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The Borel Conjecture for non-positively curved manifolds

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The Borel Conjecture for non-positively curved manifolds

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LNS

Lecture 4

The focus of this lecture is Borel's Conjecture for closed non-positively curved Riemannian manifolds of dimension $\neq 3, 4$. It is an immediate consequence of the following result "TRT".

Topological Rigidity Theorem. *(Farrell and Jones) Let M^m be a closed non-positively curved Riemannian manifold. Then the homotopy-topological structure set $\mathcal{S}(M^m \times \mathbb{D}^n, \partial)$ contains only one element when $m + n \geq 5$.*

Remark. TRT was proven for T^m ($m \geq 5$) by Wall and independently by Hsiang-Shaneson (1969). And it was proven for all closed flat Riemannian manifolds M^m ($m \geq 5$) by Farrell-Hsiang (1983).

Corollary. *Let $f : N^m \to M^m$ be a homotopy equivalence between closed manifolds where $m \neq 3, 4$. If M^m is a non-positively curved Riemannian manifold, then f is homotopic to a homeomorphism.*

Proof. This result is classical when $m = 1$ or 2. When $m \geq 5$ set $n = 0$ in TRT to conclude that N and M are h-cobordant and hence homeomorphic by the s-cobordism since $Wh(\pi_1 M) = 0$ because of the Vanishing Theorem. □

Remark 1. Gabai has recently shown that the Borel Conjecture for closed hyperbolic 3-manifolds is equivalent to the Poincaré Conjecture.

Remark 2. The Borel Conjecture for closed non-positively curved 4-manifolds M^4 is an interesting open problem which is perhaps more accessible than the 3-dimensional case. The 5-dimensional s-cobordism Theorem of Freedman and Quinn combined with TRT shows it is true when M^4 is a closed flat Riemannian manifold.

We now discuss the proof of the TRT. Throughout this lecture M^m denotes a closed (connected) non-positively curved m-dimensional Riemannian manifold. We also keep the notation from our last lecture; in particular

$$\begin{array}{ll} \tilde{M} & \text{is the universal cover of } M; \\ \Gamma & = \pi_1(M); \\ \alpha_v & \text{is the geodesic with } \dot{\alpha}_v(0) = v. \end{array}$$

And we make the simplifying assumption that M^m is orientable so that our discussion is as transparent as possible. Note there are the following two identifications since $Wh(\Gamma) = 0$:

$$\begin{aligned} L_k^s(\Gamma) &= L_k(\Gamma) \quad \text{and} \\ \mathcal{S}^s(M^m \times \mathbb{D}^n, \partial) &= \mathcal{S}(M^m \times \mathbb{D}^n, \partial) \end{aligned}$$

where $\mathcal{S}^s(\ )$ denotes the simple homotopy-topological structure set.

The following result, used to reduce TRT to a special case, is a consequence of the codimension-one splitting theorems mentioned in my first lecture.

Lemma 0. *$\mathcal{S}(M^m \times \mathbb{D}^n, \partial)$ can be identified with a subset of $\mathcal{S}(M^m \times T^n)$ provided $m+n \geq 5$; and $\mathcal{S}(M^m)$ with a subset of $\mathcal{S}(M^m \times S^1)$ provided $m \geq 5$.*

Remark. Note that $\mathcal{S}^s(N \times [0,1], \partial)$ maps to $\mathcal{S}^s(N \times S^1)$ by sending the structure

$$f : (W, \partial_0 W \amalg \partial_1 W) \to (N \times [0,1], N \times 0 \amalg N \times 1)$$

to the structure

$$\mathcal{W} \to N \times S^1$$

where $\mathcal{W}$ results from W by glueing $\partial_0 W$ to $\partial_1 W$ via the composite homeomorphism $(f|_{\partial_1 W})^{-1} \circ (f|_{\partial_0 W})$. The first identification in Lemma 0 is a n-fold elaboration of this map using that $\mathbb{D}^n = \mathbb{D}^{n-1} \times [0,1]$. The second identification sends the structure $f : N \to M$ to the structure $f \times \mathrm{id} : N \times S^1 \to M \times S^1$; which is shown in Lemma 3 (below) to be monic.

Lemma 0 together with the fact that $M^m \times T^n$ is also non-positively curved reduces the TRT to the special case where $n = 0$ and m is an *odd integer.*

Note that the main result of our second lecture, together with the (semi)-periodicity of the surgery exact sequence, yields the following short exact sequence of pointed sets

$$0 \to [M^m \times [0,1], \partial; G/\mathrm{Top}] \xrightarrow{\sigma} L_{m+1}(\Gamma) \to \mathcal{S}(M^m) \to 0.$$

Remark. The techniques developed in this lecture (and the last) give an independent proof (via the focal transfer and the geodesic flow) that the surgery sequence is short exact for non-positively curved closed manifolds M^m. This alternate proof does not use the (semi)-periodicity of the surgery sequence.

Hence it remains to show that σ is an epimorphism; which is Step 2 in the program from Lecture 1 for replacing a homotopy equivalence $f : N \to M$ with a homeomorphism. This is the most complicated step in the program and was the last to be solved. The argument accomplishing it is modeled on the one used to solve Step 3 given in the last lecture. The s-cobordism theorem was used in that argument. It's surgery analogue is the algebraic classification of normal cobordisms over M due to Wall. Given a group π, Wall algebraically defined a sequence of abelian groups $L_n(\pi)$ with $L_{n+4}(\pi) = L_n(\pi)$ for all $n \in \mathbb{Z}$. He then showed that there is a natural bijection between the equivalence classes of normal cobordisms W over $M^m \times \mathbb{D}^{n-1}$ and $L_{m+n}(\Gamma)$ with the trivial normal cobordism corresponding to 0. Denote this correspondence by

$$W \mapsto \omega(W) \in L_{m+n}(\Gamma).$$

Wall also proved the following product formula.

Let N^{4k} be a simply connected closed oriented manifold and W be a normal cobordism over $M^m \times \mathbb{D}^{n-1}$. Form a new normal cobordism $W \times N$ over $M^m \times \mathbb{D}^{n-1} \times N^{4k}$ by producting W with N, then

$$\omega(W \times N) = \mathrm{Index}(N)\omega(W).$$

Remark. Anderson's Theorem is an analogue of this result where $\chi(N)$ replaces $\mathrm{Index}(N)$.

This product formula has the following geometric consequence.

Proposition. *Let K^{4k} be a closed oriented simply connected manifold with* $\mathrm{Index}(K) = 1$. *Let $f : N \to M$ be a homotopy equivalence where N is also a closed manifold. If*

$$f \times \mathrm{id} : N \times K \to M \times K$$

is homotopic to a homeomorphism, then f is also homotopic to a homeomorphism.

Sketch of Proof. Arguing as in the proof of the main result of Lecture 2, we compare the surgery exact sequence for $\mathcal{S}(M)$ with that for $\mathcal{S}(M \times K)$. If $x \in \mathcal{S}(M)$ denotes the homotopy-topological structure $f : N \to M$, it goes to 0 in $\mathcal{S}(M \times K)$. And since the map $[M, G/\mathrm{Top}] \to [M \times K, G/\mathrm{Top}]$ is monic, x is the image of an element $\bar{x} \in L_{m+1}(\Gamma)$ which maps to an element $\hat{x} \in L_{m+1+4k}(\Gamma)$ by producting the normal cobordism with K^{4k}. But the image of $\hat{x}$ in $\mathcal{S}(M \times K)$ is represented by

$$f \times \mathrm{id} : N \times K \to M \times K$$

and is hence zero. Therefore $\hat{x}$ is in the image of the Quinn assembly map in the surgery sequence for $M \times K$. But this map factors through the assembly map

$$[M^m \times \mathbb{D}^{4k+1}, \partial; G/\mathrm{Top}] \to L_{m+4k+1}(\Gamma)$$

which is periodic of period $4k$ with $\bar{x}$ going to $\hat{x}$. This factoring can be seen using Quinn's Δ-set description of the surgery sequence or Ranicki's algebraic formulation of it. (See Jones' 3rd lecture and Ranicki's 2nd lecture.) Hence $\bar{x}$ is in the image of σ, and therefore $x = 0$. □

The complex projective plane $\mathbb{C}P^2$ is the natural candidate for K when applying this Proposition. It is important for this purpose to have the following alternate description of $\mathbb{C}P^2$. Let C_2 denote the cyclic group of order 2. It has a natural action on $S^n \times S^n$ determined by the involution $(x, y) \mapsto (y, x)$ where $x, y \in S^n$. Denote the orbit space of this action by F_n; i.e.

$$F_n = S^n \times S^n / C_2.$$

Lemma 1. $\mathbb{C}P^2 = F_2$.

Proof. Let $sl_2(\mathbb{C})$ be the set of all 2×2 matrices with complex number entries and trace zero. Since $sl_2(\mathbb{C})$ is a 3-dimensional $\mathbb{C}$-vector space, $\mathbb{C}P^2$ can be identified as the set of all equivalence classes $[A]$ of non-zero matrices $A \in sl_2(\mathbb{C})$ where A is equivalent to B iff $A = zB$ for some $z \in \mathbb{C}$. The characteristic polynomial of $A \in sl_2(\mathbb{C})$ is $\lambda^2 + \det(A)$. Consequently, A has two distinct 1-dimensional eigenspaces if $\det(A) \neq 0$, and a single 1-dimensional eigenspace if $\det(A) = 0$ and $A \neq 0$. Also, A and zA have the same eigenspaces provided $z \neq 0$. These eigenspaces correspond to points in S^2 under the identification $S^2 = \mathbb{C}P^1$. The assignment

$$[A] \mapsto \text{ the eigenspaces of } A$$

determines a homeomorphism of $\mathbb{C}P^2$ to F_2. □

Remark. The TRT was first proved in the case where M^m is a hyperbolic 3-dimensional manifold by making use of Lemma 1. It was then realized that the general result for m odd could be proven using F_{m-1} once one could handle the technical complications arising from the fact that F_k is *not* a manifold when $k > 2$. The following result is used in overcoming these complications. It shows that F_k is "very close" to being a manifold of index equal to 1 when k is even.

Lemma 2. *Let n be an even positive integer. Then F_n has the following properties.*

1. *F_n is orientable $2n$-dimensional $\mathbb{Z}[\frac{1}{2}]$-homology manifold.*

2. *F_n is simply connected*

3. $H_i(F_n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n, 2n \\ \mathbb{Z}_2 & \text{if } n < i < 2n \text{ and } i \text{ is even} \\ 0 & \text{otherwise} \end{cases}$

4. $H^i(F_n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n, 2n \\ \mathbb{Z}_2 & \text{if } n+2 < i < 2n \text{ and } i \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$

5. *The cup product pairing*

$$H^n(F_n) \otimes H^n(F_n) \to H^{2n}(F_n)$$

is unimodular and its signature is either 1 *or* -1.

Proof. There is a natural stratification of F_n consisting of two strata B and T. The bottom stratum B consists of all (agreeing) unordered pairs $\langle u, v\rangle$ where $u = v$; while the top stratum T consists of all (disagreeing) pairs $\langle u, v\rangle$ where $u \neq v$.

Note that B can be identified with S^n. Also real projective n-space $\mathbb{R}P^n$ can be identified with the set of all unordered pairs $\langle u, -u\rangle$ in F_n. It is seen that F_n is the union of "tubular neighborhoods" of S^n and $\mathbb{R}P^n$ intersecting in their boundaries. The first tubular neighborhood is a bundle over S^n with fiber the cone on $\mathbb{R}P^{n-1}$. The second tubular neighborhood is a bundle over $\mathbb{R}P^n$ with fiber $\mathbb{D}^n$. Furthermore, they intersect in the total space of the $\mathbb{R}P^{n-1}$-bundle associated to the tangent bundle of S^n. This description of F_n can be used to verify Lemma 2. □

Caveat. The fundamental class of B represents twice a generator of $H_n(F_n)$. On the other hand, if we fix a point $y_0 \in S^n$, then the map $x \mapsto \langle x, y_0\rangle$ is an embedding of S^n in F_n which represents a generator of $H_n(F_n)$.

Let $f : N \to M$ represent an element in $\mathcal{S}(M)$. Then $f \times \mathrm{id} : N \times S^1 \to M \times S^1$ represents an element in $\mathcal{S}(M \times S^1)$. This defines a map $\mathcal{S}(M) \mapsto \mathcal{S}(M \times S^1)$.

Lemma 3. *The map $\mathcal{S}(M) \mapsto \mathcal{S}(M \times S^1)$ is monic.*

Proof. Suppose $f \times \mathrm{id}$ is homotopic to a homeomorphism g via a homotopy

$$h : N \times S^1 \times [0,1] \to M \times S^1 \times [0,1]$$

where $h|_{N\times S^1\times 0} = f \times \mathrm{id}$ and $h|_{N\times S^1 \times 1} = g$. By the codimension-one splitting theorems mentioned in my first lecture, we can split h along $M \times 1 \times [0,1]$.

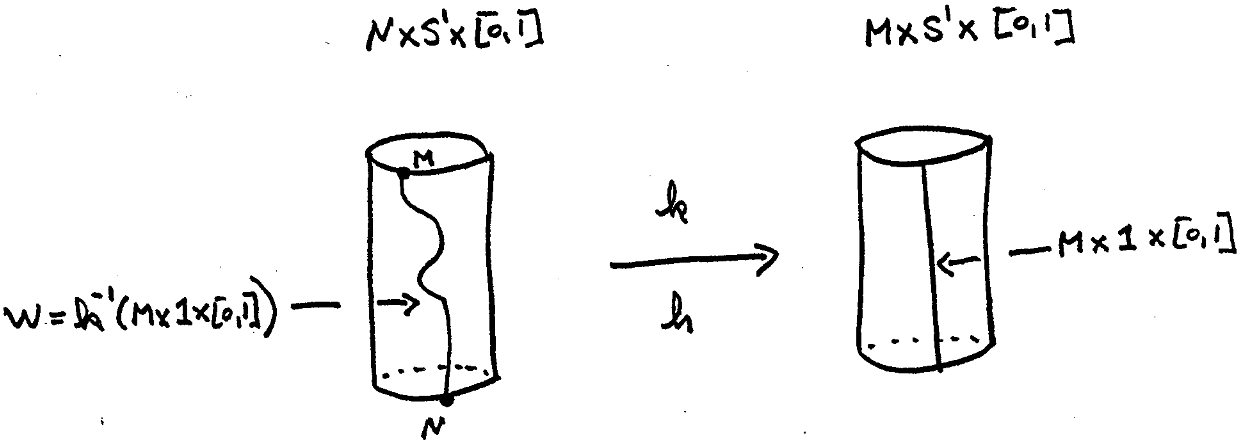

Figure 1

That is h is homotopic rel ∂ to a map k such that

$$k|_W : W \to M \times 1 \times [0,1]$$

is a homotopy equivalence where

$$W = k^{-1}(M \times 1 \times [0,1]).$$

We use that $Wh(\Gamma) = 0$ to get this. Now note that W is an h-cobordism between M and N. But W is a cylinder; again since $Wh(\Gamma) = 0$. $\square$

Remark. In order to prove the TRT, it suffices to show that $f \times \mathrm{id}$ is homotopic to a homeomorphism because of Lemma 3.

We now formulate a variant of the Proposition. This variant is used in showing that

$$f \times \mathrm{id} : N \times S^1 \to M \times S^1$$

is homotopic to a homeomorphism. There is a bundle

$$p : \mathcal{F}M \to M \times S^1$$

whose fiber over a point $(x,\theta) \in M \times S^1$ consists of all unordered pairs of unit length vectors $\langle u, v\rangle$ tangent to $M \times S^1$ at (x,θ) and satisfying the following two constraints.

1. If $u \neq v$, then both u and v are tangent to the level surface $M \times \theta$.
2. If $u = v$, then the projection $\bar{u}$ of u onto $T_\theta S^1$ points in the counterclockwise direction (or is 0).

The total space $\mathcal{F}M$ is stratified with three strata:

$$\begin{aligned}
\mathbb{B} &= \{\langle u,u\rangle \mid \bar{u} = 0\}\\
\mathbb{A} &= \{\langle u,u\rangle \mid \bar{u} \neq 0\}\\
\mathbb{T} &= \{\langle u,v\rangle \mid u \neq v\}.
\end{aligned}$$

Note that $\mathbb{B}$ is the bottom stratum and that $\mathcal{F}M - \mathbb{B}$ is the union of the two open sets $\mathbb{A}$ (auxiliary stratum) and $\mathbb{T}$ (top stratum). The restriction of p to each stratum is a sub-bundle. Let $\mathcal{F}_x$, B_x, A_x and T_x denote the fibers of these bundles over $x \in M \times S^1$; i.e.,

$$\mathcal{F}_x = p^{-1}(x),\ B_x = \mathcal{F}_x \cap \mathbb{B},\ A_x = \mathcal{F}_x \cap \mathbb{A},\ T_x = \mathcal{F}_x \cap \mathbb{T}.$$

Note that $B_x = S^{m-1}$, $A_x = \mathbb{D}^m$, $T_x \cup B_x = F_{m-1}$ and the bundle $p : \mathbb{B} \to M \times S^1$ is the pullback of the tangent unit sphere bundle of M under the projection $M \times S^1 \to M$.

The space F_{m-1} will play the role of the index one manifold K in our variant of the Proposition. Since it is unfortunately not a manifold when $m > 3$, we need to introduce the auxiliary fibers A_x. Hence the total fiber is homeomorphic to $F_{m-1} \cup \mathbb{D}^m$ where the subspace B in F_{m-1} is identified with $S^{m-1} = \partial \mathbb{D}^m$. Let

$$\mathcal{F}_f \to N \times S^1$$

denote the pullback of

$$\mathcal{F}M \to M \times S^1$$

along $f \times \mathrm{id} : N \times S^1 \to M \times S^1$ and let

$$\hat{f} : \mathcal{F}_f \to \tilde{t}M$$

be the induced bundle map. Note that the stratification of $\mathcal{F}M$ induces one on $\mathcal{F}_f$ and that $\hat{f}$ preserves strata.

We say that $\hat{f}$ is *admissibly homotopic to a split map* provided there exists a homotopy h_t, $t \in [0,1]$, with $h_0 = \hat{f}$ and satisfying the following four conditions.

1. Each h_t is strata preserving.
2. Over some closed "tubular neighborhood" $\mathcal{N}_0$ of $\mathbb{B}$ in $\mathbb{B} \cup \mathbb{T}$, each h_t is a bundle map; in particular, h_t maps fibers homeomorphically to fibers.
3. There is a larger closed "tubular neighborhood" $\mathcal{N}_1$ of $\mathbb{B}$ in $\mathbb{B} \cup \mathbb{T}$ such that h_1 is a homeomorphism over $\mathbb{B} \cup \mathbb{T} - \mathrm{Int}(\mathcal{N}_1)$ and over $\mathbb{B} \cup \mathbb{A}$.
4. Let $\rho : \mathcal{N}_1 \to M \times S^1$ denote the composition of the two bundle projections $\mathcal{N}_1 \to \mathbb{B}$ and $\mathbb{B} \to M \times S^1$. Then there is a triangulation K for $M \times S^1$ such that h_1 is transverse to $\rho^{-1}(\sigma)$ for each simplex σ of K. Furthermore
$$h_1 : h_1^{-1}(\rho^{-1}(\sigma)) \to \rho^{-1}(\sigma)$$
is a homotopy equivalence.

Remark. Conditions 3 and 4 should be hueristically replaced by the simpler and stronger condition that "h_1 is a homeomorphism". But for technical reasons we need to work instead with conditions 3 and 4.

The variant of the Proposition needed to prove the TRT is the following.

Proposition $(*)$. *The map $f : N \to M$ is homotopic to a homeomorphism provided $\hat{f}$: $\mathcal{F}_f \to \mathcal{F}M$ is admissibly homotopic to a split map.*

The proof of Proposition (∗) is basically an elaboration of the one sketched above for Proposition. It in particular uses again Quinn's Δ-set approach to the surgery exact sequence and generalizes Wall's product formula to the stratified setting above by using Lemma 2.

Proposition (∗) is the surgery theory part of the proof of the TRT. The geometry of M (in particular, its non-positive curvature) is used to show that the hypothesis of Proposition (∗) is satisfied; i.e, that $\hat{f}$ is admissibly homotopic to a split map. We now proceed to discuss how this is done.

It is a consequence of several applications of both ordinary and foliated topological control theory as discussed in Lowell Jones' lectures. Let $g : M \to N$ be a (strong) homotopy inverse to f and let h_t and k_t be (strong) homotopies of the composite $f \circ g$ to id_M and $g \circ f$ to id_N, respectively. Strong means base point preserving. It implies the following useful property.

Property (∗). For each point $x \in N$, the two paths

$$t \mapsto h_t(f(x)) \quad \text{and} \quad t \mapsto f(k_t(x))$$

are homotopic rel end points.

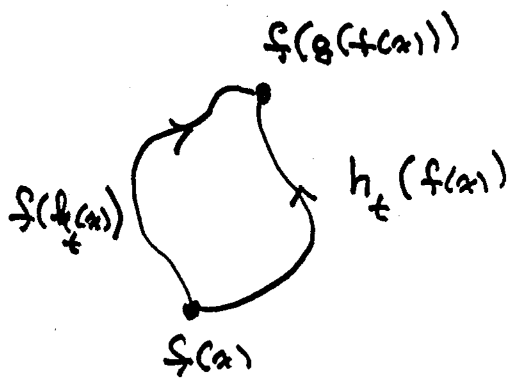

Figure 2

We may assume that N is a smooth manifold by using Kirby-Siebenmann smoothing theory. For this we need only observe that the stable topological tangent bundle of N has a real vector bundle structure since it is the pull back of TM stabilized via f because $f : M \to N$ maps to 0 in $[M, G/\mathrm{Top}]$. Therefore we may also assume that f and g are smooth maps and that both h_t and k_t are smooth homotopies.

The crucial point is to construct "good" transfers of the map g and the homotopies h_t, k_t to a map $\hat{g} : \mathcal{F}M \to \mathcal{F}_f$ and homotopies $\hat{h}_t$, $\hat{k}_t$ from $\hat{f} \circ \hat{g}$ to $\mathrm{id}_{\mathcal{F}M}$, and $\hat{g} \circ \hat{f}$ to $\mathrm{id}_{\mathcal{F}_f}$, respectively, so that control theory can be applied to admissibly homotope $\hat{f}$ to a split map. We proceed to describe what a good transfer is and then indicate how to construct one. The first requirement is that $\hat{g}$, $\hat{h}_t$ and $\hat{k}_t$ be bundle maps covering $g \times \mathrm{id}$, $h_t \times \mathrm{id}$, $k_t \times \mathrm{id}$, respectively. (Here id is the identity map on S^1.) Second, each map $\hat{g}$, $\hat{h}_t$ and $\hat{k}_t$ should preserve strata. Finally, it is necessary that a certain family $\mathcal{T}$ of paths determined by the lift is sufficiently "shrinkable". A path $\alpha : [0,1] \to \mathcal{F}M$ is in $\mathcal{T}$ if either

$$\alpha(t) = \hat{h}_t(\omega) \qquad \text{for some } \omega \in \mathcal{F}M, \quad \text{or}$$
$$\alpha(t) = \hat{f}(\hat{k}_t(\omega)) \qquad \text{for some } \omega \in \mathcal{F}_f.$$

(The family $\mathcal{T}$ is called the tracks of the transfer.) Note that each track is contained in a single stratum of $\mathcal{F}M$.

We construct good transfers by constructing their tacks $\mathcal{T}$. Since this is easier to explain when M *is negatively curved, we now make this assumption.* The construction of $\mathcal{T}$ uses (mainly) the asymptotic transfer of paths discussed in lecture 3. (The general case uses the focal transfer which, although more elementary, requires greater technical details.) Let $\mathcal{T}_1$ be the tracks determined by f, g, h_t and k_t; i.e. a curve $\alpha : [0,1] \to M$ is in $\mathcal{T}_1$ if for all $t \in [0,1]$ either

$$\begin{aligned} \alpha(t) &= h_t(x) \qquad \text{for some } \; x \in M; \;\; \text{or} \\ \alpha(t) &= f(k_t(y)) \qquad \text{for some } \; y \in N. \end{aligned}$$

Given $\gamma \in \mathcal{T}_1$ and $\omega = \langle u, v\rangle \in \mathcal{F}M$ with foot $(\gamma(0), \theta) \in M \times S^1$, we associate a lift $\omega\gamma$ of γ to a path in $\mathcal{F}M$ covering γ_θ which is the path in $M \times S^1$ defined by

$$\gamma_\theta(t) = (\gamma(t), \theta).$$

When $\omega \in \mathbb{B} \cup \mathbb{T}$, $\omega\gamma$ is defined by

$$\omega\gamma(t) = \langle u\gamma_\theta(t), v\gamma_\theta(t)\rangle$$

where $u\gamma_\theta$ and $v\gamma_\theta$ are the asymptotic transfers defined in Lecture 3. When $\omega \in \mathbb{A}$ (and hence $u = v$) $\omega\gamma$ is defined by

$$\omega\gamma(t) = \langle u(\gamma_\theta, d)(t), u(\gamma_\theta, d)(t)\rangle$$

where $u(\gamma_\theta, d)$ is the focal transfer with focal length d and chosen so that $d \to \infty$ as the angle between u and the level surface $M \times \theta$ approaches 0. Using that the asymptotic and focal transfers both satisfy properties 1-3 of lecture 3 and that property $(*)$ is satisfied by g, f, h_t, k_t; there is a natural construction of a good transfer $\hat{g}$, $\hat{h}_t$, $\hat{k}_t$ whose tracks

$$\mathcal{T} = \{\omega\gamma \mid \gamma \in \mathcal{T}_1, \omega \in \mathcal{F}M\}.$$

We now address the problem of "shrinking" the paths $\omega\gamma \in \mathcal{T}$. Since the geodesic flow g^t is defined on $\mathbb{A} \cup \mathbb{B}$, applying it to $\omega\gamma$ gives a method for making $\omega\gamma$ skinny when $\omega \in \mathbb{A} \cup \mathbb{B}$; i.e. $g^t \circ (\omega\gamma)$ is (β, ϵ)-controlled with respect to the 1-dimensional foliation of the manifold $\mathbb{A} \cup \mathbb{B}$ by the flow lines of the geodesic flow.

But the situation is different when $\omega = \langle u, v\rangle \in \mathbb{T}$. We are tempted then to "flow ω" in the direction of its arithmetic average $\frac{u+v}{2}$. But this does nothing when $u = -v$. Fortunately a different method can be used on the top stratum $\mathbb{T}$. But to describe it we need some more geometric preliminaries. We start by defining the *core* $\mathbb{P}$ of $\mathbb{T}$ by

$$\mathbb{P} = \{\langle u, -u\rangle \in \mathbb{T}\}.$$

The core is naturally identified with the total space of the projective line bundle associated to $(TM) \times S^1$. In particular there is a natural 2-sheeted covering space

$$\mathbb{B} = (SM) \times S^1 \to (\mathbb{R}P^{m-1}M) \times S^1 = \mathbb{P}$$

and the image of the geodesic line foliation of $\mathbb{B}$ gives $\mathbb{P}$ a canonical 1-dimensional foliation denoted by $\mathcal{G}$. The top strata $\mathbb{T}$ also has an *asymptotic foliation* $\mathcal{A}$ by m-dimensional leaves where each leaf of $\mathcal{A}$ is an *asymptoty class* of elements in $\mathbb{T}$. We say that elements $\omega_1 = \langle u_1, v_1 \rangle$, $\omega_2 = \langle u_2, v_2 \rangle \in \mathbb{T}$ lying over $M \times \theta$ (for some $\theta \in S^1$) are *asymptotic* provided (up to interchanging u_1 and v_1) there exist points $x, y \in \tilde{M}$ together with vectors $\tilde{u}_1, \tilde{v}_1 \in S_{(x,\theta)}(\tilde{M} \times S^1)$ and $\tilde{u}_2, \tilde{v}_2 \in S_{(y,\theta)}(\tilde{M} \times S^1)$ lying over u_1, v_1, u_2, v_2, respectively, and satisfying:

$$\begin{aligned} &\tilde{u}_1 \text{ is asymptotic to } \tilde{u}_2, \quad \text{and} \\ &\tilde{v}_1 \text{ is asymptotic to } \tilde{v}_2. \end{aligned}$$

Note that the restriction of the bundle map

$$\mathbb{T} \xrightarrow[p]{} M \times S^1 \xrightarrow[\text{proj}]{} M$$

to any leaf L of $\mathcal{A}$ is a covering space. This puts a flat structure on this bundle. And each leaf L of $\mathcal{A}$ inherits a negatively curved Riemannian metric from M via this covering projection. We call it the natural metric and note that it is compatible with the leaf topology on L.

The foliation $\mathcal{A}$ intersects the core $\mathbb{P}$ in its $\mathcal{G}$ foliation; i.e., there is a bijective correspondence between the leaves of $\mathcal{A}$ and $\mathcal{G}$ given by

$$L \mapsto L \cap \mathbb{P}, \quad L \in \mathcal{A}.$$

Also $L \cap \mathbb{P}$ is a closed subset of L in its leaf topology and is a (simple) geodesic of its natural metric. This geodesic $\mathbb{P} \cap L$ is called the *marking* of L. Furthermore, the inclusion map of $\mathbb{P} \cap L$ into L is a homotopy equivalence when L is given the leaf topology and $\mathbb{P} \cap L$ is given the subspace of L topology.

Now there is a bundle with fiber $\mathbb{R}^{m-1}$

$$\rho : \mathbb{T} \to \mathbb{P}$$

defined as follows.

For each $\omega \in \mathbb{T}$ let $L \in \mathcal{A}$ be the leaf containing ω and g be its marking. Then $\rho(\omega)$ denotes the (unique) closest point to ω on g measured inside L.

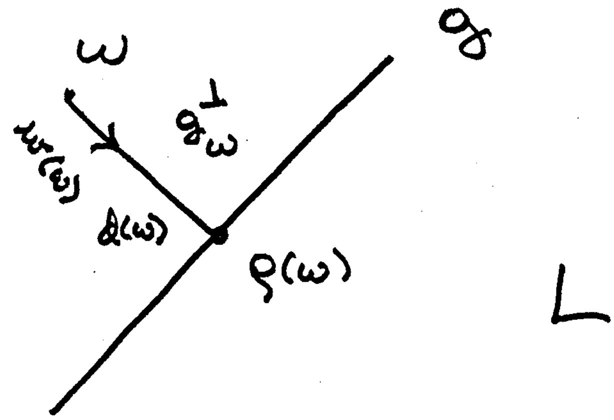

Figure 3

When $\omega \notin \mathbb{P}$, there is a unique geodesic segment $g_\omega^\perp$ in L connecting ω to $\rho(\omega)$. The unit length vector tangent to $g_\omega^\perp$ at ω which points towards $\rho(\omega)$ is denoted by $w(\omega)$. This defines a continuous vector field on $\mathbb{T} - \mathbb{P}$.

We denote the length of $g_\omega^\perp$ by $d(\omega)$. This extends to a continuous function $d : \mathbb{T} \cup \mathbb{B} \to [0, +\infty]$ when we set

$$d(\omega) = \begin{cases} 0 & \text{if } \omega \in \mathbb{P} \\ +\infty & \text{if } \omega \in \mathbb{B}. \end{cases}$$

There is also a bundle with fiber the open cone in $\mathbb{R}P^{m-1}$

$$\eta : (\mathbb{T} - \mathbb{P}) \cup \mathbb{B} \to \mathbb{B} = SM \times S^1$$

defined by

$$\eta(\omega) = \begin{cases} \omega & \text{if } \omega \in \mathbb{B} \\ dp(w(\omega)) & \text{if } \omega \in \mathbb{T} - \mathbb{P}. \end{cases}$$

Remark. We think of $\eta(\omega)$ as the *asymptotic average* of the two vectors u and v where $\omega = \langle u, v\rangle$ as opposed to their *arithmetic* average $\frac{1}{2}(u + v)$.

The vector field $w(\ )$ integrates to give an incomplete *radial flow* r^t on $\mathbb{T}$. In particular $r^t(\omega)$ is only defined for $t \in [0, d(w)]$. And there is the following important relation between r^t and g^t.

Intertwining Equation. $\eta(r^t(\omega)) = g^t(\eta(\omega))$ for all $\omega \in \mathbb{T} - \mathbb{P}$ and $t \in [0, d(\omega)]$.

We associate to each closed interval $J \subseteq [0, +\infty]$ a compact subspace W_J of $\mathbb{T} \cup \mathbb{B}$ defined by

$$W_J = d^{-1}(J).$$

If $+\infty \notin J$, then W_J is a codimension-0 submanifold of $\mathbb{T}$ with

$$\partial W_J = \begin{cases} d^{-1}(\partial J) & \text{if } 0 \notin J \\ d^{-1}(b) & \text{if } J = [0, b]. \end{cases}$$

Furthermore, we have the following:

1. If $0 \in J$ and $+\infty \notin J$, then $\rho : W_J \to \mathbb{P}$ is a fiber bundle with fiber $\mathbb{D}^m$.

2. If $+\infty \in J$ and $0 \notin J$, then $\eta : W_J \to \mathbb{B}$ is a fiber bundle with fiber the (closed) cone on $\mathbb{R}P^{m-1}$.

3. If neither 0 nor $+\infty$ is in J, then $\eta \times d : W_J \to \mathbb{B} \times J$ is a fiber bundle with fiber $\mathbb{R}P^{m-1}$.

Now fix a closed interval $I \subseteq (0, +\infty)$ containing 1 in its interior, and a very large positive real number σ together with a second closed interval R which contains $+\infty$ and is disjoint from σI. Then $[0, +\infty) - (\mathrm{Int}(R) \cup \mathrm{Int}(\sigma I))$ is the disjoint union of 2 closed intervals A and B denoted so that $0 \in A$.

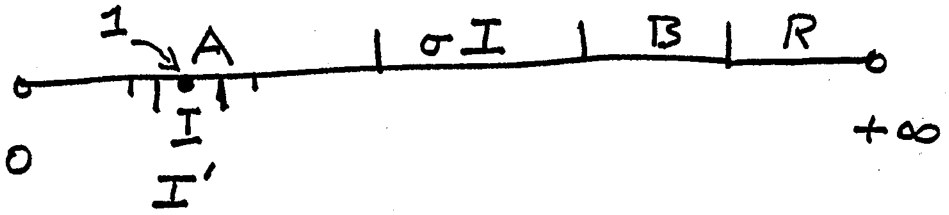

Figure 4

Fix another closed interval $I' \subseteq (0, +\infty)$ which contains I in its interior but is slightly larger and define a homeomorphism

$$\phi : W_{\sigma I'} \to W_{I'},$$

using the radial flow, by the formula

$$\phi(\omega) = r^t(\omega)$$

where $t = d(\omega) - \frac{1}{\sigma} d(\omega)$.

Note that ϕ becomes arbitrarily strongly contracting as we let $\sigma \to +\infty$. In particular if $\omega\gamma \in \mathcal{T}$ with $\omega \in W_{\sigma I}$, then $\phi \circ \omega\gamma$ is uniformly pointwise ϵ_σ-controlled in W_I with $\lim\limits_{\sigma \to \infty} \epsilon_\sigma = 0$.

Therefore we can use the ordinary control theorem to homotope $\hat{f}$ over $W_{\sigma I}$ (i.e. homotope $\hat{f}|_{\hat{f}^{-1}(W_{\sigma I})}$), in a controlled way, to a homeomorphism provided σ is large enough. This begins our construction of the admissible homotopy of $\hat{f}$ to a split map. And it gives a slightly different collection of tracks $\mathcal{T}_1$. These new tracks differ only for some of those $\gamma \in \mathcal{T}_1$ which start in $W_{\sigma I}$. And for them $\phi \circ \gamma$ is pointwise close to $\phi \circ \bar{\gamma}$ where $\bar{\gamma} \in \mathcal{T}$ is the corresponding track.

We next extend this homotopy to a homotopy of $\hat{f}$ over W_A to a homeomorphism. This is done by using the fibered and foliated version of the control theorem with respect to the fiber bundle

$$\rho : W_A \to \mathbb{P}$$

and the foliation $\mathcal{G}$ of $\mathbb{P}$. It is applicable since the fiber of ρ is $\mathbb{D}^m$ and the structure set $\mathcal{S}(\mathbb{D}^m \times \mathbb{D}^k, \partial)$ contains only one element for each $k \geq 0$. We need only check the control condition. For this, note that there exists a positive number β such that

$$\rho \circ \omega\gamma$$

is $(\beta, 0)$-controlled for each $\omega\gamma \in \mathcal{T}$ such that $\omega \in A$. And hence the tracks of $\mathcal{T}_1$ which start in A are $(\beta + \epsilon, \epsilon)$-controlled where $\epsilon \to 0$ as $\sigma \to \infty$.

Independent of these two steps, we use the foliated control theorem with respect to the foliation of $\mathbb{B} \cup \mathbb{A}$ by the orbits of the geodesic flow and the control map

$$g^\sigma : \mathbb{B} \cup \mathbb{A} \to \mathbb{B} \cup \mathbb{A}$$

to homotope $\hat{f}$ over $\mathbb{B} \cup \mathbb{A}$ to a homeomorphism. And then use the covering homotopy theorem to extend this to a homotopy of bundle maps over W_R relative to the fiber bundle

$$\eta : W_R \to \mathbb{B}$$

to a homeomorphism over W_R.

Let $\tau : B \to [0,1]$ denote the (unique) increasing linear homeomorphism, and fix a continuous function $\phi : B \to [\sigma, +\infty)$ such that

$$\phi(x) = \begin{cases} \sigma & \text{for all } x \text{ close to } B \cap R \\ (1 - \frac{1}{\sigma})x & \text{for } x \in I' \cap B. \end{cases}$$

Consider the fiber bundle

$$\xi : W_B \to \mathbb{B} \times [0,1]$$

where ξ is the composite

$$W_B \xrightarrow{\eta \times \mathrm{id}} \mathbb{B} \times B \xrightarrow{g^{\phi \circ d} \times \tau} \mathbb{B} \times [0,1]$$

i.e.,

$$\xi(x) = (g^{\phi(d(x))}(\eta(x)), \tau(d(x))).$$

Finally, we use a foliated and fibered version of the control theorem with respect to the fiber bundle $\xi : W_B \to \mathbb{B} \times [0,1]$ and the foliation of $\mathbb{B} \times [0,1]$ by the flow lines of the geodesic flow in order to extend over W_B the homotopy defined in steps 1, 2 and 3 given above. And thus complete the construction of an admissible homotopy of $\hat{f}$ to a split map. The control condition is met provided σ is sufficiently large and R is contained in a sufficiently small neighborhood of $+\infty$. The intertwining equation is used to see this. But there is one extra point to observe. The fiber of ξ is $\mathbb{R}P^{m-1}$ and $\mathcal{S}(\mathbb{R}P^{m-1} \times \mathbb{D}^k, \partial)$ usually contains more than a single element. Consequently, the control theorem only yields the weaker conclusion that the result of the homotopy is a split map rather than a homeomorphism.