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SMR1312/11

School on High-Dimensional Manifold Topology

(21 May - 8 June 2001)

The vanishing of $Wh(\pi_1 M)$ for non-positively curved manifolds M

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Lecture given at the:
School on High Dimensional Manifold Topology, Trieste
21 May - 8 June 2001

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Lecture 3

In my last lecture, I showed that step 1 in the program to replace a homotopy equivalence $f : N \to M$ between closed manifolds with a homeomorphism can be accomplished when M satisfies a certain geometric condition $(*)$. In particular, this can be done when M is a non-positively curved Riemannian manifold.

This lecture is about step 3 of the program; i.e., analyzing h-cobordisms with base M. Because of the s-cobordism theorem, this is equivalent to calculating $Wh(\pi_1 M)$ when $\dim(M) \geq 5$. The discussion will focus on the following vanishing result.

Vanishing Theorem. *(Farrell and Jones) Let M be a closed non-positively curved Riemannian manifold. Then*

$$Wh(\pi_1 M) = 0.$$

Remark. The special cases of this theorem where M is the m-torus T^m was proven by Bass-Heller-Swan (1964) and for arbitrary flat Riemannian manifolds M by Farrell-Hsiang (1978).

We need to develop a few more geometric ideas before discussing the proof of the Vanishing Theorem. Throughout this lecture M will denote a closed (connected) non-positively curved Riemannian manifold and $\tilde{M}$ is its universal cover. And we keep the geometric notation from our last lecture; in particular

$$\begin{array}{rcl} \Gamma & = & \pi_1(M). \\ \bar{M} & & \text{is the geodesic ray compactification of } \tilde{M}. \\ M(\infty) & = & \bar{M} - \tilde{M}. \\ \alpha_v & & \text{is the geodesic with } \dot{\alpha}_v(0) = v. \end{array}$$

We call a pair of vectors $u, v \in S\tilde{M}$ *asymptotic* if the two rays

$$\{\alpha_u(t) \mid t \geq 0\} \quad \text{and} \quad \{\alpha_v(t) \mid t \geq 0\}$$

are asymptotic.

Figure 1

For each pair $v \in S\tilde{M}$ and $x \in \tilde{M}$, there is a unique asymptotic vector $v(x) \in S_x\tilde{M}$. ($S_x\tilde{M}$ = unit sphere in $T_x\tilde{M}$.)

Figure 2

Furthermore the function $S\tilde{M} \times \tilde{M} \to S\tilde{M}$ defined by $(x, v) \to v(x)$ is continuous, C^1 in x, and its differential (in x) depends continuously on v. The (weakly) stable foliation of $S\tilde{M}$ has for its leaves the asymptoty classes of vectors. Note that under the bundle projection $S\tilde{M} \to \tilde{M}$ each leaf of this foliation maps diffeomorphically onto $\tilde{M}$. Since an isometry of $\tilde{M}$ sends asymptotic vectors to asymptotic vectors, this foliation induces a foliation of SM called its *(weakly) stable foliation.* Restriction of the bundle projection $SM \to M$ to any leaf L of this foliation is a covering space projection

$$L \to M.$$

And the geodesic flow $g^t : SM \to SM$ preserves the leaves of the (weakly) stable foliation.

Figure 3

The total space SN of the unit sphere bundle of a Riemannian manifold N has a natural Riemannian metric defined as follows. Let $v(t)$ be a smooth curve in SN representing a tangent vector η to SN at $v(0)$; i.e., $v(t)$ is a unit length vector field along a smooth curve $\gamma(t)$ in N. Then

$$|\eta| = \sqrt{|\dot{\gamma}(0)|^2 + |u|^2}$$

where u is the covariant derivative of $v(t)$ at $t = 0$.

We next describe the *asymptotic transfer* of a path $\gamma : [0,1] \to M$ to a path $v\gamma$ in SM where $v \in S_{\gamma(0)}M$. The asymptotic transfer sits on top of γ in the sense that the composite path $p \circ (v\gamma)$ is γ; where

$$p : SM \to M$$

denotes the bundle projection. Let L be the leaf of the (weakly) stable foliation of SM containing v. Recall that

$$p|_L : L \to M$$

is a covering space. Then $v\gamma$ is defined to be the unique lift of γ starting at v.

The following are some of the properties of the asymptotic transfer.

1. If γ is a null homotopic loop, then so is $v\gamma$.
2. If γ is a constant loop, so is $v\gamma$.
3. If γ is a C^1-curve, so is $v\gamma$.

Furthermore, if $-a^2$ is any lower bound for the sectional curvatures of M, then

$$|\dot{v\gamma}(t)| \leq \sqrt{1+a^2}\,|\dot{\gamma}(t)|$$

for each $t \in [0,1]$.

Let W be a smooth h-cobordism with base M equipped with a smooth deformation retraction h_t of W^{m+1} onto M^m. In particular $h_0 = \mathrm{id}_W$, and $r = h_1$ is a retraction of W onto M. Let $\mathcal{W}^{2m}$ be the total space of the pullback of $p : SM \to M$ via r; i.e.,

$$\mathcal{W} = \{(y,v) \in W \times SM \mid r(y) = p(v)\}.$$

Then $\mathcal{W}$ is an h-cobordism with base SM and the asymptotic transfer can be used to equip $\mathcal{W}$ with a useful C^1 deformation retraction k_t of $\mathcal{W}$ onto SM defined as follows. First associate to h_t a family of paths $\{\gamma_y \mid y \in W\}$ in M called the *tracks* of h_t. These are given by the equation

$$\gamma_y(t) = r(h_t(y)).$$

Note that each track γ_y is a smooth null homotopic loop in M based at $r(x)$. Hence, for each vector $v \in S_{r(y)}M$, the asymptotic transfer $v\gamma_y$ of γ_y to SM is a C^1 null homotopic loop based at v. Now k_t is defined by the formula

$$k_t(y,v) = (h_t(y), v\gamma_y(t))$$

where $t \in [0,1]$, $y \in M$ and $v \in S_yM$. It is important to note that the tracks of k_t are

$$\{v\gamma_{p(v)} \mid v \in SM\};$$

namely, they are all the asymptotic transfers of the tracks of h_t. Furthermore given a self-diffeomorphism $f : SM \to SM$ homotopic to id_{SM}, we can change k_t to a new C^1 deformation retraction of $\mathcal{W}$ onto SM whose tracks are

$$\{f \circ (v\gamma_{p(v)}) \mid v \in SM\}.$$

This comment applies in particular when $f = g^{t_0}$ where g^t is the geodesic flow on SM and t_0 is a fixed (large) positive real number. Which is useful because of the following consequence of Anosov's analysis of the geodesic flow.

Key Property of $v\gamma$**.** The following is true when M is *negatively curved.* Given numbers β and ϵ in $(0, +\infty)$, there exists a number $t_0 \in (0, +\infty)$ satisfying the following. Let γ be any smooth path in M whose arc length is $\leq \beta$, and v be any vector in $S_{\gamma(0)}M$. Then, for any $t \geq t_0$, the composite path $g^t \circ (v\gamma)$ is (β, ϵ)-controlled in SM with respect to the 1-dimensional foliation by the orbits of the geodesic flow.

Figure 4 indicates why this property is true. In it $\tilde{\gamma}$ is a lift of γ to $\tilde{M}$; $u \in S_{\tilde{\gamma}(0)}\tilde{M}$ is the vector lying over v; $u\tilde{\gamma}$ is the lift of $v\gamma$ to $S\tilde{M}$ starting at u, and $u(\infty) \in M(\infty)$ is the ideal point corresponding to the ray $\{\alpha_u(t) \mid t \geq 0\}$. Also $\tilde{M}$ is identified with the (weakly) stable leaf L of $S\tilde{M}$ containing u. And the lines converging to $u(\infty)$ are the flow lines of the geodesic flow which are inside of L; while the $\perp$ codimension-one submanifolds abuting to $u(\infty)$ are the horospheres inside of L; i.e. the *strongly stable leaves.*

Figure 4

Each diffeomorphism g^t, $t > 0$, of the geodesic flow preserves the family of horospheres as well as the flow lines. It is (strongly) contracting on horospheres and is an isometry on flow lines.

Remark. This Key Property of the asymptotic transfer is *not* true (in general) when M is only non-positively curved. For example it doesn't hold when M is flat since asymptotic rays are parallel in Euclidean space.

Using the above construction of a deformation retraction of $\mathcal{W}$ onto SM relative to g^{t_0}, we see that $\mathcal{W}$ is a (β, ϵ)-controlled h-cobordism over SM for a fixed positive real number β but arbitrarily small positive numbers ϵ when M is negatively curved because of the Key Property of the asymptotic transfer. Hence the Foliated Control Theorem (described by Lowell Jones in his lectures) shows that the Whitehead torsion $\tau(\mathcal{W}) = 0$. Since every element $x \in Wh(\pi_1 M)$ is the torsion $\tau(W)$ of some smooth h-cobordism with base M, the fact that $\tau(\mathcal{W}) = 0$ would show that $Wh(\pi_1 M)$ vanishes, when M is negatively curved, provided

$$\tau(\mathcal{W}) = \tau(W).$$

Unfortunately this equation is not true in general. In fact the following formula calculates $\tau(\mathcal{W})$ in terms of $\tau(W)$.

Theorem. *(D.R. Anderson 1972). Let W and $\mathcal{W}$ be h-cobordisms with bases M and $\mathcal{M}$, respectively. And let $p : \mathcal{W} \to W$ be a smooth fiber bundle with $p^{-1}(M) = \mathcal{M}$ and* dim $M > 4$. *Assume that $\pi_1(W)$ acts trivially on the integral homology groups of the fiber F of p, then*

$$p_*(\tau(\mathcal{W})) = \chi(F)\tau(W)$$

where $\chi(F)$ denotes the Euler characteristic of F and

$$p_* : Wh(\pi_1\mathcal{M}) \to Wh(\pi_1 M)$$

is the homomorphism induced by p.

Applying Anderson's theorem to the h-cobordism $\mathcal{W}$ constructed above, we see that

$$\tau(\mathcal{W}) = \begin{cases} 2\tau(W) & \text{if } m \text{ is odd} \\ 0\tau(W) = 0 & \text{if } m \text{ is even} \end{cases}$$

(provided M^m is orientable) since the fiber of $\mathcal{W} \to W$ is S^{m-1}.

To get around this difficulty we need a sub-bundle E of SM with fiber F satisfying

1. $\chi(F) = 1$;
2. E is invariant under g^t;
3. for each path γ in M and each vector $v \in E$ lying over $\gamma(0)$, $v\gamma$ is a path in E.

It unfortunately is impossible to find such a sub-bundle when M is closed because every orbit of the action of Γ on $M(\infty)$ is then dense. We are thus forced to consider a certain non-compact but complete and pinched negatively curved Riemannian manifold N^{m+1} called the *enlargement* of M^m. It is diffeomorphic to $\mathbb{R} \times M^m$ and contains M^m as a totally geodesic codimension-one subspace. In fact N is the warped product (defined by Bishop and O'Neill)

$$N = \mathbb{R} \times_{\cosh(t)} M$$

and $0 \times M$ is the totally geodesic subspace identified with M.

Figure 5

The Riemannian metric $|| \ ||$ on N is determined from the Riemannian metrics $||$ on M and $||$ on $\mathbb{R}$ by the properties

1. $\mathbb{R} \times x \perp t \times M$ for all $x \in M$, $t \in \mathbb{R}$.
2. $||v|| = \cosh(t)|v|$ if $v \in T(t \times \tilde{M})$.
3. $||v|| = |v|$ if $v \in T(\mathbb{R} \times x)$.

Let $q : N = \mathbb{R} \times M \to \mathbb{R}$ denote projection onto the first factor. Inside of SN is an *upper hemisphere* sub-bundle defined by $v \in S^+N$ iff the following set of real numbers is bounded below

$$\{q(\alpha_v(t)) \mid t \in [0, +\infty)\}.$$

(This lower bound depends on v.) That is $v \notin S^+N$ iff the geodesic $\alpha_v(t) \to$ "$-\infty$" as $t \to +\infty$. This sub-bundle satisfies the three conditions listed above; in particular its fiber is $\mathbb{D}^m$.

Now an arbitrary element $x \in Wh(\Gamma)$ can be realized as the Whitehead torsion $\tau(W)$ of a compactly supported h-cobordism with base N. And the associated h-cobordism $\mathcal{W}$ with base S^+N is (β, ϵ)-controlled for a fixed positive number β but arbitrarily small positive ϵ. Hence the Foliated Control Theorem (in one of its more sophisticated forms) together with Anderson's Theorem shows that

$$x = \tau(W) = \tau(\mathcal{W}) = 0$$

proving that $Wh(\pi_1 M) = 0$ when M is negatively curved.

To prove the general case of the Vanishing Theorem, where M is allowed to have some zero sectional curvature, we must replace the asymptotic transfer with a new *focal transfer.* It associates to each path $\gamma : [0,1] \to M$, each vector $v \in S_{\gamma(0)}M$, and every (large) positive number $d \in \mathbb{R}$ (called the *focal length* of the transfer) a path

$$v(\gamma, d) : [0,1] \to M.$$

The focal transfer satisfies properties 1-3 of the asymptotic transfer. And it satisfies the following analogue of the Key Property of $v\gamma$.

Key Property of $v(\gamma, d)$. Given M as well as numbers $\beta, \epsilon \in (0, +\infty)$, there exists a positive number t_0 ($t_0 > \beta$) satisfying the following statement for every smooth path γ in M whose arc length is $\leq \beta$ and every vector $v \in S_{\gamma(0)}M$. The composite path

$$g^d \circ v(\gamma, d)$$

is (β, ϵ)-controlled in SM with respect to the foliation given by the orbits of the geodesic flow provided $d \geq t_0$.

Remark. The focal transfer $v(\gamma, d)$ focuses when flowed a distance equal to its focal length d. When flowed farther, it gets out of focus.

To construct $v(\gamma, d)$ pick a lift $\tilde{\gamma}$ of γ to $\tilde{M}$ and let $u \in S_{\tilde{\gamma}(0)}\tilde{M}$ be the unique vector which maps to v via $d\rho$ where

$$\rho : \tilde{M} \to M$$

denotes the covering projection. Figure 6 illustrates the construction of the path $u(\tilde{\gamma}, d)$ in $S\tilde{M}$.

Figure 6

If w denotes the vector $u(\tilde{\gamma}, d)(t) \in S_{\tilde{\gamma}(t)}\tilde{M}$, then w is the unique vector such that the geodesic ray

$$\{\alpha_w(s) \mid s \geq 0\}$$

contains the point $\alpha_u(d)$. Note we must have that

$$d \geq \operatorname{diam}\{\gamma(t) \mid t \in [0,1]\}$$

for w to be necessarily defined. Since this construction is equivariant with respect to Γ, we can (and do) define the focal transfer $v(\gamma, d)$ by the equation

$$v(\gamma, d) = d\rho \circ u(\tilde{\gamma}, d).$$

The only problem with the focal transfer is that the bundle $S^+N \to N$ does not satisfy property 3 (on page 5) with respect to it. But it does except near $\partial(S^+N)$ and so the construction is slightly modified near $\partial(S^+N)$. When this is done, then the argument given above proving the Vanishing Theorem in the special case where M is negatively curved works in general after the asymptotic transfer is replaced with the focal transfer. In fact a simplification can be made in the earlier argument by using N equal to the Riemannian product

$$\mathbb{R} \times M$$

instead of the warped product

$$\mathbb{R} \times_{\cosh(t)} M.$$

We end this lecture by discussing a generalization of the Vanishing Theorem to the case where M is complete but *not* necessarily compact. Needed for this purpose is an extra geometric condition on M; namely, that M is *A-regular.*

Definition. A Riemannian manifold N is A-regular if there exists a sequence of positive real numbers $A_0, A_1, A_2, \cdots$ with $|D^n(K)| \leq A_n$. Here K is the curvature tensor and D is covariant differentiation.

Remark 1. Every closed Riemannian manifold N is A-regular. This is a consequence of an elementary continuity argument.

Remark 2. Every locally symmetric space is A-regular since $DK \equiv 0$ is one of the definitions of a locally symmetric space.

Addendum. *(Farrell and Jones 1998) Let N be any complete Riemannian manifold which is both non-positively curved and A-regular. Then $Wh(\pi_1 N) = 0$.*

Corollary 1. *$Wh(\Gamma) = 0$ for every discrete torsion-free subgroup Γ of $GL_n(\mathbb{R})$.*

Reason. Note that $\Gamma = \pi_1(N)$ where N is the double coset space

$$\Gamma \backslash GL_n(\mathbb{R})/O_n$$

which is a complete non-positively curved locally symmetric space and hence A-regular by Remark 2.

Corollary 2. *Let N be any complete and pinched negatively curved Riemannian manifold, then*

$$Wh(\pi_1 N) = 0.$$

Reason. Shi and Abresch show that the given Riemannian metric can be deformed to an A-regular one while keeping it negatively curved and complete.

The proof of the Addendum follows the same pattern as the proof of the Vanishing Theorem except that it uses the more difficult Foliated Control Theorem which Lowell Jones will discuss in his last lecture.

Let me also mention that Jones' former Ph.D. student B. Hu showed how to adapt the proof of the Vanishing Theorem to the language of Alexandroff PL-geometry thus obtaining the following result.

Theorem. *(Hu 1993) Let K be a non-positively curved finite complex, then $Wh(\pi_1 K) = 0$.*

Remark. Hu's result does not obviously include the Vanishing Theorem since Davis, Okun and Zheng have shown that *no* rank ≥ 2, irreducible, closed, non-positively curved locally symmetric space is also a non-positively curved PL-manifold.

Figure 1.

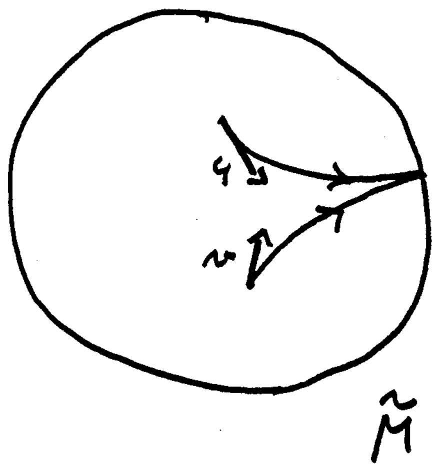


Figure 2.

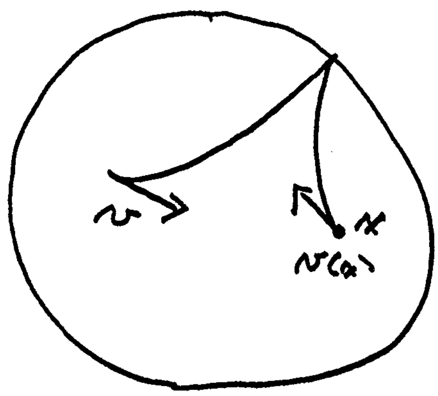
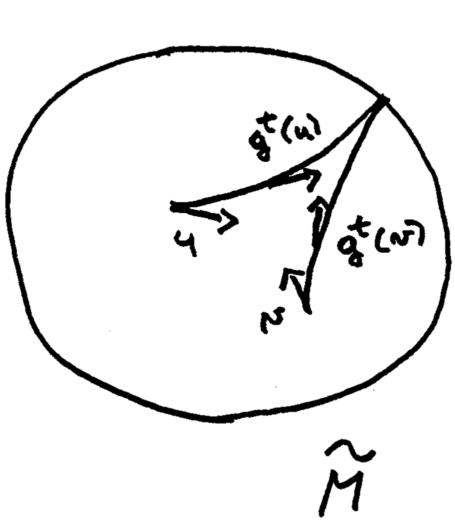


Figure 3.



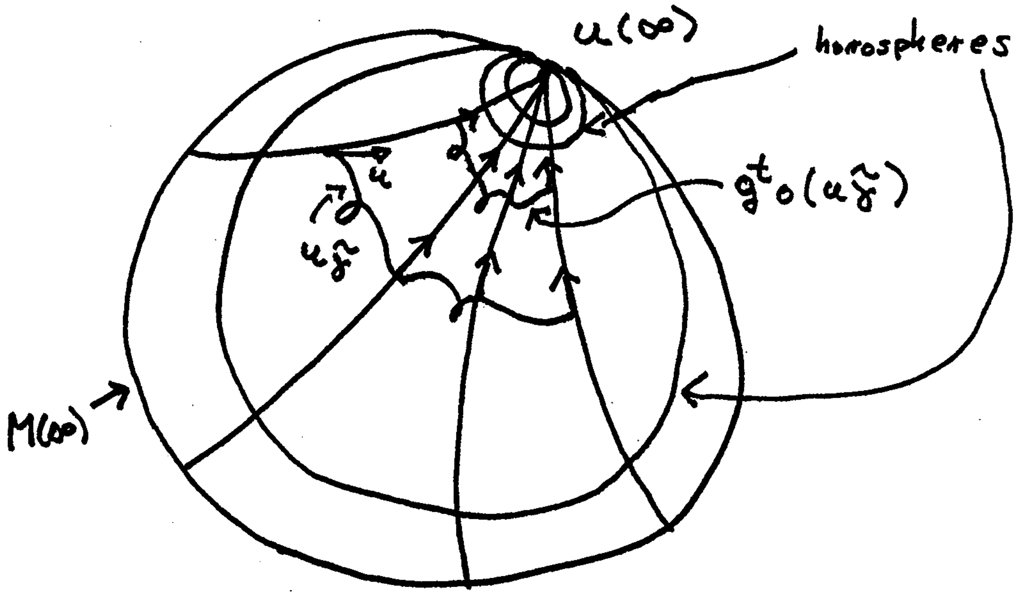
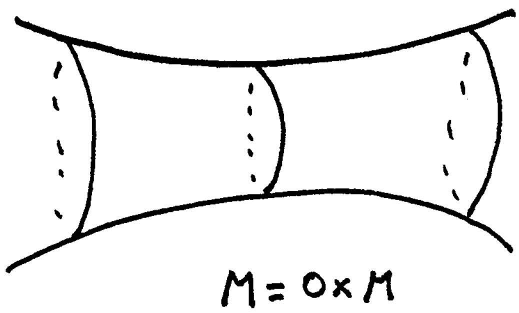
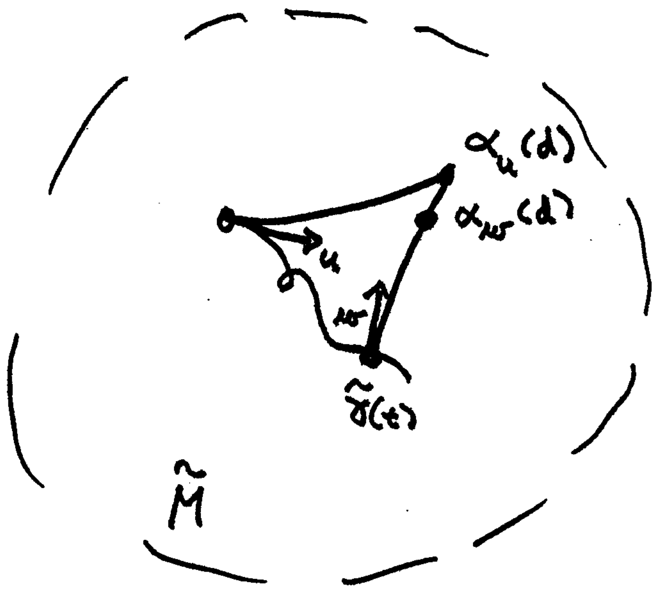


Figure 4.



$N = \mathbb{R} \times_{\cosh(t)} M$

Figure 5.



$w = u(\tilde{\gamma}, d)(t)$

Figure 6.