ccc} TM_i \oplus \underline{\mathbb{R}} \oplus \underline{\mathbb{R}^b} & \xrightarrow{\overline{f}_i \oplus \mathrm{id}_{\underline{\mathbb{R}^{b-a_i+1}}}} & \xi_i \oplus \underline{\mathbb{R}^{b-a_i+1}} \\ {\scriptstyle Tu_i \oplus n_i \oplus \mathrm{id}_{\underline{\mathbb{R}^b}}} \downarrow & & \downarrow {\scriptstyle v_i} \\ TW|_{\partial_i W} \oplus \underline{\mathbb{R}^b} & \xrightarrow[F|_{\partial_i W}]{} & \eta|_{\partial_i W} \end{array}$$

Here $Tu_i : TM_i \to TW$ is given by the differential and $n_i : \underline{\mathbb{R}} \to TW$ is the bundle monomorphism given by an inward normal field of $TW|_{\partial_i W}$. Denote by $\mathcal{N}_n^T(X)$ the set of bordism classes of normal maps of degree one with respect to the tangent bundle.

Lemma 3.52 *Let X be a connected n-dimensional finite Poincaré complex. There is a natural bijection*

$$\mathcal{N}_n(X) \;\cong\; \mathcal{N}_n^T(X).$$

Proof : We define a map $\phi_n(k) : \mathcal{N}_n(X,k) \to \mathcal{N}_n^T(X)$ as follows. Consider a normal k-map $\overline{f} : \nu(M) \to \xi$ covering the map $f : M \to X$ of degree one for some closed oriented n-dimensional manifold M with an embedding $M \to \mathbb{R}^{n+k}$. Since X is compact, we can find a bundle η together with an isomorphism $u : \eta \oplus \xi \xrightarrow{\cong} \underline{\mathbb{R}^a}$. There is an explicit isomorphism $v : \nu(M) \oplus TM \cong \underline{\mathbb{R}^{n+k}}$. We get from $\overline{f}$, u and v an isomorphism of bundles covering the identity on M

$$f^*\eta \oplus \underline{\mathbb{R}^{n+k}} \;\cong\; f^*\eta \oplus TM \oplus \nu(M) \;\cong\; f^*\eta \oplus TM \oplus f^*\xi \cong TM \oplus \underline{\mathbb{R}^a}.$$

The inverse of this isomorphism is the same as a bundle map $\overline{g} : TM \oplus \underline{\mathbb{R}^a} \to \eta \oplus \underline{\mathbb{R}^{n+k}}$ covering f. Define the image under $\phi_n(k)$ of the class $[\overline{f}, f]$ to be the class $[\overline{g}, f]$. One easily checks that this is well-defined and that the $\phi_n(k)$-s fit together to a map $\phi_n : \mathcal{N}_n(X) \to \mathcal{N}_n^T(X)$. By the analogous construction one gets an inverse. ■

Remark 3.53 Let X be a finite Poincaré complex of dimension n. Then there exists a closed manifold M, which is homotopy equivalent to X, only if there exists a normal map of degree one with target X. Namely, suppose $f : M \to X$ is a homotopy equivalence with a closed manifold as source. Choose a homotopy inverse $f^{-1} : X \to M$. Put $\xi = (f^{-1})^*TM$. Then we can cover f by a bundle map $\overline{f} : TM \to \xi$. Thus we get a normal map $(\overline{f}, f)$ of degree one with M as source and X as target. In order to solve Problem 3.1 we have to address the following problem.

Problem 3.54 *Suppose we have some normal map $(\overline{f}, f)$ from a closed manifold M to a finite Poincaré complex X. Can we change M and f leaving X fixed to get a normal map $(\overline{g}, g)$ such that g is a homotopy equivalence?*

3.4 The surgery step

In this section we explain the surgery step. We begin with a motivation.

3.4.1 Motivation for the surgery step

We first consider the CW-complex version of Problem 3.54.

Let $f : X \to Y$ be a map of CW-complexes. We want to find a procedure which changes X and f leaving Y fixed to map $f' : X' \to Y$ which is a homotopy equivalence. Of course this procedure should have the potential to carry over to the case, where X is a manifold and the resulting source X' is also a manifold. The Whitehead Theorem [65, Theorem V.3.1 and Theorem V.3.5 on page 230] says that f is a homotopy equivalence if and only if it is k-connected for all $k \geq 0$. Recall that *k-connected* means that $\pi_j(f,x) : \pi_j(X,x) \to \pi_j(Y, f(x))$ is bijective for $j < k$ and surjective for $j = k$ for all base points x in X. Hence we would expect from the procedure to get $\pi_k(f)$ to be closer and closer to be trivial for all k. It is reasonable to try to work out an inductive procedure, where f is already k-connected and we would like to make it $(k+1)$-connected. Recall that there is a long exact homotopy sequence of a map $f : X \to Y$

$$\ldots \to \pi_{l+1}(X) \to \pi_{l+1}(Y) \to \pi_{l+1}(f) \to \pi_l(X) \to \pi_l(Y) \to \ldots,$$

where $\pi_{l+1}(f) \cong \pi_{l+1}(\text{cyl}(f), X)$ consists of homotopy classes of commutative squares with j the inclusion

$$\begin{array}{ccc} S^l & \xrightarrow{q} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^{l+1} & \xrightarrow[Q]{} & Y \end{array}$$

The map f is k-connected, if and only if $\pi_l(f) = 0$ for $l \leq k$. Suppose that f is k-connected. In order to achieve that f is $(k+1)$-connected, we must arrange that $\pi_{k+1}(f) = 0$ without changing $\pi_l(f)$ for $l \leq k$. Consider an element ω in $\pi_{k+1}(f)$ given by square as above. Define X' to be the pushout

$$\begin{array}{ccc} S^k & \xrightarrow{q} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ D^{k+1} & \xrightarrow[\overline{q}]{} & X' \end{array}$$

Then the universal property of the pushout gives a map $f' : X' \to Y$. We will say that f' is obtained from f by attaching a cell.

One easily checks that $\pi_l(f') = \pi_l(f)$ for $l \leq k$ and that there is a natural map $\pi_{k+1}(f) \to \pi_{k+1}(f')$ which is surjective and whose kernel contains $\omega \in \pi_l(f)$. Recall that in general $\pi_l(f)$ is an abelian group for $l \geq 3$ and a group for $l = 2$ but carries no group structure for $l = 0, 1$. Moreover $\pi_1(X)$ acts on

$\pi_l(f)$. Hence $\pi_l(f)$ is a $\mathbb{Z}\pi_1(X)$-module for $l \geq 3$. For $l \geq 3$ the kernel of the epimorphism $\pi_{k+1}(f) \to \pi_{k+1}(f')$ is the $\mathbb{Z}\pi_1(X)$-submodule generated by ω [65, Section V.1]. We see that we can achieve by applying this construction that $\pi_{l+1}(Y)$ becomes zero. Suppose that X and Y are finite CW-complexes. One can achieve $\pi_{k+1}(f) = 0$ in a finite number of steps by the following result.

Lemma 3.55 *Let $f : X \to Y$ be a map of finite CW-complexes. Suppose that f is $(k-1)$-connected for some integer $k \geq 0$.*

1. *Suppose that $k \geq 2$ and $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ is bijective. Then $\pi_k(f)$ is a finitely generated $\mathbb{Z}\pi_1(X)$-module;*
2. *One can make f k-connected by attaching finitely many cells.*

Proof : (1) Notice that both X and Y are connected. We will identify $\pi := \pi_1(X) = \pi_1(Y)$. The map f lifts to π-map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ between the universal coverings. The obvious map $\pi_k(\widetilde{f}) \to \pi_k(f)$ is bijective and compatible with the π-operations which are given by the π-actions on the universal covering and the operation of the fundamental group on the homotopy groups. The Hurewicz homomorphism induces an isomorphism of $\mathbb{Z}\pi$-modules $\pi_k(\widetilde{f}) \xrightarrow{\cong} H_k(\widetilde{f})$ since $\widetilde{X}$ and $\widetilde{Y}$ are simply-connected [65, Corollary IV.7.10 on page 181]. In particular we see that $\pi_k(f)$ is indeed an abelian group what is true in general only for $k \geq 3$. Since f and hence $\widetilde{f}$ is $(k-1)$-connected the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{f})$ which is defined to be the mapping cone $D_* := \text{cone}_*(C_*(\widetilde{f}))$ of the $\mathbb{Z}\pi$-chain map $C_*(\widetilde{f}) : C_*(\widetilde{X}) \to C_*(\widetilde{Y})$ is $(k-1)$-connected. Thus $C_*(\widetilde{f})$ yields an exact sequence of finitely generated free $\mathbb{Z}\pi$-modules

$$0 \to \ker(c_k) \to D_k \xrightarrow{d_k} D_{k-1} \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_1} D_0 \to 0.$$

Hence $\ker(c_k)$ is a finitely generated free projective $\mathbb{Z}\pi$-modules which is stably free, i.e. after adding a finitely generated free $\mathbb{Z}\pi$-module it becomes free. Since $H_k(\widetilde{f})$ is a quotient of $\ker(c_k)$, it is finitely generated. Hence $\pi_k(f)$ is finitely generated free as $\mathbb{Z}\pi$-module.

(2) We begin with $k = 0$. Attaching zero cells means taking the disjoint union of X with finitely many points. Obviously one can achieve in this way that $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective.

Next we treat the case $k = 1$. Since X is by assumption a finite CW-complex, $\pi_0(X)$ is finite. If two path components of X are mapped to the same path component in Y, one can attach a 1-cell in the obvious manner to connect these components. Thus we can achieve that $\pi_0(f)$ is bijective. Since Y is finite, $\pi_1(Y)$ is finitely generated. By attaching 1-cells trivially to X which is the same as taking the one-point union of X with S^1, we can achieve that $\pi_1(f)$ is an epimorphism.

Next we consider $k = 2$. Since both X and Y are finite, $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ is an epimorphism of a finitely generated group onto a finitely presented group. One easily checks that the kernel of such a group homomorphism is always finitely generated. For any element in a finite set of generators we can attach 2-cells to kill these elements. The resulting map induces an isomorphism on $\pi_1(X)$. Now we can apply assertion (1). The cases $k \geq 3$ follow directly from assertion (1). This finishes the proof of Lemma 3.55. ■

Actually one can achieve in the world of CW-complexes the desired homotopy equivalence $f' : X' \to Y$ directly by the following construction. Namely consider the projection $\mathrm{pr} : \mathrm{cyl}(f) \to Y$ of the mapping cylinder of f to Y. Obviously the mapping cylinder is in general no manifold even if $f : X \to Y$ is a smooth map of closed manifolds. Neither there is a chance that the space X' obtained from X by attaching a cell is a manifold even if X is a closed manifold and f smooth. But this single step of attaching a cell can be modified so that it applies to manifolds as source such that the resulting map has a manifold as source. This will be explained next.

Suppose that M is a compact manifold of dimension n and Y is a CW-complex. Suppose that $f : M \to Y$ is a k-connected map. Consider an element $\omega \in \pi_{k+1}(f)$ represented by a diagram

$$\begin{array}{ccc} S^k & \xrightarrow{q} & M \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^{k+1} & \xrightarrow[Q]{} & Y \end{array}$$

We cannot attach a single cell to M without distroying the manifold structure. But one can glue two manifolds together along a common boundary such that the result is a manifold. Suppose that the map $q : S^k \to M$ extends to an embedding $\overline{q} : S^k \times D^{n-k} \to M$. Let $\mathrm{int}(\mathrm{im}(\overline{q}))$ be the interior of the image of $\overline{q}$. Then $M - \mathrm{int}(\mathrm{im}(\overline{q}))$ is a manifold with boundary $\mathrm{im}(\overline{q}(S^k \times S^{n-k-1})$. We can get rid of the boundary by attaching $D^{k+1} \times S^{n-k-1}$ along $\mathrm{im}(\overline{q}|_{S^k \times S^{n-k-1}}$. Call the result

$$M' := D^{k+1} \times S^{n-k-1} \cup_{\mathrm{im}(\overline{q}|_{S^k \times S^{n-k-1}}} (M - \mathrm{int}(\mathrm{im}(\overline{q}))) .$$

Choose a map $\overline{Q} : D^{k+1} \times D^{n-k} \to X$ which extends Q and $\overline{q}$. The restriction of f to $M - \mathrm{int}(\mathrm{im}(\overline{q}))$ extends to a map $f' : M' \to Y$ using $\overline{Q}|_{D^{k+1} \times S^{n-k}}$. Notice that the inclusion $M - \mathrm{int}(\mathrm{im}(\overline{q})) \to M$ is $(n-k-1)$-connected since $S^k \times S^{n-k-1} \to D^k \times D^{n-k}$ is $(n-k-1)$-connected. So the passage from M to $M - \mathrm{int}(\mathrm{im}(\overline{q}))$ will not effect $\pi_j(f)$ for $j < n-k-1$. The passage from $M - \mathrm{int}(\mathrm{im}(\overline{q}))$ to M' has the same effect as we have described in the case of CW-complexes. All in all we see that $\pi_l(f) = \pi_l(f)$ for $l \leq k$ and that there is an epimorphism $\pi_{k+1}(f) \to \pi_{k+1}(f')$ whose kernel contains ω, provided that $2(k+1) \leq n$. The condition $2(k+1) \leq n$ can be viewed as a consequence of Poincaré duality. Roughly speaking, if we change something in a manifold in dimension l, Poincaré duality forces also a change in dimension $(n-l)$. This phenomenon will cause surgery obstructions to appear.

It is important to notice that $f : M \to X$ and $f' : M' \to X$ are bordant. The relevant bordism is given by $W = D^{k+1} \times D^{n-k} \cup_{\overline{q}} M \times [0,1]$, where we think of $\overline{q}$ as an embedding $S^k \times D^{n+k} \to M \times \{1\}$. In other words, W is obtained from $M \times [0,1]$ by attaching a handle $D^{k+1} \times D^{n-k-1}$ to $M \times \{1\}$. Then M appears in W as $M \times \{0\}$ and M' as other part of the boundary of W. Define $F : W \to X$ by $f \times \mathrm{id}_{[0,1]}$ and $\overline{Q}$. Then F restricted to M and M' is f and f'.

But before we come to surgery obstructions, we must figure out, whether we arrange that q is an embedding and extends to an embedding $\overline{q}$. If $2k < n$, we can change q up to homotopy such that it becomes an embedding. In the case $2k = n$ we can change q up to homotopy to an immersion and the surgery obstruction will actually be the obstruction to change it up to homotopy into an embedding. Let us assume that q is an embedding. Then the existence of the extension $\overline{q}$ is equivalent to the triviality of the normal bundle of the embedding $g : S^k \to M$.

A priori there is no reason why this normal bundle should be trivial. At this point the bundle data attached to a normal map become useful. So far we have only considered them since it was a necessary condition to be able to solve Problem 3.1 although in Problem 3.1 no bundle maps appear. But now it will pay off that we have this bundle data available. Namely, if we want to kill an element $\omega \in \pi_{k+1}(f)$ represented by a diagram

$$\begin{array}{ccc} S^k & \xrightarrow{q} & M \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^{k+1} & \xrightarrow[Q]{} & N \end{array}$$

then $q^*TM \oplus \underline{\mathbb{R}^a}$ is isomorphic to $q^*f^*\xi = j^*Q^*\xi$ and hence is trivial since $Q^*\xi$ is a bundle over the contractible space $D^{k+1} \times D^{n-k}$. Since $\nu(S^k, M) \oplus TS^k$ is isomorphic to q^*TM, the bundle $\nu(S^k, M) \oplus \underline{\mathbb{R}}^a$ is trivial some $a \geq 0$. Suppose $2k \leq n-1$. Then the natural map $BO(n-k) \to BO(n-k+a)$ is $(k+1)$-connected. Hence $\nu(S^k, M)$ itself is trivial. So we are able to carry out one step. But we have to ensure that we can repeat this process. So we must arrange that the bundle data are also available for the resulting map $f' : M' \to X$. Therefore we must be more careful with the choice of embedding (resp. immersion) which is homotopy equivalent to g. The given bundle data should tell us which embedding we should choose. For this we need some information about embeddings and immersions which we will give next.

3.4.2 Immersions and embeddings

Given two vector bundles $\xi : E \to M$ and $\eta : F \to N$, we have so far only considered bundle maps $(\overline{f}, f) : \xi \to \eta$ which are fiberwise isomorphisms. We need to consider now more generally bundle monomorphisms, i.e. we only will require that the map is fiberwise injective. Consider two bundle monomorphism $(\overline{f}_0, f_0), (\overline{f}_1, f_1) : \xi \to \eta$. Let $\xi \times [0,1]$ be the vector bundle $\xi \times \mathrm{id} : E \times [0,1] \to M \times [0,1]$. A homotopy of bundle monomorphisms $(\overline{h}, h)$ from $(\overline{f}_0, f_0)$ to $(\overline{f}_0, f_0)$ is a bundle monomorphism $(\overline{h}, h) : \xi \times [0,1] \to \eta$ whose restriction to $X \times \{j\}$ is

$(\overline{f}_j, f_j)$ for $j = 0, 1$. Denote by $\pi_0(\text{Mono}(\xi, \eta))$ be the set of homotopy classes of bundle monomorphisms.

An immersion $f : M \to N$ is a map whose differential $Tf : TM \to TN$ is a bundle monomorphism. An immersion is locally an embedding but it is not isotopic to an embedding in general. A regular homotopy $h : M \times [0,1] \to N$ from an immersion $f_0 : M \to N$ to an immersion $f_1 : M \to N$ is a homotopy h such that $h_0 = f_0$, $h_1 = f_1$ and $h_t : M \to N$ is an immersion for each $t \in [0,1]$. Denote by $\pi_0(\text{Imm}(M,N))$ the set of regular homotopy classes of immersions from M to N. The next result is due to Whitney [66], [67].

Theorem 3.56 *Let M and N be closed manifolds of dimensions m and n. Then any map $f : M \to N$ is arbitrarily closed to an immersion provided that $2m \leq n$ and arbitrarily closed to an embedding provided that $2m < n$.*

For a proof of the following result we refer to Haefliger-Poenaru [28], Hirsch [30] and Smale [58].

Theorem 3.57 (Immersions and bundle monomorphisms) *Let M be a m-dimensional and N be a n-dimensional closed manifold.*

1. *Suppose that $1 \leq m < n$. Then taking the differential of an immersion yields a bijection*

$$T : \pi_0(\text{Imm}(M,N)) \xrightarrow{\cong} \pi_0(\text{Mono}(TM, TN));$$

2. *Suppose that $1 \leq m \leq n$ and that M has a handlebody decomposition consisting of q-handles for $q \leq n-2$. Then taking the differential of an immersion yields a bijection*

$$T : \pi_0(\text{Imm}(M,N)) \xrightarrow{\cong} \text{colim}_{a \to \infty}\, \pi_0(\text{Mono}(TM \oplus \underline{\mathbb{R}^a}, TN \oplus \underline{\mathbb{R}^a})),$$

where the colimit is given by stabilization.

Example 3.58 Theorem 3.57 (1) has the following remarkable consequence. We claim that $\pi_0(\text{Imm}(S^2, \mathbb{R}^3))$ consist of one element. We have to show that $\pi_0(\text{Mono}(TS^2, T\mathbb{R}^3))$ consists of one element. Consider bundle monomorphisms $(\overline{f}_i, f_i) : TS^2 \to T\mathbb{R}^3$ for $i = 0, 1$. Since $T_x f_i(T_x S^2) \subset T_{f_i(x)}\mathbb{R}^3$ is an oriented 2-dimensional subspace of the oriented Euclidean vector space $T_{f(x)}\mathbb{R}^3$, there is precisely one vector $v_i(x) \in T_{f_i(x)}\mathbb{R}^3$ whose norm is one and for which the orientation on $T_x f_i(T_x S^2) \oplus \mathbb{R}v(x) = T_{f_i(x)}\mathbb{R}^3$ induced by the one of $T_x S^2$ and $v(x)$ and the standard one on $T_{f_i(x)}\mathbb{R}^3$ agree. Hence we can find a orientation preserving bundle isomorphism $(\overline{g_i}, f_i) : TS^2 \oplus \underline{\mathbb{R}} \to T\mathbb{R}^3$ covering f_i and extending $\overline{f}_i$. Since f_i is homotopic to the constant map c with value 0, we can find a homotopy of bundle maps which are fiberwise orientation preserving isomorphisms from $(\overline{g}_i, f_i)$ to $(\overline{c}_i, c)$ for $i = 0, 1$. It suffices to show that $(\overline{c}_0, c)$ and $(\overline{c}_1, c)$ are strongly fiber homotopic as bundle maps (which are fiberwise isomorphisms) because then we get by restriction a homotopy of bundle monomorphism between

$(\overline{f}_0, f_0)$ and $(\overline{f}_1, f_1)$. Now $(\overline{c}_0, c)$ and $(\overline{c}_1, c)$ differ by an orientation preserving bundle automorphism covering the identity $(\overline{u}, \mathrm{id}) : S^2 \times T_0\mathbb{R}^3 \to S^2 \times T_0\mathbb{R}^3$. This is the same a map $\overline{u} : S^2 \to GL(3,\mathbb{R})^+$, where we identify $T_0\mathbb{R}^3 = \mathbb{R}^3$ and $GL(3,\mathbb{R})^+$ is the Lie group of orientation preserving linear automorphism of $\mathbb{R}^3$. The inclusion $SO(3) \to GL(3,\mathbb{R})^+$ is a homotopy equivalence by the polar decomposition. Since $\pi_2(SO(3))$ is known to be zero, there is a strong homotopy of bundle maps which are fiberwise isomorphisms and cover the identity from $(\overline{u}, id)$ to $(\mathrm{id}, \mathrm{id})$. This proves that $\pi_0(\mathrm{Imm}(S^2, \mathbb{R}^3))$ consists of precisely one element.

Let $f_0 : S^2 \to \mathbb{R}^3$ be the standard embedding. Let $f_2 : S^2 \to \mathbb{R}^3$ be its composition with the involution $i : \mathbb{R}^3 - \{0\} \to \mathbb{R}^3 - \{0\}$ which sends x to $\frac{x}{||x||^2}$. By the argument above f_0 and f_1 are regular homotopic. Notice that i is the identity on $\mathrm{im}(f_0)$. Its differential at a point $x \in \mathrm{im}(f_0)$ sends the normal vector v pointing to the origin in $\mathbb{R}^3$ to $-v$. Therefore a regular homotopy from f_0 to f_2 will turn the inside of the standard sphere $f_0 : S^2 \to \mathbb{R}^3$ to the outside.

3.4.3 The surgery step

Now we can carry out the surgery step.

Theorem 3.59 (The surgery step) *Consider a normal map* $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ *covering* $f : M \to X$ *and an element* $\omega \in \pi_{k+1}(f)$ *for* $k \leq n-2$ *for* $n = \dim(M)$. *Let* $Tj \oplus n : T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}} \to T(D^{k+1} \times D^{n-k})$ *be the bundle map covering the inclusion* j *which is given by the differential* Tj *of* j *and the inward normal field of the boundary of* $D^{k+1} \times D^{n-k}$. *Then*

1. *We can find a commutative diagram of vector bundles*

$$\begin{array}{ccc} T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}^{a+b}} & \xrightarrow{\overline{q}} & TM \oplus \underline{\mathbb{R}^{a+b}} \\ \downarrow{\scriptstyle Tj\oplus n\oplus \mathrm{id}_{\underline{\mathbb{R}^{a+b-1}}}} & & \downarrow{\scriptstyle \overline{f}} \\ T(D^{k+1} \times D^{n-k}) \oplus \underline{\mathbb{R}^{a+b-1}} & \xrightarrow[\overline{Q}]{} & \xi \oplus \underline{\mathbb{R}^b} \end{array}$$

covering a commutative diagram

$$\begin{array}{ccc} S^k \times D^{n-k} & \xrightarrow{q} & M \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^{k+1} \times D^{n-k} & \xrightarrow[Q]{} & X \end{array}$$

such that the restriction of the last diagram to $D^{k+1} \times \{0\}$ *represents* ω *and* $q : S^k \times D^{n-k} \to M$ *is an immersion;*

2. *The regular homotopy class of the immersion* q *appearing in assertion* (1) *is uniquely determined by the properties above and depends only on* ω *and* $(\overline{f}, f)$;

3. *Suppose that the regular homotopy class of the immersion q appearing in (1) contains an embedding. Then one can arrange q in assertion (1) to be an embedding. If $2k < n$, one can always find an embedding in the regular homotopy class of q;*

4. *Supose that the map q appearing in assertion (1) is an embedding.*

   Let W be the manifold obtained from $M \times [0,1]$ by attaching a handle $D^{k+1} \times D^{n-k}$ by $q : S^k \times D^{n-k} \to M = M \times \{1\}$. Let $F : W \to X$ be the map induced by $M \times [0,1] \xrightarrow{\mathrm{pr}} M \xrightarrow{f} X$ and $Q : D^k \times D^{k+1} \to X$. After possibly stabilizing $\overline{f}$ the bundle maps $\overline{f}$ and $\overline{Q}$ induce a bundle map $\overline{F} : TW \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ covering $F : W \to X$. Thus we get a normal map $(\overline{F}, F) : TW \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ which extends $(\overline{f} \oplus (f \times \mathrm{id}_{\mathbb{R}^b}), f) : TM \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$. The normal map $(\overline{f}', f') : TM' \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ obtained by restricting of $(\overline{F}, F)$ to $\partial W - M \times \{0\} =: M'$ is a normal map of degree one which is normally bordant to $(\overline{f}, f)$ and has as underlying manifold $M' = M - \mathrm{int}(q(S^k \times D^{n-k})) \cup_q D^k \times S^{n-k-1}$.

Proof : (1) Choose a commutative diagram of smooth maps

$$\begin{array}{ccc} S^k & \xrightarrow{q'} & M \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle f} \\ D^{k+1} & \xrightarrow[Q']{} & X \end{array}$$

representing ω. Of course it can be extended to a diagram

$$\begin{array}{ccc} S^k \times D^{n-k} & \xrightarrow{q} & M \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^{k+1} \times D^{n-k} & \xrightarrow[Q]{} & X \end{array}$$

Since $D^{k+1} \times D^{n-k+1}$ is contractible, we can find a bundle map $\overline{Q} : TD^{k+1} \times D^{n-k} \oplus \underline{\mathbb{R}^{a-1}} \to \xi$ covering Q. There is precisely one bundle map $\overline{q}$ covering q such that the following diagram commutes

$$\begin{array}{ccc} T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{q}} & TM \oplus \underline{\mathbb{R}^a} \\ {\scriptstyle Tj \oplus n \oplus \mathrm{id}_{\underline{\mathbb{R}^{a-1}}}}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ T(D^{k+1} \times D^{n-k+1}) \oplus \underline{\mathbb{R}^{a-1}} & \xrightarrow[\overline{Q}]{} & \xi \end{array}$$

Suppose that $k \leq n-2$. From Theorem 3.57 (2) we get an immersion $q_0 : S^k \times D^{n-k} \to M$, such that $(Tq_0, q_0) : T(S^k \times D^{n-k}) \to TM$ defines in $\mathrm{colim}_{c \to \infty} \pi_0(\mathrm{Mono}(TS^k \times D^{n-k} \oplus \underline{\mathbb{R}^c}, TM \oplus \underline{\mathbb{R}^s}))$ the same element as $(\overline{q}, q)$. Now after possibly stabilization $\overline{f}$ and thus enlarging a to $a+b$ we can achieve the desired diagram by a cofibration argument.

(2) The stable homotopy class of $T\overline{q}$ is uniquely determined by the commutativity of the diagram of vector bundles

$$\begin{array}{ccc} T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}^{a+b}} & \xrightarrow{T\overline{q}\oplus(\overline{q}\times \mathrm{id}_{\mathbb{R}^{a+b}})} & TM \oplus \underline{\mathbb{R}^{a+b}} \\ {\scriptstyle Tj\oplus n\oplus \mathrm{id}_{\mathbb{R}^{a+b-1}}}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ T(D^{k+1} \times D^{n-k}) \oplus \underline{\mathbb{R}^{a+b-1}} & \xrightarrow[\overline{Q}]{} & \xi \end{array}$$

since $D^{k+1} \times D^{n+k}$ is contractible and hence $(\overline{Q}, Q)$ is unique up to fiber homotopy. Now apply Theorem 3.57 (2).

(3) By a cofibration argument we can use a regular homotopy from q to an embedding q_0 to homotop all the diagrams to diagrams for q_0. If $2k < n$, we can find an embedding q_0 which is arbitrary close to q by Theorem 3.56 and hence regular homotopic to q since the condition being an immersion is an open condition.

(4) We leave it to the reader to check that the construction of $(\overline{F}, F)$ makes sense. ■

Definition 3.60 *Consider a normal map $(\overline{f}, f) : M \to N$ and an element $\omega \in \pi_{k+1}(f)$ for $k \leq n-2$ for $n = \dim(M)$. We call the normal map $(\overline{f}', f') : TM' \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ appearing in Theorem 3.59 (4) the result of surgery on $(\overline{f}, f)$ and ω if it exists. Sometimes we call the step from $(\overline{f}, f)$ to $(\overline{f}, f)$ a* surgery step.

We conclude from Lemma 3.55, from the discussion of the effect of surgery after Lemma 3.55 and from Theorem 3.59

Theorem 3.61 *Let X be a connected finite n-dimensional Poincaré complex. Let $\overline{f} : TM \oplus \underline{\mathbb{R}^a} \to \xi$ be a normal map of degree one covering $f : M \to X$. Then we can carry out a finite sequence of surgery steps to obtain a normal map of degree one $\overline{g} : TN \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ covering $g : N \to X$ such that $(\overline{f}, f)$ and $(\overline{g}, g)$ are normally bordant and g is k-connected, where $n = 2k$ or $n = 2k+1$.*

Recall that we want to address Problem 3.54. The strategy we have developed so far is to do surgery to change our normal map into a homotopy equivalence. Theorem 3.61 gives us some hope to carry out this program successfully, at least we can get a highly connected map. So we can give the final version of the surgery problem.

Problem 3.62 (Surgery problem) *Suppose we have some normal map $(\overline{f}, f)$ from a closed manifold M to a finite Poincaré complex X. Can we change M and f leaving X fixed by finitely many surgery steps to get a normal map $(\overline{g}, g)$ from a closed manifold N to X such that g is a homotopy equivalence?*

Remark 3.63 Suppose that X appearing in Problem 3.62 is orientable and of dimension $n = 4k$. Notice that the surgery step does not change the normal bordism class. In particular the manifolds M and N are oriented bordant. Hence Lemma 3.18 implies that M and N have the same signature. This implies that $\text{sign}(M) - \text{sign}(X)$ is an invariant of the normal bordism class and is not changed by a surgery step, where for non-connected M we mean by $\text{sign}(M)$ the sum of the signatures of the components of M. Since the signature is also an oriented homotopy invariant, we see an obstruction to solve the Surgery Problem 3.62, namely $\text{sign}(M) - \text{sign}(X)$ must be zero. We will see that this is the only obtruction if X is a simply connected orientable $4k$-dimensional Poincaré complex for $k \geq 2$. If X is not simply connected, the vanishing of $\text{sign}(M) - \text{sign}(X)$ will not be sufficient, more complicated surgery obstructions will occur.

3.5 Miscellaneous

Chapter 4

The algebraic surgery obstruction

Introduction

In this chapter we want to give the solution to the surgery Problem 3.62, whether we can change a normal map $(\overline{f}, f)$ from a closed manifold M to a finite Poincaré complex X by finitely many surgery steps to get a normal map $(\overline{f'}, f')$ from a closed manifold N to X such that f' is a homotopy equivalence. We have already seen in Theorem 3.61 that we can make f k-connected if $n = 2k$ or $n = 2k + 1$. So it remains to achieve that f is $(k + 1)$-connected because then f is a homotopy equivalence by Poincaré duality. Of course we want to do further surgery on elements in $\pi_{k+1}(f)$ to make f $(k + 1)$-connected. It will turn out that this is not possible in general. We will encounter an obstruction, the so called surgery obstruction. It takes values in the so called L-groups which are defined in terms of forms and formations.

Let us start with the case $n = 2k$ for $k \geq 3$. Then the problem will be that we cannot do surgery on each element ω in $\pi_{k+1}(f)$. The main obstacle is that an immersion $f : S^k \to M$ associated to ω may not be regularly homotopic to an embedding. This assumption appears in Theorem 3.59 (3). If we put f in general position, we may encounter double points. We have to figure out whether we can get rid of these double points. The main tool will be the Whitney trick which allows to get rid of two of the double points under certain algebraic conditions. In Section 4.1 we introduce intersection numbers and selfintersection numbers for immersions $S^k \to M$ and show that the selfintersection number of f is trivial if and only f is regularly homotopic to an embedding provided that $k \geq 3$ (see Theorem 4.8).

We will explain in Section 4.2 that the intersection pairing λ and the self-intersection numbers $\mu(f)$ for pointed immersions $f : S^k \to M$ are linked to one another. They together define the structure of a non-degenerate $(-1)^k$-quadratic form on the surgery kernel $K_k(\widetilde{M})$. We will show that we are able to kill the surgery kernel $K_k(\widetilde{M})$ by finitely many surgery steps if and only if this non-degenerate $(-1)^k$)-quadratic form is isomorphic to a standard $(-1)^k$-hyperbolic form $H_{(-1)^k}(R^b)$ for some b after adding a standard $(-1)^k$-hyperbolic form $H_{(-1)^k}(R^a)$ for some a (see Theorem 4.27). This leads in a natural way to the definition of the even-dimensional L-groups in Section 4.3 and of the surgery obstruction in even dimensions in Section 4.4.

In the odd-dimensional case $n = 2k + 1$ the embedding question has always a positive answer because of $2k < n$. Hence one can always do surgery on an element in $\pi_{k+1}(f)$. But it is not clear that one can find appropriate elements so that after doing surgery on them the surgery kernel is trivial. The problem is that surgery on such an element simultaneously affects $K_k(\widetilde{M})$ and $K_{k+1}(\widetilde{M})$ since these are related by Poincaré duality. We will explain the definition of the relevant odd-dimensional L-groups in Section 4.5 and we will only sketch the definition and the proof of the main properties of the surgery obstruction in Section 4.6.

Our main concern is not the surgery Problem 3.62 but the question whether two closed manifolds are diffeomorphic. We have explained the surgery program in Remark 1.5. We will explain in Section 4.7 why this forces us to consider also normal maps whose underlying map $(f, \partial f) : (M, \partial M) \to (X, \partial X)$ is a map from a compact manifold with boundary to a Poincaré pair such that ∂f is already a homotopy equivalence. In this situation the aim of surgery is to change f into a homotopy equivalence without changing ∂f. The relevant modification of the surgery obstruction will be introduced. Because of the appearance of the Whitehead torsion in the s-cobordism Theorem 1.1 we are also forced to take Whitehead torsion into account. So we want to achieve that f is a simple homotopy equivalence provided that $(X, \partial X)$ is a simple pair and ∂f is a simple homotopy equivalence. This will lead to the definition of simple L-groups and the simple surgery obstruction.

4.1 Intersection and selfintersection pairings

4.1.1 Intersections of immersions

We are facing the problem to decide whether we can change an immersion $f : S^k \to M$ within its regular homotopy class to an embedding, where M is a closed manifold of dimension $n = 2k$. This problem occurs when we want to carry out a surgery step in the middle dimension (see Theorem 3.59). We first deal with the necessary algebraic obstructions and then address the question whether their vanishing is also sufficient.

We fix base points $s \in S^k$ and $b \in M$ and assume that M is connected and $k \geq 2$. We will consider pointed immersions (f, w), i.e. an immersion $f : S^k \to M$ together with a path w from b to $f(s)$. A pointed regular homotopy from (f_0, w_0) to (f_1, w_1) is a regular homotopy $h : S^k \times [0,1] \to M$ from $h_0 = f_0$ to $h_1 = f_1$ such that $w_0 * h(s, ?)$ and w_1 are homotopic paths relative end points. Here $h(s, ?)$ is the path from $f_0(s)$ to $f_1(s)$ given by restricting h to $\{s\} \times [0,1]$.

Denote by $I_k(M)$ the set of pointed homotopy classes of pointed immersions from S^k to M. We need the paths to define the structure of an abelian group on $I_k(M)$. The sum of $[(f_0, w_0)]$ and $[(f_1, w_1)]$ is given by the connected sum along the path $w_0^- * w_1$ from $f_0(s)$ to $f_1(s)$. The zero element is given by the composition of the standard embedding $S^k \to \mathbb{R}^{k+1} \subset \mathbb{R}^{k+1} \times \mathbb{R}^{k-1} = \mathbb{R}^n$ with some embedding $\mathbb{R}^n \subset M$ and any path w from b to the image of s. The inverse of the class of (f, w) is the class of $(f \circ a, w)$ for any base point preserving diffeomorphism $a : S^k \to S^k$ of degree -1.

The fundamental group $\pi = \pi_1(M, b)$ operates on $I_k(M)$ by composing the path w with a loop at b. Thus $I_k(M)$ inherits the structure of a $\mathbb{Z}\pi$-module.

Next we want to define the *intersection pairing*

$$\lambda : I_k(M) \times I_k(M) \to \mathbb{Z}\pi. \tag{4.1}$$

For this purpose we will have to fix an orientation of T_bM at b. Consider $\alpha_0 = [(f_0, w_0)]$ and $\alpha_1 = [(f_1, w_1)]$ in $I_k(M)$. Choose representatives (f_0, w_0) and (f_1, w_1). We can arrange without changing the pointed regular homotopy class that $D = \operatorname{im}(f_0) \cap \operatorname{im}(f_1)$ is finite, for any $y \in D$ both the preimage $f_0^{-1}(y)$ and the preimage $f_1^{-1}(y)$ consists of precisely one point and for any two points x_0 and x_1 in S^k with $f_0(x_0) = f_1(x_1)$ we have $T_{x_0}f_0(T_{x_0}S^k) + T_{x_1}f_1(T_{x_1}S^k) = T_{f_0(x_0)}M$. Consider $d \in D$. Let x_0 and x_1 in S^k be the points uniquely determined by $f_0(x_0) = f_1(x_1) = d$. Let u_i be a path in S^k from s to x_i. Then we obtain an element $g(d) \in \pi$ by the loop at b given by the composition $w_1 * f_1(u_1) * f_0(u_0)^- * w_0^-$. Recall that we have fixed an orientation of T_bM. The fiber transport along the path $w_0 * f(u_0)$ yields an isomorphism $T_bM \cong T_dM$ which is unique up to isotopy. Hence T_dM inherits an orientation from the given orientation of T_bM. The standard orientation of S^k determines an orientation on $T_{x_0}S^k$ and $T_{x_1}S^k$. We have the isomorphism of oriented vector spaces

$$T_{x_0}f_0 \oplus T_{x_1}f_1 : T_{x_0}S^k \oplus T_{x_1}S^k \xrightarrow{\cong} T_dM.$$

Define $\epsilon(d) = 1$ if this isomorphisms respects the orientations and $\epsilon(d) = -1$ otherwise. The elements $g(d) \in \pi$ and $\epsilon(d) \in \{\pm 1\}$ are independent of the choices of u_0 and u_1 since S^k is simply connected for $k \geq 2$. Define

$$\lambda(\alpha_0, \alpha_1) := \sum_{d \in D} \epsilon(d) \cdot g(d).$$

Lift $b \in M$ to a base point $\widetilde{b} \in \widetilde{M}$. Let $\widetilde{f_i} : S^k \to \widetilde{M}$ be the unique lift of f_i determined by w_i and $\widetilde{b}$ for $i = 0, 1$. Let $\lambda_{\mathbb{Z}}(\widetilde{f_0}, \widetilde{f_1})$ be the $\mathbb{Z}$-valued intersection number of $\widetilde{f_0}$ and $\widetilde{f_1}$. This is the same as the algebraic intersection number of the classes in the k-th homology with compact support defined by $\widetilde{f_0}$ and $\widetilde{f_1}$ which obviously depends only on the homotopy classes of $\widetilde{f_0}$ and $\widetilde{f_1}$. Then

$$\lambda(\alpha_0, \alpha_1) = \sum_{g \in \pi} \lambda_{\mathbb{Z}}(\widetilde{f_0}, l_{g^{-1}} \circ \widetilde{f_1}) \cdot g, \tag{4.2}$$

where $l_{g^{-1}}$ denotes left multiplication with g^{-1}. This shows that $\lambda(\alpha_0, \alpha_1)$ depends only on the pointed regular homotopy classes of (f_0, w_0) and (f_1, w_1).

In the sequel we use the $w = w_1(M)$-twisted involution on $\mathbb{Z}\pi$ which sends $\sum_{g\in\pi} a_g \cdot g$ to $\sum_{g\in\pi} w(g) \cdot a_g \cdot g^{-1}$ One easily checks

Lemma 4.3 *For $\alpha, \beta, \beta_1, \beta_2 \in I_k(M)$ and $u_1, u_2 \in \mathbb{Z}\pi$ we have*

$$\begin{aligned} \lambda(\alpha, \beta) &= (-1)^k \cdot \overline{\lambda(\beta, \alpha)}; \\ \lambda(\alpha, u_1 \cdot \beta_1 + u_2 \cdot \beta_2) &= u_1 \cdot \lambda(\alpha, \beta_1) + u_2 \cdot \lambda(\alpha, \beta_2). \end{aligned}$$

Remark 4.4 Suppose that the normal bundle of the immersion $f : S^k \to M$ has a nowhere vanishing section. (In our situation it actually will be trivial.) Suppose that f is regular homotopic to an embedding g. Then the normal bundle of g has a nowhere vanishing section σ. Let g' be the embedding obtained by moving g a little bit in the direction of this normal vector field σ. Choose a path w_f from $f(s)$ to b. Then for appropriate paths w_g and $w_{g'}$ we get pointed embeddings (g, w_g) and $(g', w_{g'})$ such that the pointed regular homotopy classes of (f, w), (g, w_g) and $(g', w_{g'})$ agree. Since g and g' have disjoint images, we conclude

$$\lambda([f, w], [f, w]) = 0.$$

Hence the vanishing of $\lambda([f, w], [f, w])$ is a necessary condition for finding an embedding in the regular homotopy class of f, provided that the normal bundle of f has a nowhere vanishing section. It is not a sufficient condition. To get a sufficient condition we have to consider self-intersections what we will do next.

4.1.2 Selfintersections of immersions

Let $\alpha \in I_k(M)$ be an element. Let (f, w) be a pointed immersion representing α. Recall that we have fixed base points $s \in S^k$, $b \in M$ and an orientation of T_bM. Since we can find arbitrarily close to f an immersion which is in general position, we can assume without loss of generality that f itself is in general position. This means that there is a finite subset D of $\mathrm{im}(f)$ such that $f^{-1}(y)$ consists of precisely two points for $y \in D$ and of precisely one point for $y \in \mathrm{im}(f) - D$ and for two points x_0 and x_1 in S^k with $x_0 \neq x_1$ and $f(x_0) = f(x_1)$ we have $T_{x_0}f(T_{x_0}S^k) + T_{x_1}f(T_{x_1}S^k) = T_{f_0(x_0)}M$. Now fix for any $d \in D$ an ordering $x_0(d), x_1(d)$ of $f^{-1}(d)$. Analogously to the construction above one defines $\epsilon(x_0(d), x_1(d)) \in \{\pm 1\}$ and $g(x_0(d), x_1(d)) \in \pi = \pi_1(M, b)$. Consider the element $\sum_{d\in D} \epsilon(x_0(d), x_1(d)) \cdot g(x_0(d), x_1(d))$ of $\mathbb{Z}\pi$. It does not only depend on f but also on the choice of the ordering of $f^{-1}(d)$ for $d \in D$. One easily checks that the change of ordering of $f^{-1}(d)$ has the following effect for $w = w_1(M) : \pi \to \{\pm 1\}$

$$\begin{aligned} g(x_1(d), x_0(d)) &= g(x_0(d), x_1(d))^{-1}; \\ w(g(x_1(d), x_0(d))) &= w(g(x_0(d), x_1(d))); \\ \epsilon(x_1(d), x_0(d)) &= (-1)^k \cdot w(g(x_0(d), x_1(d))) \cdot \epsilon(x_0(d), x_1(d)); \\ \epsilon(x_1(d), x_0(d)) \cdot g(x_1(d), x_0(d)) &= (-1)^k \cdot \epsilon(x_0(d), x_1(d)) \cdot \overline{g(x_0(d), x_1(d))}. \end{aligned}$$

Define an abelian group, where we use the w-twisted involution on $\mathbb{Z}\pi$

$$Q_{(-1)^k}(\mathbb{Z}\pi, w) \quad := \quad \mathbb{Z}\pi/\{u - (-1)^k \cdot \overline{u} \mid u \in \mathbb{Z}\pi\}. \tag{4.5}$$

Define the *selfintersection element*

$$\mu(\alpha) \quad := \left[\sum_{d \in D} \epsilon(x_0(d), x_1(d)) \cdot g(x_0(d), x_1(d))\right] \quad \in Q_{(-1)^k}(\mathbb{Z}\pi, w). \tag{4.6}$$

The passage from $\mathbb{Z}\pi$ to $Q_{(-1)^k}(\mathbb{Z}\pi, w)$ ensures that the definition is independent of the choice of the order on $f^{-1}(d)$ for $d \in D$. It remains to show that it depends only on the pointed regular homotopy class of (f, w). Let h be a pointed regular homotopy from (f, w) to (g, v). We can arrange that h is in general position. In particular each immersion h_t is in general position and comes with a set D_t. The collection of the D_t-s yields a bordism W from the finite set D_0 to the finite set D_1. Since W is a compact one-dimensional manifold, it consists of circles and arcs joining points in $D_0 \cup D_1$ to points in $D_0 \cup D_1$. Suppose that the point e and the point e' in $D_0 \cup D_1$ are joint by an arc. Then one easily checks that their contributions to

$$\begin{aligned} \mu(f, w) - \mu(g, w) \quad := \quad & \Bigg[\sum_{d_0 \in D_0} \epsilon(x_0(d_0), x_1(d_0)) \cdot g(x_0(d_0), x_1(d_0)) \\ & - \sum_{d_1 \in D_1} \epsilon(x_0(d_1), x_1(d_1)) \cdot g(x_0(d_1), x_1(d_1))\Bigg] \end{aligned}$$

cancel out. This implies $\mu(f, w) = \mu(g, w)$.

Lemma 4.7 *Let $\mu : I_k(M) \to Q_{(-1)^k}(\mathbb{Z}\pi, w)$ be the map given by the selfintersection element (see* (4.6)*) and let $\lambda : I_k(M) \times I_k(M) \to \mathbb{Z}\pi$ be the intersection pairing (see* (4.1)*). Then*

1. *Let $(1 + (-1)^k \cdot T) : Q_{(-1)^k}(\mathbb{Z}\pi, w) \to \mathbb{Z}\pi$ be the homomorphism of abelian groups which sends $[u]$ to $u + (-1)^k \cdot \overline{u}$. Denote for $\alpha \in I_k(M)$ by $\chi(\alpha) \in \mathbb{Z}$ the Euler number of the normal bundle $\nu(f)$ for any representative (f, w) of α with respect to the orientation of $\nu(f)$ given by the standard orientation on S^k and the orientation on f^*TM given by the fixed orientation on T_bM and w. Then*
$$\lambda(\alpha, \alpha) = (1 + (-1)^k \cdot T)(\mu(\alpha)) + \chi(\alpha) \cdot 1;$$
2. *We get for* $\mathrm{pr} : \mathbb{Z}\pi \to Q_{(-1)^k}(\mathbb{Z}\pi, w)$ *the canonical projection and $\alpha, \beta \in I_k(M)$:*
$$\mu(\alpha + \beta) - \mu(\alpha) - \mu(\beta) = \mathrm{pr}(\lambda(\alpha, \beta));$$
3. *For $\alpha \in I_k(M)$ and $u \in \mathbb{Z}\pi$ we get with respect to the obvious $\mathbb{Z}\pi$-bimodule structure on $Q_{(-1)^k}(\mathbb{Z}\pi, w)$*
$$\mu(u \cdot \alpha) = u\mu(\alpha)\overline{u}.$$

Proof : (1) Represent $\alpha \in I_k(M)$ by a pointed immersion (f,w) which is in general position. Choose a section σ of $\nu(f)$ which meets the zero section transversally. Notice that then the Euler number satisfies

$$\nu(f) = \sum_{y \in N(\sigma)} \epsilon(y),$$

where $N(\sigma)$ is the (finite) set of zero points of σ and $\epsilon(y)$ is a sign which depends on the local orientations. We can arrange that no zero of σ is the preimage of an element in the set of double points D_f of f. Now move f a little bit in the direction of this normal field σ. We obtain a new immersion $g : S^k \to M$ with a path v from b to $g(s)$ such that (f,w) and (g,v) are pointed regularly homotopic.

We want to compute $\lambda(\alpha,\alpha)$ using the representatives (f,w) and (g,v). Notice that any point in $d \in D_f$ corresponds to two distinct points $x_0(d)$ and $x_1(d)$ in the set $D = \mathrm{im}(f) \cap \mathrm{im}(g)$ and any element $n \in N(\sigma)$ corresponds to one point $x(n)$ in D. Moreover any point in D occurs as $x_i(d)$ or $x(n)$ in a unique way. Now the contribution of d to $\lambda([(f,w)],[(g,v)])$ is $\epsilon(d) \cdot g(d) + (-1)^k \cdot \epsilon(d) \cdot \overline{g(d)}$ and the contribution of $n \in N(\sigma)$ is $\epsilon(n) \cdot 1$. Now assertion (1) follows. The elementary proof of assertions (2) and (3) is left to the reader. This finishes the proof of Lemma 4.7. ■

Theorem 4.8 *Let M be a compact connected manifold of dimension $n = 2k$. Fix base points $s \in S^k$ and $b \in M$ and an orientation of T_bM. Let (f,w) be a pointed immersion of S^k in M. Suppose that $k \geq 3$. Then (f,w) is pointed homotopic to a pointed immersion (g,v) for which $g : S^k \to M$ is an embedding, if and only $\mu(f) = 0$.*

Proof : If f is represented by an embedding, then $\mu(f,w) = 0$ by definition. Suppose that $\mu(f,w) = 0$. We can assume without loss of generality that f is in general position. Since $\mu(f) = 0$, we can find d and e in the set of double points D_f of f and a numeration $x_0(d), x_1(d)$ of $f^{-1}(d)$ and $x_0(e), x_1(e)$ of $f^{-1}(e)$ such that

$$\begin{aligned} \epsilon(x_0(d), x_1(d)) &= -\epsilon(x_0(e), x_1(e)); \\ g(x_0(d), x_1(d)) &= g(x_0(e), x_1(e)). \end{aligned}$$

Therefore we can find arcs u_0 and u_1 in S^k such that $u_0(0) = x_0(d)$, $u_0(1) = x_0(e)$, $u_1(0) = x_1(d)$ and $u_1(1) = x_1(e)$, the path u_0 and u_1 are disjoint from one another, $f(u_0((0,1)))$ and $f(u_1((0,1)))$ do not meet D_f and $f(u_0)$ and $f(u_1)$ are homotopic relative endpoints. We can change u_0 and u_1 without destroying the properties above and find a smooth map $U : D^2 \to M$ whose restriction to S^1 is an embedding and is given on the upper hemisphere S^1_+ by u_0 and on the lower hemisphere S^1_- by u_1 and which meets $\mathrm{im}(f)$ transversally. There is a compact neighborhood K of S^1 such that $U|_K$ is an embedding. Since $k \geq 3$ we can find arbitrarily close to U' an embedding which agrees with U on K. Hence we can assume without loss of generality that U itself is an embedding. The

Whitney trick (see [46, Theorem 6.6 on page 71], [67]) allows to change f within its pointed regular homotopy class to a new pointed immersion (g, v) such that $D_g = D_f - \{d, e\}$. By iterating this process we achieve $D_f = \emptyset$. ■

Remark 4.9 The condition $\dim(M) \geq 5$ which arises in high-dimensional manifold theory ensures in the proof of Theorem 4.8 that $k \geq 3$ and hence we can arrange U to be an embedding. If $k = 2$, one can achieve that U is an immersion but not necessarily an embedding. This is the technical reason why surgery in dimension 4 is much more complicated than in dimensions ≥ 5.

4.2 Kernels and forms

4.2.1 Symmetric forms and surgery kernels

For the rest of this section we fix a normal map of degree one $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering $f : M \to X$, where M is a closed connected manifold of dimension n and X is a connected finite Poincaré complex of dimension n. Suppose that f induces an isomorphism on the fundamental groups. Fix a base point $b \in M$ together with lifts $\widetilde{b} \in \widetilde{M}$ of b and $\widetilde{f(b)} \in \widetilde{X}$ of $f(b)$. We identify $\pi = \pi_1(M, b) = \pi(X, f(b))$ by $\pi_1(f, b)$. The choice of $\widetilde{b}$ and $\widetilde{f(b)}$ determine π-operations on $\widetilde{M}$ and on $\widetilde{X}$ and a lift $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ which is π-equivariant.

Definition 4.10 *Let $K_k(\widetilde{M})$ be the kernel of the $\mathbb{Z}\pi$-map $H_k(\widetilde{f}) : H_k(\widetilde{M}) \to H_k(\widetilde{X})$. Denote by $K^k(\widetilde{M})$ be the cokernel of the $\mathbb{Z}\pi$-map $H^k(\widetilde{f}) : H^k(\widetilde{X}) \to H^k(\widetilde{M})$ which is the $\mathbb{Z}\pi$-map induced by $C^*(\widetilde{f}) : C^*(\widetilde{X}) \to C^*(\widetilde{M})$.*

Lemma 4.11 1. *The cap product with $[M]$ induces isomorphisms*

$$? \cap [M] : K^{n-k}(\widetilde{M}) \xrightarrow{\cong} K_k(\widetilde{M});$$

2. *Suppose that f is k-connected. Then there is the composition of natural $\mathbb{Z}\pi$-isomorphisms*

$$h_k : \pi_{k+1}(f) \xrightarrow{\cong} \pi_{k+1}(\widetilde{f}) \xrightarrow{\cong} H_{k+1}(\widetilde{f}) \xrightarrow{\cong} K_k(\widetilde{M});$$

3. *Suppose that f is k-connected and $n = 2k$. Then there is a natural $\mathbb{Z}\pi$-homomophism*

$$t_k : \pi_k(f) \to I_k(M).$$

Proof : (1) The following diagram commutes and has isomorphisms as vertical arrows

$$\begin{array}{ccc} H^{n-k}(\widetilde{M}) & \xleftarrow{H^{n-k}(\widetilde{f})} & H^{n-k}(\widetilde{X}) \\ {\scriptstyle ?\cap[M]}\downarrow\cong & & \cong\downarrow{\scriptstyle ?\cap[X]} \\ H_k(\widetilde{M}) & \xrightarrow[H_k(\widetilde{f})]{} & H_k(\widetilde{X}) \end{array} \qquad (4.12)$$

Hence the composition $K_k(\widetilde{M}) \to H_k(\widetilde{M}) \xrightarrow{(? \cap [M])^{-1}} H^{n-k}(\widetilde{M}) \to K^{n-k}(\widetilde{M})$ is bijective.

(2) The commutative square (4.12) above implies that $H_k(\widetilde{f}) : H_k(\widetilde{M}) \to H_k(\widetilde{X})$ is split surjective. We conclude from the long exact sequence of $C_*(\widetilde{f})$ that the boundary map

$$\partial : H_{k+1}(\widetilde{f}) := H_{k+1}(\text{cone}(C_*(\widetilde{f}))) \to H_k(\widetilde{M})$$

induces an isomorphism

$$\partial_{k+1} : H_{k+1}(\widetilde{f}) \xrightarrow{\cong} K_k(\widetilde{M}).$$

Since f and hence $\widetilde{f}$ is k-connected, the Hurewicz homomorphism

$$\pi_{k+1}(\widetilde{f}) \xrightarrow{\cong} H_{k+1}(\widetilde{f})$$

is bijective [65, Corollary IV.7.10 on on page 181]. The canonical map

$$\pi_{k+1}(\widetilde{f}) \to \pi_{k+1}(f)$$

is bijective. The composition of the maps above or their inverses yields a natural isomorphism $h_k : \pi_{k+1}(f) \to K_k(\widetilde{M})$.

(3) is analogous to the proof of Theorem 3.59 (2) which was based on Theorem 3.57 provided one takes the base loops into account. Notice that an element in $\pi_{k+1}(f, b)$ is given by a commutative diagram

$$\begin{array}{ccc} S^k & \xrightarrow{q} & M \\ \downarrow & & \downarrow \\ D^{k+1} & \xrightarrow[Q]{} & X \end{array}$$

together with a path w from b to $f(s)$ for a fixed base point $s \in S^k$. ■

Suppose that $n = 2k$. The Kronecker product $\langle\ ,\ \rangle : H^k(\widetilde{M}) \times H_k(\widetilde{M}) \to \mathbb{Z}\pi$ is induced by the evaluation pairing $\hom_{\mathbb{Z}\pi}(C_p(\widetilde{M}), \mathbb{Z}\pi) \times C_p(\widetilde{M}) \to \mathbb{Z}\pi$ which sends (ϕ, x) to $\phi(x)$. It induces a pairing

$$\langle\ ,\ \rangle : K^k(\widetilde{M}) \times K_k(\widetilde{M}) \to \mathbb{Z}\pi.$$

Together with the isomorphism

$$? \cap [M] : K^{n-k}(\widetilde{M}) \xrightarrow{\cong} K_k(\widetilde{M});$$

of Theorem 4.11 (1) it yields the *intersection pairing*

$$s : K_k(\widetilde{M}) \times K_k(\widetilde{M}) \to \mathbb{Z}\pi. \tag{4.13}$$

We get from Lemma 4.11(2) and (3) a $\mathbb{Z}\pi$-homomorphism

$$\alpha : K_k(\widetilde{M}) \quad \to \quad I_k(\widetilde{M}). \tag{4.14}$$

We leave it to the reader to check

Lemma 4.15 *The following diagram commutes, where the upper pairing is defined in* (4.13)*, the lower pairing in* (4.1) *and the left vertical arrows in* (4.14)

$$\begin{array}{ccc} K_k(\widetilde{M}) \times K_k(\widetilde{M}) & \xrightarrow{s} & \mathbb{Z}\pi \\ {\scriptstyle \alpha\times\alpha}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ I_k(M) \times I_k(M) & \xrightarrow[\lambda]{} & \mathbb{Z}\pi \end{array}$$

The pairing s of (4.13) is the prototype of the following algebraic object which can be defined for any ring R with involution and $\epsilon \in \{\pm 1\}$.

Definition 4.16 *An* ϵ-symmetric form (P, ϕ) *over an associative ring* R *with unit and involution is a finitely generated projective* R*-module* P *together with a* R*-map* $\phi : P \to P^*$ *such that the composition* $P \xrightarrow{e(P)} (P^*)^* \xrightarrow{\phi^*} P$ *agrees with* $\epsilon \cdot \phi$*. Here and elsewhere* $e(P)$ *is the canonical isomorphism sending* $p \in P$ *to the element in* $(P^*)^*$ *given by* $P^* \to R, f \mapsto \overline{f(p)}$*. We call* (P, ϕ) non-degenerate *if* ϕ *is an isomorphism.*

We will sometimes identify P and $(P^*)^*$ by $e(P)$ and denote for an R-map $f : P \to P^*$ the composition $P \xrightarrow{e(P)} (P^*)^* \xrightarrow{f^*} P$ by $f^* : P \to P^*$, and analogously for $f : P^* \to P$.

There are obvious notions of isomorphisms and direct sums of ϵ-symmetric forms.

We can rewrite (P, ϕ) as pairing

$$\lambda : P \times P \to \mathbb{Z}\pi, \qquad (p, q) \mapsto \phi(p)(q).$$

Then the condition that ϕ is R-linear becomes the condition

$$\begin{aligned} \lambda(p, r_1 \cdot q_1 + r_2 \cdot q_2,) &= r_1 \cdot \lambda(p, q_1) + r_2 \cdot \lambda(p, q_2); \\ \lambda(r_1 \cdot p_1 + r_2 \cdot p_2, q) &= \lambda(p_1, q) \cdot \overline{r_1} + \lambda(p_2, q) \cdot \overline{r_2}. \end{aligned}$$

The condition $\phi = \epsilon \cdot \phi^* \circ e(P)$ translates to $\lambda(q, p) \ = \ \epsilon \cdot \overline{\lambda(p, q)}$.

Example 4.17 Let P be a finitely generated projective R-module. The *standard hyperbolic* ϵ*-symmetric form* $H^\epsilon(P)$ is given by the $\mathbb{Z}\pi$-module $P \oplus P^*$ and the R-isomorphism

$$\phi : (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\mathrm{id} \oplus e(P)} P^* \oplus (P^*)^* = (P \oplus P^*)^*.$$

If we write it as a pairing we obtain

$$(P \oplus P^*) \times (P \oplus P^*) \to R, \qquad ((p, \phi), (p', \phi')) \ \mapsto \ \phi(p') + \epsilon \cdot \phi'(p).$$

An example of a non-degenerate $(-1)^k$-symmetric form over $\mathbb{Z}\pi$ with the w-twisted involution is $K_k(\widetilde{M})$ with the pairing s of (4.13), provided that f is k-connected and $n = 2k$. We have to show that $K_k(\widetilde{M})$ is finitely generated projective.

Lemma 4.18 *Let D_* be a finite projective R-chain complex. Suppose for a fixed integer k that $H_i(D_*) = 0$ for $i < r$. Suppose that $H^{r+1}(\hom_R(D_*, V)) = 0$ for any R-module V. Then $\operatorname{im}(d_{r+1})$ is a direct summand in D_r and there are canonical exact sequences*

$$0 \to \ker(d_r) \to D_r \to D_{r-1} \to \ldots \to D_0 \to 0$$

and

$$0 \to \operatorname{im}(d_{r+1}) \to \ker(d_r) \to H_r(D_*) \to 0.$$

In particular $H_r(D_*)$ is finitely generated projective. If $H_i(D_*) = 0$ for $i > r$, we obtain an exact sequence

$$\ldots \to D_{r+3} \xrightarrow{d_{r+3}} D_{r+2} \xrightarrow{d_{r+2}} D_{r+1} \xrightarrow{d_{r+1}} \operatorname{im}(d_{r+1}) \to 0.$$

Proof : If we apply the assumption $H^{r+1}(\hom_R(D_*, V)) = 0$ in the case $V = \operatorname{im}(d_{r+1})$, we obtain an exact sequence

$$\hom_{\mathbb{Z}\pi}(D_r, \operatorname{im}(d_{r+1})) \xrightarrow{\hom_{\mathbb{Z}\pi}(d_{r+1}, \mathrm{id})} \hom_{\mathbb{Z}\pi}(D_{r+1}, \operatorname{im}(d_{r+1}))$$
$$\xrightarrow{\hom_{\mathbb{Z}\pi}(d_{r+2}, \mathrm{id})} \hom_{\mathbb{Z}\pi}(D_{r+2}, \operatorname{im}(d_{r+1})).$$

Since $d_{r+1} \in \hom_{\mathbb{Z}\pi}(D_{r+1}, \operatorname{im}(d_{r+1}))$ is mapped to zero under $\hom_{\mathbb{Z}\pi}(d_{r+2}, \mathrm{id})$, we can find a R-homomorphism $\rho : D_r \to \operatorname{im}(d_{r+1})$ with $\rho \circ d_{r+1} = d_{r+1}$. Hence $\operatorname{im}(d_{r+1})$ is a direct summand in D_r. The other claims are obvious. ■

A R-module V is called *stably finitely generated free* if for some non-negative integer l the R-module $V \oplus R^l$ is a finitely generated free R-module.

Lemma 4.19 *If $f : X \to Y$ is k-connected for $n = 2k$ or $n = 2k+1$, then $K_k(\widetilde{M})$ is stably finitely generated free.*

Proof : We only give the proof for $n = 2k$. The proof for $n = 2k+1$ is along the same lines using Poincare duality for the kernels (see Lemma 4.11 (1)).

Consider the finitely generated free $\mathbb{Z}\pi$-chain complex $D_* := \operatorname{cone}(C_*(\widetilde{f}))$. Its homology $H_p(D_*)$ is by definition $H_p(\widetilde{f})$. Since f is k-connected and D_* is projective, there is a R-subchain complex $E_* \subset D_*$ such that E_* is finite projective, $E_i = 0$ for $i \leq k$ and the inclusion $E_* \to D_*$ is a homology equivalence and hence an R-chain homotopy equivalence. Namely, take $E_i = D_i$ for $i \geq k+2$, $E_{k+1} = \ker(d_{k+1})$ and $E_i = 0$ for $i \leq k$. We get from the commutative square (4.12) a $\mathbb{Z}\pi$-chain homotopy equivalence $D^{n+1-*} \to D_*$. This implies for any R-module V since $\hom_R(D_*, V)$ is chain homotopy equivalent to $\hom_R(E_*, V)$.

$$\begin{aligned} H^{n+1-i}(\hom_R(D_*, V)) &= 0 && \text{for } i \leq k; \\ H_i(D) &= 0 && \text{for } i \geq n+1-k; \end{aligned}$$

We conclude from Lemma 4.11.

$$\begin{aligned} H_{p+1}(D_*) &\cong \begin{cases} K_k(\widetilde{M}) & \text{, if } p = k; \\ 0 & \text{, if } p \neq k; \end{cases} \\ H^{k+2}(\hom_D(D_*, R)) &= 0. \end{aligned}$$

Now apply Lemma 4.18 to D_*. ■

Example 4.20 Consider the normal map $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}}^a \to \xi$ covering the k-connected map of degree one $f : M \to N$ of closed n-dimensional manifolds for $n = 2k$. If we do surgery on the zero element in $\pi_{k+1}(f)$, then the effect on M is that M is replaced by the connected sum $M' = M \sharp (S^k \times S^k)$. The effect on $K_k \widetilde{M})$ is that it is replaced by $K_k(\widetilde{M'}) = K_k(\widetilde{M}) \oplus (\mathbb{Z}\pi \oplus \mathbb{Z}\pi)$. The intersection pairing on this new kernel is the orthogonal sum of the given intersection pairing on $K_k(\widetilde{M})$ together with the standard hyperbolic symmetric form $H^{(-1)^k}(\mathbb{Z}\pi)$. In particular we can arrange by finitely many surgery steps on the trivial element in $\pi_{k+1}(f)$ that $K_k(\widetilde{M})$ is a finitely generated free $\mathbb{Z}\pi$-module.

4.2.2 Quadratic forms and surgery kernels

We have already seen that it will not be enough to study intersections of different immersions, we must also deal with selfintersections of one immersion. We have seen in Lemma 4.7 that we can enrich the intersection pairing by the self-intersection pairing. This leads to the following algebraic analogon for an associative ring R with unit and involution and $\epsilon \in \{\pm 1\}$. For a finitely generated projective R-module P define an involution of R-modules

$$T : \hom_R(P, P^*) \to \hom(P, P^*) \qquad f \mapsto f^* \circ e(P) \tag{4.21}$$

where $e(P) : P \to (P^*)^*$ is the canonical isomorphism.

Definition 4.22 *Let P be a finitely generated projective R-module. Define*

$$\begin{aligned} Q^\epsilon(P) &:= \ker\left((1 - \epsilon \cdot T) : \hom_R(P, P^*) \to \hom_R(P, P^*)\right); \\ Q_\epsilon(P) &:= \operatorname{coker}\left((1 - \epsilon \cdot T) : \hom_R(P, P^*) \to \hom_R(P, P^*)\right). \end{aligned}$$

Let

$$(1 + \epsilon \cdot T) : Q_\epsilon(P) \to Q^\epsilon(P)$$

be the homomorphism which sends the class represented by $f : P \to P^*$ to the element $f + \epsilon \cdot T(f)$.

*A ϵ-*quadratic form *(P, ψ) is a finitely generated projective R-module P together with an element $\psi \in Q_\epsilon(P)$. It is called* non-degenerate *if the associated ϵ-symmetric form $(P, (1+\epsilon \cdot T)(\psi))$ is non-degenerate, i.e. $(1+\epsilon \cdot T)(\psi) : P \to P^*$ is bijective.*

There is an obvious notion of direct sum of two ϵ-quadratic forms. An isomorphism $f : (P, \psi) \to (P', \psi')$ of two ϵ-quadratic forms is an R-isomorphism $f : P \xrightarrow{\cong} P'$ such that the induced map

$$Q_\epsilon(P') \to Q_\epsilon(P), \qquad [\phi : P' \to (P')^*] \mapsto [f^* \circ \phi \circ f : P \to P^*]$$

sends ψ' to ψ.

We can rewrite this as follows. An ϵ-quadratic form (P, ϕ) is the same as a triple (P, λ, μ) consisting of pairing

$$\lambda : P \times P \to R$$

satisfying

$$\begin{aligned} \lambda(p, r_1 \cdot q_1 + r_2 \cdot q_2,) &= r_1 \cdot \lambda(p, q_1) + r_2 \cdot \lambda(p, q_2); \\ \lambda(r_1 \cdot p_1 + r_2 \cdot p_2, q) &= \lambda(p_1, q) \cdot \overline{r_1} + \lambda(p_2, q) \cdot \overline{r_2}; \\ \lambda(q, p) &= \epsilon \cdot \overline{\lambda(p, q)}. \end{aligned}$$

and a map

$$\mu : P \to Q_\epsilon(R) = R/\{r - \epsilon \cdot \overline{r} \mid r \in R\}$$

satisfying

$$\begin{aligned} \mu(rp) &= r\mu(p)\overline{r}; \\ \mu(p + q) - \mu(p) - \mu(q) &= \mathrm{pr}(\lambda(p, q)); \\ \lambda(p, p) &= (1 + \epsilon \cdot T)(\mu(p)), \end{aligned}$$

where $\mathrm{pr} : R \to Q_\epsilon(R)$ is the projection and $(1 + \epsilon \cdot T) : Q_\epsilon(R) \to R$ the map sending the class of r to $r + \epsilon \cdot \overline{r}$. Namely, put

$$\begin{aligned} \lambda(p, q) &= ((1 + \epsilon \cdot T)(\psi))(p))(q); \\ \mu(p) &= \psi(p)(p). \end{aligned}$$

Example 4.23 Let P be a finitely generated projective R-module. The *standard hyperbolic ϵ-quadratic form $H_\epsilon(P)$* is given by the $\mathbb{Z}\pi$-module $P \oplus P^*$ and the class in $Q_\epsilon(P \oplus P^*)$ of the R-homomorphism

$$\phi : (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\mathrm{id} \oplus e(P)} P^* \oplus (P^*)^* = (P \oplus P^*)^*.$$

The ϵ-symmetric form associated to $H_\epsilon(P)$ is $H^\epsilon(P)$.

Example 4.24 An example of a non-degenerate $(-1)^k$-quadratic form over $\mathbb{Z}\pi$ with the w-twisted involution is given as follows, provided that f is k-connected and $n = 2k$. Namely, take $K_k(\widetilde{M})$ with the pairing s of (4.13) and the map

$$t : K_k(\widetilde{M}) \xrightarrow{\alpha} I_k(M) \xrightarrow{\mu} Q_{(-1)^k}(\mathbb{Z}\pi, w), \tag{4.25}$$

where $\mu : I_k(M) \to Q_{(-1)^k}(\mathbb{Z}\pi, w)$ is defined in (4.6) and α is defined in (4.14).

Remark 4.26 Suppose that $1/2 \in R$. Then the homomorphism

$$(1+\epsilon \cdot T) : Q_\epsilon(P) \xrightarrow{\cong} Q^\epsilon(P) \qquad [\psi] \mapsto [\psi + \epsilon \cdot T(\psi)]$$

is bijective. The inverse sends $[u]$ to $[u/2]$. Hence any ϵ-symmetric form carries a unique ϵ-quadratic structure. Hence there is no difference between the symmetric and the quadratic setting if 2 is invertible in R.

The next result is the key step in translating the geometric question, whether we can change a normal map by a finite sequence of surgery steps into a homotopy equivalence to an algebraic question about quadratic forms. It will lead in a natural way to the definition of the surgery obstruction groups $L_{2k}(R)$.

Theorem 4.27 *Consider the normal map $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}}^a \to \xi$ covering the k-connected map of degree one $f : M \to N$ of closed connected n-dimensional manifolds for $n = 2k$. Suppose that $k \geq 3$ and that for the non-degenerate $(-1)^k$-quadratic form $(K_k(\widetilde{M}), s, t)$ there are integers $u, v \geq 0$ together with an isomorphism of non-degenerate $(-1)^k$-quadratic forms*

$$(K_k(\widetilde{M}), s, t) \oplus H_{(-1)^k}(\mathbb{Z}\pi^u) \;\cong\; H_{(-1)^k}(\mathbb{Z}\pi^v).$$

Then we can perform a finite number of surgery steps resulting in a normal map of degree one $(\overline{g}, g) : TM' \oplus \underline{\mathbb{R}}^{a+b} \to \xi \oplus \underline{\mathbb{R}^b}$ such that $g : M' \to X$ is a homotopy equivalence.

Proof : If we do a surgery step on the trivial element in $\pi_{k+1}(f)$, we have explained the effect on $(K_k(\widetilde{M}), t)$ in Example 4.20. The effect on the quadratic form $(K_k(\widetilde{M}), s, t)$ is analogous, one adds a copy of $H_{(-1)^k}(\mathbb{Z}\pi)$. Hence we can assume without loss of generality that the non-degenerate quadratic form $(K_k(\widetilde{M}), s, t)$ is isomorphic to $H_{(-1)^k}(\mathbb{Z}\pi^v)$. Thus we can choose a $\mathbb{Z}\pi$-basis $\{b_1, b_2, \ldots b_v, c_1, c_2, \ldots c_v\}$ for $K_k(\widetilde{M})$ such that

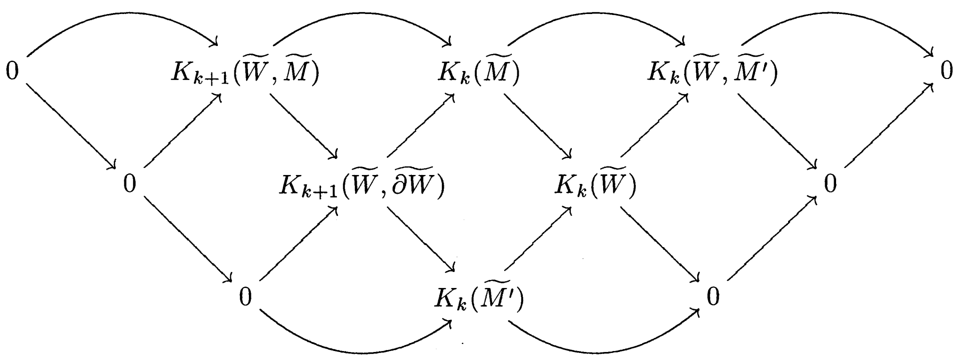
$$\begin{array}{lcll} s(b_i, c_i) & = & 1 & i \in \{1, 2, \ldots v\}; \\ s(b_i, c_j) & = & 0 & i, j \in \{1, 2, \ldots v\},\ i \neq j; \\ s(b_i, b_j) & = & 0 & i, j \in \{1, 2, \ldots v\}; \\ s(c_i, c_j) & = & 0 & i, j \in \{1, 2, \ldots v\}; \\ t(b_i) & = & 0 & i \in \{1, 2, \ldots v\}. \end{array}$$

Notice that f is a homotopy equivalence if and only if the number v is zero. Hence it suffices to explain how we can lower the number v to $(v-1)$ by a surgery step on an element in $\pi_{k+1}(f)$. Of course our candidate is the element ω in $\pi_{k+1}(f)$ which corresponds under the isomorphism $h : \pi_{k+1}(f) \to K_k(\widetilde{M})$ (see Lemma 4.11 (2)) to the element b_v. By construction the composition

$$\pi_{k+1}(f) \xrightarrow{t_k} I_k(M) \xrightarrow{\mu} Q_{(-1)^k}(\mathbb{Z}\pi, w)$$

of the maps defined in (4.6) and in Lemma 4.11 (3) sends ω to zero. Now Theorem 3.59 and Theorem 4.8 ensure that we can perform surgery on ω. Notice that the assumption $k \geq 3$ and the quadratic structure on the kernel become relevant exactly at this point. Finally it remains to check whether the effect on $K_k(\widetilde{M})$ is the desired one, namely, that we get rid of one of the hyperbolic summands $H_\epsilon(\mathbb{Z}\pi)$, or equivalently, v is lowered to $v-1$.

We have explained earlier that doing surgery yields not only a new manifold M' but also a bordism from M to M'. Namely, take $W = M \times [0,1] \cup_{S^k \times D^{n-k}} D^{k+1} \times D^{n-k}$, where we attach $D^{k+1} \times D^{n-k}$ by an embedding $S^k \times D^{n-k} \to M \times \{1\}$, and $M' := \partial W - M$, where we identify $M = M \times \{0\}$. The manifold W comes with a map $F : W \to X \times [0,1]$ whose restriction to M is the given map $f : M = M \times \{0\} \to X = X \times \{0\}$ and whose restriction to M' is a map $f' : M' \to X \times \{1\}$. The definition of the kernels makes also sense for pair of maps. We obtain an exact braid combining the various long exact sequences of pairs



The $(k+1)$-handle $D^{k+1} \times D^{n-k}$ defines an element ϕ^{k+1} in $K_{k+1}(\widetilde{W},\widetilde{M})$ and the associated dual k-handle (see (1.25)) defines an element $\psi^k \in K_k(\widetilde{W},\widetilde{M'})$. These elements constitute a $\mathbb{Z}\pi$-basis for $K_{k+1}(\widetilde{W},\widetilde{M}) \cong \mathbb{Z}\pi$ and $K_k(\widetilde{W},\widetilde{M'}) \cong \mathbb{Z}\pi$. The $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{M}) \to K_k(\widetilde{M})$ maps ϕ to b_v. The $\mathbb{Z}\pi$-homomorphism $K_k(\widetilde{M}) \to K_k(\widetilde{W},\widetilde{M'})$ sends x to $s(b_v,x) \cdot \psi^k$. Hence we can find elements b_1', b_2' , …, b_v' and c_1', c_2' , …, c_{v-1}' in $K_{k+1}(\widetilde{W},\widetilde{\partial W})$ uniquely determined by the property that b_i' is mapped to b_i and c_i' to c_i under the $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{\partial W}) \to K_k(\widetilde{M})$. Moreover, these elements form a $\mathbb{Z}\pi$-basis for $K_{k+1}(\widetilde{W},\widetilde{\partial W})$ and the element ϕ^{k+1} is mapped to b_v' under the $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{M}) \to K_{k+1}(\widetilde{W},\widetilde{\partial W})$. Define b_i'' and c_i'' for $i = 1,2,\ldots(v-1)$ to be the image of b_i' and c_i' under under the $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{\partial W}) \to K_k(\widetilde{M'})$. Then $\{b_i'' \mid i = 1,2\ldots(v-1)\} \coprod \{c_i'' \mid i = 1,2\ldots(v-1)\}$ is a $\mathbb{Z}\pi$-basis for $K_k(\widetilde{M'})$. One easily checks for the quadratic structure (s',t') on $K_k(\widetilde{M'})$

$$\begin{array}{lllll}
s'(b_i'',c_i'') & = & s(b_i,c_i) & = & 1 \qquad i \in \{1,2,\ldots(v-1)\}; \\
s'(b_i'',c_j'') & = & s(b_i,c_j) & = & 0 \qquad i,j \in \{1,2,\ldots(v-1)\},\ i \neq j; \\
s'(b_i'',b_j'') & = & s(b_i,b_j) & = & 0 \qquad i,j \in \{1,2,\ldots(v-1)\}; \\
s'(c_i'',c_j'') & = & s(c_i,c_j) & = & 0 \qquad i,j \in \{1,2,\ldots(v-1)\}; \\
t'(b_i'') & = & t(b_i) & = & 0 \qquad i \in \{1,2,\ldots(v-1)\}.
\end{array}$$

This finishes the proof of Theorem 4.27. ■

4.3 Even dimensional L-groups

Next we define in even dimensions the abelian group, where our surgery obstruction will take values in.

Definition 4.28 *Let R be an associative ring with unit and involution. For an even integer $n = 2k$ define the abelian group $L_n(R)$ called the n-th* quadratic L-group *of R by to be the abelian group of equivalence classes $[(F, \psi)]$ of non-degenerate $(-1)^k$-quadratic forms (F, ψ) whose underlying R-module F is a finitely generated free R-module with respect to the following equivalence relation. We call (F, ψ) and (F', ψ') equivalent if and only if there exists integers $u, u' \geq 0$ and an isomorphism of non-degenerate ϵ-quadratic forms*

$$(F, \psi) \oplus H_\epsilon(R)^u \cong (F', \psi') \oplus H_\epsilon(R)^{u'}.$$

Addition is given by the sum of two ϵ-quadratic forms. The zero element is represented by $[H_\epsilon(R)^u]$ for any integer $u \geq 0$. The inverse of $[F, \psi]$ is given by $[F, -\psi]$.

A morphism $u : R \to S$ of rings with involution induces homomorphism $u_* : L_k(R) \to L_k(S)$ for $k = 0, 2$ by induction. One easily checks $(u \circ v)_* = u_* \circ v_*$ and $(\mathrm{id}_R)_* = \mathrm{id}_{L_k(R)}$ for $k = 0, 2$.

Before we come to the surgery obstruction, we will present a criterion for an ϵ-quadratic form (P, ψ) to represent zero in $L_{1-\epsilon}(R)$ which we will later need at several places. Let (P, ψ) be a ϵ-quadratic form. A *subLagrangian* $L \subset P$ is a R-submodule such that the inclusion $i : L \to P$ is split injective, the image of ψ under the map $Q_n(i) : Q_n(P) \to Q_n(L)$ is zero and L is contained in his annihilator $L^\perp$ which is by definition the kernel of

$$P \xrightarrow{(1+\epsilon \cdot T)(\psi)} P^* \xrightarrow{i^*} L^*.$$

A subLagrangian $L \subset P$ is called *Lagrangian* if $L = L^\perp$. Equivalenty, a Lagrangian $L \subset P$ is a R-submodule L with inclusion $i : L \to P$ such that the sequence

$$0 \to L \xrightarrow{i} P \xrightarrow{i^* \circ (1+\epsilon \cdot T)(\psi)} P^* \to 0.$$

is exact.

Lemma 4.29 *Let (P, ψ) be an ϵ-quadratic form. Let $L \subset P$ be a subLagrangian. Then L is a direct summand in $L^\perp$ and ψ induces the structure of a non-degenerate ϵ-quadratic form $(L/L^\perp, \psi^\perp/\psi)$. Moreover, the inclusion $i : L \to P$ extends to an isomorphism of ϵ-quadratic forms*

$$H_\epsilon(L) \oplus (L^\perp/L, \psi^\perp/\psi) \xrightarrow{\cong} (P, \psi).$$

In particular a non-degenerate ϵ-quadratic form (P, ψ) is isomorphic to $H_\epsilon(Q)$ if and only if it contains a Lagrangian $L \subset P$ which is isomorphic as R-module to Q. The analogous statement holds for ϵ-symmetric forms.

Proof : Choose an R-homomorphism $s : L^* \to P$ such that $i^* \circ (1 + \epsilon \cdot T)(\psi) \circ s$ is the identity on L^*. Our first attempt is the obvious split injection $i \oplus s : L \oplus L^* \to P$. The problem is that it is not necessarily compatible with the ϵ-quadratic structure. To be compatible with the ϵ-quadratic structure it is necessary to be compatible with the ϵ-symmetric symmetric structure, i.e. the following diagram must commute

$$\begin{array}{ccc} L \oplus L^* & \xrightarrow{i \oplus s} & P \\ \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \Big\downarrow & & \Big\downarrow \psi + \epsilon \cdot T(\psi) \\ L^* \oplus L & \xleftarrow[i^* + s^*]{} & P^* \end{array}$$

The diagram above commutes if and only if $s^* \circ (\psi + \epsilon \cdot \psi^*) \circ s = 0$. Notice that s is not unique, we can replace s by $s' = s + i \circ v$ for any R-map $v : L^* \to L$. For this new section s' the diagram above commutes if and only if $s^* \circ (\psi + \epsilon \cdot T(\psi) \circ s + v^* + \epsilon \cdot v = 0$. This suggests to take $v = -\epsilon s^* \circ \psi \circ s$. Now one easily checks that

$$g := i \oplus (s - \epsilon \cdot i \circ s^* \circ \psi \circ s) : L \oplus L^* \to P$$

is split injective and compatible with the ϵ-quadratic structures and hence induces a morphism $g : H_\epsilon(L) \to (P, \psi)$ of ϵ-quadratic forms.

Let $\mathrm{im}(g)^\perp$ be the annihilator of $\mathrm{im}(g)$. Denote by $j : \mathrm{im}(g) \to P$ the inclusion. We obtain an isomorphism of ϵ-quadratic forms

$$g \oplus j : H_\epsilon(L) \oplus (\mathrm{im}(g)^\perp, j^* \circ \psi \circ j) \to (P, \psi).$$

The inclusion $L^\perp \to \mathrm{im}(g)^\perp$ induces an isomorphism $h : L^\perp/L \to \mathrm{im}(g)^\perp$. Let $\psi^\perp/\psi$ be the ϵ-quadratic structure on $L^\perp/L$ for which h becomes an isomorphism of ϵ-quadratic forms. This finishes the proof in the quadratic case. The proof in the symmetric case is analogous. ■

Finally we state the computation of the even-dimensional L-groups of the integers. Consider an element (P, ϕ) in $L_0(\mathbb{Z})$. By tensoring over $\mathbb{Z}$ with $\mathbb{R}$ and only taking the symmetric structure into account we obtain a non-degenerate symmetric $\mathbb{R}$-bilinear pairing $\lambda : \mathbb{R} \otimes_{\mathbb{Z}} P \times \mathbb{R} \otimes_{\mathbb{Z}} P \to \mathbb{R}$. It turns out that its signature is always divisible by eight.

Theorem 4.30 *The signature defines an isomorphism*

$$\frac{1}{8} \cdot \mathrm{sign} : L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}, \qquad [P, \psi] \mapsto \frac{1}{8} \cdot \mathrm{sign}(\mathbb{R} \otimes_{\mathbb{Z}} P, \lambda).$$

Consider a non-degenerate quadratic form (P, ψ) over the field $\mathbb{F}_2$ of two elements. Write (P, ψ) as a triple (P, ϕ, μ) as explained above. Choose any symplectic basis $\{b_1, b_2, \dots b_{2m}\}$ for P, where symplectic means that $\lambda(x_i, x_j)$ is 1 if $i - j = m$ and 0 otherwise. Define the *Arf invariant* of (P, ψ) by

$$\operatorname{Arf}(P, \psi) := \sum_{i=1}^{m} \mu(b_i) \cdot \mu(b_{i+m}) \qquad \in \mathbb{Z}/2. \tag{4.31}$$

The Arf invariant defines an isomorphism

$$\operatorname{Arf} : L_2(\mathbb{F}_2) \xrightarrow{\cong} \mathbb{Z}/2.$$

The change of rings homomorphism $\mathbb{Z} \to \mathbb{F}_2$ induces an isomorphism

$$L_2(\mathbb{Z}) \xrightarrow{\cong} L_2(\mathbb{F}_2).$$

Theorem 4.32 *The Arf invariant defines an isomorphism*

$$\operatorname{Arf} : L_2(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/2, \qquad [(P, \psi)] \mapsto \operatorname{Arf}(\mathbb{F}_2 \otimes_{\mathbb{Z}} (P, \psi)).$$

For more information about forms over the integers and the Arf invariant we refer for instance to [9], [49].

4.4 The surgery obstruction in even dimensions

Consider a normal map of degree one $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering $f : M \to X$, where M is a closed oriented manifold of dimension n and X is a connected finite Poincaré complex of dimension n for even $n = 2k$. To these data we want to assign an element $\sigma(\overline{f}, f) \in L_n(\mathbb{Z}\pi, w)$ such that the following holds

Theorem 4.33 (Surgery obstruction theorem in even dimensions) *We get under the conditions above:*

1. *Suppose $k \geq 3$. Then $\sigma(\overline{f}, f) = 0$ in $L_n(\mathbb{Z}\pi, w)$ if and only if we can do a finite number of surgery steps to obtain a normal map $(\overline{f'}, f) : TM' \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ which covers a homotopy equivalence $f' : M' \to X$;*

2. *The surgery obstruction $\sigma(\overline{f}, f)$ depends only on the normal bordism class of $(\overline{f}, f)$.*

We first explain the definition of the surgery obstruction $\sigma(\overline{f}, f)$. By finitely many surgery steps we can achieve that f is k-connected (see Theorem 3.61). By a finite number of surgery steps in the middle dimension we can achieve that the surgery kernel $K_k(\widetilde{M})$ is a finitely generated free $\mathbb{Z}\pi$-module (see Example 4.20). We have already explained in Example 4.24 that $K_k(\widetilde{M})$ carries the structure $(K_k(\widetilde{M}), t, s)$ of a non-degenerate $(-1)^k$-quadratic form. We want to define the *surgery obstruction* of $(\overline{f}, f)$

$$\sigma(\overline{f}, f) \quad := \quad [K_k(\widetilde{M}), t, s] \quad \in L_n(\mathbb{Z}\pi, w). \tag{4.34}$$

We have to show that this is independent of the surgery steps we have performed to make f k-connected and $K_k(\widetilde{M})$ finitely generated free. Notice that surgery does not change the normal bordism class. Hence it suffices to show that $\sigma(\overline{f}, f)$ and $\sigma(\overline{f'}, f')$ define the same element in $L_n(\mathbb{Z}\pi, w)$ if $f : M \to X$ and $f' : M' \to X$ are k-connected and normally bordant and their kernels $K_k(\widetilde{M})$ and $K_k(\widetilde{M'})$ are finitely generated free $\mathbb{Z}\pi$-modules. Notice that this also will prove Theorem 4.33 (2).

Consider a normal bordism $(\overline{F}, F) : TW \oplus \underline{\mathbb{R}^a} \to \eta$ covering a map $F : W \to X \times [0,1]$ such that ∂W is $M^- \cup M'$, $F(M^-) \subset X \times \{0\}$, $F(M') \subset X \times \{1\}$, the restriction of $\overline{F}$ to M^- is $\overline{f}$ and the restriction of $\overline{F}$ to M' is $\overline{f'}$. By surgery on the interior of W we can change W and F leaving ∂W and $F|_{\partial W}$ fixed such that F is k-connected. The proof of Theorem 3.61 carries directly over. By a handle subtraction argument (see [64, Theorem 1.4 on page 14, page 50]) we can achieve that $K_k(\widetilde{W}, \widetilde{\partial W}) = 0$. This handle subtraction leaves M fixed but may change M'. But the change on the surgery kernel of M' is adding a standard ϵ-hyperbolic form which does not change the class in the L-group. Moreover, f, f' and F remain k-connected. So we can assume without loss of generality that $K_i(\widetilde{W}, \widetilde{\partial W}) = 0$ for $i \le k$ and $K_i(\widetilde{\partial W})) = 0$ for $i \le k-1$. We know already that $K_k(\widetilde{\partial W})$ is stably finitely generated free by Lemma 4.18. A similar argument shows that $K_{k+1}(\widetilde{W}, \widetilde{\partial W})$ and $K_k(\widetilde{W})$ are stably finitely generated free.

We obtain an exact sequence of $\mathbb{Z}\pi$-modules

$$0 \to K_{k+1}(\widetilde{W}, \widetilde{\partial W}) \xrightarrow{\partial_{k+1}} K_k(\widetilde{\partial W}) \xrightarrow{K_k(\widetilde{i})} K_k(\widetilde{W}) \to 0. \tag{4.35}$$

where $\widetilde{\partial W}$ and $\widetilde{i} : \widetilde{\partial W} \to \widetilde{W}$ come from the pullback construction applied to the universal covering $\widetilde{W} \to W$ and the inclusion $i : \partial W \to W$.

The strategy of the proof is to show that the image of $\partial_{k+1} : K_{k+1}(\widetilde{W}, \widetilde{\partial W}) \to K_k(\widetilde{\partial W})$ is a Lagrangian for the non-degenerate ϵ-quadratic form $K_k(\widetilde{\partial W})$. Before we do this we explain how the claim follows then. Notice that $\partial W = M^- \coprod M'$. Since F, f and f' are k-connected and $k \ge 2$, we get identifications $\pi = \pi_1(X) = \pi_1(M) = \pi_1(M') = \pi_1(W)$ and $\widetilde{\partial W}$ is the disjoint union of the universal coverings $\widetilde{M}$ and $\widetilde{M'}$. Hence we get in $L_{1-\epsilon}(\mathbb{Z}\pi, w)$

$$\begin{aligned}[(K_k(\widetilde{\partial W}), t'', s'')] &= [(K_k(M), -t, -s)] + [(K_k(M'), s', t')] \\ &= -[(K_k(M), t, s)] + [(K_k(M'), s', t')].\end{aligned}$$

If we can construct the Lagrangian above for $(K_k(\widetilde{\partial W}), t'', s'')$, we conclude $[(K_k(\widetilde{\partial W}), t'', s'')] = 0$ in $L_{1-\epsilon}(R)$ from Lemma 4.29. This implies the desired equation in $L_{1-\epsilon}(\mathbb{Z}\pi, w)$

$$[(K_k(M), -t, -s)] \;=\; [(K_k(M'), s', t')].$$

It remains to show that $\mathrm{im}(\partial_{k+1})$ is a Lagrangian.

This is rather easy for the symmetric structure. We first show for any $x, y \in K_{k+1}(\widetilde{W}, \widetilde{\partial W})$ that $s(\partial_{k+1}(x), \partial_{k+1}(y)) = 0$. The following diagram commutes and has isomorphisms as vertical arrows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{k+1}(\widetilde{W}, \widetilde{\partial W}) & \xrightarrow{\partial_{k+1}} & K_k(\widetilde{\partial W}) & \xrightarrow{K_k(\widetilde{i})} & K_k(\widetilde{W}) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle ?\cap[W,\partial W]} & & \uparrow{\scriptstyle ?\cap[\partial W]} & & \uparrow{\scriptstyle ?\cap[W,\partial W]} & & \\ 0 & \longrightarrow & K^k(\widetilde{W}) & \xrightarrow{K^k(\widetilde{i})} & K^k(\widetilde{\partial W}) & \xrightarrow{\delta^k} & K^{k+1}(\widetilde{W}, \widetilde{\partial W}) & \longrightarrow & 0 \end{array}$$

We have for $x, y \in K_{k+1}(\widetilde{W}, \widetilde{\partial W})$

$$\begin{aligned} s(\partial_{k+1}(x), \partial_{k+1}(y)) &= \langle (? \cap [\partial W])^{-1} \circ \partial_{k+1}(x), \partial_{k+1}(y) \rangle \\ &= \langle (K^k(\widetilde{i}) \circ (? \cap [W, \partial W])^{-1}(x), \partial_{k+1}(y) \rangle \\ &= \langle (\delta^k \circ K^k(\widetilde{i}) \circ (? \cap [W, \partial W])^{-1}(x), y \rangle \\ &= \langle (0 \circ (? \cap [W, \partial W])^{-1}(x), y \rangle \\ &= 0. \end{aligned}$$

Now suppose for $x \in K_k(\widetilde{M})$ that $s(x, \partial_{k+1}(y)) = 0$ for all $y \in K_{k+1}(\widetilde{W}, \widetilde{\partial W})$. We must show $x \in \mathrm{im}(\partial_{k+1})$. This is equivalent to $(? \cap [W, \partial W])^{-1} \circ K_k(\widetilde{i})(x) = 0$. Since $K_p(\widetilde{W}, \widetilde{\partial W}) = 0$ for $p \leq k$ and finitely generated projective for $p = k+1$, the canonical map

$$K^{k+1}(\widetilde{W}, \widetilde{\partial W}) \xrightarrow{\cong} \hom_{\mathbb{Z}\pi}(K_{k+1}(\widetilde{W}, \widetilde{\partial W}), \mathbb{Z}\pi), \qquad \alpha \mapsto \langle \alpha, ? \rangle$$

is bijective by an elementary chain complex argument or by the universal coefficient spectral sequence. Hence the claim follows from the following calculation for $y \in K_{k+1}(\widetilde{M})$

$$\begin{aligned} \langle (? \cap [W, \partial W])^{-1} \circ K_k(\widetilde{i})(x), y \rangle &= \langle \delta^k \circ (? \cap [\partial W])^{-1}(x), y \rangle \\ &= \langle (? \cap [\partial W])^{-1}(x), \partial_{k+1}(y) \rangle \\ &= 0. \end{aligned}$$

Thus we have shown that $\mathrm{im}(\partial_{k+1})$ is a Lagrangian for the non-degenerate ϵ-symmetric form $[K_k(\widetilde{\partial}), s)]$. It remains to show that it is also a Lagrangian for the non-degenerate ϵ-quadratic form $[K_k(\widetilde{\partial}), s)]$. In other words, we must show that t vanishes on $\mathrm{im}(\partial_{k+1})$. We sketch the idea of the proof.

Consider $x \in K_{k+1}(\widetilde{W}, \widetilde{\partial W})$. We can find a smooth map $(g, \partial g) : (S, \partial S) \to (\widetilde{W}, \widetilde{\partial W})$ such that S is obtained from S^{k+1} by removing a finite number of open embedded disjoint discs D^{k+1} and the image of the fundamental class $[S, \partial S]$ under the map on homology induced by $(g, \partial g)$ is x. Moreover we can assume that $\partial g : \partial S \to \widetilde{\partial W}$ is an immersion and g is in general position. We have to show that $\mu(\partial g)$ is zero, where $\mu(\partial g)$ the sum of the self intersection numbers of

$(\partial g)|_C : C \to \widetilde{W}$ for $C \in \pi_0(\partial S)$. Since g is in general position, the set of double points consists of circles which do not concern us and arcs whose end points are on $\partial\widetilde{W}$. Now one shows for each arc that the contributions of its two end points to the self intersection number $\mu(\partial g)$ cancel out. This proves $\mu(\partial_{k+1}(x)) = 0$. This finishes the proof that the surgery obstruction is well-defined and depends only on the normal bordism class. Thus we have proven Theorem 4.33 (2). Assertion (1) of Theorem 4.33 is a direct consequence of Theorem 4.27. This finishes the proof of Theorem 4.33. ■

Now we can give a rather complete answer to Problem 3.1 and Problem 3.62 for even-dimensions and in the simply connected case.

Theorem 4.36 *1. Let $(\overline{f}, f)$ be normal map from a closed manifold M to a simply connected finite Poincaré complex X of dimension $n = 4k \geq 5$. Then we can change M and f leaving X fixed by finitely many surgery steps to get a normal map $(\overline{g}, g)$ from a closed manifold N to X such that g is a homotopy equivalence if and only if* $\operatorname{sign}(M) = \operatorname{sign}(X)$*;*

2. Let $(\overline{f}, f)$ be normal map from a closed manifold M to a simply connected finite Poincaré complex X of dimension $n = 4k+2 \geq 5$. Then we can change M and f leaving X fixed by finitely many surgery steps to get a normal map $(\overline{g}, g)$ from a closed manifold N to X such that g is a homotopy equivalence if and only if the Arf invariant taking values in $\mathbb{Z}/2$ vanishes;

3. Let X be a simply connected finite Poincaré complex of dimension $n = 4k \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle $\xi : E \to X$, i.e. the set of normal invariants $\mathcal{T}_n(X)$ is non-empty, such that

$$\langle \mathcal{L}(\xi)^{-1}, [X]\rangle \ = \ \operatorname{sign}(X);$$

4. Let X be a simply-connected finite connected Poincaré complex of dimension $n = 4k+2 \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle $\xi : E \to X$, i.e. the set of normal invariants $\mathcal{T}_n(X)$ is non-empty such that the Arf invariant of the associated surgery problem, which takes values in $\mathbb{Z}/2$, vanishes.

Proof : (1) Because of Theorem 4.33 and Theorem 4.30 we have to show for a $2k$-connected normal map of degree one $f : M \to X$ from a closed simply connected oriented manifold M of dimension $n = 4k$ to a simply connected Poincaré complex X of dimension $n = 4k$ that for the non-degenerate symmetric bilinear form $\mathbb{R} \otimes_{\mathbb{Z}} (K_{2k}(M), \lambda)$ induced by the intersection pairing we get

$$\operatorname{sign}\left(\mathbb{R} \otimes_{\mathbb{Z}} (K_{2k}(M), \lambda)\right) \ = \operatorname{sign}(M) - \operatorname{sign}(X).$$

This follows from elementary considerations about signatures and from the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{2k}(M) & \longrightarrow & H_{2k}(M) & \xrightarrow{H_{2k}(f)} & H_{2k}(X) & \longrightarrow & 0 \\
 & & {\scriptstyle ?\cap[M]}\uparrow & & {\scriptstyle ?\cap[M]}\uparrow & & {\scriptstyle ?\cap[X]}\uparrow & & \\
0 & \longleftarrow & K^{2k}(M) & \longleftarrow & H^{2k}(M) & \xleftarrow[H^{2k}(f)]{} & H^{2k}(X) & \longleftarrow & 0
\end{array}$$

(2) follows from Theorem 3.49, Theorem 4.33 and Theorem 4.32.

(3) follows from (1) and the Hirzebruch signature formula which implies for a surgery problem $(\overline{f}, f) : \nu(M) \to \xi$ covering $f : M \to X$ which is obtained from a reduction ξ of the Spivak normal fibration of X

$$\operatorname{sign}(M) \;=\; \langle \mathcal{L}(TM), [M] \rangle \;=\; \langle \mathcal{L}(\nu(M) \oplus \underline{\mathbb{R}^a})^{-1}, [M] \rangle \;=\; \langle \mathcal{L}(\xi)^{-1}, [X] \rangle.$$

(4) follows from Theorem 3.49 and (2). ■

4.5 Formations and odd dimensional L-groups

In this subsection we explain the algebraic objects which describe the surgery obstruction and will be the typical elements in the surgery obstruction group in odd dimensions. Throughout this section R will be an associative ring with unit and involution and $\epsilon \in \{\pm 1\}$.

Definition 4.37 *An* ϵ-quadratic formation $(P, \psi; F, G)$ *consists of a non-degenerate ϵ-quadratic form (P, ψ) together with two Lagrangians F and G.*

An isomorphism $f : (P, \psi; F, G) \to (P', \psi'; F,', G')$ of ϵ-quadratic formations is an isomorphism $f : (P, \psi) \to (P', \psi')$ of non-degenerate ϵ-quadratic forms such that $f(F) = F'$ and $f(G) = G'$ holds.

Definition 4.38 *The* trivial ϵ-quadratic formation *associated to a finitely generated projective R-module P is the formation $(H_\epsilon(P); P, P^*)$. A formation $(P, \psi; F, G)$ is called* trivial *if it isomorphic to the trivial ϵ-quadratic formation associated to some finitely generated projective R-module. Two formations are* stably isomorphic *if they become isomorphic after taking the direct sum of trivial formations.*

Remark 4.39 We conclude from Lemma 4.29 that any formation is isomorphic to a formation of the type $(H_\epsilon(P); P, F)$ for some Lagrangian $F \subset P \oplus P^*$. Any automorphism $f : H_\epsilon(P) \xrightarrow{\cong} H_\epsilon(P)$ of the standard hyperbolic ϵ-quadratic form $H_\epsilon(P)$ for some finitely generated projective R-module P defines a formation by $(H_\epsilon(P); P, f(P))$.

Consider a formation $(P, \psi; F, G)$ such that P, F and G are finitely generated free and suppose that R has the property that R^n and R^m are R-isomorphic if and only if $n = m$. Then $(P, \psi; F, G)$ is stably isomorphic to $(H_\epsilon(Q); Q, f(Q))$ for some finitely generated free R-module Q by the following argument. Because of Lemma 4.29 we can choose isomorphisms of non-degenerate ϵ-quadratic forms $f : H_\epsilon(F) \xrightarrow{\cong} (P, \psi)$ and $g : H_\epsilon(G) \xrightarrow{\cong} (P, \psi)$ such that $f(F) = F$ and $g(G) = G$. Since $F \cong R^a$ and $G \cong R^b$ by assumption and $R^{2a} \cong F \oplus F^* \cong P \cong G \oplus G^* \cong R^{2b}$, we conclude $a = b$. Hence we can choose an R-isomorphism $u : F \to G$. Then we obtain an isomorphism of non-degenerate ϵ-quadratic forms by the composition

$$v : H_\epsilon(F) \xrightarrow{H_\epsilon(u)} H_\epsilon(G) \xrightarrow{g} (P, \psi) \xrightarrow{f^{-1}} H_\epsilon(F)$$

and an isomorphism of ϵ-quadratic formations

$$f : (H_\epsilon(F); F, v(F)) \xrightarrow{\cong} (P, \psi; F, G).$$

Recall that $K_1(R)$ is defined in terms of automorphisms of finitely generated free R-modules. Hence it is plausible that the odd-dimensional L-groups will be defined in terms of formations which is essentially the same as in terms of automorphisms of the standard hyperbolic form over a finitely generated free R-module.

Definition 4.40 *Let (P, ψ) be a (not necessarily non-degenerate) $(-\epsilon)$-quadratic form. Define its* boundary $\partial(P, \psi)$ *to be the ϵ-quadratic formation $(H_\epsilon(P); P, \Gamma_\psi)$, where Γ_ψ is the Lagrangian given by the image of the R-homomorphism*

$$P \to P \oplus P^*, \qquad x \mapsto (x, (1 - \epsilon \cdot T)(\psi)(x)).$$

One easily checks that Γ_ψ appearing in Definition 4.40 is indeed a Lagrangian. Two Lagrangians F, G of a non-degenerate ϵ-quadratic form (P, ψ) are called *complementary* if $F \cap G = \{0\}$ and $F + G = P$.

Lemma 4.41 *Let $(P, \psi; F, G)$ be an ϵ-quadratic formation. Then:*

1. *$(P, \psi; F, G)$ is trivial if and only F and G are complementary to one another;*
2. *$(P, \psi; F, G)$ is isomorphic to a boundary if and only if there is a Lagrangian $L \subset P$ such that L is a complement of both F and G;*
3. *There is an ϵ-quadratic formation $(P', \psi'; F', G')$ such that $(P, \psi; F, G) \oplus (P', \psi'; F', G')$ is a boundary;*
4. *An $(-\epsilon)$-quadratic form (Q, μ) is non-degenerate if and only if its boundary is trivial.*

Proof : (1) The inclusions of F and G in P induce an R-isomorphism $f : F \oplus G \to P$. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : F \oplus G \to (F \oplus G)^* = F^* \oplus G^*$$

be $f^* \circ \psi \circ f$ for some representative $\psi : P \to P^*$ of $\psi \in Q_\epsilon(P)$ and let $\psi' \in Q_\epsilon(F \oplus G)$ be the associated class. Then f is an isomorphism of non-degenerate ϵ-quadratic forms $(F \oplus G, \psi') \to (P, \psi)$ and F and G are Lagrangians in $(F \oplus G, \psi')$. This implies that the isomorphism $\psi' + \epsilon \cdot T(\psi')$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \epsilon \cdot T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + \epsilon \cdot a^* & b + \epsilon c^* \\ c + \epsilon b^* & d + \epsilon d^* \end{pmatrix} = \begin{pmatrix} 0 & e \\ f & 0 \end{pmatrix}$$

Hence $(b + \epsilon \cdot c^*) : G \to F^*$ is an isomorphism. Define an R-isomorphism

$$u : \begin{pmatrix} 1 & 0 \\ 0 & (b + \epsilon \cdot c^*)^{-1} \end{pmatrix} : F \oplus F^* \xrightarrow{\cong} P.$$

One easily checks that u defines an isomorphism of formations

$$u : (H_\epsilon(F); F, F^*) \xrightarrow{\cong} (F \oplus G, \psi'; F, G).$$

(2) One easily checks that for an $(-\epsilon)$-quadratic form (P, ϕ) the Lagrangian P^* in its boundary $\partial(P, \psi) := (H_\epsilon(P); P, \Gamma_\psi)$ is complementary to both P and Γ_ψ. Conversely, suppose that $(P, \psi; F, G)$ is an ϵ-quadratic formation such that there exists a Lagrangian $L \subset P$ which is complementary to both F and G. By the argument appearing in the proof of assertion (1) we find an isomorphism of ϵ-quadratic formations

$$f : (H_\epsilon(F); F, F^*) \xrightarrow{\cong} (P, \psi; F, L)$$

which is the identity on F. The preimage $G' := f^{-1}(G)$ is a Lagrangian in $H_\epsilon(F)$ which is complementary to F^*. Write the inclusion of G' into $F \oplus F^*$ as $(a, b) : G' \to F \oplus F^*$. Consider the $(-\epsilon)$-quadratic form (F, ψ'), where $\psi' \in Q_{-\epsilon}(F)$ is represented by $b \circ a^* : F \to F^*$. One easily checks that its boundary is precisely $(H_\epsilon(F); F, G')$ and f induces an isomorphism of ϵ-quadratic formations

$$\partial(F, \psi') = (H_\epsilon(F); F, G') \xrightarrow{\cong} (P, \psi; F, G).$$

(3) Because of Lemma 4.29 we can find Lagrangians F' and G' such that F and F' are complementary and G and G' are complementary. Put $(P', \psi'; F', G') = (P, -\psi, F', G')$. Then $M = \{(p, p) \mid p \in P\} \subset P \oplus P$ is a Lagrangian in the direct sum

$$(P, \psi; F, G) \oplus (P', \psi'; F', G') = (P \oplus P, \psi \oplus (-\psi), F \oplus F', G \oplus G')$$

which is complementary to both $F \oplus F'$ and $G \oplus G'$. Hence the direct sum is isomorphic to a boundary by assertion (2).

(4) The Lagrangian Γ_ψ in the boundary $\partial(Q, \mu) := H_\epsilon(P); P, \Gamma_\psi)$ is complementary to P if and only if $(1 - \epsilon \cdot T)(\mu) : P \to P^*$ is an isomorphism. This finishes the proof of Lemma 4.41. ■ ■

Now we can define the odd-dimensional surgery groups.

Definition 4.42 *Let R be an associative ring with unit and involution. For an odd integer $n = 2k+1$ define the abelian group $L_n(R)$ called the* n-th quadratic L-group *of R to be the abelian group of equivalence classes $[(P,\psi;F,G)]$ of $(-1)^k$-quadratic formations $(P,\psi;F,G)$ such that P, F and G are finitely generated free with respect to the following equivalence relation. We call $(P,\psi;F,G)$ and $(P',\psi';F',G')$ equivalent if and only if there exists $(-\epsilon)$-quadratic forms (Q,μ) and (Q',μ') for finitely generated free R-modules Q and Q' and finitely generated free R-modules S and S' together with an isomorphism of ϵ-quadratic formations*

$$\begin{aligned}(P,\psi;F,G) \oplus \partial(Q,\mu) \oplus (H_\epsilon(S);S,S^*) \\ \cong \quad (P',\psi';F',G') \oplus \partial(Q',\mu') \oplus (H_\epsilon(S');S',(S')^*).\end{aligned}$$

Addition is given by the sum of two ϵ-quadratic forms. The zero element is represented by $\partial(Q,\mu) \oplus (H_\epsilon(S);S,S^*)$ for any $(-\epsilon)$-quadratic forms (Q,μ) for finitely generated free R-modules Q and any finitely generated free R-modules S. The inverse of $[(P,\psi;F,G)]$ is represented by $(P,-\psi;F',G')$ for any choice of Lagrangians F' and G' in $H_\epsilon(P)$ such that F and F' are complementary and G and G' are complementary.

A morphism $u : R \to S$ of rings with involution induces homomorphisms $u_* : L_k(R) \to L_k(S)$ for $k = 1,3$ by induction. One easily checks $(u \circ v)_* = u_* \circ v_*$ and $(\mathrm{id}_R)_* = \mathrm{id}_{L_k(R)}$ for $k = 1,3$.

The odd-dimensional L-groups of the ring of integers vanish.

Theorem 4.43 *We have $L_{2k+1}(\mathbb{Z}) = 0$.*

4.6 The surgery obstruction in odd dimensions

Consider a normal map of degree one $(\overline{f},f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering $f : M \to X$, where M is a closed oriented manifold of dimension n and X is a connected finite Poincaré complex of dimension n for odd $n = 2k+1$. To these data we want to assign an element $\sigma(\overline{f},f) \in L_n(\mathbb{Z}\pi, w)$ such that the following holds

Theorem 4.44 (Surgery obstruction theorem in odd dimensions) *We get under the conditions above:*

1. *Suppose $k \geq 2$. Then $\sigma(\overline{f},f) = 0$ in $L_n(\mathbb{Z}\pi,w)$ if and only if we can do a finite number of surgery steps to obtain a normal map $(\overline{f'},f') : TM' \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ covering a homotopy equivalence $f' : M' \to X$;*
2. *The surgery obstruction $\sigma(\overline{f},f)$ depends only on the normal bordism class of $(\overline{f},f)$.*

We can arrange by finitely many surgery steps that f is k-connected (see Theorem 3.61). Consider a normal bordism $(\overline{F},F) : TW \oplus \underline{\mathbb{R}^b} \to \eta$ covering a map $F : W \to X$ of degree one from the given normal map $(\overline{f},f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$

covering $f : M \to X$ to a new k-connected normal map $(\overline{f'}, f') : TM' \oplus \underline{\mathbb{R}^{a'}} \to \xi'$ covering $f' : M' \to X$. We finally want to arrange that f' is a homotopy equivalence. By applying Theorem 3.61 to the interior of W without changing ∂W we can arrange that W is $(k+1)$-connected. The kernels fit into exact sequences

$$0 \to K_{k+1}(\widetilde{M'}) \xrightarrow{K_{k+1}(\widetilde{j})} K_{k+1}(\widetilde{W}) \xrightarrow{K_{k+1}(\widetilde{i'})} K_{k+1}(\widetilde{W}, \widetilde{M'}) \xrightarrow{\partial_{k+1}} K_k(\widetilde{M'}) \to 0,$$

where $j : M' \to W$ and $i' : W \to (W, M')$ are the inclusions. Hence f' is a homotopy equivalence if and only if $K_{k+1}(\widetilde{i'}) : K_{k+1}(\widetilde{W}) \to K_{k+1}(\widetilde{W}, \widetilde{M'})$ is bijective. Therefore we must arrange that $K_{k+1}(\widetilde{i'}) : K_{k+1}(\widetilde{W}) \to K_{k+1}(\widetilde{W}, \widetilde{M'})$ is bijective.

We can associate to the normal bordism $(\overline{F}, F) : TW \oplus \underline{\mathbb{R}^b} \to \eta$ from the given k-connected normal map of degree one $(\overline{f}, f)$ to another k-connected normal map of degree one $(\overline{f'}, f')$ a $(-1)^k$-quadratic formation $(H_{(-1)^k}(F); F, G)$ as follows. The underlying non-degenerate $(-1)^k$-quadratic form is $H_{(-1)^k}(F)$ for $F := K_{k+1}(\widetilde{W}, \widetilde{M'})$. The first Lagrangian is F. The second Lagrangian G is given by the image of the map

$$\begin{aligned}(K_{k+1}(\widetilde{i'}), u \circ (? \cap [W, \partial W])^{-1} \circ K_{k+1}(\widetilde{i}) : K_{k+1}(\widetilde{W}) \\ \to F \oplus F^* = K_{k+1}(\widetilde{W}, \widetilde{M'}) \oplus K_{k+1}(\widetilde{W}, \widetilde{M'})^*,\end{aligned}$$

where $i : W \to (W, M)$ is the inclusion, $(? \cap [W, \partial W]) : K^{k+1}(\widetilde{W}, \widetilde{M'}) \xrightarrow{\cong} K_{k+1}(\widetilde{W}, \widetilde{M})$ is the Poincaré isomorphism and u is the canonical map

$$u : K^{k+1}(\widetilde{W}, \widetilde{M'}) \xrightarrow{\cong} K_{k+1}(\widetilde{W}, \widetilde{M'})^*, \qquad \alpha \mapsto \langle \alpha, ? \rangle$$

which is bijective by an elementary chain complex argument or by the universal coefficient spectral sequence.

What is the relation between this formation and the problem whether f' is a homotopy equivalence. Suppose that f' is a homotopy equivalence. Then $K_{k+1}(\widetilde{i'}) : K_{k+1}(\widetilde{W}) \to K_{k+1}(\widetilde{W}, \widetilde{M'})$ is an isomorphism. Define a $(-1)^{k+1}$-quadratic form (H, ψ) by $H = K_{k+1}(\widetilde{W})$ and

$$\begin{aligned}\psi : H := K_{k+1}(\widetilde{W}) \xrightarrow{K_{k+1}(\widetilde{i})} K_{k+1}(\widetilde{W}, \widetilde{M}) \xrightarrow{(? \cap [W, \partial W])^{-1}} K^{k+1}(\widetilde{W}, \widetilde{M'}) \\ \xrightarrow{u} K_{k+1}(\widetilde{W}, \widetilde{M'})^* \xrightarrow{K_{k+1}(\widetilde{i'})^*} K_{k+1}(\widetilde{W})^* = H^*.\end{aligned}$$

One easily checks that the isomorphism $K_{k+1}(\widetilde{i'}) : K_{k+1}(\widetilde{W}) \to K_{k+1}(\widetilde{W}, \widetilde{M'})$ induces an isomorphism of $(-1)^k$-quadratic formations

$$\partial(H, \psi) \xrightarrow{\cong} (H_{(-1)^k}(F), F, G).$$

Hence we see that f' is a homotopy equivalence only if $(H_{(-1)^k}(F); F, G)$ is isomorphic to a boundary.

One decisive step is to prove that the class of this ϵ-quadratic formation formation $(H_{(-1)^k}(F);, F, G)$ in $L_{2k+1}(\mathbb{Z}\pi, w)$ is independent of the choice of the $(k+1)$-connected nullbordism. We will not give the proof of this fact. This fact enables us to define the *surgery obstruction* of $(\overline{f}, f)$

$$\sigma(\overline{f}, f) \quad := \quad [(H_\epsilon(F); F, G)] \quad \in L_n(\mathbb{Z}\pi, w), \tag{4.45}$$

where $(H_{\epsilon(F)}; F, G)$ is the ϵ-quadratic formation associated to any normal bordism of degree one $(\overline{F}, F)$ from a normal map $(f_0, \overline{f_0})$ to some normal map $(f', \overline{f'})$ such that $(f_0, \overline{f_0})$ and $(f', \overline{f'})$ are k-connected, F is $(k+1)$-connected and $(f_0, \overline{f_0})$ is obtained from the original normal map of degree one $(\overline{f}, f)$ by surgery below the middle dimension. From the discussion above it is clear that the vanishing of $\sigma(\overline{f}, f)$ is a necessary condition for the existence of a normal map $(\overline{f'}, f')$ which is normally bordant to f and whose underlying map is a homotopy equivalence. We omit the proof that for $k \geq 2$ this condition is also sufficient. This finishes the outline of the proof of Theorem 4.44.

More details can be found in [64, chapter 8]. The reason why we have been very brief in the even-dimensional case is that the proofs are more complicated but not as illuminating as in the even-dimensional case and that a reader who is interested in details should directly read the approach using Poincaré complexes of Ranicki (see [51], [51], [52]). This approach is more conceptual and treats the even-dimensional and odd-dimensional case simultaneously.

Now we can give a rather complete answer to Problem 3.1 and Problem 3.62 for odd dimensions in the simply connected case.

Theorem 4.46 *1. Suppose we have some normal map $(\overline{f}, f)$ from a closed manifold M to a simply connected finite Poincaré complex X of odd dimension $n = 2k+1 \geq 5$. Then we can always change M and f leaving X fixed by finitely many surgery steps to get a normal map $(\overline{g}, g)$ from a closed manifold N to X such that g is a homotopy equivalence;*

2. Let X be a simply-connected finite connected Poincaré complex of odd dimension $n = 2k+1 \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle $\xi : E \to X$, i.e. the set of normal invariants $\mathcal{T}_n(X)$ is non-empty.

Proof : (1) follows from Theorem 4.44 and Theorem 4.43.

(2) follows from assertion 1 together with Theorem 3.49. ■

4.7 Variations of the surgery obstruction and the L-groups

So far we have dealt with Problem 3.1 when a topological space is homotopy equivalent to a closed manifold. This question has motivated and led us to the notions of a finite Poincaré complex, of its Spivak normal fibration, of a normal map of degree one, of L-groups and of the surgery obstruction. This problem is certainly interesting but not our ultimative goal. We have already mentioned in Remark 1.5 our main goal which is to decide whether two closed manifolds are diffeomorphic and we have discussed the so called surgery program which is a strategy to attack it. The surgery program will lead to the surgery sequence in the next chapter (see Theorem 5.12). The surgery program suggests that we have to consider the following variations of the surgery obstruction and the L-groups which will play a role when establishing the surgery sequence.

Consider the third step (3) in the surgery program which we have explained in Remark 1.5. There are a cobordism $(W; M, N)$ and a map (F, f, id) : $(W; M, N) \to (N \times [0,1]; N \times \{0\}, N \times \{1\})$ for f a homotopy equivalence is given and we want to change F into a homotopy equivalence without changing W on the boundary. Thus we have to deal also with surgery problems $(\overline{g}, g)$ where the underlying map $g : V \to X$ is a pair of maps $(g, \partial g) : (V, \partial V) \to (X, \partial X)$ from a compact oriented manifold V with boundary ∂V to a pair of finite Poincaré complexes $(X, \partial X)$ such that ∂f is already a homotopy equivalence. This will be done in Section 4.7.1. If this procedure is successful we obtain a h-cobordism W from M to N. We would like to know whether W is relative M diffeomorphic to $M \times [0,1]$. This would imply that M and N are diffeomorphic. Because of the s-Cobordism Theorem 1.1 this comes down to the problem to control the Whitehead torsion of the h-cobordism. The Whitehead torsion of the h-cobordism is trivial if and only if the Whitehead torsion of $\tau(F)$ and of $\tau(f)$ in $\mathrm{Wh}(\pi_1(X))$ agree (see Theorem 2.1). We may modify the first step (1) of the surgery program appearing in Remark 1.5 by requiring that f is a simple homotopy equivalence. The existence of a simple homotopy equivalence $f : M \to N$ is a necessary condition for M and N to be diffeomorphic. This means that we must modify our surgery obstruction so that its vanishing means that we get a simple homotopy equivalence $F : W \to X \times [0,1]$. This will be outlined in Section 4.7.2.

4.7.1 Surgery obstructions for manifolds with boundary

We want to extend the notion of a normal map from closed manifolds to manifolds with boundary. The underlying map f is a map of pairs $(f, \partial f)$: $(M, \partial M) \to (X, \partial X)$, where M is a compact oriented manifold with boundary ∂M and $(X, \partial X)$ is a finite Poincaré pair, the degree of f is one and $\partial f : \partial M \to \partial X$ is required to be a homotopy equivalence. The bundle data are unchanged, they consist of a vector bundle ξ over X and a bundle map $\overline{f} : TM \oplus \underline{\mathbb{R}^a} \to \xi$. Next we explain what a nullbordism in this setting means.

A *manifold triad* $(W; \partial_0 W, \partial_1 W)$ of dimension n consists of a compact manifold W of dimension n whose boundary decomposes as $\partial W = \partial_0 W \cup \partial_1 W$ for two compact submanifolds $\partial_0 W$ and $\partial_1 W$ with boundary such that $\partial_0 W \cap \partial_1 W = \partial(\partial_0 W) = \partial(\partial_1 W)$. A *Poincaré triad* $(X; \partial_0 X, \partial_1 X)$ of dimension n consists of a finite Poincaré pair $(X, \partial X)$ of dimension n together with $(n-1)$-dimensional finite subcomplexes $\partial_0 X$ and $\partial_1 X$ such that $\partial X = \partial_0 X \cup \partial_1 X$, the intersection $\partial_0 X \cap \partial_1 X$ is $(n-2)$-dimensional, ∂X is equipped with the structure of a Poincaré complex induced by Poincaré pair structure on $(X, \partial X)$ and both $(\partial_0 X, \partial_0 X \cap \partial_1 X)$ and $(\partial_1 X, \partial_0 X \cap \partial_1 X)$ are equipped with the structure of a Poincaré pair induced by the given structure of a Poincaré complex on ∂X. We use the convention that the fundamental class of $[X, \partial X]$ is send to $[\partial X]$ under the boundary homomorphism $H_n(X, \partial X) \xrightarrow{\partial_n} H_{n-1}(\partial X)$ and $[\partial X]$ is sent to $([\partial_0 X, \partial_0 X \cap \partial_1 X], -[\partial_1 X, \partial_0 X \cap \partial_1 X])$ under the composite

$$\begin{aligned} H_{n-1}(\partial X) &\to H_{n-1}(\partial_0 X \cup \partial_1 X, \partial_0 X \cap \partial_1 X) \\ &\cong H_{n-1}(\partial_0 X, \partial_0 X \cap \partial_1 X) \oplus H_{n-1}(\partial_1 X, \partial_0 X \cap \partial_1 X). \end{aligned}$$

A manifold triad $(W; \partial_0 W, \partial_1 W)$ together with an orientation for W is a Poincaré triad. We allow that $\partial_0 X$ or $\partial_1 X$ or both are empty.

A normal nullbordism $(\overline{F}, F) : TW \oplus \underline{\mathbb{R}^b} \to \eta$ for a normal map of degree one $(\overline{f}, f)$ with underlying map $(f, \partial f) : (M, \partial M) \to (X, \partial X)$ consists of the following data. Put $n = \dim(M)$. We have a compact oriented $(n+1)$-dimensional manifold triad $(W; \partial_0 W, \partial_1 W)$, a finite $(n+1)$-dimensional Poincaré triad $(Y; \partial_0 Y, \partial Y)$, a map $(F; \partial_0 F, \partial_1 F) : (W; \partial_0 W, \partial_1 W) \to (Y; \partial_0 Y, \partial_1 Y)$, an orientation preserving diffeomorphism $(u, \partial u) : (M, \partial M) \to (\partial_0 W, \partial(\partial_0 W))$ and an orientation preserving homeomorphism $(v, \partial v) : (X, \partial X) \to (\partial_0 Y, \partial_0 Y \cap \partial_1 Y)$. We require that F has degree one, $\partial_1 F$ is a homotopy equivalence and $\partial_0 F \circ u = v \circ f$. Moreover, we have a bundle map $\overline{F} : TW \oplus \underline{\mathbb{R}^b} \to \eta$ covering F and a bundle map $\overline{v} : \xi \oplus \underline{\mathbb{R}^c} \to \eta$ covering v such that $\overline{F}$, Tu and $\overline{v}$ fit together.

We call two such normal maps $(\overline{f}, f)$ and $(\overline{f'}, f')$ normally bordant if the disjoint union of them after changing the orientation for M appearing in $(\overline{f}, f)$ possesses a normal nullbordism.

The definition and the main properties of the surgery obstruction carry over from normal maps for closed manifolds to normal maps for compact manifolds with boundary. The main reason is that we require $\partial f : \partial M \to \partial X$ to be a homotopy equivalence so that the surgery kernels "do not feel the boundary". All arguments such as making a map highly connected by surgery steps and intersection pairings and selfintersection can be carried out in the interior of M without affecting the boundary. Thus we get

Theorem 4.47 (Surgery obstruction theorem for manifolds with boundary) *Let $(\overline{f}, f)$ be a normal map of degree one with underlying map $(f, \partial f) : (M, \partial M) \to (X, \partial X)$ such that ∂f is a homotopy equivalence. Put $n = \dim(M)$. Then:*

1. *We can associate to it its surgery obstruction*

$$\sigma(\overline{f}, f) \in L_n(\mathbb{Z}\pi, w). \tag{4.48}$$

2. *The surgery obstruction depends only on the normal bordism class of $(\overline{f}, f)$;*

3. *Suppose* $n \geq 5$. *Then* $\sigma(\overline{f}, f) = 0$ *in* $L_n(\mathbb{Z}\pi, w)$ *if and only if we can do a finite number of surgery steps on the interior of* M *leaving the boundary fixed to obtain a normal map* $(\overline{f'}, f') : TM' \oplus \underline{\mathbb{R}^{a'}} \to \xi'$ *which covers a homotopy equivalence of pairs* $(f', \partial f') : (M', \partial M') \to (X, \partial X)$ *with* $\partial M' = \partial M$ *and* $\partial f' = \partial f$.

4.7.2 Surgery obstructions and Whitehead torsion

Next we want to modify the L-groups and the surgery obstruction so that the surgery obstruction is the obstruction to achieve a simple homotopy equivalence.

We begin with the L-groups. It is clear that this requires to take equivalence classes of basis into account. Suppose that we have specified a subgroup $U \subset K_1(R)$ such that U is closed under the involution on $K_1(R)$ coming from the involution of R and contains the image of the change of ring homomorphism $K_1(\mathbb{Z}) \to K_1(R)$.

Two basis B and B' for the same finitely generated free R-module V are called *U-equivalent* if the change of basis matrix defines an element in $K_1(R)$ which belongs to U. Notice that the U-equivalence class of a basis B is unchanged if we permute the order of elements of B. We call an R-module V *U-based* if V is finitely generated free and we have chosen a U-equivalence class of basis.

Let V be a stably finitely generated free R-module. A *stable basis* for V is a basis B for $V \oplus R^u$ for some integer $u \geq 0$. Denote for any integer v the direct sum of the basis B and the standard basis S^a for R^a by $B \coprod S^a$ which is a basis for $V \oplus R^{u+a}$. Let C be a basis for $V \oplus R^v$. We call the stable basis B and C *stably U-equivalent* if and only if there is an integer $w \geq u, v$ such that $B \coprod S^{w-u}$ and $C \coprod S^{w-v}$ are U-equivalent basis. We call a R-module V *stably U-based* if V is stably finitely generated free and we have specified a stable U-equivalence class of stable basis for V.

Let V and W be stably U-based R-modules. Let $f : V \oplus R^a \xrightarrow{\cong} W \oplus R^b$ be a R-isomorphism. Choose a non-negative integer c together with basis for $V \oplus R^{a+c}$ and $W \oplus R^{b+c}$ which represent the given stable U-equivalence classes of basis for V and W. Let A be the matrix of $f \oplus \mathrm{id}_{R^c} : V \oplus R^{a+c} \xrightarrow{\cong} W \oplus R^{b+c}$. It defines an element [A] in $K_1(R)$. Define the *U-torsion*

$$\tau^U(f) \quad \in \quad K_1(R)/U \tag{4.49}$$

by the class represented by $[A]$. One easily checks that $\tau(f)$ is independent of the choices of c and the basis and depends only on f and the stable U-basis for V and W. Moreover, one easily checks

$$\begin{aligned} \tau^U(g \circ f) &= \tau^U(g) + \tau^U(f); \\ \tau^U \begin{pmatrix} f & 0 \\ u & v \end{pmatrix} &= \tau^U(f) + \tau^U(v); \\ \tau^U(\mathrm{id}_V) &= 0 \end{aligned}$$

for R-isomorphisms $f : V_0 \xrightarrow{\cong} V_1$, $g : V_1 \xrightarrow{\cong} V_2$ and $v : V_3 \xrightarrow{\cong} V_4$ and an R-homomorphism $u : V_0 \to V_4$ of stably U-based R-modules V_i. Let C_* be a contractible stably U-based finite R-chain complex, i.e. a contractible R-chain complex C_* of stably U-based R-modules which satisfies $C_i = 0$ for $|i| > N$ for some integer N. The definition of Whitehead torsion in (2.7) carries over to the definition of the *U-torsion*

$$\tau^U(C_*) \quad = \quad [A] \qquad \in K_1(R)/U. \tag{4.50}$$

Analogously we can associate to a R-chain homotopy equivalence $f : C_* \to D_*$ of stably U-based finite R-chain complexes its *U-torsion* (cf (2.8))

$$\tau^U(f_*) \quad := \quad \tau(\mathrm{cone}_*(f_*)) \qquad \in K_1(R)/U. \tag{4.51}$$

Theorem 2.9 carries over to U-torsion in the obvious way.

We will consider U-based ϵ-quadratic forms (P, ψ), i.e. ϵ-quadratic forms whose underlying R-module P is a U-based finitely generated free R-module such that U-torsion of the isomorphism $(1 + \epsilon \cdot T)(\psi) : P \xrightarrow{\cong} P^*$ is zero in $K_1(R)/U$. An isomorphism $f : (P, \psi) \to (P', \psi')$ of U-based ϵ-quadratic forms is U-simple if the U-torsion of $f : P \to P'$ vanishes in $K_1(R)/U$. Notice that the ϵ-quadratic form $H_\epsilon(R)$ inherits a basis from the standard basis of R. The sum of two stably U-based ϵ-quadratic forms is again a stably U-based ϵ-quadratic form. It is worthwhile to mention the following simple version of Lemma 4.29.

Lemma 4.52 *Let (P, ψ) be a U-based ϵ-quadratic form. Let $L \subset P$ be a Lagrangian such that L is U-based and the U-torsion of the following 2-dimensional U-based finite R-chain complex*

$$0 \to L \xrightarrow{i} P \xrightarrow{i^* \circ (1+\epsilon \cdot T)(\psi)} L^* \to 0$$

vanishes in $K_1(R)/U$. Then the inclusion $i : L \to P$ extends to a U-simple isomorphism of ϵ-quadratic forms

$$H_\epsilon(L) \xrightarrow{\cong} (P, \psi).$$

Next we give the simple version of the even-dimensional L-groups.

Definition 4.53 *Let R be an associative ring with unit and involution. For $\epsilon \in \{\pm 1\}$ define $L^U_{1-\epsilon}(R)$ to be the abelian group of equivalence classes $[(F, \psi)]$ of U-based non-degenerate ϵ-quadratic forms (F, ψ) with respect to the following equivalence relation. We call (F, ψ) and (F', ψ') equivalent if and only if there exists integers $u, u' \geq 0$ and a U-simple isomorphism of non-degenerate ϵ-quadratic forms*

$$(F, \psi) \oplus H_\epsilon(R)^u \cong (F', \psi') \oplus H_\epsilon(R)^{u'}.$$

Addition is given by the sum of two ϵ-quadratic forms. The zero element is represented by $[H_\epsilon(R)^u]$ for any integer $u \geq 0$. The inverse of $[F, \psi]$ is given by $[F, -\psi]$.

For an even integer n define the abelian group $L_n^U(R)$ called the n-th U-decorated quadratic L-group *of R by*

$$L_n^U(R) \;:=\; \begin{cases} L_0^U(R) & \text{if } n \equiv 0 \mod 4; \\ L_2^U(R) & \text{if } n \equiv 2 \mod 4. \end{cases}$$

A U-based ϵ-quadratic formation $(P,\psi;F,G)$ consists of an ϵ-quadratic formation $(P,\psi;F,G)$ such that (P,ψ) is a U-based non-degenerate ϵ-quadratic form, the Lagrangians F and G are U-based R-modules and the U-torsion of the following two contractible U-based finite R-chain complexes

$$0 \to F \xrightarrow{i} P \xrightarrow{i^* \circ (1+\epsilon\cdot T)(\psi)} F^* \to 0$$

and

$$0 \to G \xrightarrow{j} P \xrightarrow{j^* \circ (1+\epsilon\cdot T)(\psi)} G^* \to 0$$

vanish in $K_1(R)/U$, where $i : F \to P$ and $j : G \to P$ denote the inclusions. An isomorphism $f : (P,\psi;F,G) \to (P',\psi';F',G')$ of U-based ϵ-quadratic formations is U-simple if underlying the U-torsion of the induced R-isomorphisms $P \xrightarrow{\cong} P'$, $F \xrightarrow{\cong} F'$ and $G \xrightarrow{\cong} G'$ vanishes in $K_1(R)/U$. Notice that the trivial ϵ-quadratic formation $(H_\epsilon(R^u);R^u,(R^u)^*)$ inherits a U-basis from the standard basis on R^u. Given a U-based $(-\epsilon)$-quadratic form (Q,ψ), its boundary $\partial(Q,\psi)$ is a U-based ϵ-quadratic formation. Obviously the sum of two U-based ϵ-quadratic formations is again a U-based ϵ-quadratic formation. Next we give the simple version of the odd-dimensional L-groups.

Definition 4.54 *Let R be an associative ring with unit and involution. For $\epsilon \in \{\pm 1\}$ define $L_{2-\epsilon}(R)$ to be the abelian group of equivalence classes $[(P,\psi;F,G)]$ of U-based ϵ-quadratic formations $(P,\psi;F,G)$ with respect to the following equivalence relation. We call two U-based ϵ-quadratic formations $(P,\psi;F,G)$ and $(P',\psi';F',G')$ equivalent if and only if there exists U-based $(-\epsilon)$-quadratic forms (Q,μ) and (Q',μ') and non-negative integers u and u' together with a U-simple isomorphism of ϵ-quadratic formations*

$$\begin{aligned} &(P,\psi;F,G) \oplus \partial(Q,\mu) \oplus (H_\epsilon(R^u);R^u,(R^u)^*) \\ &\qquad \cong \;\; (P',\psi';F',G') \oplus \partial(Q',\mu') \oplus (H_\epsilon(R^{u'});R^{u'},(R^{u'})^*). \end{aligned}$$

Addition is given by the sum of two ϵ-quadratic forms. The zero element is represented by $\partial(Q,\mu) \oplus (H_\epsilon(R^u);R^u,(R^u)^*)$ for any U-based $(-\epsilon)$-quadratic form (Q,μ) and non-negative integer u. The inverse of $[(P,\psi;F,G)]$ is represented by $(P,-\psi;F',G')$ for any choice of stably U-based Lagrangians F' and G' in $H_\epsilon(P)$ such that F and F' are complementary and G and G' are complementary and the U-torsion of the the obvious isomorphism $F \oplus F' \xrightarrow{\cong} P$ and $F \oplus F' \xrightarrow{\cong} P$ vanishes in $K_1(R)/U$.

For an odd integer n define the abelian group $L_n^U(R)$ called the n-th U-decorated quadratic L-group *of R*

$$L_n^U(R) \;:=\; \begin{cases} L_1^U(R) & \text{if } n \equiv 1 \mod 4; \\ L_3^U(R) & \text{if } n \equiv 3 \mod 4. \end{cases}$$

Notation 4.55 *If $R = \mathbb{Z}\pi$ with the w-twisted involution and $U \subset K_1(\mathbb{Z}\pi)$ is the abelian group of elements of the shape $(\pm g)$ for $g \in \pi$, then we write*

$$\begin{aligned} L_n^s(\mathbb{Z}\pi, w) &:= L_n^U(\mathbb{Z}\pi, w); \\ L_n^h(\mathbb{Z}\pi, w) &:= L_n(\mathbb{Z}\pi, w); \end{aligned}$$

where $L_n(R)$ is the L-group introduced in Definitions 4.28 and 4.42 and $L_n^U(R)$ is the L-group introduced in Definitions 4.53 and 4.54. The L-groups $L_n^s(\mathbb{Z}\pi, w)$ are called simple quadratic L-groups.

Let $(\overline{f}, f)$ be a normal map of degree one with underlying map $(f, \partial f) : (M, \partial M) \to (X, \partial X)$ such that $(X, \partial X)$ is a simple finite Poincaré complex and ∂f is a simple homotopy equivalence. Next we want to explain how the definition of the surgery obstruction of (4.48) can be modified to the simple setting. Notice that the difference between the L-groups $L_n^h(\mathbb{Z}\pi, w)$ and the simple L-groups $L_n^h(\mathbb{Z}\pi, w)$ is the additional structure of a U-basis. The definition of the simple surgery obstruction

$$\sigma(\overline{f}, f) \quad \in \quad L_n^s(\mathbb{Z}\pi, w). \tag{4.56}$$

is the same as the one in (4.48) except that we must explain how the various surgery kernels inherit a U-basis.

The elementary proof of the following lemma is left to the reader. Notice that for any stably U-based R-module V and element $x \in K_1(R)/U$ we can find another stable U-basis C for V such that the U-torsion $\tau^U(\mathrm{id} : (V, B) \to (V, C)$ is x. This is not true in the unstable setting. For instance, there exists a ring R with an element $x \in K_1(R)/U$ for U the image of $K_1(\mathbb{Z}) \to K_1(R)$ such that x cannot be represented by a unit in R, in other words x is not the U-torsion of any R-automorphism of R.

Lemma 4.57 *Let C_* be a contractible finite stably free R-chain complex and r be an integer. Suppose that each chain module C_i with $i \neq r$ comes with a stable U-basis. Then C_r inherits a preferred stable U-basis which is uniquely defined by the property that the U-torsion of C_* vanishes in $K_1(R)/U$.*

Now Lemma 4.18 has the following version in the simple homotopy setting.

Lemma 4.58 *Let D_* be a stably U-based finite R-chain complex. Suppose for a fixed integer k that $H_i(D_*) = 0$ for $i \neq r$. Suppose that $H^{r+1}(\mathrm{hom}_R(D_*, V)) = 0$ for any R-module V. Then $H_r(D_*)$ is stably finitely generated free and inherits a peferred stable U-basis.*

Proof : combine Lemma 4.57 and Lemma 4.18. ∎

Now we can prove the following version of Lemma 4.19

Lemma 4.59 *If $f : X \to Y$ is k-connected for $n = 2k$ or $n = 2k + 1$, then $K_k(\widetilde{M})$ is stably finitely generated free and inherits a preferred stable U-basis.*

Proof : The proof is the same as the one of the proof of Lemma 4.19 except that we apply Lemma 4.58 instead of Lemma 4.18. ■

In the proof that the surgery obstruction is well-defined, one has to show that the U-equivalence classes of basis on the surgery kernels appearing in the short exact sequence (4.35) are compatible with the short exact sequence in the sense of Lemma 4.57. This follows from the following lemma whose elementary proof is left to the reader.

Lemma 4.60 *Let $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{q_*} E_* \to 0$ be a U-based exact sequence of U-based finite R-chain complexes. Here U-based exact means that for each $p \geq 0$ the U-torsion of the 2-dimensional U-based finite R-chain complex $0 \to C_p \xrightarrow{i_p} D_p \xrightarrow{q_p} E_p \to 0$ vanishes in $K_1(R)/U$. Let r be a fixed integer. Suppose that $H_i(C_*) = H_i(E_*) = 0$ for $i \neq r$ and $H^{r+1}(\hom_R(C_*, V)) = H^{r+1}(\hom_R(C_*, V)) = 0$ holds for any R-module V. Equip $H_r(C_*)$ and $H_r(E_*)$ with the U-equivalence class of stable basis defined in Lemma 4.58.*

Then $H_i(D_*) = 0$ for $i \neq r$ and $H^{r+1}(\hom_R(D_*, V)) = 0$ holds for any R-module V. We obtain a short exact sequence

$$0 \to H_r(C_*) \xrightarrow{H_p(i_*)} H_r(D_*) \xrightarrow{H_r(q_*)} H_r(E_*) \to 0.$$

The U-equivalence class of stable basis on $H_r(D_*)$ obtained from Lemma 4.58 applied to this exact sequence and the U-equivalence class of stable basis on $H_r(D_*)$ obtained from Lemma 4.58 applied to D_* agree.

Next we can give the simple version of the surgery obstruction theorem. Notice that simple normal bordism class means that in the definition of normal nullbordisms the pairs $(Y, \partial Y)$, $(\partial_0 Y, \partial_0 Y \cap \partial_1 Y)$ and $(\partial_1 Y, \partial_0 Y \cap \partial_1 Y)$ are required to be simple finite Poincaré pairs and the map $\partial_1 F : \partial_1 M \to \partial_1 Y$ is required to be a simple homotopy equivalence.

Theorem 4.61 (Simple surgery obstruction theorem for manifolds with boundary) *Let $(\overline{f}, f)$ be a normal map of degree one with underlying map $(f, \partial f) : (M, \partial M) \to (X, \partial X)$ such that $(X, \partial X)$ is a simple finite Poincaré complex ∂f is a simple homotopy equivalence. Put $n = \dim(M)$. Then:*

1. *The simple surgery obstruction depends only on the simple normal bordism class of $(\overline{f}, f)$;*

2. *Suppose $n \geq 5$. Then $\sigma(\overline{f}, f) = 0$ in $L^s_n(\mathbb{Z}\pi, w)$ if and only if we can do a finite number of surgery steps on the interior of M leaving the boundary fixed to obtain a normal map $(\overline{f'}, f') : TM' \oplus \underline{\mathbb{R}}^{a'} \to \xi'$ which covers a simple homotopy equivalence of pairs $(f', \partial f') : (M', \partial M') \to (X, \partial X)$ with $\partial M' = \partial M$ and $\partial f' = \partial f$.*

4.8 Miscellaneous

A guide for the calculation of the L-groups for finite groups is presented by Hambleton and Taylor [14], where further references are given.

Chapter 5

The surgery exact sequence

Introduction

In this section we introduce the exact surgery sequence (see Theorem 5.12). It is the realization of the surgery program which we have explained in Remark 1.5. The surgery exact sequence is the main theoretical tool in solving the classification problem of manifolds of dimensions greater or equal to five.

5.1 The structure set

Definition 5.1 *Let X be a closed oriented manifold of dimension n. We call two orientation preserving simple homotopy equivalences $f_i : M_i \to X$ from closed oriented manifolds M_i of dimension n to X for $i = 0, 1$ equivalent if there exists an orientation preserving diffeomorphism $g : M_0 \to M_1$ such that $f_1 \circ g$ is homotopic to f_0. The* simple structure set *$\mathcal{S}_n^s(X)$ of X is the set of equivalence classes of orientation preserving simple homotopy equivalences $M \to X$ from closed oriented manifolds of dimension n to X. This set has a preferred base point, namely the class of the identity* $\mathrm{id} : X \to X$.

The simple structure set $\mathcal{S}_n^s(X)$ is the basic object in the study of manifolds which are diffeomorphic to X. Notice that a simple homotopy equivalence $f : M \to X$ is homotopic to a diffeomorphism if and only if it represents the base point in $\mathcal{S}_n^s(X)$. A manifold M is oriented diffeomorphic to N if and only if for some orientation preserving simple homotopy equivalence $f : M \to N$ the class of $[f]$ agrees with the preferred base point. Some care is necessary since it may be possible that a given orientation preserving simple homotopy equivalence $f : M \to N$ is not homotopic to a diffeomorphism although M and N are diffeomorphic. Hence it does not suffice to compute $\mathcal{S}_n^s(N)$, one also has to understand the operation of the group of homotopy classes of simple orientation preserving selfequivalences of N on $\mathcal{S}_n^s(N)$. This can be rather complicated in general. But it will be no problem in the case $N = S^n$, because any orientation preserving selfhomotopy equivalence $S^n \to S^n$ is homotopic to the identity.

There is also a version of the structure set which does not take Whitehead torsion into account.

Definition 5.2 *Let X be a closed oriented manifold of dimension n. We call two orientation preserving homotopy equivalences $f_i : M_i \to X$ from closed oriented manifolds M_i of dimension n to X for $i = 0, 1$ equivalent if there is a manifold triad $(W; \partial_0 W, \partial_1 W)$ with $\partial_0 W \cap \partial_1 W = \emptyset$ and an orientation preserving homotopy equivalence of triads $(F; \partial_0 F, \partial_1 F) : (W; \partial_0 W, \partial_1 W) \to (X \times [0,1]; X \times \{0\}, X \times \{1\})$ together with orientation preserving diffeomorphisms $g_0 : M_0 \to \partial_0 W$ and $g_1 : M_1^- \to \partial_0 W$ satisfying $\partial_i F \circ g_i = f_i$ for $i = 0, 1$. (Here M_1^- is obtained from M_1 by reversing the orientation.) The* structure set *$\mathcal{S}_n^h(X)$ of X is the set of equivalence classes of orientation preserving homotopy equivalences $M \to X$ from a closed oriented manifold M of dimension n to X. This set has a preferred base point, namely the class of the identity* $\mathrm{id} : X \to X$.

Remark 5.3 If we would require in Definition 5.2 the homotopy equivalences F, f_0 and f_1 to be simple homotopy equivalences, we would get the simple structure set $\mathcal{S}_n^s(X)$ of Definition 5.1, provided that $n \geq 5$. We have to show that the two equivalence relations are the same. This follows from the s-cobordism Theorem 1.1. Namely, W appearing in Definition 5.2 is a h-cobordism and is even a s-cobordims if we require F, f_0 and f_1 to be simple homotopy equivalences (see Theorem 2.1). Hence there is a diffeomorphism $\Phi : \partial_0 W \times [0,1] \to W$ inducing the obvious identification $\partial_0 W \times \{0\} \to \partial_0 W$ and some orientation preserving diffeomorphism $\phi_1 : (\partial_0 W)^- = (\partial_0 W \times \{1\})^- \to \partial_1 W$. Then $\phi : M_0 \to M_1$ given by $g_1^{-1} \circ \phi_1 \circ g_0$ is an orientation preserving diffeomorphism such that $f_1 \circ \phi$ is homotopic to f_0. The other implication is obvious.

Remark 5.4 As long as we are dealing with smooth manifolds, there is in general no canonical group structure on the structure set.

5.2 Realizability of surgery obstructions

In this section we explain that any element in the L-groups can be realized as the surgery obstruction of a normal map $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map $f : (M, \partial M) \to (Y, \partial Y)$ if we allow M to have non-empty boundary ∂M.

Theorem 5.5 *Suppose $n \geq 5$. Consider a connected compact oriented manifold X possibly with boundary ∂X. Let π be its fundamental group and let $w : \pi \to \{\pm 1\}$ be its orientation homomorphism. Consider an element $x \in L_n(\mathbb{Z}\pi, w)$.*

Then we can find a normal map of degree one

$$(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to TX \times [0,1] \oplus \underline{\mathbb{R}^a}$$

covering a map of triads

$$f = (f; \partial_0 f, \partial_1 f) : (M; \partial_0 M, \partial_1 M) \to (X \times [0,1], X \times \{0\} \cup \partial X \times [0,1], X \times \{1\})$$

with the following properties:

1. *∂f_0 is a diffeomorphism and $\overline{f}|_{\partial_0 M}$ is given by $T(\partial f_0) \oplus \mathrm{id}_{\underline{\mathbb{R}^{a+1}}}$;*
2. *$\partial_1 f$ is a homotopy equivalence;*
3. *The surgery obstruction $\sigma(\overline{f}, f)$ in $L_n(\mathbb{Z}\pi, w)$ (see Theorem 4.48) is the given element x.*

The analogous statement holds for $x \in L_n^s(\mathbb{Z}\pi, w)$ if we require $\partial_1 f$ to be a simple homotopy equivalence and we consider the simple surgery obstruction (see (4.56)*).*

Proof : We give at least the idea of the proof in the case $n = 2k$, more details can be found in [64, Theorem 5.3 on page 53, Theorem 6.5 on page 66].

Recall that the element $x \in L_n(\mathbb{Z}\pi, w)$ is represented by a non-degenerate $(-1)^k$-quadratic form (P, ψ) with P a finitely generated free $\mathbb{Z}\pi$-module. We fix such a representative and write it as a triple (P, μ, λ) as explained in Subsection 4.2.2. Let $\{b_1, b_r, \ldots b_r\}$ be a $\mathbb{Z}\pi$-basis for P.

Choose r disjoint embeddings $j_i : D^{2k-1} \to X - \partial X$ into the interior of X for $i = 1, 2, \ldots r$. Let $F^0 : S^{k-1} \times D^k \to D^{2k-1}$ be the standard embedding. Define embeddings $F_i^0 : S^{k-1} \times D^k \to X$ by the composition of the standard embedding F^0 and j_i for $i = 1, 2, \ldots r$. Let $f_i^0 : S^{k-1} \to X$ be the restriction of F_i to $S^{k-1} \times \{0\} \subset S^{k-1} \times D^k$. Fix base points $b \in X - \partial X$ and $s \in S^{k-1}$ and paths w_i from b to $f_i^0(s)$ for $i = 1, 2, \ldots r$. Thus each f_i^0 is a pointed immersion. Next we construct regular homotopies $\eta_i : S^{k-1} \to X$ from f_i^0 to a new embedding f_i^1 which are modelled upon (P, μ, λ). These regular homotopies define associated immersions $\eta_i' : S^{k-1} \times [0,1] \to X \times [0,1]$, $(x,t) \mapsto (\eta_i(x), t)$. Since these immersions η_i' are embeddings on the boundary, we can define their intersection and selfintersection number as in (4.1) and (4.6).

We want to achieve that the intersection number of η_i' and η_j' for $i < j$ is $\lambda(b_i, b_j)$ and the selfintersection number $\mu(\eta_i')$ is $\mu(b_i)$ for $i = 1, 2, \ldots r$. Notice that these numbers are additive under stacking regular homotopies together. Hence it suffices to explain how to introduce a single intersection with value $\pm g$ between η_i' and η_j' for $i < j$ or a selfintersection for η_i' with value $\pm g$ for $i \in \{1, 2, \ldots, r\}$ without introducing other intersections or selfintersections. We explain the construction for introducing an intersection between η_i' and η_j' for $i < j$, the construction for a selfintersection is analogous. This construction will only change η_i, the other regular homotopies η_j for $j \neq i$ will be unchanged.

Join $f_i^0(s)$ and $f_j^0(s)$ by a path v such that v is an embedding, does not meet any of the f_i^0-s and the the composition $w_j * v * w_i^-$ represents the given element $g \in \pi$. Now there is an obvious regular homotopy from f_i^0 to another embedding along the path v inside a small tubular neighborhood $N(v)$ of v which leaves

f_i^0 fixed outside a small neighborhood $U(s)$ of s and moves $f_i^0|_{U(s)}$ within this small neighborhood $N(v)$ along v very close to $f_i^1(s)$. Now use a disc D^k which meets f_j^0 transversally at the origin to move f_i^0 further, thus introducing the desired intersection with value $\pm g$. Inside the disc the move looks like pushing the upper hemisphere of the boundary down to the lower hemisphere of the boundary, thus having no intersection with the origin at any point of time with one exception. One has to check that there is enough room to realize both possible signs.

Since f_i^1 is regularly homotopic to f_i^0 which is obtained from the trivial embeddings F_i^0, we can extend f_i^1 to an embedding $F_i^1 : S^{k-1} \times D^k \to X - \partial X$. Now attach for each $i \in \{1, 2 \ldots, r\}$ a handle $(F_i^1) = D^k \times D^k$ to $X \times [0,1]$ by $F_i^1 \times \{1\} : S^{k-1} \times D^k \to X \times \{1\}$. Let M be the resulting manifold. We obtain a manifold triad $(M; \partial_0 M, \partial_1 M)$ if we put $\partial_0 M = X \times 0 \cup \partial X \times [0,1]$ and $\partial_1 M = \partial M - \operatorname{int}(\partial_0 X)$. There is an extension of the identity $\mathrm{id} : X \times [0,1] \to X \times [0,1]$ to a map $f : M \to X \times [0,1]$ which induces a map of triads $(f; \partial_0 f, \partial_1 f)$ such that $\partial_0 f$ is a diffeomorphism. This map is covered by a bundle map $\overline{f} : TM \oplus \underline{\mathbb{R}^a} \to TX \oplus \underline{\mathbb{R}}^a$ which is given on $\partial_0 M$ by the differential of $\partial_0 f$. These claims follow from the fact that f_i^1 is regularly homotopic to f_i^0 and the f_i^0 were trivial embeddings so that we can regard this construction as surgery on the identity map $X \to X$. The kernel $K_k(\widetilde{M}) = K_k(\widetilde{M}, \widetilde{X \times [0,1]})$ has a preferred basis corresponding to the cores $D^k \times \{0\} \subset D^k \times D^k$ of the new handles (F_i^1). These cores can be completed to immersed spheres S_i by adjoining the images of the η_i' in $X \times [0,1]$ and finally the obvious discs D^k in $X \times \{0\}$ whose boundaries are given by $f_i^0(S^{k-1})$. Then the intersection of S_i with S_j is the same as the intersection of η_i' with η_j' and the selfintersection of S_i is the selfintersection of η_i' with itself. Hence by construction $\lambda(S_i, S_j) = \lambda(b_i, b_j)$ for $i < j$ and the selfintersection number $\mu(S_i')$ is $\mu(b_i)$ for $i = 1, 2, \ldots r$. This implies $\lambda(S_i, S_j) = \lambda(b_i, b_j)$ for $i, j \in \{1, 2, \ldots, r\}$ (see Subsection 4.2.2). The isomorphism $P \to P^*$ associated to the given non-degenerate $(-1)^k$-quadratic form can be identified with $K_k(\widetilde{M}) \to K_k(\widetilde{M}, \widetilde{\partial_1 M})$ if we use for $K_k(\widetilde{M}, \widetilde{\partial_1 M})$ the basis given by the cocores of the handles (F_i^1). Hence $K_k(\widetilde{\partial_1 M}) = 0$ and we conclude that $\partial_1 f$ is a homotopy equivalence. By definition and construction $\sigma(\overline{f}, f)$ in $L_n(\mathbb{Z}\pi, w)$ is the class of the non-degenerated $(-1)^k$-quadratic form (P, μ, λ). This finishes the outline of the proof of Theorem 5.5. ■

Remark 5.6 It is not true that for any closed oriented manifold N of dimension n with fundamental group π and orientation homomorphism $w : \pi \to \{\pm 1\}$ and any element $x \in L_n(\mathbb{Z}\pi, w)$ there is a normal map $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map of closed oriented manifolds $f : M \to N$ of degree one such that $\sigma(\overline{f}, f) = x$. Notice that in Theorem 5.5 the target manifold $X \times [0,1]$ is not closed. The same remark holds for $L_n^s(\mathbb{Z}\pi, w)$.

5.3 The surgery exact sequence

Now we can establish one of the main tools in the classification of manifolds, the surgery exact sequence. For this purpose we have to extend the Definition 3.51 of a normal map for closed manifolds to manifolds with boundary. Let $(X, \partial X)$ be a compact oriented manifold of dimension n with boundary ∂X. We consider normal maps of degree one $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ with underlying map $(f, \partial f) : (M, \partial M) \to (X, \partial X)$ for which $\partial f : \partial M \to \partial X$ is a diffeomorphism. A normal nullbordism for $(\overline{f}, f)$ consists of a bundle map

$$(\overline{F}, F) : TW \oplus \underline{\mathbb{R}^b} \to \eta$$

with underlying map of manifold triads of degree one

$$(F; \partial_0 F, \partial_1 F) : (W; \partial_0 W, \partial_1 W) \to (X \times [0,1]; X \times \{0\}, \partial X \times [0,1] \cup X \times \{1\})$$

together with a diffeomorphism

$$g : (M, \partial M) \to (\partial_0 W, \partial_0 W \cap \partial_1 W)$$

covered by a bundle map $\overline{g} : TM \oplus \underline{\mathbb{R}^{a+b+1}} \to TW \oplus \underline{\mathbb{R}^{a+b}}$ and a bundle isomorphism

$$(\overline{i_0}, i_0) : \xi \oplus \underline{\mathbb{R}^{b+1}} \to \eta$$

covering the inclusion $i_0 : X \to X \times [0,1]$, $x \mapsto (x, 0)$ such that the composition $(\overline{F}, F) \circ (\overline{g}, g)$ agrees with $(\overline{i_0}, i_0) \circ (\overline{f} \oplus \mathrm{id}_{\underline{\mathbb{R}^{b+1}}}, f)$ and $\partial_1 F : \partial_1 W \to \partial X \times [0,1] \cup X \times \{1\}$ is a diffeomorphism. Notice that here the target of the bordism is $X \times [0,1]$, we have allowed in Theorem 4.47 (2) a more general notion of normal bordism, where the target manifold also could vary.

Definition 5.7 *Let $(X, \partial X)$ be a compact oriented manifold of dimension n with boundary ∂X. Define the* set of normal maps to $(X, \partial X)$

$$\mathcal{N}_n(X, \partial X)$$

to be the set of normal bordism classes of normal maps of degree one $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ with underlying map $(f, \partial f) : (M, \partial M) \to (X, \partial X)$ for which $\partial f : \partial M \to \partial X$ is a diffeomorphism.

Notice that this definition uses tangential bundle data. One could also use normal bundle data (see Lemma 3.52).

Let X be a closed oriented connected manifold of dimension $n \geq 5$. Denote by π its fundamental group and by $w : \pi \to \{\pm 1\}$ its orientation homomorphism. Let $\mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\})$ and $\mathcal{N}_n(X)$ be the set of normal maps of degree one as introduced in Definition 5.5. Let $\mathcal{S}^s_n(X)$ be the structure set of Definition 5.1. Denote by $L^s_n(\mathbb{Z}\pi)$ the simple surgery obstruction group (see Notation 4.55). Denote by

$$\sigma : \mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\}) \to L^s_{n+1}(\mathbb{Z}\pi, w); \quad (5.8)$$

$$\sigma : \mathcal{N}_n(X) \to L^s_n(\mathbb{Z}\pi, w) \quad (5.9)$$

the maps which assign to the normal bordism class of a normal map of degree one its simple surgery obstruction (see (4.56)). This is well-defined by Theorem 4.61 (1). Let

$$\eta : \mathcal{S}_n^s(X) \quad \to \quad \mathcal{N}_n(X) \tag{5.10}$$

be the map which sends the class $[f] \in \mathcal{S}_n^s(X)$ represented by a simple homotopy equivalence $f : M \to X$ to the normal bordism class of the following normal map of degree one. Choose a homotopy inverse $f^{-1} : X \to M$ and a homotopy $h : \mathrm{id}_M \simeq f^{-1} \circ f$. Put $\xi = (f^{-1})^* TM$. Up to isotopy of bundle maps there is precisely one bundle map $(\overline{h}, h) : TM \times [0,1] \to TM$ covering $h : M \times [0,1] \to M$ whose restriction to $TM \times \{0\}$ is the identity map $TM \times \{0\} \to TM$. The restriction of $\overline{h}$ to $X \times \{1\}$ induces a bundle map $\overline{f} : TM \to \xi$ covering $f : M \to X$. Put $\eta([f]) := [(\overline{f}, f)]$. One easily checks that the normal bordism class of $(\overline{f}, f)$ depends only on $[f] \in \mathcal{N}_n^s(X)$ and hence that η is well-defined. Next we define an action of the abelian group $L_{n+1}^s(\mathbb{Z}\pi, w)$ on the structure set $\mathcal{S}_n^s(X)$

$$\rho : L_{n+1}^s(\mathbb{Z}\pi, w) \times \mathcal{S}_n^s(X) \quad \to \quad \mathcal{S}_n^s(X). \tag{5.11}$$

Fix $x \in L_{n+1}^s(\mathbb{Z}\pi, w)$ and $[f] \in \mathcal{N}_n^s(X)$ represented by a simple homotopy equivalence $f : M \to X$. By Theorem 5.5 we can find a normal map $(\overline{F}, F)$ covering a map of triads $(F; \partial_0 F, \partial_1 F) : (W; \partial_0 W, \partial_1 W) \to (M \times [0,1], M \times \{0\}, M \times \{1\})$ such that $\partial_0 F$ is a diffeomorphism and $\partial_1 F$ is a simple homotopy equivalence and $\sigma(\overline{F}, F) = u$. Then define $\rho(x, [f])$ by the class $[f \circ \partial_1 F : \partial_1 W \to X]$. We have to show that this is independent of the choice of $(\overline{F}, F)$. Let $(\overline{F'}, F')$ be a second choice. We can glue W' and W^- together along the diffeomorphism $(\partial_0 F)^{-1} \circ \partial_0 F' : \partial_1 W' \to \partial_1 W$ and obtain a normal bordisms from $(\overline{F}|_{\partial_1 W}, \partial_1 F)$ to $(\overline{F'}|_{\partial_1 W'}, \partial_1 F')$. The simple obstruction of this normal bordism is

$$\sigma(\overline{F'}, F') - \sigma(\overline{F}, F) \;=\; x - x \;=\; 0.$$

Because of Theorem 4.61 (2) we can perform surgery relative boundary on this normal bordism to arrange that the reference map from it to $X \times [0,1]$ is a simple homotopy equivalence. In view of Remark 5.3 this shows that $f \circ \partial_1 F$ and $f \circ \partial_1 F'$ define the same element in $\mathcal{S}_n^s(X)$. One easily checks that this defines a group action since the surgery obstruction is additive under stacking normal bordism together. The next result is the main result of this chapter and follows from the definitions and Theorem 4.61 (2)

Theorem 5.12 (The surgery exact sequence) *Under the conditions and in the notation above the so called* surgery sequence

$$\mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi, w) \xrightarrow{\partial} \mathcal{S}_n^s(X) \xrightarrow{\eta} \mathcal{N}_n(X) \xrightarrow{\sigma} L_n^s(\mathbb{Z}\pi, w)$$

is exact for $n \geq 5$ in the following sense. An element $z \in \mathcal{N}_n(X)$ lies in the image of η if and only if $\sigma(z) = 0$. Two elements $y_1, y_2 \in \mathcal{S}_n^s(X)$ have the same image under η if and only if there exists an element $x \in L_{n+1}^s(\mathbb{Z}\pi, w)$ with

$\rho(x, y_1) = y_2$. For two elements x_1, x_2 in $L^s_{n+1}(\mathbb{Z}\pi)$ we have $\rho(x_1, [\mathrm{id} : X \to X]) = \rho(x_2, [\mathrm{id} : X \to X])$ if and only if there is $u \in \mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\})$ with $\sigma(u) = x_1 - x_2$.

There is an analogous surgery exact sequence

$$\mathcal{N}^h_{n+1}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma} L^h_{n+1}(\mathbb{Z}\pi, w) \xrightarrow{\partial} \mathcal{S}^h_s(X) \xrightarrow{\eta} \mathcal{N}_n(X) \xrightarrow{\sigma} L^h_n(\mathbb{Z}\pi, w)$$

where $\mathcal{S}^h(X)$ is the structure set of Definition 5.2 and $L^h_n(\mathbb{Z}\pi, w) := L_n(\mathbb{Z}\pi, w)$ have been introduced in Definitions 4.28 and 4.42.

Remark 5.13 The surgery sequence of Theorem 5.12 can be extended to infinity to the left.

5.4 Miscellaneous

One can also develop surgery theory in the PL-category or in the topological category [36]. This requires to carry over the notions of vector bundles and tangent bundles to these categories. There are analogues of the sets of normal invariants $\mathcal{N}^{PL}_n(X)$ and $\mathcal{N}^{TOP}_n(X)$ and the structure sets $\mathcal{S}^{PL,h}_n(X)$, $\mathcal{S}^{PL,s}_n(X)$ $\mathcal{S}^{TOP,h}_n(X)$ and $\mathcal{S}^{TOP,s}_n(X)$. There are analogues PL and TOP of the group O. Theorem 3.46 and Theorem 3.49 (see also Remark 3.50) carry over to the PL-category and the topological category.

Theorem 5.14 *Let X be a connected finite n-dimensional Poincaré complex. Suppose that $\mathcal{N}^{PL}_n(X)$ is non-empty. Then there is a canonical group structure on the set $[X, G/PL]$ of homotopy classes of maps from X to G/PL and a transitive free operation of this group on $\mathcal{N}^{PL}_n(X)$. The analogue statement holds for TOP instead of PL.*

There are analogoues of the surgery exact sequence (see Theorem 5.12) for the PL-category and the topological category.

Theorem 5.15 (The surgery exact sequence) *There is a surgery sequence*

$$\mathcal{N}^{PL}_{n+1}(X\times[0,1], X\times\{0,1\}) \xrightarrow{\sigma} L^s_{n+1}(\mathbb{Z}\pi, w) \xrightarrow{\partial} \mathcal{S}^{PL,s}_n(X) \xrightarrow{\eta} \mathcal{N}^{PL}_n(X) \xrightarrow{\sigma} L^s_n(\mathbb{Z}\pi, w)$$

which is exact for $n \geq 5$ in the sense of Theorem 5.12. There is an analogous surgery exact sequence

$$\mathcal{N}^{PL}_{n+1}(X\times[0,1], X\times\{0,1\}) \xrightarrow{\sigma} L^h_{n+1}(\mathbb{Z}\pi, w) \xrightarrow{\partial} \mathcal{S}^{PL,h}_n(X) \xrightarrow{\eta} \mathcal{N}^{PL}_n(X) \xrightarrow{\sigma} L^h_n(\mathbb{Z}\pi, w)$$

The analogue sequences exists in the topological category.

Notice that the surgery obstruction groups are the same in the smooth category, PL-category and in the topological category. Only the set of normal invariants and the structure sets are different. The set of normal invariants in the smooth category, PL-category or topological category do not depend on the decoration h and s, whereas the structure sets and the surgery obstruction groups depend on the decoration h and s. In particular the structure set depends on both the choice of category and choice of decoration.

As in the smooth setting the surgery sequence above can be extended to infinity to the left.

Some interesting constructions can be carried out in the topological category which do not have smooth counterparts. An algebraic surgery sequence is constructed in [53, §14, §18] and identified with the geometric surgery sequence above in the topological category. Given a finite Poincaré complex X of dimension ≥ 5, a single obstruction, the so called total surgery obstruction, is constructed in [53, §17]. It vanishes if and only if X is homotopy equivalent to a closed topological manifold. It combines the two stages of the obstruction we have seen before, namely, the problem whether the Spivak normal fibration has a reduction to a TOP-bundle and whether the surgery obstruction of the associated normal map is trivial.

Chapter 6

Homotopy spheres

Introduction

Recall that S^n is the standard sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$. We will equip it with the structure of a smooth manifold for which the canonical inclusion $S^n \to \mathbb{R}^{n+1}$ is an embedding of smooth manifolds. We use the orientation on S^n which is compatible with the isomorphism $T_xS^n \oplus \nu(S^n, \mathbb{R}^n) \xrightarrow{\cong} T_x\mathbb{R}^n$, where we use on $\nu_x(S^n, \mathbb{R}^n)$ the orientation coming from the normal vector field pointing to the origin and on $T_x\mathbb{R}^n$ the standard orientation. This agrees with the convention that $S^n = \partial D^{n+1}$ inherits its orientation from D^{n+1}. A *homotopy n-sphere* Σ is a closed oriented n-dimensional smooth manifold which is homotopy equivalent S^n. The Poincaré Conjecture says that any homotopy n-sphere Σ is oriented homeomorphic to S^n and is known to be true for all dimensions except $n = 3$. In this chapter we want to solve the problem how many oriented diffeomorphism classes of homotopy n-spheres exist for given n. For $n \neq 3$ this is the same as determining how many different oriented smooth structures exist on S^n. This is a beautiful and illuminating example. It shows how the general surgery methods which we have developed so far apply to a concrete problem. It illustrates what kind of input from homotopy theory and algebra is needed for the final solution.

The following theorem summarizes what we will prove in this chapter. Here Θ^n denotes the abelian group of oriented h-cobordism classes of oriented homotopy n-spheres, $bP^{n+1} \subset \Theta^n$ is the subgroup of those homotopy n-spheres which bound a stably parallizable compact manifold, $J_n : \pi_n(SO) \to \pi_n^s$ denotes the J-homomorphism and B_n is the n-th Bernoulli number. These notions and the proof of the next result will be presented in this chapter (see Theorem 6.39, Corollary 6.43, Theorem 6.44, Theorem 6.46 and Theorem 6.56.

Theorem 6.1 (Classification of homotopy spheres) *1. Let $k \geq 2$ be an integer. Then bP^{4k} is a finite cyclic group of order*

$$\frac{3-(-1)^k}{2} \cdot 2^{2k-2} \cdot (2^{2k-1} - 1) \cdot \text{numerator}(B_k/(4k));$$

2. *Let $k \geq 1$ be an integer. Then bP^{4k+2} is trivial or isomorphic to $\mathbb{Z}/2$. We have*

$$bP^{4k+2} = \begin{cases} \mathbb{Z}/2 & 4k+2 \neq 2^l - 2, k \geq 1; \\ 0 & 4k+2 \in \{6, 14, 30, 62\}. \end{cases}$$

3. *If $n = 4k+2$ for $k \geq 2$, then there is an exact sequence*

$$0 \to \Theta^n \to \operatorname{coker}(J_n) \to \mathbb{Z}/2.$$

If $n = 4k$ for $k \geq 2$ or $n = 4k+2$ with $4k+2 \neq 2^l - 2$, then

$$\Theta^n \cong \operatorname{coker}(J_n);$$

4. *Let $n \geq 5$ be odd. Then there is an exact sequence*

$$0 \to bP^{n+1} \to \Theta^n \to \operatorname{coker}(J_n) \to 0.$$

If $n \neq 2^l - 3$, the sequence splits.

The following table taken from [35, pages 504 and 512] gives the orders of the finite groups Θ^n, bP^{n+1} and Θ^n/bP^{n+1} as far as they are known in low dimensions. The values in dimension 1 and 2 come from obvious adhoc computations. The computation of Θ^4 requires some additional analysis which we will not present here. Notice that $\Theta^4 = 1$ does *not* mean that any homotopy 4-sphere is diffeomorphic to S^4 (see Lemma 6.2).

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Θ^n	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256
bP^{n+1}	1	1	?	1	1	1	28	1	2	1	992	1	1	1	8128
Θ^n/bP^{n+1}	1	1	1	1	1	1	1	2	4	6	1	1	2	2	2

The basic paper about homotopy spheres is the one of Kervaire and Milnor [35] which contains a systematic study and can be viewed as the beginning of surgery theory. Nearly all the results presented here are taken from this paper. Another survey article about homotopy spheres has been written by Lance [13] and Levine [38].

6.1 The group of homotopy spheres

Define the *n-th group of homotopy spheres* Θ^n as follows. Elements are oriented h-cobordism classes $[\Sigma]$ of oriented homotopy n-spheres Σ, where Σ and Σ' are called oriented h-cobordant if there is an oriented h-cobordism $(W, \partial_0 W, \partial_1 W)$ together with orientation preserving diffeomorphisms $f : \Sigma \to \partial_0 W$ and $f' : (\Sigma')^- \to \partial_1 W$. The addition is given by the connected sum. The zero element is represented by S^n. The inverse of $[\Sigma]$ is given by $[\Sigma^-]$, where Σ^- is obtained from Σ by reversing the orientation.

Obviously Θ^n becomes an abelian semi-group by the connected sum. It remains to check that $[\Sigma^-]$ is an inverse of $[\Sigma]$. It is easy to see that for a homotopy n-sphere Σ that there is an h-cobordism W from the connected sum $\Sigma\sharp\Sigma^-$ to S^n. Add to $\Sigma \times [0,1]$ a handle $D^1 \times D^n$ by an orientation reversing embedding $S^0 \times D^n \times \{0,1\} \to \Sigma \times \{0,1\}$ which meets both components of $\Sigma \times \{0,1\}$. Delete the interior of a trivially embedded disk $D^n \subset \Sigma \times (0,1)$. The result is the desired h-cobordism W.

The fundamental group π of a homotopy n-sphere is trivial for $n \geq 2$. This implies that the orientation homomorphism w is always trivial and $\mathrm{Wh}(\pi) = \{0\}$. Hence it does not matter whether we work with simple homotopy equivalences or homotopy equivalence since $\mathcal{S}_n^s(S^n) = \mathcal{S}_n^h(S^n)$ for $n \geq 5$ and $L_n^s(\mathbb{Z}\pi) = L_n^h(\mathbb{Z}\pi) = L_n(\mathbb{Z})$ holds. Therefore we will omitt the decoration h or s for the remainder of this chapter.

Lemma 6.2 *Let $\overline{\Theta}^n$ be the set of oriented diffeomorphism classes $[\Sigma]$ of oriented homotopy n-spheres Σ. The forgetful map*

$$f : \overline{\Theta^n} \to \Theta^n$$

is bijective for $n \neq 3, 4$.

Proof : This follows from the s-cobordism Theorem 1.1 for $n \geq 5$ and by obvious adhoc computations for $n \leq 2$. ■

Lemma 6.3 *There is a natural bijection*

$$\alpha : \mathcal{S}_n^s(S^n) \xrightarrow{\cong} \overline{\theta}^n \qquad [f : M \to S^n] \mapsto [M].$$

If $n \neq 3$, there is an obvious bijection

$$\{\textit{smooth oriented structures on } S^n\}/\textit{oriented diffeomorphic} \xrightarrow{\cong} \overline{\Theta}^n.$$

Proof : For any homotopy n-sphere Σ there is up to homotopy precisely one map $f : \Sigma \to S^n$ of degree one. For $n \neq 3$ the Poincaré Conjecture is true which says that any homotopy n-sphere is homeomorphic to S^n. ■

In dimension 3 there is no difference between the topological and smooth category. Any closed topological 3-manifold M carries a unique smooth structure. Hence any manifold which is homeomorphic to S^3 is automatically diffeomorphic to S^3. The Poincaré Conjecture is at the time of writting open in dimension 3. It is not known whether any closed 3-manifold which is homotopy equivalent to S^3 is homeomorphic to S^3.

Definition 6.4 *A manifold M is called* stably parallizable *if $TM \oplus \underline{\mathbb{R}^a}$ is trivial for some $a \geq 0$.*

Definition 6.5 *Let $bP^{n+1} \subset \Theta^n$ be the subset of elements $[\Sigma]$ for which Σ is oriented diffeomorphic to the boundary ∂M of a stably parallizable compact manifold M.*

Lemma 6.6 *The subset $bP^{n+1} \subset \Theta^n$ is a subgroup of Θ^n. It is the preimage under the composition*

$$\Theta^n \xrightarrow{(f\circ\alpha)^{-1}} \mathcal{S}_n(S^n) \xrightarrow{\eta} \mathcal{N}_n(S^n)$$

of the base point $[\mathrm{id} : TS^n \to TS^n]$ *in* $\mathcal{N}_n(S^n)$, *where f is the bijection of Lemma 6.2 and α is the bijection of Lemma 6.3.*

Proof : Suppose that Σ bounds W and Σ' bounds W' for stably parallizable manifolds W and W'. Then the boundary connected sum $W\sharp W'$ is stably parallizable and has $\Sigma\sharp\Sigma'$ as boundary. This shows that $bP^{n+1} \subset \Theta^{n+1}$ is a subgroup.

Consider an element $[f : \Sigma \to S^n]$ in $\eta^{-1}([\mathrm{id} : TS^n \to TS^n])$. Then there is a normal bordism from a normal map $(\overline{f}, f) : T\Sigma \oplus \underline{\mathbb{R}^a} \to \xi$ covering $f : \Sigma \to S^n$ to the normal map $\mathrm{id} : TS^n \to TS^n$. This normal bordism is given by a bundle map $(\overline{F}, F) : TW \oplus \underline{\mathbb{R}^{a+b}} \to \eta$ covering a map of triads $(F; \partial_0 F, \partial_1 F) : (W; \partial_0 W, \partial_1 W) \to (S^n \times [0,1]; S^n \times \{0\}, S^n \times \{1\})$, and bundle isomorphisms $(\overline{u}, u) : T\Sigma \oplus \underline{\mathbb{R}^{a+b+1}} \to TW \oplus \underline{\mathbb{R}^b}$ covering an orientation preserving diffeomorphism $u : \Sigma \to \partial_0 W$, $(\overline{u'}, u') : TS^n \oplus \underline{\mathbb{R}^{a+b+1}} \to TW \oplus \underline{\mathbb{R}^b}$ covering an orientation preserving diffeomorphism $u' : (S^n)^- \to \partial_1 W$, $(\overline{v}, v) : \xi \oplus \underline{\mathbb{R}^{b+1}} \to \eta$ covering the obvious map $v : S^n \to S^n \times \{0\}$ and $(\overline{v'}, v') : TS^n \oplus \underline{\mathbb{R}^{a+b+1}} \to \eta$ covering the obvious map $v' : S^n \to S^n \times \{1\}$ such that $(\overline{F}, F) \circ (\overline{u}, u) = (\overline{v}, v) \circ (\overline{f} \oplus \mathrm{id}_{\underline{\mathbb{R}^{b+1}}})$ and $(\overline{F}, F) \circ (\overline{u'}, u') = (\overline{v'}, v')$ holds. Then $D^{n+1} \cup_{u'} W$ is a manifold whose boundary is oriented diffeomorphic to Σ by u and for which the bundle data above yield a stable isomorphism $TW \oplus \underline{\mathbb{R}^{a+b}} \to \underline{\mathbb{R}^{n+1+a+b}}$. Hence $[\Sigma]$ lies in bP^{n+1}.

Conversely, consider $[\Sigma]$ such there exists a stably parallizable manifold W together with an orientation preserving diffeomorphism $u : \Sigma \to \partial W$. We can assume without loss of generality that W is connected. Choose an orientation preserving homotopy equivalence $f : \Sigma \to S^n$. We can extend $\partial_0 F := f \circ u^{-1} : \partial W \to S^n$ to a smooth map $F : W \to D^{n+1}$. Since f has degree one, the map $(F, \partial F) : (W, \partial W) \to (D^{n+1}, S^n)$ has degree one. Let $y \in D^{n+1} - S^n$ be a regular value. Then the degree of F is the finite sum $\sum_{x \in F^{-1}(y)} \epsilon(x)$, where $\epsilon(x) = 1$, if $T_x F : T_x W \to T_y D^n$ preserves the induced orientations and $\epsilon(x) = -1$ otherwise. If two points x_1 and x_2 in $F^{-1}(y)$ satisfy $\epsilon(x_1) \neq \epsilon(x_2)$, one can change F up to homotopy relative ∂W such that $F^{-1}(y)$ contains two points less than before. Thus one can arrange that $F^{-1}(y)$ consists of precisely one point x and that $T_x F : T_x W \to T_x D^{n+1}$ is orientation preserving. Then one can change F up to homotopy in a small neigborhood of x such that there is an embedded disk $D_o^{n+1} \subset W - \partial W$ such that F induces a diffeomorphism $D_0^{n+1} \to F(D_0^{n+1})$ and no point outside D_0^{n+1} is mapped to $F(D_0^{n+1})$. Define $V = W - \mathrm{int}(D_0^{n+1})$. Then F induces a map also denoted by $F : V \to D^{n+1} - F(D_0^{n+1})$. If we identify $D^{n+1} - F(D_0^{n+1})$ with $S^n \times [0,1]$ by an orientation preserving diffeomorphism, we obtain a map of triads $(F; \partial_0 F, \partial_1 F) : (V; \partial_0 V, \partial_1 V) \to (S^{n+1} \times [0,1], S^n \times \{0\}, S^n \times \{1\})$ together with diffeomorphisms $u : \Sigma \to \partial_0 V$, $v : S^n \to S^n \times \{0\}$, $u' : S^n \to \partial_0 V$, $v' : S^n \to S^n \times \{1\}$ such that $\partial_1 F$ is an orientation preserving diffeomorphism and $F \circ u = v \circ f$ and $F \circ u' = v'$. Now one covers everything with appropriate bundle data to obtain a normal bordism from $(\overline{f}, f) : T\Sigma \oplus \underline{\mathbb{R}^a} \to \xi$ to $\text{id} : TS^n \to TS^n$. This shows $\eta \circ \alpha^{-1} \circ f^{-1}([\Sigma]) = [\text{id} : TS^n \to TS^n]$. ■

6.2 The surgery sequence for homotopy spheres

In this section we examine the surgery sequence (see Theorem 5.12) in the case of the sphere. In contrast to the general case we will obtain a long exact sequence of abelian groups. We have to introduce the following bordism groups.

Definition 6.7 *A* stable framing *of a closed oriented manifold M of dimension n is a strong bundle isomorphism $\overline{u} : TM \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{n+a}}$ for some $a \geq 0$ which is compatible with the given orientation. (Recall that strong means that $\overline{f}$ covers the identity.) An* almost stable framing *of a closed oriented manifold M of dimension n is a choice of a point $x \in M$ together with a strong bundle isomorphism $\overline{u} : TM|_{M-\{x\}} \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{n+a}}$ for some $a \geq 0$ which is compatible with the given orientation on $M - \{x\}$.*

Of course any stably framed manifold is in particular an almost stably framed manifold. A homotopy n-sphere Σ is an almost stably framed manifold since for any point $x \in \Sigma$ the complement $\Sigma - \{x\}$ is contractible and hence $T\Sigma|_{\Sigma-\{x\}}$ is trivial. We will later show the non-trivial fact that any homotopy n-sphere is stably parallizable, i.e. admits a stable framing. The standard sphere S^n inherits its standard stable framing from its embedding to $\mathbb{R}^{n+1}$.

A stably framed nullbordism for a stably framed manifold $(M, \overline{u})$ is a compact manifold W with a stable framing $\overline{U} : TW \oplus \underline{\mathbb{R}^{a+b}} \xrightarrow{\cong} \underline{\mathbb{R}^{n+1+a+b}}$ and a bundle isomorphism $(\overline{v}, v) : TM \oplus \underline{\mathbb{R}^{a+1+b}} \xrightarrow{\cong} TW \oplus \underline{\mathbb{R}^{a+b}}$ coming from the differentia of an orientation preserving diffeomorphism $v : M \to \partial W$ such that $\overline{U} \circ \overline{v} = \overline{u} \oplus \text{id}_{\underline{\mathbb{R}^{b+1}}}$. Now define the notion of a stably framed bordism from a stably framed manifold $(M, \overline{u})$ to another a stably framed manifold $(M', \overline{u'})$ to be a stably framed nullbordism for the disjoint union of $(M^-, \overline{u}^-))$ and $(M', \overline{u'})$, where M^- is obtained from M by reversing the orientation and $\overline{u}^-$ is the composition

$$\overline{u}^- : TM \oplus \underline{\mathbb{R}^a} \xrightarrow{\overline{u}} \overline{\mathbb{R}^{a+n}} = \underline{\mathbb{R}} \oplus \underline{\mathbb{R}^{a+n-1}} \xrightarrow{-\text{id}_{\underline{\mathbb{R}}} \oplus \text{id}_{\underline{\mathbb{R}^{a+n-1}}}} \underline{\mathbb{R}} \oplus \underline{\mathbb{R}^{a+n-1}} = \overline{\mathbb{R}^{a+n}}.$$

Consider two almost stably framed manifolds $(M, x, \overline{u} : TM|_{M-\{x\}} \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{a+n}})$ and $(M', x', \overline{u'} : TM'|_{M'-\{x'\}} \oplus \underline{\mathbb{R}^{a'}} \xrightarrow{\cong} \underline{\mathbb{R}^{a'+n}})$. An almost stably framed bordism from the first to the second consists of the following data. There is a compact oriented $(n+1)$-dimensional manifold triad $(W; \partial_0 W, \partial_1 W)$ with $\partial_0 W \cap \partial_1 W = \emptyset$ together with an embedding $j : ([0,1]; \{0\}, \{1\}) \to (W; \partial_0 W, \partial_1 W)$ such that j is transversal at the boundary. We also need a strong bundle isomorphism $\overline{U} : TW_{W-\text{im}(j)} \oplus \underline{\mathbb{R}^b} \xrightarrow{\cong} \underline{\mathbb{R}^{n+1+b}}$ for some $b \geq a, a'$.

Furthermore we require the existence of a bundle isomorphism $(\overline{v}, v) : TM \oplus \underline{\mathbb{R}^{b+1}} \xrightarrow{\cong} TW \oplus \underline{\mathbb{R}^b}$ coming from the differential an orientation preserving diffeomorphism $v : M^- \to \partial_0 W$ with $v(x) = j(0)$ and of a bundle isomorphism $(\overline{v'}, v') : TM \oplus \underline{\mathbb{R}^{b+1}} \xrightarrow{\cong} TW \oplus \underline{\mathbb{R}^b}$ coming from the differential of an orientation preserving diffeomorphism $v' : M' \to \partial_1 W$ with $v'(x') = j(1)$ such that $\overline{U} \circ \overline{v} = \overline{u}^- \oplus \mathrm{id}_{\underline{\mathbb{R}^{b-a+1}}}$ and $\overline{U} \circ \overline{v'} = \overline{u'} \oplus \mathrm{id}_{\underline{\mathbb{R}^{b+1-a'}}}$ holds.

Definition 6.8 *Let Ω_n^{fr} be the abelian group of stably framed bordism classes of stably framed closed oriented manifolds of dimension n. This becomes an abelian group by the disjoint union. The zero element is represented by S^n with its standard stable framing. The inverse of the class of $(M, \overline{u})$ is represented by the class of $(M^-, \overline{u} \oplus (-\mathrm{id}_{\underline{\mathbb{R}}}))$.*

Let Ω_n^{alm} be the abelian group of almost stably framed bordism classes of almost stably framed closed oriented manifolds of dimension n. This becomes an abelian group by the connected sum at the preferred base points. The zero element is represented by S^n with the base point $s = (1, 0, \ldots, 0)$ with its standard stable framing restricted to $S^n - \{s\}$. The inverse of the class of $(M, x, \overline{u})$ is represented by the class of $(M^-, x, \overline{u} \oplus (-\mathrm{id}_{\underline{\mathbb{R}}}))$.

Lemma 6.9 *There are canonical bijections of pointed sets*

$$\begin{aligned} \beta : \mathcal{N}_n(S^n) &\xrightarrow{\cong} \Omega_n^{\mathrm{alm}}; \\ \gamma : \mathcal{N}_{n+1}(S^n \times [0,1], S^n \times \{0,1\}) &\xrightarrow{\cong} \mathcal{N}_{n+1}(S^{n+1}). \end{aligned}$$

Proof : Consider an element $r \in \mathcal{N}_n(S^n)$ represented by a normal map $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map of degree one $f : M \to S^n$. Since f has degree one, one can change $(\overline{f}, f)$ by a homotopy such that $f^{-1}(s)$ consists of one point $x \in M$ for a fixed point $s \in S^n$. Since $S^n - \{s\}$ is contractible, $\xi_{S^n - \{s\}}$ admits a trivilization which is unique up to isotopy. It induces together with $\overline{f}$ an almost stable framing on (M, x). The class of (M, x) with this stable framing in Ω_n^{alm} is defined to be the image of r under β.

The inverse β^{-1} of β is defined as follows. Let $r \in \Omega_n^{\mathrm{alm}}$ be represented by the almost framed manifold $(M, x, \overline{u})$. Let $c : M \to S^n$ be the collaps map for a small embedded disk $D^n \subset M$ with origin x. By construction c induces a diffeomorphism $c|_{\mathrm{int}(D^n)} : \mathrm{int}(D^n) \to S^n - \{s\}$ and maps $M - \mathrm{int}(D^n)$ to $\{s\}$ for fixed $s \in S^n$. The almost stable framing $\overline{u}$ yields a bundle map $\overline{c'} : TM|_{M-\{x\}} \oplus \underline{\mathbb{R}^a} \to \underline{\mathbb{R}^{n+a}}$ covering $c|_{M-\{x\}} : M - \{x\} \to S^n - \{c(x)\}$. Since D^n is contractible, we obtain a bundle map unique up to isotopy $\overline{c''} : TD^n \oplus \underline{\mathbb{R}^a} \to \underline{\mathbb{R}^{n+a}}$ covering $c|_{D^n} : D^n \to S^n$. The composition of the inverse of the restriction of $\overline{c''}$ to $\mathrm{int}(D^n) - \{x\}$ and of the restriction of $\overline{c'}$ to $\mathrm{int}(D^n) - \{x\}$ yields a strong bundle automorphism of the trivial bundle $\underline{\mathbb{R}^{n+a}}$ over $S^n - \{s, c(x)\}$. Let ξ be the bundle obtained by glueing the trivial bundle $\underline{\mathbb{R}^{a+n}}$ over $S^n - \{s\}$ and the trivial bundle $\underline{\mathbb{R}^{a+n}}$ over $S^n - \{c(x)\}$ together using this bundle automorphism over $S^n - \{s, c(x)\}$. Then $\overline{c'}$ and $\overline{c''}$ fit together to a bundle map $\overline{c} : TM \oplus \underline{\mathbb{R}^{n+a}} \to \xi$ covering c. Define the image of r under β^{-1} to be the class of $(\overline{c}, c)$.

Consider $r \in \mathcal{N}_{n+1}(S^n \times [0,1], S^n \times \{0,1\})$ represented by a normal map $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering $(f, \partial f) : (M, \partial M) \to (S^n \times [0,1], S^n \times \{0,1\})$. Recall that ∂f is a diffeomorphism. Hence one can form the closed manifold $N = M \cup_{\partial f : \partial M \to S^n \times \{0,1\}} D^{n+1} \times \{0,1\}$. The map f and the identity on $D^{n+1} \times \{0,1\}$ induce a map of degree one $g : N \to S^n \times [0,1] \cup_{S^n \times \{0,1\}} D^{n+1} \times \{0,1\} \cong S^{n+1}$. Define the bundle η over $S^n \times [0,1] \cup_{S^n \times \{0,1\}} D^{n+1} \times \{0,1\} \cong S^{n+1}$ by glueing ξ and $T(D^{n+1} \times \{0,1\}) \oplus \underline{\mathbb{R}^{a-1}}$ together over $S^n \times \{0,1\}$ by the strong bundle isomorphism

$$\xi|_{S^{n-1} \times \{0,1\}} \xrightarrow{(\overline{f}|_{S^{n-1} \times \{0,1\}})^{-1}} TM|_{\partial M} \oplus \underline{\mathbb{R}^a} = T(\partial M) \oplus \underline{\mathbb{R}^{a+1}}$$

$$\xrightarrow{T\partial f} T(S^{n-1} \times \{0,1\}) \oplus \underline{\mathbb{R}^{a+1}} = T(D^{n+1} \times \{0,1\}) \oplus \underline{\mathbb{R}^a}.$$

Then $\overline{f}$ and $\mathrm{id} : T(D^{n+1} \times \{0,1\}) \oplus \underline{\mathbb{R}^a} \to T(D^{n+1} \times \{0,1\}) \oplus \underline{\mathbb{R}^a}$ fit together yielding a bundle map $\overline{g} : TN \oplus \underline{\mathbb{R}^a} \to \eta$ covering g. Define the image of r under γ by the class of $(\overline{g}, g)$. We leave it to the reader to construct the inverse of γ which is similar to the construction in Lemma 6.6 but now two embedded discs instead of one embedded disc are removed. ■

Next we want to construct a long exact sequence of abelian groups

$$\ldots \to \Omega_{n+1}^{\mathrm{alm}} \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}) \xrightarrow{\partial} \Theta^n \xrightarrow{\eta} \Omega_n^{\mathrm{alm}} \xrightarrow{\sigma} L_n(\mathbb{Z}) \to \ldots$$

The map

$$\sigma : \Omega_{n+1}^{\mathrm{alm}} \to L_{n+1}(\mathbb{Z})$$

is given by the composition

$$\Omega_{n+1}^{\mathrm{alm}} \xrightarrow{\beta^{-1}} \mathcal{N}_{n+1}(S^{n+1}) \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}),$$

where β is the bijection of Lemma 6.9 and $\sigma : \mathcal{N}_n(S^n) \to L_n(\mathbb{Z})$ is given by the surgery obstruction and has already appeared in the surgery sequence (see Theorem 5.12). The map

$$\partial : L_{n+1}(\mathbb{Z}) \to \Theta^n$$

is the composition of the inverse of the bijection $\alpha \circ f : \mathcal{S}_n(S^n) \xrightarrow{\cong} \Theta^n$ coming from Lemma 6.2 and Lemma 6.3 and the map $\partial : L_{n+1}(\mathbb{Z}) \to \mathcal{N}_n(S^n)$ of the surgery sequence (see Theorem 5.12). The map

$$\eta : \Theta^n \to \Omega_n^{\mathrm{alm}} \tag{6.10}$$

sends the class of a homotopy sphere Σ to the class of $(\Sigma, x, \overline{u})$, where x is any point in Σ and the stable framing of $T\Sigma|_{\Sigma - \{x\}}$ comes from the fact that $\Sigma - \{x\}$ is contractible. This map η corresponds to the map η appearing in the surgery exact sequence (see Theorem 5.12) under the identification $\alpha \circ f : \mathcal{S}_n(S^n) \xrightarrow{\cong} \Theta^n$ coming from Lemma 6.2 and Lemma 6.3.

We leave it to the reader to check that all these maps are homomorphisms of abelian groups. The surgery sequence (see Theorem 5.12) implies

Theorem 6.11 *The long sequence of abelian groups which extends infinitely to the left*

$$\ldots \to \Omega_{n+1}^{\mathrm{alm}} \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}) \xrightarrow{\partial} \Theta^n \xrightarrow{\eta} \Omega_n^{\mathrm{alm}} \xrightarrow{\sigma} L_n(\mathbb{Z}) \xrightarrow{\partial} \ldots \xrightarrow{\eta} \Omega_5^{\mathrm{alm}} \xrightarrow{\sigma} L_5(\mathbb{Z})$$

is exact.

Recall that we have shown in Theorem 4.30, Theorem 4.32 and Theorem 4.43 that there are isomorphisms

$$\frac{1}{8} \cdot \mathrm{sign} : L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$$

and

$$\mathrm{Arf} : L_2(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/2$$

and that $L_{2i+1}(\mathbb{Z}) = 0$ for $i \in \mathbb{Z}$. Consider a normal map $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map of degree one $(f, \partial f) : (M, \partial M) \to (X, \partial X)$ of oriented compact $4i$-dimensional manifolds such that X is simply connected and ∂f a homotopy equivalence. Then the isomorphism $\frac{1}{8} \cdot \mathrm{sign} : L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$ sends $\sigma(\overline{f}, f)$ to $\frac{1}{8} \cdot \mathrm{sign}(M \cup_{\partial f} X^-)$. Hence Theorem 6.11 and Lemma 6.6 imply

Corollary 6.12 *There are for $i \geq 2$ and $j \geq 3$ short exact sequences of abelian groups*

$$0 \to \Theta^{4i} \xrightarrow{\eta} \Omega_{4i}^{\mathrm{alm}} \xrightarrow{\frac{\mathrm{sign}}{8}} \mathbb{Z} \xrightarrow{\partial} bP^{4i} \to 0$$

and

$$0 \to \Theta^{4i-2} \xrightarrow{\eta} \Omega_{4i-2}^{\mathrm{alm}} \xrightarrow{\mathrm{Arf}} \mathbb{Z}/2 \xrightarrow{\partial} bP^{4i-2} \to 0$$

and

$$0 \to bP^{2j} \to \Theta^{2j-1} \xrightarrow{\eta} \Omega_{2j-1}^{\mathrm{alm}} \to 0.$$

Here the map

$$\frac{\mathrm{sign}}{8} : \Omega_{4i}^{\mathrm{alm}} \to \mathbb{Z}$$

sends $[M]$ to the signature $\mathrm{sign}(M)$ of M and $\mathrm{Arf} : \Omega^{\mathrm{alm}} \to \mathbb{Z}/2$ sends $[M]$ to the Arf invariant of the normal map $\beta^{-1}([M]) \in \mathcal{N}_n(S^n)$ for β the bijection appearing in Lemma 6.9.

6.3 The J-homomorphism and stably and almost stably framed bordism

By Theorem 6.11 we have reduced the computation of Θ^n to computations about Ω_n^{alm} and certain maps to $\mathbb{Z}$ and $\mathbb{Z}/2$ given by the signature and the Arf invariant. This reduction is essentially due to the surgery machinery. The rest of the computation will essentially be homotopy theory. First we try to understand Ω_*^{fr} geometrically. There is an obvious forgetful map

$$f : \Omega_n^{\mathrm{fr}} \to \Omega_n^{\mathrm{alm}}. \tag{6.13}$$

Define the group homomorphism

$$\partial : \Omega_n^{\mathrm{alm}} \quad \to \quad \pi_{n-1}(SO) \tag{6.14}$$

as follows. Given $r \in \Omega_n^{alm}$ choose a representative $(M, x, \overline{u} : TM|_{M-\{x\}} \oplus \underline{\mathbb{R}^a} \to \underline{\mathbb{R}^{n+a}})$. Let $D^n \subset M$ be an embedded disk with origin x. Since D^n is contractible, we obtain a strong bundle isomorphism unique up to isotopy $\overline{v} : TM|_{D^n} \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{a+n}}$. The composition of the inverse of the restriction of $\overline{u}$ to $S^{n-1} = \partial D^n$ and of the restriction of $\overline{v}$ to S^{n-1} is an orientation preserving bundle automorphism of the trivial bundle $\overline{\mathbb{R}^{a+n}}$ over S^{n-1}. This is the same as a map $S^{n-1} \to SO(n+a)$. It composition with the canonical map $SO(n+a) \to SO$ represents an element in $\pi_{n-1}(SO)$ which is defined to be the image of r under $\partial : \Omega_n^{\mathrm{alm}} \to \pi_{n-1}(SO)$. Let

$$\overline{J} : \pi_n(SO) \quad \to \quad \Omega_n^{\mathrm{fr}} \tag{6.15}$$

be the group homomorphism which assigns to the element $r \in \pi_n(SO)$ represented by a map $\overline{u} : S^n \to SO(n+a)$ the class of S^n with the stable framing $TS^n \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{a+n}}$ which is induced by the standard trivialization

$$TS^n \oplus \underline{\mathbb{R}} \cong T(\partial D^{n+1}) \oplus \nu(\partial D^{n+1}, D^{n+1}) \cong TD^{n+1}|_{\partial D^{n+1}} \cong \underline{\mathbb{R}^{n+1}}$$

and the strong bundle automorphism of the trivial bundle $\overline{\mathbb{R}^{a+n}}$ over S^n given by $\overline{u}$. One easily checks

Lemma 6.16 *The following sequence is a long exact sequence of abelian groups*

$$\ldots \xrightarrow{\partial} \pi_n(SO) \xrightarrow{\overline{J}} \Omega_n^{fr} \xrightarrow{f} \Omega_n^{\mathrm{alm}} \xrightarrow{\partial} \pi_{n-1}(SO) \xrightarrow{\overline{J}} \Omega_{n-1}^{fr} \xrightarrow{f} \ldots.$$

Next we want to interpret the exact sequence appearing in Lemma 6.16 homotopy theoretically. We begin with the homomorphism $\overline{J}$. Notice that there is a natural bijection

$$\tau' : \mathrm{colim}_{k\to\infty}\, \Omega_n(\underline{\mathbb{R}^k} \to \{*\}) \quad \xrightarrow{\cong} \quad \Omega_n^{\mathrm{fr}} \tag{6.17}$$

which is defined as follows. Consider an element

$$x = [(M, i : M \to \mathbb{R}^{n+k}, \mathrm{pr} : M \to \{*\}, \overline{u} : \nu(i) \to \underline{\mathbb{R}^k})] \ \in \ \Omega_n(\underline{\mathbb{R}^k} \to \{*\})$$

There is a canonical strong isomorphism $\overline{u'} : TM \oplus \nu(i) \xrightarrow{\cong} \underline{\mathbb{R}^{n+k}}$. From $\underline{u}$ and $\underline{u'}$ we get an isomorphism $\underline{v} : TM \oplus \underline{\mathbb{R}^k} \to \underline{\mathbb{R}^{n+k}}$ covering the projection $M \to \{*\}$. Define $\tau'(x)$ by the class of $(M, \overline{v})$. The Thom space $\mathrm{Th}(\underline{\mathbb{R}^{n+k}})$ is S^{n+k}. Hence the Pontrjagin-Thom construction (see Theorem 3.26) induces an isomorphism

$$P : \mathrm{colim}_{k\to\infty}\, \Omega_n(\underline{\mathbb{R}^k} \to \{*\}) \ \cong \ \mathrm{colim}_{k\to\infty}\, \pi_{n+k}(S^k). \tag{6.18}$$

Notice that the *stable n-th homotopy group* of a space X is defined by

$$\pi_n^s(X) \ := \ \mathrm{colim}_{k\to\infty}\, \pi_{n+k}(S^k \wedge X) \tag{6.19}$$

The stable *n-stem* is defined by

$$\pi_n^s \ := \pi_n^s(S^0) \ := \ \operatorname{colim}_{k\to\infty} \pi_{n+k}(S^k). \tag{6.20}$$

Thus the isomorphism τ' of (6.17) and the isomorphism P of (6.18) yield an isomorphism

$$\tau : \Omega_n^{\mathrm{fr}} \xrightarrow{\cong} \pi_n^s. \tag{6.21}$$

Next we explain the so called *Hopf construction* which defines for spaces X, Y and Z a map

$$H : [X \times Y, Z] \ \to \ [X * Y, \Sigma Z] \tag{6.22}$$

as follows. Recall that the join $X * Y$ is defined by $X \times Y \times [0,1]/\sim$, where $\sim$ is given by $(x,y,0) \sim (x',y,0)$ and $(x,y,1) \sim (x,y',1)$, and that the (unreduced) suspension ΣZ is defined by $Z \times [0,1]/\simeq$, where $\sim$ is given by $(z,0) \sim (z',0)$ and $(z,1) \sim (z',1)$. Given $f : X \times Y \to Z$, let $H(f) : X * Y \to \Sigma Z$ be the map induced by $f \times \mathrm{id} : Y \times [0,1] \to Z \times [0,1]$. Consider the following composition

$$[S^n, SO(k)] \to [S^n, \mathrm{aut}(S^{k-1})] \to [S^n \times S^{k-1}, S^{k-1}]$$

$$\xrightarrow{H} [S^n * S^{k-1}, \Sigma S^{k-1}] = [S^{n+k}, S^k].$$

Notice that $\pi_1(SO(k))$ acts trivially on $\pi_n(SO(k))$ and $\pi_1(S^k)$ acts trivially on $\pi_{n+k}(S^k)$ for $k, n \geq 1$. Hence no base point questions arise in the next definition.

Definition 6.23 *The composition above induces for $n, k \geq 1$ homomorphisms of abelian groups*

$$J_{n,k} : \pi_n(SO(k)) \to \pi_{n+k}(S^k).$$

Taking the colimit for $k \to \infty$ induces the so called J-homomorphism

$$J_n : \pi_n(SO) \to \pi_n^s.$$

One easily checks

Lemma 6.24 *The composition of the homomorphism $\overline{J} : \pi_n(SO) \to \Omega_n^{\mathrm{fr}}$ of (6.15) with the isomorphism $\tau : \Omega_n^{\mathrm{fr}} \xrightarrow{\cong} \pi_n^s$ of (6.21) is the J-homomorphism $J_n : \pi_n(SO) \to \pi_n^s$ of Definition 6.23.*

The homotopy groups of O are 8-periodic and given by

i mod 8	0	1	2	3	4	5	6	7
$\pi_i(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Notice that $\pi_i(SO) = \pi_i(O)$ for $i \geq 1$ and $\pi_0(SO) = 1$. The first stable stems are given by

n	0	1	2	3	4	5	6	7	8	9
π_n^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$

The *Bernoulli numbers* B_n for $n \geq 1$ are defined by

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n \geq 1} \frac{(-1)^{n+1} \cdot B_n}{(2n)!} \cdot (z)^{2n}. \tag{6.25}$$

The first values are given by

n	1	2	3	4	5	6	7	8
B_n	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$	$\frac{3617}{510}$

The next result is a deep theorem due to Adams [1, Theorem 1.1, Theorem 1.3 and Theorem 1.5].

Theorem 6.26 *1. If* $n \neq 3 \mod 4$, *then the J-homomorphism* $J_n : \pi_n(SO) \to \pi_n^s$ *is injective;*

2. The order of the image of the J-homomorphism $J_{4k-1} : \pi_{4k-1}(SO) \to \pi_{4k-1}^s$ *is* denominator$(B_k/4k)$, *where* B_k *is the k-th Bernoulli number.*

6.4 Computation of bP^{n+1}

In this section we want to compute the subgroups $bP^{n+1} \subset \Theta^n$ (see Definition 6.5).

We have introduced the bijection $\beta : \mathcal{N}_n(S^n) \xrightarrow{\cong} \Omega_n^{\mathrm{alm}}$ in Lemma 6.9 and the map $\partial : \Omega_n^{alm} \to \pi_{n-1}(SO)$ in (6.14). Let

$$\delta(k) : \pi_n(BSO(k)) \xrightarrow{\cong} \pi_{n-1}(SO(k)) \tag{6.27}$$

be the boundary map in the long exact homotopy sequence associated to the fibration $SO(k) \to ESO(k) \to BSO(k)$. It is an isomorphism since $ESO(k)$ is contractible. It can be described as follows. Consider $x \in \pi_n(BSO(k))$. Choose a representative $f : S^n \to BSO(k)$ for some k. If $\gamma_k \to BSO(k)$ is the universal k-dimensional oriented vector bundle, $f^*\gamma_k$ is a k-dimensional oriented vector bundle over S^n. Let S^n_- be the lower and S^n_+ be the upper hemisphere and $S^{n-1} = S^n_- \cap S^n_+$. Since the hemispheres are contractible, we obtain an up to isotopy unique strong bundle isomorphisms $\overline{u}_- : f^*\gamma_k|_{S^n_-} \xrightarrow{\cong} \underline{\mathbb{R}^k}$ and $\overline{u}_+ : f^*\gamma_k|_{S^n_+} \xrightarrow{\cong} \underline{\mathbb{R}^k}$. The composition of the inverse of the restriction of $\overline{u}_-$ to S^{n-1} with the restriction of $\overline{u}_+$ to S^{n-1} is a bundle automorphism of the trivial bundle $\underline{\mathbb{R}^k}$ over S^{n-1} which is the same as map $S^{n-1} \to SO(k)$. Define its class in $\pi_{n-1}(SO(k))$ to be the image of x under $\delta(k)^{-1} : \pi_n(BSO(k)) \to \pi_{n-1}(SO(k))$. Analogously we get an isomorphism

$$\delta : \pi_n(BSO) \xrightarrow{\cong} \pi_{n-1}(SO). \tag{6.28}$$

Define a map

$$\gamma : \mathcal{N}_n(S^n) \to \pi_n(BSO) \tag{6.29}$$

by sending the class of the normal map of degree one $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map $f : M \to S^n$ to the the class represented by the classifying map $f_\xi : S^n \to BSO(n+k)$ of ξ. One easily checks

Lemma 6.30 *The following diagram commutes*

$$\begin{array}{ccc} \Omega_n^{alm} & \xrightarrow{\partial} & \pi_{n-1}(SO) \\ \beta^{-1}\downarrow & & \downarrow\delta^{-1} \\ \mathcal{N}_n(S^n) & \xrightarrow[\gamma]{} & \pi_n(BSO) \end{array}$$

The next ingredient is the Hirzebruch signature formula. It says for a closed oriented manifold M of dimension $n = 4k$ that its signature can be computed in terms of the L-class $\mathcal{L}(M)$ by

$$\operatorname{sign}(M) = \langle \mathcal{L}(M), [M] \rangle. \tag{6.31}$$

The L-class is a cohomology class which is obtained from inserting the Pontrjagin classes $p_i(TM)$ into a certain polynomial $L(x_1, x_2, \ldots x_k)$. The L-polynomial $L(x_1, x_2, \ldots x_n)$ is the sum of $s_k \cdot x_k$ and terms which do not involve x_k, where s_k is given in terms of the Bernoulli numbers B_k by

$$s_k := \frac{2^{2k} \cdot (2^{2k-1} - 1) \cdot B_k}{(2k)!}. \tag{6.32}$$

Assume that M is almost stably parallizable. Then for some point $x \in M$ the restriction of the tangent bundle TM to $M - \{x\}$ is stably trivial and hence has trivial Pontrjagin classes. Since the inclusion induces an isomorphism $H^p(M) \xrightarrow{\cong} H^p(M - \{x\})$ for $p \leq n-2$, we get $p_i(M) = 0$ for $i \leq k-1$. Hence (6.31) implies for a closed oriented almost stably parallizable manifold M of dimension $4k$

$$\operatorname{sign}(M) = s_k \cdot \langle p_k(TM), [M] \rangle. \tag{6.33}$$

We omit the proof of the next lemma which is based on certain homotopy theoretical computations (see for instance [38, Theorem 3.8 on page 76]).

Lemma 6.34 *Let $n = 4k$. Then there is an isomorphism*

$$\phi : \pi_{n-1}(SO) \xrightarrow{\cong} \mathbb{Z}.$$

Define a map

$$p_k : \pi_n(BSO) \to \mathbb{Z}$$

by sending the element $x \in \pi_n(BSO)$ represented by a map $f : S^n \to BSO(m)$ to $\langle p_k(f^*\gamma_m), [S^n] \rangle$ for $\gamma_m \to BSO(m)$ the universal bundle. Let $\delta : \pi_n(BSO) \to \pi_{n-1}(SO)$ be the isomorphism of (6.27). *Put*

$$t_k := \tfrac{3-(-1)^k}{2} \cdot (2k-1)! \tag{6.35}$$

Then

$$t_k \cdot \phi = p_k \circ \delta^{-1}.$$

Lemma 6.36 *The following diagram commutes for $n = 4k$*

$$\begin{array}{ccc} \Omega_n^{\mathrm{alm}} & \xrightarrow{\frac{\mathrm{sign}}{8}} & \mathbb{Z} \\ \partial \downarrow & & \uparrow \frac{s_k \cdot t_k}{8} \cdot \mathrm{id} \\ \pi_{n-1}(SO) & \xrightarrow[\cong]{\phi} & \mathbb{Z} \end{array}$$

where $\frac{\mathrm{sign}}{8}$ is the homomorphism appearing in Corollary 6.12, the homomorphism ∂ has been defined in in (6.14) *and the isomorphism ϕ is taken from* (6.34).

Proof : Consider an almost stably parallizable manifold M of dimension $n = 4k$. We conclude from Lemma 6.30 and Lemma 6.34

$$\begin{aligned} \frac{s_k \cdot t_k}{8} \cdot \phi \circ \partial([M]] &= \frac{s_k}{8} \cdot p_k \circ \delta^{-1} \circ \partial([M]] \\ &= \frac{s_k}{8} \cdot p_k \circ \gamma \circ \beta^{-1}([M]]. \end{aligned} \tag{6.37}$$

By definition the composition

$$\Omega_n^{\mathrm{alm}} \xrightarrow{\beta^{-1}} \mathcal{N}_n(S^n) \xrightarrow{\gamma} \pi_n(SO) \xrightarrow{p_k} \mathbb{Z}$$

sends the class of M to $\langle p_k(\xi), [S^n] \rangle$ for a bundle ξ over S^n for which there exists a bundle map $(\bar{c}, c) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map $c : M \to S^n$ of degree one. This implies

$$\begin{aligned} \frac{s_k}{8} \cdot p_k \circ \gamma \circ \beta^{-1}([M]) &= \frac{s_k}{8} \cdot \langle p_k(\xi), [S^n] \rangle \\ &= \frac{s_k}{8} \cdot \langle p_k(\xi), c_*([M]) \rangle \\ &= \frac{s_k}{8} \cdot \langle c^*(p_k(\xi)), [M] \rangle. \\ &= \frac{s_k}{8} \cdot \langle p_k(TM), [M] \rangle \end{aligned} \tag{6.38}$$

Now the claim follows from (6.33), (6.37) and (6.38). ■

Theorem 6.39 *Let $k \geq 2$ be an integer. Then bP^{4k} is a finite cyclic group of order*

$$\begin{aligned} &\frac{s_k \cdot t_k}{8} \cdot \left| \mathrm{im} \left(J_{4k-1} : \pi_{4k-1}(SO) \to \pi_{4k-1}^s \right) \right| \\ &= \frac{1}{8} \cdot \frac{2^{2k} \cdot (2^{2k-1} - 1) \cdot B_k}{(2k)!} \cdot \frac{3 - (-1)^k}{2} \cdot (2k-1)! \cdot \mathrm{denominator}(B_k/4k) \\ &= \frac{3 - (-1)^k}{2} \cdot 2^{2k-2} \cdot (2^{2k-1} - 1) \cdot \mathrm{numerator}(B_k/(4k)). \end{aligned}$$

Proof : This follows from Theorem 6.11, Theorem 6.26 (2) and Lemma 6.36. ■

Next we treat the case $n = 4k+2$ for $k \geq 1$ Let

$$\mathrm{Arf} : \pi^s_{4k+2} \quad \to \quad \mathbb{Z}/2 \tag{6.40}$$

be the composition of the inverse of the Pontrjagin-Thom isomorphism $\tau : \Omega^{\mathrm{fr}}_n \xrightarrow{\cong} \pi^s_n$ of (6.21), the forgetful homomorphism $f : \Omega^{\mathrm{fr}}_{4k+2} \to \Omega^{\mathrm{alm}}_{4k+2}$ of (6.13) and the map $\mathrm{Arf} : \Omega^{\mathrm{alm}}_{4k+2} \to \mathbb{Z}/2$ appearing in Corollary 6.12.

Theorem 6.41 *Let $k \geq 3$. Then bP^{4k+2} is a trivial group if the homomorphism* $\mathrm{Arf} : \pi^s_{4k+2} \to \mathbb{Z}/2$ *of* (6.40) *is surjective and is $\mathbb{Z}/2$ if the homomorphism* $\mathrm{Arf} : \pi^s_{4k+2} \to \mathbb{Z}/2$ *of* (6.40) *is trivial.*

Proof : We conclude from Lemma 6.16, Lemma 6.24 and and Theorem 6.26 (1) that the forgetful map $f : \Omega^{\mathrm{fr}}_{4k+2} \to \Omega^{\mathrm{alm}}_{4k+2}$ is surjective. Now the claim follows from Corollary 6.12. ■

The next result is due to Browder [8].

Theorem 6.42 *The homomorphism* $\mathrm{Arf} : \pi^s_{4k+2} \to \mathbb{Z}/2$ *of* (6.40) *is trivial if* $2k+1 \neq 2^l - 1$

The homomorphism $\mathrm{Arf} : \pi^s_{4k+2} \to \mathbb{Z}/2$ of (6.40) is also known to be non-trivial for $4k+2 \in \{6, 14, 30, 62\}$ (combine [8], [4] and [43]). Hence Theorem 6.41 and Theorem 6.42 imply

Corollary 6.43 *The group bP^{4k+2} is trivial or isomorphic to $\mathbb{Z}/2$. We have*

$$bP^{4k+2} = \begin{cases} \mathbb{Z}/2 & 4k+2 \neq 2^l - 2, k \geq 1; \\ 0 & 4k+2 \in \{6, 14, 30, 62\}. \end{cases}$$

We conclude from Corollary 6.12 and Lemma 6.6 Theorem 6.11

Theorem 6.44 *We have for $k \geq 3$*

$$bP^{2k+1} = 0.$$

Theorem 6.45 *For $n \geq 1$ any homotopy n-sphere Σ is stably parallizable.*

For an almost parallizable manifold M the image of its class $[M] \in \Omega^{\mathrm{alm}}_n$ under the homomorphism $\partial : \Omega^{\mathrm{alm}}_n \to \pi_n(SO(n-1))$ is exactly the obstruction to extend the almost stable framing to a stable framing. Recall that any homotopy n-sphere is almost stably parallizable. The map ∂ is trivial for $n \neq 0 \mod 4$ by Lemma 6.16, Lemma 6.24, Theorem 6.26 (1). If $n = 0 \mod 4$, the claim follows from Lemma 6.16, Lemma 6.24 and and Lemma 6.36 since the signature of a homotopy n-sphere is trivial. ■

6.5 Computation of Θ^n/bP^{n+1}

In this chapter we compute Θ^n/bP^{n+1}.

Theorem 6.46 *1. If $n = 4k+2$, then there is an exact sequence*

$$0 \to \Theta^n/bP^{n+1} \to \operatorname{coker}(J_n : \pi_n(SO) \to \pi_n^s) \to \mathbb{Z}/2;$$

2. If $n \neq 2 \mod 4$ or if $n = 4k+2$ with $2k+1 \neq 2^l - 1$, then

$$\Theta^n/bP^{n+1} \;\cong\; \operatorname{coker}(J_n : \pi_n(SO) \to \pi_n^s).$$

Proof : Lemma 6.16, Lemma 6.24, Theorem 6.26 (1) and Lemma 6.36 imply

$$\begin{array}{lcll}
\ker\left(\partial : \Omega_n^{\mathrm{alm}} \to \pi_{n-1}(SO)\right) & = & \Omega_n^{\mathrm{alm}} & n \neq 0 \mod 4; \\
\ker\left(\partial : \Omega_n^{\mathrm{alm}} \to \pi_{n-1}(SO)\right) & = & \ker\left(\frac{\mathrm{sign}}{8} : \Omega_n^{\mathrm{alm}} \to \mathbb{Z}\right) & n = 0 \mod 4; \\
\ker\left(\partial : \Omega_n^{\mathrm{alm}} \to \pi_{n-1}(SO)\right) & = & \operatorname{coker}\left(J_n : \pi_n(SO) \to \pi_n^{\mathrm{fr}}\right). &
\end{array}$$

Now the claim follows from Corollary 6.12 and Theorem 6.42. ■

6.6 The Kervaire-Milnor braid

We have established in Lemma 6.16 the long exact sequence

$$\ldots \xrightarrow{\partial} \pi_n(SO) \xrightarrow{J} \Omega_n^{fr} \xrightarrow{f} \Omega_n^{\mathrm{alm}} \xrightarrow{\partial} \pi_{n-1}(SO) \xrightarrow{J} \Omega_{n-1}^{fr} \xrightarrow{f} \ldots.$$

The long exact sequence

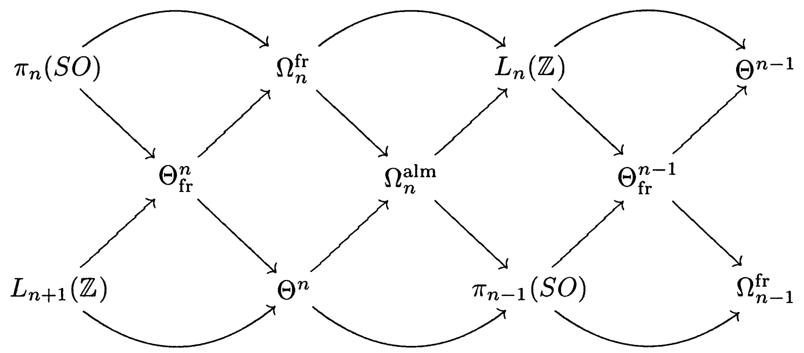
$$\ldots \to L_{n+1}(\mathbb{Z}) \to \Theta^n \to \Omega_n^{\mathrm{alm}} \to L_n(\mathbb{Z}) \to \Theta^{n-1} \to \ldots$$

is taken from Theorem 6.11. Denote by Θ_{fr}^n the abelian group of stably framed h-cobordism classes of stably framed homotopy n-spheres. There is a long exact sequence

$$\ldots \to L_{n+1}(\mathbb{Z}) \to \Theta_{\mathrm{fr}}^n \to \Omega_n^{\mathrm{fr}} \to L_n(\mathbb{Z}) \to \Theta_{\mathrm{fr}}^{n-1} \to \ldots$$

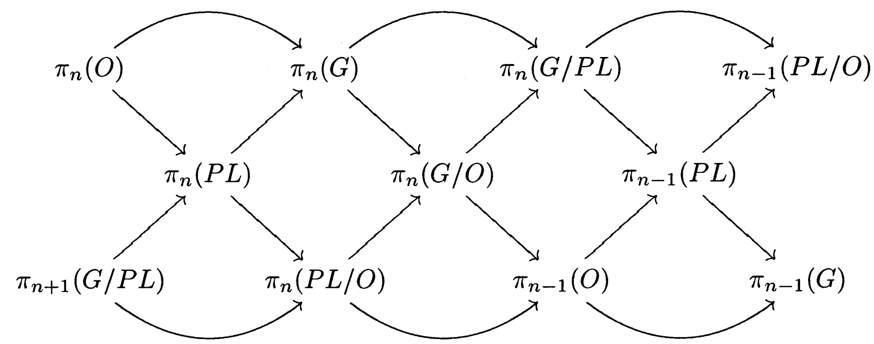
which is defined as follows. The map $\Theta_{\mathrm{fr}}^n \to \Omega_n^{\mathrm{fr}}$ assign to the class of a framed homotopy sphere its class in Ω_n^{fr}. The map $\Omega_n^{\mathrm{fr}} \to L_n(\mathbb{Z})$ assigns to a class $[M]$ in Ω_n^{fr} the surgery obstruction of the normal map $(\overline{c}, c) : TM \oplus \underline{\mathbb{R}^a} \to \underline{\mathbb{R}^{n+a}}$ for any map of degree one $c : M \to S^n$, where $\overline{c}$ is given by the framing on M. The map $L_{n+1}(\mathbb{Z}) \to \Theta_{\mathrm{fr}}^n$ assigns to $x \in L_{n+1}(\mathbb{Z})$ the class of the framed homotopy n-sphere $(\Sigma, \overline{v} : T\Sigma \oplus \underline{\mathbb{R}^a} \to \underline{\mathbb{R}^{a+n}})$, for which there is a normal map of degree one $(\overline{U}, U) : TW \oplus \underline{\mathbb{R}^{a+b}} \to \underline{\mathbb{R}^{n+a+b}}$ covering a map of triads $U : (W; \partial_0 W, \partial_1 W) \to (S^n \times [0,1]; S^n \times \{0\}, S^n \times \{1\})$ and a bundle map $(\overline{u_0}, u_0) : T\Sigma \oplus \underline{\mathbb{R}^{a+b+1}} \to TW_0 \oplus \underline{\mathbb{R}^{a+b}}$ covering the orientation preserving diffeomorphism $u_0 : \Sigma \to \partial_0 W$ such that $\overline{U} \circ \overline{u_0} = \overline{v} \oplus \mathrm{id}_{\underline{\mathbb{R}^{b+1}}}$, U induces a diffeomorphism $\partial_1 W \to S^n$ and the surgery obstruction associated to $(\overline{U}, U)$ is the given element $x \in L_{n+1}(\mathbb{Z})$.

Theorem 6.47 (The Kervaire-Milnor braid) *The long exact sequences above fit together to an exact braid for* $n \geq 5$



We have already introduced BO and BG and have defined G/O as the homotopy fiber of this map. There is also a PL-version of BO called BPL . We can define analogously spaces G/PL and PL/O and fibrations $G/PL \to BPL \to BO$ and $PL/G \to BPL \to BG$. Since $\Omega BO \simeq O$, $\Omega BPL \simeq BPL$ and $\Omega BG \simeq G$ holds, we get fibrations $O \to G \to G/O$, $PL \to G \to PL/G$ and $O \to PL \to PL/O$. Notice that for an inclusion of topological groups $H \subset K$ there is an obvious fibration $H \to K \to K/H$ and the fibrations above are in this spirit. But we have to use the classifying spaces since for instance G is not a group and we cannot talk about the homogeneous space G/PL. More information about these spaces and their homotopy theoretic properties can be found for instance in [42].

Theorem 6.48 (Homotopy theoretic interpretation of the Kervaire-Milnor braid) *The long exact homotopy sequences of these three fibrations above yield an exact braid*



which is for $n \geq 5$ *isomorphic to the Kervaire-Milnor braid of Theorem 6.47*

Proof : At least we explain how the two braids are related by isomorphisms. Since O/SO is the discrete group $\{\pm 1\}$, the inclusion induces an isomorphism

$$\pi_n(SO) \xrightarrow{\cong} \pi_n(O) \qquad \text{for } n \geq 1.$$

The Pontrjagin-Thom isomorphism $\tau : \Omega_n^{\mathrm{fr}} \xrightarrow{\cong} \pi_n^s$ of (6.21) and the canonical isomorphism $\pi_n(G) \cong \pi_n^s$ yield an isomorphism

$$\Omega_n^{\mathrm{fr}} \xrightarrow{\cong} \pi_n(G).$$

The isomorphism $\beta : \mathcal{N}_n(S^n) \xrightarrow{\cong} \Omega_n^{\mathrm{alm}}$ of Lemma 6.9 together with the bijection $\pi_n(G/O) \xrightarrow{\cong} \mathcal{N}_n(S^n)$ coming from the operation of Theorem 3.46 and Lemma 3.49 and the preferred base point $[\mathrm{id} : TS^n \to TS^n]$ in $\mathcal{N}_n(S^n)$ induce a bijection

$$\Omega_n^{\mathrm{alm}} \xrightarrow{\cong} \pi_n(G/O).$$

One can also develop surgery theory in the PL-category instead of the smooth category. We have the surgery exact sequence in the PL-category (see Theorem 5.15.

$$\ldots \to \pi_{n+1}(G/PL) \to L_{n+1}(\mathbb{Z}) \to \mathcal{S}_n^{PL}(S^{n+1}) \to \pi_n(G/PL) \to L_n(\mathbb{Z}) \to \ldots$$

Since the Poincare Conjecture is true for $n \geq 5$ and any PL-homeomorphism $f : S^n \to S^n$ can be extended to a PL-homeomorphism $F : D^{n+1} \to D^{n+1}$ by coning off (what is not possible in the smooth category), any homotopy n-sphere is PL-homeomorphic to S^n for $n \geq 5$. Hence $\mathcal{S}_n^{PL}(S^n) = \{*\}$ for $n \geq 5$. This is the main ingredient in the proof for $n \geq 5$, the low dimensional cases follow from direct computations, that we obtain isomorphisms of abelian groups

$$\pi_n(G/PL) \xrightarrow{\cong} L_n(\mathbb{Z}) \qquad n \geq 1$$

There is an isomorphism

$$\Theta^n \xrightarrow{\cong} \pi_n(PL/O) \qquad n \neq 3$$

which is defined as follows. Let Σ be a homotopy n-sphere. As explained above, there is an orientation preserving PL-homeomorphism $h : \Sigma \to S^n$. Fix a classifying map $f_{S^n} : S^n \to BPL$ for the PL-tangent bundle of S^n. We obtain a classifying map $f_\Sigma : \Sigma \to BO$ of the smooth tangent bundle of Σ together with a homotopy $h : Bi \circ f_\Sigma \simeq f_{S^n}$, where $Bi : BO \to BPL$ is the canonical map. The pair (f_Σ, h) yields a map $S^n \to PL/O$, since PL/O is the homotopy fiber of $Bi : BO \to BPL$. The bijectivity of this map for $n \geq 5$ follows from the five lemma and the comparision of the surgery exact sequence with the long homotopy sequence associated to the fibration $PL/O \to G/O \to G/PL$. There is an isomorphism

$$\Theta^n_{\mathrm{fr}} \xrightarrow{\cong} \pi_n(PL) \qquad n \neq 3, 4$$

which is defined as follows. As above we get a pair (f_Σ, h). The framing yields also a homotopy $g : f_\Sigma \simeq c$ for c the constant map. Since PL can be viewed as the homotopy fiber of the obvious map $PL/O \to BO$, these data yield a map $S^n \to PL$. ■

6.7 Miscellaneous

Remark 6.49 We have shown in Theorem 6.39 that bP^{4k} for $k \geq 2$ is a finite cyclic group. If we go through the construction again, an explicit generator can be constructed as follows. Recall that $\frac{\text{sign}}{8} : L_{4k}(\mathbb{Z}) \to \mathbb{Z}$ is an isomorphism. Let W be any oriented stably parallizable manifold of dimension $4k$ whose boundary is a homotopy sphere and whose intersection pairing on $H_{2k}(W)$ has signature 8. Then $\partial_1 W$ is an exotic sphere representing a generator of bP^{4k}. Such manifolds W can be explicitly constructed by a plumbing construction (see for instance [9, Theorem V.2.9 on page 122]).

Example 6.50 The first example of an *exotic sphere* , i.e. a closed manifold which is homeomorphic but not diffeomorphic to S^n, was constructed by Milnor [44]. See also [12]. The construction and the detection that it is an exotic sphere is summarized below.

There is an isomorphism

$$\begin{array}{ccl} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\cong} & \pi_3(SO(4)) \\ (h,j) & \mapsto & \omega(h,j) : S^3 \to SO(4) \quad x \mapsto \left(y \mapsto x^h \cdot y \cdot x^j\right) \end{array}$$

where we identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ by $(a,b,c,d) \mapsto a + bi + cj + dk$ and $x^h \cdot y \cdot x^j$ is to be understood with respect to the multiplication in $\mathbb{H}$. Since $\pi_3(SO(4)) \cong \pi_4(BSO(4))$, each pair $(h,j) \in \mathbb{Z} \oplus \mathbb{Z}$ determines an oriented vector bundle $E(h,j)$ with Riemannian metric over S^4 unique up to orientation and Riemannian metric preserving isomorphism. The Euler number and the first Pontrjagin class of $E(j,h)$ are given by

$$\begin{array}{rcl} \chi(E(h,j)) & = & h + j; \\ \langle p_1(E(h,j)), [S^4] \rangle & = & 2(h-j). \end{array}$$

The Gysin sequence shows that the sphere bundle $SE(h,j)$ is a homotopy 7-sphere if and only if $\chi(E(h,j)) = h + j = 1$.

Let k be any odd integer. Let $W(k)$ be the disk bundle $D(E((1+k)/2, (1-k)/2))$ and $\Sigma(k)$ be $\partial W(k) = S(E((1+k)/2, (1-k)/2))$. Then $\Sigma(k)$ is a homotopy 7-sphere. Next we recall Milnor's argument why $\Sigma(k)$ cannot be diffeomorphic to S^7.

The obvious embedding $i : S^4 \to W(k)$ given by the zero section is a homotopy equivalence and $i^*TW(k)$ is isomorphic to $TS^4 \oplus E((1+k)/2, (1-k)/2)$. Hence

$$\langle i^* p_1(TW(k)), [S^4] \rangle = 2k.$$

Suppose that there exists a diffeomorphism $\Sigma(k) \to S^7$. Then we can form the closed oriented smooth 8-dimensional manifold $M(k) = W(k) \cup_f D^8$. Let $j : W(k) \to M(k)$ be the inclusion. Since the inclusion $j \circ i : S^4 \to M(k)$ induces an isomorphism on H_4, the signature of $M(k)$ is one. The Hirzebruch signature Theorem says $1 = \text{sign}(M) = \langle \mathcal{L}(M), [M] \rangle$. Since

$$\mathcal{L}(M) = \frac{7}{45} \cdot p_2(TM) - \frac{1}{45} \cdot p_1(TM)^2$$

we conclude

$$1 = \langle \frac{7}{45} \cdot p_2(TM) - \frac{1}{45} p_1(TM)^2, [M] \rangle = \frac{7}{45} \langle p_2(TM), [M] \rangle - \frac{1}{45} \cdot \langle p_1(TM)^2, [M] \rangle$$

$$= \frac{7}{45} \langle p_2(TM), [M] \rangle - \frac{1}{45} \cdot \langle i^* j^* p_1(TM), [S^4] \rangle^2 = \frac{7}{45} \langle p_2(TM), [M] \rangle - \frac{4k^2}{45}.$$

Since $\langle p_2(TM), [M] \rangle$ is an integer, we conclude $k^2 = 1 \mod 7$. Hence $\Sigma(k)$ is an exotic homotopy 7-sphere if $k^2 \neq 1 \mod 7$.

Remark 6.51 Milnor's example 6.50 above fits into the general context as follows. Recall that $bP^8 = \{0\}$ and that we have an isomorphism

$$\frac{\text{sign}}{8} : \Theta^7 \to \mathbb{Z}/28$$

which sends $[\Sigma]$ to $\text{sign}(W)/8$ for any stably parallizable manifold W whose boundary is oriented diffeomorphic to Σ. If $\Sigma(k)$ is the oriented homotopy 8-sphere of Example 6.50, then the isomorphism above sends $[\Sigma(k)]$ to $(1 - k^2) \in \mathbb{Z}/28$.

Example 6.52 Let $W^{2n-1}(d)$ be the subset of $\mathbb{C}^{n+1}$ consisting of those points $(z_0, z_1, \ldots, z_n)$ which satisfy the equations $z_o^d + z_1^2 + \ldots z_n^2 = 0$ and $||z_0||^2 + ||z_1||^2 + \ldots + ||z_n||^2 = 1$. These turns out to be smooth submanifolds and are called *Brieskorn varieties* (see [6]), [32]). Suppose that d and n are odd, Then $W^{2n-1}(d)$ is a homotopy $(2n-1)$-sphere. It is diffeomorphic to the standard sphere S^{2n-1} if $d = \pm 1 \mod 8$ and it is an exotic sphere representing the generator of bP^{2n} if $d = \pm 3 \mod 8$ [6, page 11]. In general one can study the intersection $K = f^{-1}(0) \cap \{z \in \mathbb{C}^{n+1} \mid ||z|| = \epsilon\}$ for a polynomial $f(z_0, z_1, \ldots, z_n)$ with an isolated singularity at the origin and examine when K is a homotopy sphere and when K is an exotic sphere [48, §8, §9].

Remark 6.53 Let Σ be a homotopy n-sphere for $n \geq 5$. Let $D_0^n \to \Sigma$ and $D_1^n \to \Sigma$ be two disjoint embedded discs. Then $W = \Sigma - (\text{int}(D_0^n) \coprod \text{int}(D_1^n))$ is a simply-connected h-cobordism. By the h-cobordism Theorem 1.2 there is a diffeomorphism $(F, \text{id}, f) : \partial D_0^n \times [0,1], \partial D_0^n \times \{0\}, \partial D_0^n \times \{1\}) \to (W, \partial D_0^n, \partial D_1^n)$. Hence Σ is oriented diffeomorphic to $D^n \cup_{f: S^{n-1} \to S^{n-1}} (D^n)^-$ for some orientation preserving diffeomorphism $f : S^{n-1} \to S^{n-1}$. If f is isotopic to the identity, Σ is oriented diffeomorphic to S^n. Hence the existence of exotic spheres shows the existence of selfdiffeomorphisms of S^{n-1} which are homotopic but not isotopic to the identity.

The next result is due to Berger and Klingenberg. Its proof and the proof of the following theorem can be found for instance in [20, Theorem 6.1 on page 106, Theorem 7.16 on page 126].

Theorem 6.54 (Sphere theorem) *Let M be a complete simply connected Riemannian manifold whose sectional curvature is pinched by $1 \geq \sec(M) > \frac{1}{4}$. Then M is homeomorphic to the standard sphere.*

Theorem 6.55 (Differentiable sphere theorem) *There exists a constant δ with $1 > \delta \geq \frac{1}{4}$ with the following property: if M is a complete simply connected Riemannian manifold whose sectional curvature is pinched by $1 \geq \sec(M) > \delta$. then M is diffeomorphic to the standard sphere.*

Brumfield and Frank [11] have shown

Theorem 6.56 *For $n \neq 2^k - 2$ or $n \neq 2^k - 3$ the sequence*

$$0 \to bP^{n+1} \to \Theta^n \to \text{im}\,((\theta^n \to \text{coker}(J_n)) \to 0$$

splits where the map $\Theta^n \to \text{im}\,(\Theta^n \to \text{coker}(J_n))$ comes from the map $\Theta^n \to \text{coker}(J_n)$ appearing in Theorem 6.46.

Kirby and Siebenmann [36, Theorem 5.5 in Essay V on page 251] (see also [56]) have proven

Theorem 6.57 *The space TOP/PL is an Eilenberg MacLane space of type $(\mathbb{Z}/2, 3)$.*

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Notation

Index