united nations
educational, scientific
and cultural
organization

international atomic
energy agency

the
abdus salam
international centre for theoretical physics

SMR1312/14

School on High-Dimensional Manifold Topology

(21 May - 8 June 2001)

Some calculations of π_n (Top M), π_n (Diff M) and other applications

F.T. Farrell
Binghamton University
Department of Mathematical Sciences
Binghamton, New York 13902-6000
U.S.A.

Some calculations of $\pi_n(\text{Top } M)$, $\pi_n(\text{Diff } M)$ and other applications

F.T. Farrell[†]

[†] *Binghamton University, Department of Mathematical Sciences, Binghamton, New York 13902-6000, USA*

Lecture given at the:
School on High Dimensional Manifold Topology, Trieste
21 May - 8 June 2001

LNS

Lecture 5

Recall (Lecture 3) that the Vanishing Theorem showing $Wh(\pi_1 M) = 0$ extends to complete, A-regular, non-positively curved Riemannian manifolds M. Likewise there is a version of the Topological Rigidity Theorem (TRT) valid for such manifolds which we proceed to formulate.

Let M be an arbitrary manifold; i.e. it can be non-compact and can have non-empty boundary. We say that M is *topologically rigid* if it has the following property. Let

$$h : (N, \partial N) \to (M, \partial M)$$

be any proper homotopy equivalence where N is another manifold. Suppose there exists a compact subset $C \subseteq N$ such that the restriction of h to $\partial N \cup (N - C)$ is a homeomorphism. Then there exists a proper homotopy

$$h_t : (N, \partial N) \to (M, \partial M)$$

from h to a homeomorphism and a perhaps larger compact subset K of N such that the restrictions of h_t and h to $\partial N \cup (N - K)$ agree for all $t \in [0,1]$. (When M and N are closed, this just says that a homotopy equivalence $h : N \to M$ is homotopic to a homeomorphism.)

Addendum to TRT. *(Farrell and Jones 1998). Let M^m be an arbitrary aspherical manifold with $m \geq 5$. Suppose $\pi_1(M)$ is isomorphic to the fundamental group of an A-regular complete non-positively curved Riemannian manifold. (This happens for example when $\pi_1(M)$ is isomorphic to a torsion-free discrete subgroup of $GL_n(\mathbb{R})$.) Then M is topologically rigid. In particular, every A-regular complete non-positively curved Riemannian manifold of* dim ≥ 5 *is topologically rigid.*

The special case of this Addendum where M is an A-regular complete non-positively curved Riemannian manifold is proved by an argument very close to that made in Lecture 4 for TRT. But stronger control theorems are needed when M is not closed; in particular when the injectivity radius at a point $x \in M$ goes to 0 as $x \to \infty$. These control theorems were discussed by Lowell Jones in his last lecture. The general case of the Addendum follows from this special case and the version of the surgery sequence for arbitrary spaces developed by Andrew Ranicki in his lectures; in particular that the assembly map in homology

$$A_* : H_*(M; \mathcal{L}) \to L_*(\pi_1 M, w)$$

is uniquely determined by the homotopy type of M and the orientation data $w : \pi_1(M) \to \mathbb{Z}_2$.

This Addendum even has (perhaps unexpectedly) consequences beyond, what follows from TRT, for closed manifolds. We now discuss some of these.

Corollary 1. *Let N and M be a pair of closed complete affine flat manifolds with* $\dim(M) \neq 3, 4$. *If $\pi_1(N) \simeq \pi_1(M)$, then N and M are homeomorphic (via a homeomorphism inducing this isomorphism).*

Corollary 1 is an affine analogue of the classical Bieberbach Theorem valid for Riemannian flat manifolds. We note that Corollary 1 does *not* follow from the TRT proved in Lecture 4 since there are closed complete affine flat manifolds M which *cannot* support a Riemannian metric of non-positive curvature. For example $M^3 = \mathbb{R}^3/\Gamma$ does *not* where Γ is the group generated by the three affine motions α, β and γ of $\mathbb{R}^3$ with

$$\begin{aligned}
\alpha(x,y,z) &= (x+1,y,z)\\
\beta(x,\gamma,z) &= (x,y+1,z)\\
\gamma(x,y,z) &= (x+y,2x+3y,z+1).
\end{aligned}$$

Since Γ is solvable but not virtually abelian, Yau's thesis shows that M cannot support a non-positively curved Riemannian metric. But Corollary 1 does follow from the Addendum to TRT since M is aspherical and $\pi_1(M)$ is a discrete subgroup of Aff$(\mathbb{R}^m)$ which is a closed subgroup of $GL_{m+1}(\mathbb{R})$; namely

$$\text{Aff}(\mathbb{R}^m) = \left\{ A \in GL_{m+1}(\mathbb{R}) \,\middle|\, A_{m+1,i} = \begin{cases} 0 & i \leq m \\ 1 & i = m+1 \end{cases} \right\}$$

We next use this Addendum to verify a special case of a well known conjecture of C.T.C. Wall.

Conjecture. (Wall) Let Γ be a torsion-free group which contains a subgroup of finite index isomorphic to the fundamental group of a closed aspherical manifold. Then Γ is the fundamental group of a closed aspherical manifold.

Corollary 2. *Let M^m be a closed (connected) non-positively curved Riemannian manifold and Γ be a torsion-free group which contains $\pi_1(M)$ as a subgroup with finite index. Assume that $m \neq 3,4$, then the deck transformation action of $\pi_1(M)$ on the universal cover $\tilde{M}$ extends to a topological action of Γ on $\tilde{M}$. Consequently Wall's Conjecture is true in this case since $\tilde{M}/\Gamma$ is a closed apsherical manifold with $\pi_1(\tilde{M}/\Gamma) = \Gamma$.*

Remark. When $\tilde{M}$ is a symmetric space without 1 or 2 dimensional factors, Γ embeds in its isometry group Iso$(\tilde{M})$ extending $\pi_1(M) \subseteq$ Iso$(\tilde{M})$; this is a consequence of Mostow's Strong Rigidity Theorem. When $m = 2$, Corollary 2 is a consequence of a result due to Eckmann, Linnell and Muller; our proof only applies to the situation $m \geq 5$.

In proving Corollary 2 we can clearly make the simplifying assumptions that M is orientable and $\pi_1(M)$ is normal in Γ. We now use an important trick due to Serre where he constructs a natural, properly discontinuous action of Γ via isometries on the Riemannian product $\mathcal{M}^{sm}$ of s-copies of $\tilde{M}$

$$\mathcal{M} = \tilde{M} \times \tilde{M} \times \cdots \times \tilde{M}$$

where $s = [\pi_1 M : \Gamma]$. (Serre's construction is a kind of geometric co-induced representation.) Note that $\mathcal{M}^{sm}$ is A-regular and non-positively curved since M^m is. Hence $N^{sm} = \mathcal{M}/\Gamma$ is a complete (but *not* closed) A-regular non-positively curved manifold with $\pi_1(N) = \Gamma$. Thus the Addendum to TRT applies to $N^{sm} \times \mathbb{D}^k$ for all $k \geq 0$. From this we conclude that Ranicki's periodic assembly map

$$A_* : H_*(B\Gamma, \mathcal{L}) \to L_*(\Gamma)$$

is an isomorphism. Also the Vanishing Theorem applies showing that

$$Wh(\Gamma) = 0 = \tilde{K}_0(\mathbb{Z}\Gamma).$$

And Ranicki, by reworking the existence part of surgery theory, has shown that when this happens $B\Gamma$ is homotopically equivalent to a closed manifold K^m provided $B\pi$ is for some subgroup π of finite index in Γ. In this case, we can take $\pi = \pi_1(M)$.

Let $\hat{K}$ be the cover of K corresponding to $\pi_1(M)$. And note that K is homotopically equivalent to M since both are aspherical and have the same fundamental group. Therefore $\hat{K}$ is homeomorphic to M by the TRT. Consequently $\tilde{K} = \tilde{M}$ and the deck transformation action of $\Gamma = \pi_1(K)$ on $\tilde{M}$ is the desired extension of the action by $\pi_1(M)$. Q.E.D.

Corollary 2 can be applied to obtain positive information about the following generalization of the classical Nielsen Problem. Let $\mathrm{Top}(M)$ denote the group of all homeomorphisms of a manifold M and denote the group of all outer automorphisms of $\pi_1(M)$ by $\mathrm{Out}(\pi_1 M)$.

Generalized Nielsen Problem. (GNP) Let M be a closed aspherical manifold and F be a finite subgroup of $\mathrm{Out}(\pi_1 M)$. Does F split back to $\mathrm{Top}(M)$; i.e., does there exist a finite subgroup $\bar{F}$ of $\mathrm{Top}(M)$ which maps isomorphically onto F under the natural homomorphism

$$\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M)?$$

Remark. There are cases where this is impossible. One necessary extra condition is that there exist an extension

$$1 \to \pi_1(M) \to \Gamma \to F \to 1$$

inducing the embedding $F \subseteq \mathrm{Out}(\pi_1 M)$. F. Raymond and L. Scott gave an example where this condition is not satisfied. In their example M is a nilmanifold. There is a natural exact sequence

$$1 \to \mathrm{Center}(\Gamma) \to \Gamma \underset{\phi}{\longrightarrow} \mathrm{Aut}(\Gamma) \underset{\psi}{\longrightarrow} \mathrm{Out}(\Gamma) \to 1$$

where $\phi(\gamma)$ is conjugation by γ. Let $\Gamma_F = \psi^{-1}(F)$. When $\mathrm{Center}(\Gamma) = 1$

$$1 \to \Gamma \to \Gamma_F \to F \to 1$$

is the necessary extension mentioned in this Remark.

Corollary 3. *The finite group F of the GNP splits back to* $\mathrm{Top}(M)$ *under the following extra assumptions:*

1. $\mathrm{Center}(\pi_1 M) = 1$.

2. M is a non-positively curved Riemannian manifold.

3. $\dim(M) \neq 3, 4$.

4. Γ_F is torsion-free.

Remark 1. Conditions 1 and 2 are satisfied when M is negatively curved.

Remark 2. When $\dim(M) = 2$, this result is due to Eckmann, Linnell and Muller (1981).

Remark 3. When $\tilde{M}$ is a symmetric space without 1 or 2 dimensional metric factors, this result, due to Mostow (1973), is true even with conditions 1, 3 and 4 dropped.

Remark 4. Corollary 3 remains true when condition 2 is replaced by the weaker condition that $\pi_1(M)$ is isomorphic to the fundamental group of a complete, A-regular non-positively curved Riemannian manifold. This is because Corollary 2 is also true under the same weakening of its hypotheses.

To prove Corollary 3, note that Γ_F satisfies the hypotheses for the group Γ in Corollary 2. Hence Γ_F acts on $\tilde{M}$ extending the action of $\pi_1(M)$ by deck transformations. The image of this action in $\mathrm{Top}(M)$ is the subgroup $\bar{F}$ asked for in GNP. Q.E.D.

There is also the related question of whether the natural homomorphism

$$\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M)$$

is onto?

Corollary 4. *Let M^m be a closed aspherical manifold. Assume that $m \neq 3, 4$ and that $\pi_1(M)$ is isomorphic to the fundamental group of a complete, A-regular, non-positively curved Riemannian manifold. Then the natural homomorphism* $\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M)$ *is a surjection.*

Corollary 4 is classical for $m = 2$ or 1. And for $m \geq 5$, it follows immediately from the Addendum to TRT since every outer automorphism of $\pi_1(M)$ is induced by a self homotopy equivalence of M; cf. Hurewicz's result mentioned in Lecture 1. Q.E.D.

Remark. When $\tilde{M}$ is a symmetric space without 1 or 2 dimensional metric factors, Corollary 4 is due to Mostow (1973).

Give the group $\mathrm{Top}(M)$ the compact open topology and let its closed subgroup $\mathrm{Top}_0(M)$ be the kernel of the natural continuous homomorphism (analyzed in Corollary 4) to the discrete group $\mathrm{Out}(\pi_1 M)$. $\mathrm{Top}_0(M)$ is *not* in general the connected component of the identity element in $\mathrm{Top}(M)$. However the following is true.

Corollary 5. *Let M^m be a closed (connected) non-positively curved Riemannian manifold with $m > 10$. Then*

$$\begin{aligned}
\pi_0(\mathrm{Top}_0\ M) &= \mathbb{Z}_2^\infty, \\
\pi_1(\mathrm{Top}\ M) \otimes \mathbb{Q} &= \mathrm{Center}(\pi_1 M) \otimes \mathbb{Q}, \quad \textit{and} \\
\pi_n(\mathrm{Top}\ M) \otimes \mathbb{Q} &= 0 \quad \textit{if}\ \ 1 < n \leq \frac{(m-7)}{3}.
\end{aligned}$$

Remark. There is in particular the following exact sequence

$$1 \to \mathbb{Z}_2^\infty \to \pi_0(\mathrm{Top}\ M) \to \mathrm{Out}(\pi_1 M) \to 1.$$

And $\mathbb{Z}_2^\infty$ denotes the direct sum of a countably infinite number of copies of $\mathbb{Z}_2$.

The proof of Corollary 5 depends not only on the Addendum to TRT but also on the following result "PIT" concerning the stable topological pseudo-isotopy functor $\mathcal{P}(\ )$. Recall that this functor was defined and discussed earlier in lectures by Tom Goodwillie, Lowell Jones and Frank Quinn.

Pseudo Isotopy Theorem. *(Farrell and Jones) Let M be a closed (connected) non-positively curved Riemannian manifold. Then, for all n,*

$$\begin{aligned} \pi_n(\mathcal{P}(M)) \otimes \mathbb{Q} &= 0 \quad \textit{and} \\ \pi_0(\mathcal{P}(M)) &= \mathbb{Z}_2^\infty. \end{aligned}$$

We will discuss the ideas behind the proof of PIT after first using it in proving Corollary 5.

For this we need to introduce the auxiliary spaces $G(M)$ and $\overline{\text{Top}}(M)$. Let $G(M)$ denote the H-space of all self-homotopy equivalences of M; note that $\text{Top}(M)$ is a subspace of $G(M)$. The semisimplicial group $\overline{\text{Top}}(M)$ of blocked homeomorphisms of M can be interpolated between $\text{Top}(M)$ and $G(M)$. A typical k-simplex of $\overline{\text{Top}}(M)$ consists of a homeomorphism

$$h : \Delta^k \times M \to \Delta^k \times M$$

such that $h(\Delta \times M) = \Delta \times M$ for each face Δ of Δ^k, where Δ^k is the standard k-simplex.

Let $G(M)/\text{Top}(M)$ and $\overline{\text{Top}}(M)/\text{Top}(M)$ denote the homotopy fiber of the map

$$B\ \text{Top}(M) \to BG(M) \quad \text{and} \quad B\ \text{Top}(M) \to B\ \overline{\text{Top}}(M),$$

respectively. Because of Frank Quinn's function space interpretation of the surgery exact sequence (discussed in Lowell Jone's lectures) the relative homotopy groups of the map

$$\overline{\text{Top}}(M) \to G(M)$$

can be identified with the groups

$$\mathcal{S}(M \times \mathbb{D}^n, \partial).$$

And these all vanish because of the TRT; consequently the following is true.

Fact 1. $G(M)/\text{Top}(M)$ and $\overline{\text{Top}}(M)/\text{Top}(M)$ have the same weak homotopy type.

Now the homotopy groups of $G(M)$ are easy to calculate. They are

Fact 2.

$$\pi_n(G(M)) = \begin{cases} \text{Out}(\pi_1 M) & \text{if } n = 0 \\ \text{Center}(\pi_1 M) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

Since the calculation for $n \geq 2$ is particularly easy to do, we sketch it. Let

$$f : S^n \times M \to M$$

represent an element in $\pi_n(G(M))$. To show this element is zero, we need to extend f to a map

$$\hat{f} : \mathbb{D}^{n+1} \times M \to M.$$

The construction of $\hat{f}$ is by an elementary obstruction theory argument. Fix a triangulation of M and assume $\hat{f}$ has already been defined over $\mathbb{D}^{n+1}\times\sigma$ for all simplices σ with $\dim(\sigma) < k$. Let σ be a k-simplex and identify $\mathbb{D}^{n+1}\times\sigma$ with $\mathbb{D}^{n+k+1}$. Then $\hat{f}|_{\partial\mathbb{D}^{n+k+1}}$ has already been defined and represents an element of $\pi_{n+k}(M)$ which vanishes since M is aspherical. Therefore $\hat{f}$ extends over $\mathbb{D}^{n+1}\times\sigma$. It is shown in this way that $\pi_n(G(M)) = 0$ when $n \geq 2$.

It therefore remains to analyze $\overline{\mathrm{Top}}(M)/\mathrm{Top}(M)$. Which can be done in terms of $\mathcal{P}(M)$ by using the following result of Hatcher.

Theorem. *(Hatcher) When $m > 10$ ($m = \dim M$) there is a spectral sequence converging to*

$$\pi_{p+q+1}(\overline{\mathrm{Top}}(M)/\mathrm{Top}(M))$$

with

$$E^2_{pq} = H_p(\mathbb{Z}_2; \pi_1(\mathcal{P}(M)))$$

in the stable range $q \leq \frac{(m+p-7)}{3}$.

Remark. This result depends on Igusa's Stability Theorem for pseudo-isotopy spaces which Tom Goodwillie discussed in an earlier lecture.

Combining Hatcher's Theorem and PIT together with Facts 1 and 2 yields that

Fact 3.

$$\pi_n(\mathrm{Top}(M))\otimes\mathbb{Q} = \begin{cases} 0 & \text{if } 1 \leq n \leq \frac{m-7}{3} \\ \mathrm{Center}(\pi_1 M)\otimes\mathbb{Q} & \text{if } m = 1 \end{cases}$$

and the following exact sequence:

$$\mathrm{Center}(\pi_1 M) \to H_0(\mathbb{Z}_2; \mathbb{Z}_2^\infty) \to \pi_0(\mathrm{Top}(M)) \to \mathrm{Out}(\pi_1 M).$$

Since the kernel of $\pi_0(\mathrm{Top}(M)) \to \mathrm{Out}(\pi_1 M)$ is $\pi_0(\mathrm{Top}(M))$, this exact sequence can be rewritten as

Fact 4.

$$\mathrm{Center}(\pi_1 M) \to H_0(\mathbb{Z}_2; \mathbb{Z}_2^\infty) \to \pi_0(\mathrm{Top}(M)) \to 0.$$

Define a homomorphism $d : \mathbb{Z}_2^\infty \to \mathbb{Z}_2^\infty$ by

$$d(x) = x + \bar{x}$$

where $x \mapsto \bar{x}$ denotes the action of the generator of $\mathbb{Z}_2$ on $\mathbb{Z}_2^\infty$. Then the formula

$$H_0(\mathbb{Z}_2, \mathbb{Z}_2^\infty) = \mathbb{Z}_2^\infty/\mathrm{image}(d)$$

is the definition of $H_0(\mathbb{Z}_2, \mathbb{Z}_2^\infty)$. We claim that $\mathbb{Z}_2^\infty/\mathrm{image}(d)$ cannot be a finite group. If it were, then $\mathbb{Z}_2^\infty/\ker(d)$ would also be finite since

$$\ker(d) \supseteq \mathrm{image}(d).$$

(Note that $d^2 = 0$ since $\mathbb{Z}_2^\infty$ has exponent 2.) But image(d) is isomorphic to $\mathbb{Z}_2^\infty/\ker(d)$. And the finiteness of both image(d) and $\mathbb{Z}_2^\infty/\mathrm{image}(d)$ would imply that $\mathbb{Z}_2^\infty$ is also finite, which is a contradiction. Since $H_0(\mathbb{Z}_2; \mathbb{Z}_2^\infty)$ is thus a countable infinite group of exponent 2, it must be isomorphic to $\mathbb{Z}_2^\infty$. We therefore rewrite the sequence in Fact 4 as

Fact 5.
$$\mathrm{Center}(\pi_1 M) \to \mathbb{Z}_2^\infty \to \pi_0(\mathrm{Top}(M)) \to 0.$$

Now Lawson and Yau showed that $\mathrm{Center}(\pi_1 M^m)$ is finitely generated. (In fact it is isomorphic to $\mathbb{Z}^n$ where $n \leq m$.) Hence Fact 5 implies that $\pi_0(\mathrm{Top}_0(M))$ is a countably infinite group of exponent 2, and therefore it is isomorphic to $\mathbb{Z}_2^\infty$. This result together with Fact 3 proves Corollary 5.

We now return to a discussion of PIT. Its proof follows the pattern established in proving the Vanishing Theorem (cf. Lecture 3). The main difference is that the corresponding foliated control theorem is obstructed since $\mathcal{P}(S^1)$ is *not* contractible. So we get a calculation instead of a vanishing theorem. Key ingredients for this calculation are ideas developed by Frank Quinn which were discussed in his and Lowell Jones' lectures.

We formulate a more precise result than PIT; namely a weak version of the Isomorphism Conjectures which Wolfgang Lueck talked about in one of his lectures. For the rest of this lecture M denotes a closed (connected) non-positively curved Riemannian manifold, $\tilde{M}$ its universal cover, and $\Gamma = \pi_1(M)$ its group of deck transformations. Fix a universal space $\mathcal{E}$ for Γ relative to the class $\mathcal{C}$ of all virtually cyclic subgroups of Γ.

Theorem. *(Farrell and Jones) There exists a spectral sequence converging to $\pi_{p+q}(\mathcal{P}(M))$ with $E^2_{pq} = H_p(\mathcal{E}/\Gamma; \pi_q(\mathcal{P}(\tilde{M}/\Gamma_\sigma)))$.*

Remark. In this theorem Γ_σ denotes the subgroup of Γ fixing a cell σ of $\mathcal{E}$. And $H_p(\mathcal{E}/\Gamma; \pi_q(\mathcal{P}(\tilde{M}/\Gamma_\sigma)))$ is the p-th homology group of a chain complex whose p-th chain group is the direct sum of the groups $\pi_q(\mathcal{P}(\tilde{M}/\Gamma_\sigma))$ where σ varies over a set S_p of p-cells of $\mathcal{E}$. The set S_p contains exactly one p-cell from each Γ-orbit of p-cells.

To deduce PIT from this result, we must analyze the spectral sequence. Note first that Γ_σ is either infinite cyclic or trivial since Γ is torsion-free. Therefore $\tilde{M}/\Gamma_\sigma$ is homotopically equivalent to either the circle S^1 or a point $*$ since $\tilde{M}/\Gamma_\sigma$ is aspherical. And there is the following important calculation:

Calculation 1. (a) $\pi_n(\mathcal{P}(*)) = 0$ for all n,

(b) $\pi_n(\mathcal{P}(S^1)) \otimes \mathbb{Q} = 0$ for all n,

(c) $\pi_0(\mathcal{P}(S^1)) = \mathbb{Z}_2^\infty$.

Calculation (a) is a consequence of Alexander's Trick discussed in Lecture 1. Calculation (b) is due to Waldhausen, and (c) is due to Waldhausen and Igusa. Calculations (b) and (c) are deep results related to Tom Goodwillie's Lectures 1 and 2. Because of (a) and (b), $E^2_{pq} \otimes \mathbb{Q} = 0$. Hence the Theorem yields that $\pi_n(\mathcal{P}(M)) \otimes \mathbb{Q} = 0$; which is the first assertion of PIT.

Our Theorem also yields that

$$\pi_0(\mathcal{P}(M)) = H_0(\mathcal{E}/\Gamma; \pi_0(\mathcal{P}(\tilde{M}/\Gamma_\sigma))).$$

Since we can pick $\mathcal{E}$ to be a countable CW-complex (because Γ is countable) this equation together with Calculations (a) and (c) imply that $\pi_0(\mathcal{P}(M))$ is a quotient group of $\mathbb{Z}_2^\infty$; i.e. is a countable abelian group of exponent 2. To complete the proof of PIT, it remains to show that $\pi_0(\mathcal{P}(M))$ is an infinite group. We will only show this when M is negatively curved, since the general case depends on constructing a universal space $\mathcal{E}$ for Γ with better properties than the abstract construction. This geometric construction of $\mathcal{E}$ uses strongly the assumption that M is closed and non-positively curved. But, when M is negatively curved, the fact that $\pi_0(\mathcal{P}(M))$ is infinite is an immediate consequence of the following Assertion.

Assertion. *Assume M is negatively curved and let $\gamma : S^1 \to M$ represent a non-trivial element $[\gamma] \in \pi_1(M)$. Then*

$$\mathcal{P}(\gamma)_\# : \pi_0(\mathcal{P}(S^1)) \to \pi_0(\mathcal{P}(M))$$

is monic where $\mathcal{P}(\gamma) : \mathcal{P}(S^1) \to \mathcal{P}(M)$ is the functorially induced map.

We indicate the proof of this Assertion under the simplifying assumption that M is orientable. To do this we construct a transfer map

$$\tau : \mathcal{P}(M) \to \mathcal{P}(S^1)$$

such that $\tau \circ \mathcal{P}(\gamma)$ is homotopic to $\mathrm{id}_{\mathcal{P}(S^1)}$. The Assertion is clearly a consequence of this. Our construction uses ideas from Lecture 2. We first define a map

$$P(M) \to P(\bar{M}).$$

(Recall that $\bar{M} = \tilde{M} \cup M(\infty)$ is homeomorphic to $\mathbb{D}^m$.) This is done by sending the pseudo-isotopy f to the pseudo-isotopy $\bar{f}$ where

$$\bar{f}(x) = \begin{cases} x & \text{if } x \in M(\infty) \times [0,1] \\ \tilde{f}(x) & \text{if } x \in \tilde{M} \times [0,1] \end{cases}$$

and $\tilde{f}$ is the unique lift of f such that $\tilde{f}|_{\tilde{M}\times 0} = \mathrm{id}_{\tilde{M}\times 0}$. This pseudo-isotopy $\bar{f}$ is "well-defined" because Cartan's Theorem shows that property 2 of Condition $(*)$ holds (cf. Lecture 2). To be precise, $\bar{f}$ is only well defined after we collapse $x \times [0,1]$, $x \in M(\infty)$, to the single point x. But this quotient space can be identified with $\bar{M} \times [0,1]$. Note that $\bar{f}$ is Γ-equivariant. Let S be the infinite cyclic subgroup of Γ generated by $[\gamma]$. There are exactly two points S^+ and S^- on $M(\infty)$ fixed by S since $[\gamma]$ can be represented by a closed geodesic (because M is compact). Furthermore S acts freely and properly discontinuously on $\bar{M} - \{S^+, S^-\} = M_S$ and hence $\bar{f}$ induces a pseudo-isotopy

$$\hat{f} \in P(M_S/S).$$

But M_S/S is homeomorphic to $S^1 \times \mathbb{D}^{m-1}$ since M is orientable. The function $f \mapsto \hat{f}$ mapping

$$P(M) \to P(S^1 \times \mathbb{D}^{m-1})$$

stabilizes to give the desired transfer τ. Q.E.D.

We end our lectures by giving an analogue of Corollary 5 true for Diff(M).

Corollary 6. *Suppose that M^m is orientable, $m > 10$ and $1 < n \leq \frac{(m-7)}{3}$. Then*

$$\pi_n(\mathrm{Diff}(M)) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } m \text{ is even} \\ \displaystyle\bigoplus_{j=1}^{\infty} H_{(n+1)-4j}(M, \mathbb{Q}) & \text{if } m \text{ is odd.} \end{cases}$$

Furthermore, $\pi_1(\mathrm{Diff}(M)) \otimes \mathbb{Q} = \mathrm{Center}(\pi_1 M) \otimes \mathbb{Q}$.

Corollary 6 is an immediate consequence of the following result combined with TRT, PIT and the Vanishing Theorem.

Theorem. *(Farrell and Hsiang 1978) Let N^m be a closed aspherical manifold such that*

$$\begin{aligned} \mathcal{S}(N^m \times \mathbb{D}^k, \partial = 0 & \quad \text{for all } k \geq 0, \\ \pi_k(\mathcal{P}(N)) \otimes \mathbb{Q} = 0 & \quad \text{for all } k \geq 0, \\ Wh(\pi_1(N) \times \mathbb{Z}^k) = 0 & \quad \text{for all } k \geq 0. \end{aligned}$$

Then for $1 \leq n \leq (\frac{m-7}{3})$

$$\pi_n(\mathrm{Diff}(N)) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } n > 1,\ n \text{ even} \\ \displaystyle\bigoplus_{j=1}^{\infty} H_{(n+1)-4j}(N, \mathbb{Q}) & \text{if } n > 1,\ n \text{ odd} \\ \mathrm{Center}(\pi_1 N) \otimes \mathbb{Q} & \text{if } n = 1. \end{cases}$$