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Notes on

Exotic aspherical manifolds

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by

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Contents

1. The geometric realization of a simplicial complex. A *simplicial complex* L consists of a set I (called the *vertex set*) and a collection of finite subsets $\mathcal{S}(L)$ of I such that

(a) $\emptyset \in \mathcal{S}(L)$

(b) for each $i \in I$, $\{i\} \in \mathcal{S}(L)$, and

(c) if $\sigma \in \mathcal{S}(L)$ and $\tau < \sigma$ then $\tau \in \mathcal{S}(L)$. An element σ of $\mathcal{S}(L)$ is called a *simplex*; its *dimension* is defined by $\dim \sigma = \mathrm{Card}(\sigma) - 1$.

Let us assume that $I = \{1, 2, \ldots, m\}$. The *standard* $(m-1)$*-simplex* on I, denoted by Δ^{m-1}, is the convex polytope in $\mathbb{R}^m$ defined by intersecting the positive quadrant (defined by $x_i \geq 0$ for all $i \in I$) with the hyperplane $\Sigma x_i = 1$. A vertex of Δ^{m-1} is an element e_i of the standard basis for $\mathbb{R}^m$. The poset of faces of Δ^{m-1}, denoted $\mathcal{F}(\Delta^{m-1})$, is isomorphic to the poset of all nonempty subsets of I. This gives us a simplicial complex which we will also denote by Δ^{m-1}. (Usual practice is to blur the distinction between a simplicial complex and its geometric realization.)

If σ is a nonempty subset of I $(= \{1, \ldots m\})$, then let Δ_σ denote the face of Δ^{m-1} spanned by $\{e_i\}_{i \in \sigma}$.

If L is a simplicial complex with vertex set I, then its *geometric realization* is defined to be the union of all subspaces of Δ^{m-1} of the form Δ_σ for some $\sigma \in \mathcal{S}(L)$. The geometric realization will also be denoted L.

2. Cubical cell complexes. As before, $I = \{1, \ldots, m\}$. The *standard* m*-dimensional cube* is the convex polytope $[\text{-}1,1]^m \subset \mathbb{R}^m$. For each subset σ of I let $\mathbb{R}^\sigma$ denote the linear subspace spanned by $\{e_i\}_{i \in \sigma}$ and let $\square_\sigma$ denote the standard cube in $\mathbb{R}^\sigma$. (If $\sigma = \emptyset$, then $\mathbb{R}^\emptyset = \square_\emptyset = \{0\}$.) The faces of $[\text{-}1,1]^m$ which are parallel to $\square_\sigma$ have the form $v + \square_\sigma$ for some vertex v of $[\text{-}1,1]^m$.

Next we want to describe the poset of nonempty faces of $[\text{-}1,1]^m$. For each $i \in I$, let $r_i : [\text{-}1,1]^m \to [\text{-}1,1]^m$ denote the orthogonal reflection across the hyperplane $x_i = 0$. Let J be the group of symmetries of $[\text{-}1,1]^m$ generated by $\{r_i\}_{i \in I}$. Then J is isomorphic to $(\mathbb{Z}/2)^m$. The group J acts simply transitively on the vertex set of $[\text{-}1,1]^m$. The stabilizer of a face $v + \square_\sigma$ is the subgroup J_σ generated by $\{r_i\}_{i \in \sigma}$. Hence, the poset of nonempty faces of $[\text{-}1,1]^m$ is isomorphic to the poset of cosets

$$\coprod_{\sigma \subset I} J/J_\sigma$$

Roughly speaking, a cubical cell complex P is a regular cell complex in which each cell is combinatorially isomorphic to a standard cube of some dimension. More precisely, P consists of a poset $\mathcal{F}(= \mathcal{F}(P))$ such that for each $c \in \mathcal{F}$ the subposet $\mathcal{F}_{\leq c}$ is isomorphic to the poset of nonempty faces of $[\text{-}1,1]^k$; k is the *dimension* of c. (Here $\mathcal{F}_{\leq c} = \{x \in \mathcal{F} | x \leq c\}$.) The elements of $\mathcal{F}$ are called *cells*. A vertex of P is synonymous with a 0-dimensional cell. By definition the *link* of a vertex v in P, denoted by $Lk(v, P)$, is the subposet $\mathcal{F}_{>v}$ of all cells which are strictly greater than v (i.e., which have v as a vertex.) For example, if v is a vertex of [-1,1]^m, then $Lk(v,$ [-1,1]$^m)$ is the simplicial complex Δ^{m-1}. It follows that the link of a vertex in any cubical cell complex is a simplicial complex.

The geometric realization of a cubical complex P can be defined by pasting together standard cubes, one for each element of $\mathcal{F}$. A neighborhood of a vertex v in (the geometric realization of) P is homeomorphic to the cone on $Lk(v, P)$.

3. The cubical complex P_L. Given a simplicial complex L with vertex set I, we shall now define a subcomplex P_L of [-1,1]^m. The vertex set of P_L will be the same as that of [-1,1]^m. The main property of P_L will be that the link of each of its vertices is isomorphic to L. The construction is very similar to the way in which we realized L as a subcomplex of Δ^{m-1}.

By definition, P_L is the union of all faces of [-1,1]^m which are parallel to $\square_\sigma$ for some $\sigma \in \mathcal{S}(L)$. Hence, the poset of cells of P_L can be identified with

$$\coprod_{\sigma\in\mathcal{S}(L)} J/J_\sigma.$$

Examples. 1) If $L = \Delta^{m-1}$, then $P_L =$[-1,1]^m.

2) If $L = \partial(\Delta^{m-1})$, then P_L is the boundary of an m-cube i.e., P_L is homeomorphic to S^{m-1}.

3) If L is the disjoint union of m points, then P_L is the 1-skeleton of an m-cube.

4) If $m = 3$ and L is the disjoint union of a 1-simplex and a point then P_L is the subcomplex of the 3-cube consisting of the top and bottom faces and the 4 vertical edges.

¿From the fact that a neighborhood of a vertex in P_L is homeomorphic to the cone on L we get the following.

Proposition. *If L is homeomorphic to S^{n-1}, then P_L is an n-manifold.*

Proof. The cone on S^{n-1} is homeomorphic to an n-disk.

4. The universal cover of P_L and the group W_L. Let $\tilde{P}_L$ denote the universal cover of P_L. The cubical cell structure on P_L lifts to a cubical structure on $\tilde{P}_L$. The group J ($\cong (\mathbb{Z}/2)^m$) acts on P_L. Let W_L denote the group of all lifts of elements of J to $\tilde{P}_L$ and let $\varphi : W_L \to J$ be the homomorphism induced by the projection $\tilde{P}_L \to P_L$. We have a short exact sequence,

$$1 \to \pi_1(P_L) \to W_L \xrightarrow{\varphi} J \to 1.$$

We will use the notation:

$$\Gamma_L = \pi_1(P_L).$$

Since J acts simply transitively on the vertex set of P_L, the group W_L acts simply transitively on the vertex set of $\tilde{P}_L$. It follows that the 2-skeleton of $\tilde{P}_L$ is the Cayley 2-complex associated to a presentation of W_L. (In particular, the 1-skeleton is the Cayley graph associated to a set of generators.) Next, we use this observation to write down a presentation for W_L.

The vertex set of P_L can be identified with J. Fix a vertex v of P_L (corresponding to the identity element in J). Let $\tilde{v}$ be a lift of v in $\tilde{P}_L$. The 1-cells at v or at $\tilde{v}$ correspond to vertices of L, i.e., to elements of $\{1, \ldots, m\}$. The reflection r_i stabilizes the i^{th} 1-cell at v. Let s_i denote the unique lift of r_i which stabilizes the i^{th} 1-cell at $\tilde{v}$. Since s_i^2 fixes $\tilde{v}$ and covers the identity on P_L, it follows that $s_i^2 = 1$. Suppose σ is a 1-simplex of L connecting vertices i and j. The corresponding 2-cell at $\tilde{v}$ is then a square with edges labelled successively by s_i, s_j, s_i, s_j. Hence, we get a relation $(s_i s_j)^2 = 1$ for each 1-simplex $\{i, j\}$ of L.

Let $\hat{W}_L$ denote the group defined by this presentation, i.e., a set of generators for $\hat{W}_L$ is $\{s_i\}_{i \in I}$ and the relations are given by: $s_i^2 = 1$, for all $i \in I$, and $(s_i s_j)^2 = 1$ whenever $\{i, j\}$ is a 1-simplex of L. The Cayley 2-complex associated to this presentation maps to the 2-skeleton of $\tilde{P}_L$ by a covering projection. Since $\tilde{P}_L$ is simply connected, this covering projection is a homomorphism and the natural homomorphism $\hat{W}_L \to W_L$ is an isomorphism. Therefore, we have proved the following.

Proposition. *W_L has a presentation as described above.*

Remarks. 1) A group with a presentation of the above form is a Coxeter group, in fact, a "right-angled" Coxeter group. (In a general Coxeter group we allow relations of the form $(s_i s_j)^{m_{ij}} = 1$ where the integer m_{ij} can be > 2.

2) Examining the presentation, we see that the abelianization of W_L is J. Thus, Γ_L is the commutator subgroup of W_L.

For each subset σ of I let W_σ denote the subgroup generated by $\{s_i\}_{i \in \sigma}$. If $\sigma \in \mathcal{S}(L)$, then W_σ is the stabilizer of the corresponding cube in $\tilde{P}_L$ which contains $\tilde{v}$. It follows that the poset of cells of $\tilde{P}_L$ is isomorphic to the poset of cosets,

$$\coprod_{\sigma \in \mathcal{S}(L)} W_L / W_\sigma .$$

5. When is $\tilde{P}_L$ contractible?

Definition. A simplicial complex L is a *flag complex* if any finite set of vertices which are pairwise connected by edges spans a simplex of L.

Theorem 5.1. *$\tilde{P}_L$ is contractible if and only if L is a flag complex.*

Corollary 5.2. *P_L is aspherical if and only if L is a flag complex.*

Before sketching the proof of this theorem we make a few comments on the nature of flag complexes.

An *incidence relation* on a set I is a symmetric and antireflexive relation R. A *flag* in I is defined to be a finite subset of I of pairwise incident elements. The poset Flag (R) of nonempty flags in I is then a simplicial complex with vertex set I. It is obviously a flag complex. Conversely, any flag complex arises from this construction. (Indeed, given a flag complex L define two vertices to be incident if they are connected by an edge; if R denotes this incidence relation, then $L =$ Flag (R).)

Given a poset, we can symmetrize the partial order relation to get an incidence relation. A flag is then a finite chain of elements in the poset. Given a poset $\mathcal{P}$ let Flag ($\mathcal{P}$) denote the flag complex of chains in $\mathcal{P}$. If $\mathcal{F}$ is the poset of nonempty cells in a regular convex cell complex P, then Flag ($\mathcal{F}$) can be identified the poset of simplices in the barycentric subdivision of P. It follows that the condition of being a flag complex does not restrict the topological type of L: it can be any polyhedron.

Example. A *polygon* is a simplicial complex homeomorphic to S^1. It is a *k-gon* if it has k edges. A k-gon is a flag complex if and only if $k > 3$.

Remarks. 1) Gromov has used the terminology that a simplicial complex L satisfies the "no Δ condition" to mean that it is a flag complex. The idea is that L is a flag complex if and only if it has no "missing simplices".

2) A flag complex is a simplicial complex which is, in a certain sense, determined by its 1-skeleton. Indeed, suppose Λ is a 1-dimensional simplicial complex. Then Λ determines an incidence relation on its vertex set. The associated flag complex L is constructed by filling in the missing simplices corresponding to the complete subgraphs of Λ. Thus, Λ is the 1-skeleton of L.

3) The graph Λ also provides the data for a presentation of a right-angled Coxeter group $W(= W_\Lambda)$. In this case, the flag complex L is called the *nerve* of the Coxeter group.

6. Nonpositive curvature. The notion of "nonpositive curvature" makes sense for a more general class of metric spaces than Riemannian manifolds. A *geodesic* in a metric space X is a path $\gamma : [a, b] \to X$ which is an isometric embedding. X is called a *geodesic space* if any two points can be connected by a geodesic segment. A *triangle* in a geodesic space X is the image of three geodesic segments meeting at their endpoints. Given a triangle T in X, there is a triangle T^* in $\mathbb{R}^2$ with the same edge lengths. T^* is called a *comparison triangle* for T. To each point $x \in T$ there is a corresponding point $x^* \in T^*$. The triangle T is said to *satisfy the $CAT(0)$-inequality*, if given any two points $x, y \in T$ we have $d(x, y) \leq d(x^*, y^*)$. The space X is *nonpositively* curved if the $CAT(0)$-inequality holds for all sufficiently small triangles. X is a $CAT(0)$-*space* (or a *Hadamard space*) if it is complete and if the $CAT(0)$-inequality holds for all triangles in X. It follows immediately from the definitions that there is a unique geodesic between any two points in a $CAT(0)$-space and from this that any $CAT(0)$-space is contractible. Gromov proved that the universal cover of a complete nonpositively curved geodesic space is $CAT(0)$. Hence, any such nonpositively curved space is aspherical.

Next, suppose that P is a connected cubical cell complex. There is a natural piecewise Euclidean metric on P. Roughly speaking, it is defined by declaring each cell of P to be (locally) isometric to a standard Euclidean cube (of edge length 2). More precisely, the distance between two points $x, y \in P$ is defined to be the infimum of the lengths of all piecewise linear curves connecting x to y. It then can be shown that P is a complete geodesic space. Gromov proved the following.

Gromov's Lemma. *A cubical cell complex P is nonpositively curved if and only if the link of each of its vertices is a flag complex.*

A corollary is that the cubical complex $\tilde{P}_L$ constructed previously is $CAT(0)$ if and only if L is a flag complex. In particular, this gives a proof (in one direction) of the theorem from the previous section: if L is a flag complex, then $\tilde{P}_L$ is contractible.

7. Another construction of P_L and $\tilde{P}_L$. The group J $(= (\mathbb{Z}/2)^m)$ acts as a group generated by reflection on $[-1,1]^m$. The subspace $[0,1]^m$ is a fundamental domain. (Also, $[0,1]^m$ can be identified with the orbit space of the J-action.) Let e be the vertex $(1, \dots, 1) \in [0,1]^m$ and for each subset σ of I $(= \{1, \dots, m\})$ let $\Box_\sigma^+ = (e + \Box_\sigma) \cap [0,1]^m$.

The corresponding fundamental domain K for the J-action on P_L is given by

$$K = P_L \cap [0,1]^m.$$

Thus,

$$K = \bigcup_{\sigma \in S(L)} \Box_\sigma^+$$

For each $i \in I$ let $[0,1]_i^m$ denote the intersection of the fixed point set of r_i with $[0,1]^m$, i.e., $[0,1]_i^m = [0,1]^m \cap \{x_i = 0\}$. Set $K_i = K \cap [0,1]_i^m$ and call it a *mirror* of K. It is not difficult to see that $\cup K_i$ can be identified with the barycentric subdivision of L so that K_i is the closed star of the vertex i in this barycentric subdivision. Thus, K is homeomorphic to the cone on L.

For each $x \in K$, let $\sigma(x) = \{i \in I | x \in K_i\}$. The space P_L can be constructed by pasting together copies of K, one for each element of J. More precisely, define an equivalence relation $\sim$ on $J \times K$ by $(j, x) \sim (j', x')$ if and only if $x = x'$ and $j^{-1}j' \in J_{\sigma(x)}$. Then

$$P_L \cong (J \times K) / \sim .$$

In other words, $\sim$ is the equivalence relation generated by identifying $j \times K_i$ with $jr_i \times K_i$, for all $i \in I$.

Similarly,

$$\tilde{P}_L = (W_L \times K)/\sim$$

where the equivalence relation on $\tilde{P}_L$ is defined in an analogous fashion.

Let $\hat{L}$ denote the flag complex determined by the 1-skeleton of L. Thus, $\hat{L}$ is the nerve of $W_L : \sigma \in \mathcal{S}(\hat{L})$ if and only if W_σ is finite. For each $\sigma \in \mathcal{S}(\hat{L})$, set

$$K^\sigma = \bigcup_{i \in \sigma} K_i$$

We are now in position to prove two lemmas which imply the theorem in Section 5.

Lemma 7.1. *The following conditions are equivalent.*

(i) L is a flag complex

(ii) For each $\sigma \in \mathcal{S}(\hat{L}), K^\sigma$ is contractible.

(iii) For each $\sigma \in \mathcal{S}(\hat{L}), K^\sigma$ is acyclic.

Proof. If L is a flag complex, then $L = \hat{L}$ and K^σ can be identified with a closed regular neighborhood of σ in the barycentric subdivision of L. Hence, (i)$\Rightarrow$ (ii). The implication (ii)$\Rightarrow$(iii) is obvious. If $\sigma \in \mathcal{S}(\hat{L})$ is a k-simplex such that $\partial\sigma \subset L$ but $\sigma \not\subset L$, then a computation shows that K^σ has the homology of a $(k-1)$-sphere. Hence, (iii)$\Rightarrow$(i).

For each $w \in W$, let $\sigma(w) = \{i \in I | \ell(ws_i) < \ell(w)\}$. Here $\ell : W \to \mathbb{N}$ denotes word length. Thus, $\sigma(w)$ is the set of letters with which a minimal word for w can end. Geometrically, it indexes the set of mirrors of wK such that the adjacent chamber ws_iK is one chamber closer to the base chamber K. The following is a basic fact about Coxeter groups (the proof of which we will omit).

Lemma 7.2. *For each $w \in W_L, \sigma(w) \in \mathcal{S}(\hat{L})$.*

Now we can sketch the proof that $\tilde{P}_L$ is contractible if and only if L is a flag complex.

Proof. We can compute the homology of $\tilde{P}_L$. Order the elements of $W_L : w_1, w_2, \ldots,$ so that $\ell(w_{k+1}) \geq \ell(w_k)$. Set $Y_k = w_1K \cup \cdots \cup w_kK$. Then $w_kK \cap Y_{k-1} \cong K^{\sigma(w_k)}$. Hence, $H_*(Y_k, Y_{k-1}) \cong H_*(K, K^{\sigma(w_k)}) \cong \tilde{H}_{*-1}(K^{\sigma(w_k)})$, since K is contractible. The exact sequence of the pair (Y_k, Y_{k-1}) gives

$$\to H_*(Y_{k-1}) \to H_*(Y_k) \to H_*(K, K^{\sigma(w_k)})$$

The map $H_*(Y_k) \to H_*(K, K^{\sigma(w_k)})$ is a split surjection. Indeed, a splitting $\varphi_* : H_*(K, K^{\sigma(w_k)}) \to H_*(Y_k)$ can be defined by the formula:

$$\varphi(\alpha) = \sum_{u \in W_{\sigma(w_k)}} (-1)^{\ell(u)} w_k u \alpha$$

where α is a relative cycle in $C_*(K, K^{\sigma(w_k)})$. It is then easy to see that $\varphi(\alpha)$ is a cycle and that the induced map on homology is a splitting. Thus, $H_*(Y_k) \cong H_*(Y_{k-1}) \oplus H_*(K, K^{\sigma(w_k)})$, and consequently

$$\tilde{H}_*(\tilde{P}_L) \cong \bigoplus_{i=1}^{\infty} \tilde{H}_*(K, K^{\sigma(w_k)})$$

Thus, $\tilde{P}_L$ is acyclic if and only if each $K^{\sigma(w_k)}$ is acyclic. The theorem follows from Lemmas 1 and 2.

Henceforth we shall always assume that L is a flag complex. Moreover, we shall use the notation Σ_L instead of $\tilde{P}_L$.

8. The reflection group trick. Next we modify the construction of the previous section.

Suppose X is a space and that ∂X is a subspace which is homeomorphic to a polyhedron. Let L be a triangulation of ∂X as a flag complex with vertex set I. For each $i \in I$, put $X_i = K_i$ where K_i is the previously defined subcomplex of the barycentric subdivision of L. In other words, X_i is the closed star of the vertex i in the barycentric subdivision of L $(= \partial X)$. Let J, W_L and Γ_L be the groups defined previously. Set

$$P_L(X) = (J \times X)/\sim \text{ and}$$
$$\Sigma_L(X) = (W_L \times X)/\sim .$$

The equivalence relations are defined almost exactly as before.

Next we record a few elementary properties of this construction. The orbit space of the J-action on $P_L(X)$ is X. Let $r : P_L(X) \to X$ be the orbit map. Since X can also be regarded as a subspace of $P_L(X)$ (namely as the image of $1 \times X$), we have the following theorem.

Theorem 8.1. *The map $r : P_L(X) \to X$ is a retraction.*

Corollary 8.2. *$\pi_1(X)$ is a retract of $\pi_1(P_L(X))$.*

Theorem 8.3. *If X is a compact n-dimensional manifold with boundary $(= \partial X)$, then $P_L(X)$ is a closed n-manifold.*

Proof. For each $x \in X$, let $\sigma(x) = \{i \in I | x \in X_I\}$. A neighborhood of x in ∂X has the form $\mathbb{R}^{n-k} \times \mathbb{R}_+^{\sigma(x)}$ where $k = \text{Card}(\sigma(x))$, where $\mathbb{R}_+^{\sigma(x)}$ denotes the positive quadrant where all coordinates are nonnegative. It follows that a neighborhood of $(1, x)$ in $P_L(X)$ has the form $\mathbb{R}^{n-k} \times (J_{\sigma(x)} \times \mathbb{R}_+^{\sigma(x)})/\sim$ which is homeomorphic to $\mathbb{R}^n$. Thus, $P_L(X)$ is an n-manifold.

We also have that $\Sigma_L(X) \to P_L(X)$ is a regular covering with group of deck transformation Γ_L. The proof of the next proposition is the same as the proof of Theorem 5.1 given in the previous section.

Proposition 8.4. *(i) If X is simply connected, then $\Sigma_L(X)$ is the universal cover of $P_L(X)$ and hence, $\pi_1(P_L(X)) = \Gamma_L$.*

(ii) If X is contractible, then $\Sigma_L(X)$ is contractible.

9. Aspherical manifolds not covered by Euclidean space. Suppose Y is a reasonable space (for example, suppose Y is a locally compact, locally path connected, second countable Hausdorff space). Also, suppose Y is not compact. A *neighborhood* of *infinity* in Y is the complement of a compact set. Y is *one-ended* if every neighborhood of infinity contains a connected neighborhood of infinity. A one-ended space Y is *simply connected at infinity* if for any compact subset $C \subset Y$ and any loop γ in $Y - C$ there is a larger compact subset $C' \supset C$ such that γ is null-homotopic in $Y - C'$.

For example, $\mathbb{R}^n$ is one-ended for $n \geq 2$ and simply connected at infinity for $n \geq 3$. The following characterization of Euclidean space was proved by Stallings for $n \geq 5$ and by Freedman for $n = 4$. For $n = 3$ the corresponding result is not known.

Theorem. *(Stallings, Freedman) Let M^n be a contractible n-manifold, $n \geq 4$. Then M^n is homeomorphic to $\mathbb{R}^n$ if and only if it is simply connected at infinity.*

In certain circumstances it is possible to define a "fundamental group at infinity" for a one-ended space Y. Suppose that $C_1 \subset C_2 \subset \ldots$ is an exhaustive sequence of compact subsets (i.e., $Y = \cup C_i$). This gives an inverse system of fundamental groups, $\pi_1(Y - C_1) \leftarrow \pi_1, (Y - C_2) \leftarrow \ldots$. An inverse sequence of groups,

$$G_1 \leftarrow G_2 \leftarrow \ldots,$$

is *Mittag-Leffler* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that the image of G_k in G_n is the same for all $k \geq f(n)$. The space Y is *semistable* if there is such an inverse system of fundamental groups which is Mittag-Leffler. If this condition holds for one such system, then it holds for all and the resulting inverse limit is independent of the choice of base points. Hence, if Y is semistable we define its *fundamental group at infinity*, denoted by $\pi_1^\infty(Y)$, to be the inverse limit, $\varprojlim \pi_1(Y - C_k)$.

Definition. A closed manifold N^n is a *homology n-sphere* if $H_*(N^n) \cong H_*(S^n)$

For $n \geq 3$ there are many examples of homology spheres N^n which are not simply connected (however, $\pi_1(N^n)$ must be a perfect group).

The next result is an easy consequence of surgery theory for $n \geq 5$. It was proved by Freedman for $n = 4$.

Theorem. *Let N^{n-1} be a homology $(n-1)$-sphere. Then there is a compact contractible n-manifold with boundary X such that $\partial X = N^{n-1}$.*

Now suppose that a flag complex L is a homology $(n-1)$-sphere and that X^n is a contractible n-manifold with $\partial X = L$. By Theorem 8.3 and Proposition 8.4, $P_L(X)$ is an aspherical n-manifold. Its universal cover $\Sigma_L(X)$ is a contractible n-manifold.

Proposition 9.1. *Suppose as above that L is a homology $(n-1)$-sphere cobounding a contractible manifold X. If L is not simply connected, then $\Sigma_L(X)$ is not simply connected at infinity.*

Proof. As in the proof at the end of Section 7, order the elements of W_L: $w_1, w_2, \dots$ so that $\ell(w_{k+1}) \geq \ell(w_k)$ and put $Y_k = w_1 X \cup \cdots \cup w_k X$. Then Y_k is a contractible manifold with boundary. Let $\overline{Y}_k$ denote the complement of a open collared neighborhood of its boundary. Then it is easy to see that $\Sigma_L(X) - \overline{Y}_k$ is homotopy equivalent to ∂Y_k. Moreover, since $w_k X \cap Y_{k-1} \cong K^{\sigma(w_k)}$, which is an $(n-1)$-disk, it follows that ∂Y_k is the connected sum of k copies of ∂X $(= L)$. Hence, if $n \geq 3$, π_1 (∂Y_k) is the free product of k copies of $\pi_1(L)$ and the inverse system of fundamental groups is,

$$\pi_1(L) \leftarrow \pi_1(L) * \pi_1(L) \leftarrow \dots$$

Thus, $\pi_1^\infty(\Sigma_L(X))$ is not trivial.

As a corollary of the construction we have proved the following

Theorem 9.2. *For each $n \geq 4$, there is a closed aspherical n-manifold (of the form $P_L(X)$) such that its universal cover is not homeomorphic to $\mathbb{R}^n$.*

10. The reflection group trick, continued. As before, $(X, \partial X)$ is a pair of spaces and L is a triangulation of ∂X as a flag complex.

Theorem 10.1. *If X is aspherical, then so is $P_L(X)$.*

Proof. It suffices to show that the covering space $\Sigma_L(X)$ is aspherical. Order the elements of W_L as before and set $Y_k = w_1 X \cup \cdots \cup w_k X$. Thus, Y_k is formed by gluing on a copy of X to Y_{k-1}. Since $w_k X \cap Y_{k-1}$ is contractible. Y_k is homotopy equivalent to $Y_{k-1} \vee X$ and hence, to $X \vee \cdots \vee X$ (k times). Since X is aspherical, so is Y_k. Since $\Sigma_L(X)$ is the increasing union of the Y_k, $\Sigma_L(X)$ is also aspherical.

Definition. A group π is of *type F* if its classifying space $B\pi$ has the homotopy type of a finite complex. ($B\pi$ is also called the "$K(\pi, 1)$-complex.")

If B is homotopy equivalent to a finite CW complex, then we can "thicken" it to a compact manifold with boundary. This means that we can find a compact manifold with boundary X which is homotopy equivalent to B. The proof goes as follows. First, up to homotopy, we can assume that B is a finite simplicial complex.

The next step is to piecewise linearly embed B in some Euclidean space $\mathbb{R}^n$. So, we can assume B is a subcomplex of same triangulation of $\mathbb{R}^n$. Finally, possibly after taking barycentric subdivisions, we can replace B by its regular neighborhood in $\mathbb{R}^n$. This is X.

The "reflection group trick" can then be summarized as follows. Start with a group π of type F. After thickening we may assume that $B\pi$ is a compact manifold with boundary X. Triangulate ∂X as a flag complex L. Then $P_L(X)$ is a closed aspherical manifold which retracts onto $B\pi$.

11. The Isomorphism Conjecture.

The Borel Conjecture. *Suppose $(M, \partial M)$ and $(M', \partial M')$ are aspherical manifolds with boundary and $f : (M, \partial M) \to (M', \partial M')$ is a homotopy equivalence such that $f|_{\partial M}$ is a homeomorphism. Then f is homotopic rel ∂M to a homeomorphism.*

In the case where $\partial M = \emptyset$, the Borel Conjecture implies that two closed aspherical manifolds with the same fundamental group are homeomorphic.

Suppose $(X, \partial X)$ is a pair of finite complexes with $\pi_1(X) = \pi$. Then $(X, \partial X)$ is a *Poincaré pair* of dimension n if there is a $\mathbb{Z}\pi$-module D which is isomorphic to $\mathbb{Z}$ as an abelian group and a homology class $\mu \in H_n(X, \partial X; D)$ so that for any $\mathbb{Z}\pi$-module A, cap product with μ defines an isomorphism: $H^i(X; A) \cong H_{n-i}(X, \partial X; D \otimes A)$. If $\partial X = \emptyset$, then X is a *Poincaré complex*. A group π of type F is a *Poincaré duality group of dimension n* (or a PD^n-group) if $B\pi$ is a Poincaré complex.

The PD^n-group Conjecture. . *Suppose π is a group of type F and that $(X, \partial X)$ is a Poincaré pair with X homotopy equivalent to $B\pi$ and with ∂X a manifold. Then $(X, \partial X)$ is homotopy equivalent rel ∂X to a compact manifold with boundary.*

A weak version of this conjecture replaces the word manifold by "ANR homology manifold."

In the absolute case, where $\partial X = \emptyset$, this conjecture asserts that for any PD^n-group π of type F, $B\pi$ is homotopy equivalent to a closed manifold.

As others will explain in their lectures, there is an important space in surgery theory denoted by G/TOP. Its homotopy groups are 4-periodic and are given by the formula:

$$\pi_i(G/TOP) = \begin{cases} \mathbb{Z}, & i \equiv 0(4) \\ \mathbb{Z}/2, & i \equiv 2(4) \\ 0, & \text{otherwise} \end{cases}$$

Moreover, the 4-fold loop space, $\Omega_4(\mathbb{Z} \times G/TOP)$, is homotopy equivalent to $\mathbb{Z} \times G/TOP$. It follows that $\mathbb{Z} \times G/TOP$ defines a spectrum $\mathbb{L}$ and a generalized homology theory $H_*(X; \mathbb{L})$. It is almost, but not quite, true that

$$H_n(X; \mathbb{L}) \cong \bigoplus_k H_{n-4k}(X; \mathbb{Z}) \oplus H_{n-4k-2}(X; \mathbb{Z}/2).$$

Quinn has defined an "assembly map" $A_n : H_n(X; \mathbb{L}) \to L_n(\mathbb{Z}\pi)$ where $\pi = \pi_1(X)$ and where $L_n(\mathbb{Z}\pi)$ denotes Wall's surgery group for $\mathbb{Z}\pi$. These groups are also known to be 4-periodic.

The Isomorphism Conjecture. *Suppose π is a group of type F. Then the assembly map $A_* : H_*(B\pi; \mathbb{L}) \to L_*(\mathbb{Z}\pi)$ is an isomorphism.*

Conceivably, the Isomorphism Conjecture could be true for any torsion-free group π.

In dimension ≥ 5 it is known that for a given group π the truth of the Isomorphism Conjecture is equivalent to the truth of both the Borel Conjecture and (the weak version of) the PD^n-group Conjecture.

Theorem 11.1. *The Isomorphism Conjecture is true for the fundamental groups of all closed aspherical manifolds if and only if it is true for all groups of type F.*

Proof. We use the reflection group trick. Suppose π is a group of type F. Thicken $B\pi$ to a manifold with boundary X and triangulate ∂X as a flag complex L. Then $P_L(X)$ is a closed aspherical manifold. Let $r : P_L(X) \to X$ be the retraction from Theorem 8.1 and let $G = \pi_1(P_L(X))$. We have the following commutative diagram:

$$\begin{array}{ccc} H_n(P_L(X); \mathbb{L}) & \to & L_n(\mathbb{Z}G) \\ i \uparrow\downarrow r_* & & i_* \uparrow\downarrow r_* \\ H_n(X; \mathbb{L}) & \to & L_N(\mathbb{Z}\pi). \end{array}$$

Hence, if the arrow on the top row is an isomorphism, so is the arrow on the bottom.

Remark. What this argument shows is that, in dimensions ≥ 5, if we have a counterexample to the relative version of the Borel Conjecture or to the relative version of the PD^n-group Conjecture, then the reflection group trick will provide us with a counterexample in the absolute case. For example, suppose $f : (M, \partial M) \to (M', \partial M')$ is a counterexample to the Borel Conjecture. We might as well assume that $\partial M = \partial M' = L$ and that $f|_{\partial M} = \mathrm{id}$. Then f induces a homotopy equivalence $P_L(M) \to P_L(M')$ which is not homotopy equivalent to a homeomorphism. Similarly, if $(X, \partial X)$ is a counterexample to the PD^n-group Conjecture and $L = \partial X$, then $G = \pi_1(P_L(X))$ is a PD^n-group which is not the fundamental group of a closed aspherical manifold.

12. Aspherical manifolds which cannot be smoothed. Suppose $(X, \partial X)$ is a compact aspherical n-manifold with boundary and that ∂X is triangulable. Suppose further that the Spivak normal fibration of X does not lift to a linear vector bundle. (In other words, a certain map $X \to BG$ does not lift to BO.) Apply the reflection group trick with $L = \partial X$. Since X is codimension 0 in $P_L(X)$ and X is a retract of $P_L(X)$, the Spivak normal fibration of $P_L(X)$ cannot lift to linear vector bundle. Hence, $P_L(X)$ is not homotopy equivalent to a smooth manifold. J-C. Hausmann and I showed that there exist examples of such X for each $n \geq 13$, thereby proving the following.

Theorem 12.1. *In each dimension ≥ 13, there is a closed aspherical manifold not homotopy equivalent to a smooth manifold.*

We will prove a stronger result in Section 16.

13. Further applications of the reflection group trick. The next two results were proved by G. Mess.

Theorem 13.1. *(Mess). For each $n \geq 4$, there is a closed aspherical n-manifold whose fundamental group is not residually finite.*

Theorem 13.2. *(Mess). For each $n \geq 4$, there is a closed aspherical n-manifold whose fundamental group contains an infinitely divisible abelian group.*

On the other hand, it is known that there are no such examples in dimension 3.

The proofs of both theorems are similar. By a theorem of R. Lyndon, if π is a finitely generated 1-relator group and if the relation cannot be written as a proper power of another word, then the presentation 2-complex for π is aspherical. In particular, any such π is of type F with a 2-dimensional $B\pi$. This 2-complex can then be thickened to a compact 4-manifold. For Theorem 13.1 take π to be the Baumslag-Solitar group $< a, b | ab^2a^{-1} = b^3 >$. It is known that π is not residually finite; hence, neither is any group which contains it. For Theorem 13.2 take π to be the Baumslag-Solitar group $< a, b | aba^{-1} = b^2 >$. The centralizer of b in this group is isomorphic to a copy of the dyadic rationals.

S. Weinberger has pointed out that some of the original examples of finitely presented groups with unsolvable word problem are groups of type F. Since any group which retracts onto such a group also has unsolvable word problem, the reflection group trick gives the following.

Theorem 13.3. *(Weinberger). There are closed aspherical manifolds the fundamental groups of which have unsolvable word problem.*

14. Hyperbolization. "Hyperbolization" refers to certain constructions, invented by Gromov, for converting any cell complex into an aspherical polyhedron (in fact, into a nonpositively curved polyhedron). One of the key features of such constructions is that they preserve local structure. Thus, hyperbolization will convert an n-manifold into an aspherical n-manifold. In this section we describe one such construction, Gromov's "Möbius band hyperbolization procedure," which given a cubical cell complex P produces an aspherical cubical cell complex $h(P)$. Before giving the definition, we discuss some properties of the construction: 1) The construction is functorial in the following sense: if $f : P \to Q$ is an embedding onto a subcomplex, then there is an induced embedding $h(f) : h(P) \to h(Q)$ onto a subcomplex.

2) The cubical complexes P and $h(P)$ have the same vertex set. Moreover, for each vertex v, $Lk(v, h(P))$ is the barycentric subdivision of $Lk(v, P)$.

In particular, it follows that if P is an n-manifold, then so is $h(P)$. Property 2) also shows that the link of each vertex in $h(P)$ is a flag complex. Hence, by Gromov's Lemma (stated in Section 6), the piecewise Euclidean metric on $h(P)$ is nonpositively curved. Consequently, we have the following property.

3) $h(P)$ is aspherical.

By functoriality, each cube $\Box$ in P is converted into a subspace $h(\Box)$ of $h(P)$ (called a "hyperbolized cell"). There is a map $c : h(P) \to P$, unique up to homotopy, which is the identity on the vertex set and which takes each hyperbolized cell to the corresponding cell of P. The map c has the following two properties.

4) c induces a surjection on homology groups with coefficients in $\mathbb{Z}/2$.

5) $c_* : \pi_1(h(P)) \to \pi_1(P)$ is surjective.

We also list one final property.

6) If P is an n-manifold, then there is an (unoriented) cobordism between P and $h(P)$.

A corollary of 6) is the following.

Theorem 14.1. *(Gromov). Every triangulable manifold is cobordant to an aspherical manifold.*

The definition of $h(P)$ is by induction on dim$P$. If dim$P \leq 1$, then $h(P) = P$. Suppose that the construction has been defined for all cubical complexes of dimension $< n$ and that properties 1) and 2) hold. Let $\Box^n$ denote the standard n-cube and let $a : \Box^n \to \Box^n$ denote the antipodal map (a is also called the "central symmetry" of $\Box^n$). By induction, $h(\partial\Box^n)$ has been defined and by 1) the isomorphism a induces an involution $h(a) : h(\partial\Box^n) \to h(\partial\Box^n)$. The quotient space $h(\partial\Box^n)/(\mathbb{Z}/2)$ is also a cubical complex. We define $h(\Box^n)$ to be the canonical interval bundle over $h(\partial\Box^n)/(\mathbb{Z}/2)$, i.e.,

$$h(\Box^n) = [\text{-}1,1] \times_{\mathbb{Z}/2} h(\partial\Box^n)$$

where $\mathbb{Z}/2$ acts on the first factor via $t \to -t$ and on the second via $h(a)$.

Since the restriction of this interval bundle to each cell of $h(\partial\Box^n)/\mathbb{Z}/2)$ is a trivial bundle, $h(\Box^n)$ naturally has the structure of a cubical complex: each new cell is the product of [-1,1] with a cell of $h(\partial\Box^n)/(\mathbb{Z}/2)$. If follows that if v is a vertex of $\Box^n$, then there is a new $(k+1)$-simplex in $Lk(v, h(\Box^n))$ for each k-simplex in $Lk(v, h(\partial\Box^n))$. Thus, $Lk(v, h(\Box^n))$ is the cone on $Lk(v, h(\partial\Box^n))$. Using induction, this implies that $Lk(v, h(\Box^n))$ is the barycentric subdivision of Δ^{n-1}.

We note that the boundary of $h(\Box^n)$ is canonically identified with $h(\partial\Box^n)$. (The identification is canonical because a lies in the center of the automorphism group of the cube.)

Let $P^{(k)}$ denote the k-skeleton of a cubical complex P. If $\dim P = n$, then $h(P)$ is defined by attaching, for each n-cell $\Box^n$ in P, a copy of $h(\Box^n)$ to the subcomplex $h(\partial\Box^n)$ of $h(P^{(n-1)})$ via the canonical identification.

Given an embedding $f : P \to Q$ with restriction $f^{(n-1)} : P^{(n-1)} \to Q^{(n-1)}$, the map $h(f) : h(P) \to h(Q)$ is induced by the map on each new cell which is the product of the identity map of [-1,1] with $h(f^{(n-1)})$. It is clear that the new links are barycentric subdivisions of the old ones. The proof of properties 3) and 4) are straightforward and are left to the reader.

To check property 5), let $\tilde{P}$ denote the universal cover of P and let $\pi = \pi_1(P)$. By functorality, π acts freely on $h(\tilde{P})$ and $h(\tilde{P}) \to h(\tilde{P})/\pi$ is a covering projection. In fact, it is clear that $h(\tilde{P})/\pi \cong h(P)$. This defines an epimorphism $\varphi : \pi_1(h(P)) \to \pi$. In fact, it is not hard to see that φ is just the homomorphism induced by the canonical map $c : h(P) \to P$.

To check 6), suppose that M is triangulated manifold. Let CM denote the cone on M, i.e., $CM = (M \times [0,1])/\sim$ where $(x,0) \sim (x',0)$ for all $x, x' \in M$. Let c denote the cone point. There is a standard method for subdividing each simplex of CM into cubes. This gives CM the structure of a cubical complex. The link of c in $h(CM)$ is isomorphic to the barycentric subdivision of M. Hence, removing a small regular neighborhood of c from $h(CM)$ we obtain a cobordism between M and $h(M)(= h(M \times 1))$.

Remark. There is a slight variation of the above which provides a sort of "relative hyperbolization procedure." Suppose Q is a subcomplex of P. Let $p : \tilde{P} \to P$ be the universal cover. Let $\hat{P}$ be the cubical complex formed by attaching a cone to each component of $p^{-1}(Q)$ in $\tilde{P}$. Let $h(\hat{P})_0$ denote the complement of small regular neighborhoods of the cone points in $h(\hat{P})$. The fundamental group $\pi = \pi_1(P)$ acts on $h(\hat{P})_0$, so we define

$$h(P,Q) = h(\hat{P})_0/\pi,$$

In general, $h(P,Q)$ will not be aspherical. However, if Q is aspherical and if $\pi_1(Q) \to \pi_1(P)$ is injective, then the link of each cone point in $h(\hat{P})$ is contractible (since it is a copy of the universal cover of Q). Hence, $h(\hat{P})$ and $h(\hat{P})_0$ will be homotopy equivalent. So, in the above situation, $h(P,Q)$ is aspherical.

Not only did Gromov state Theorem 14.1, he also asserted that if two closed aspherical manifolds M_1 and M_2 were cobordant, then one could choose the cobordism to be an aspherical manifold with boundary. Although I don't know how to prove Gromov's assertion, the construction above does prove the following.

Theorem 14.2. *Suppose that there is a triangulable cobordism N^{n+1} between two closed aspherical manifolds M_1^n and M_2^n such that for $i = 1, 2$, $\pi_1(M_i^n) \to \pi_1(N^{n+1})$ is injective. Then there is an aspherical cobordism between M_1^n and M_2^n with the same property.*

Remark. Let $M^n = h(\partial \square^{n+1})/(\mathbb{Z}/2))$. The manifolds M^n occur in nature. Indeed, consider real projective space $\mathbb{R}P^n$ and the collection of all its coordinate hyperplanes, defined by the equations $x_i = 0$, for $1 \leq i \leq n+1$. Then M^n is the manifold resulting from blowing up (in the sense of algebraic geometry) all projective subspaces which are intersections of such coordinate hyperplanes. It follows from this that M^n can also be described as the "closure of a generic torus orbit on a real flag manifold." More precisely, the flag manifold is the homogeneous space $SL(n+1; \mathbb{R})/B$ where B denotes the Borel subgroup of upper triangular matrices. The real "torus" H is the subgroup of diagonal matrices in $SL(n+1, \mathbb{R})$. Thus, $H \cong (\mathbb{R}^*)^n$. An H-orbit on the homogeneous space is "generic" if it is the orbit of a flag which is in general position with respect to the coordinate hyperplanes in $\mathbb{R}^{n+1}$ (in other words, it is generic if its stabilizer in H is trivial). It can then be shown that the closure of a generic H-orbit is homeomorphic to M^n.

15. An orientable hyperbolization procedure. The trouble with the Möbius band procedure is that the hyperbolized cells are not orientable. Gromov gave a second construction which remedied this. We explain it below.

The rough idea behind any hyperbolization procedure is this. We first give some functorial procedure for hyperbolizing cells. Then, given a cell complex Λ, we define its hyperbolization by gluing together hyperbolized cells in the same combinatorial pattern as the cells of Λ.

Gromov's second procedure can be applied to any finite dimensional simplicial complex Λ. The result will be an aspherical (in fact, nonpositively curved) cubical cell complex $h(\Lambda)$. The construction will have analogous properties to properties 1) through 6) in the previous section. In addition, it will have the following two properties.

7) the natural map $c : h(\Lambda) \to \Lambda$ induces on surjection on integral homology groups.

8) If Λ is a manifold, then c pulls back the stable tangent bundle of Λ to the stable tangent bundle of $h(\Lambda)$.

The definition of the construction will again be by induction on dimension. In order to define the hyperbolization of an n-dimensional simplicial complex Λ, we first need to define the hyperbolization of an n-simplex, $h(\Delta^n)$. To complete the definition of $h(\Lambda)$ we need to have some fixed identification of each n-simplex in Λ with the standard n-simplex. Thus, we need to assume that Λ admits a "folding map" $p : \Lambda \to \Delta^n$, that is, a simplicial map p which restricts to an injection on each simplex. If we replace Λ with its barycentric subdivision Λ' then it always admits such a folding map. Once we have such a p, $h(\Lambda)$ is defined to be the fiber product of $p : \Lambda \to \Delta^n$ and $c : h(\Delta^n) \to \Delta^n$. In other words, $h(\Lambda) = \{(x, y) \in \Lambda \times h(\Delta^n) | p(x) = c(y)\}$.

Each transposition of two vertices of Δ^n induces a reflection on Δ^n. In order to make such a reflection into a simplicial isomorphism it is necessary to pass to the barycentric subdivision $(\Delta^n)'$.

If $\dim\Lambda \leq 1$, then, by definition, $h(\Lambda) = \Lambda$. Suppose that $h(\Lambda)$ has been defined for simplicial complexes of dimension $< n$. Choose a reflection $r : (\partial\Delta^n)' \to (\partial\Delta^n)'$. By functorality, we have an induced involution $h(r) : h((\partial\Delta^n)') \to h((\partial\Delta^n)')$. The involution $h(r)$ acts as a reflection on the $(n-1)$-manifold $h((\partial\Delta^n)')$ and its fixed point set separates $h((\partial\Delta^n)')$ into two "half-spaces" which we denote by H_+ and H_-. Then $h(\Delta^n)$ is defined to be

$$(h((\partial\Delta)') \times [-1,1])/\sim$$

where the equivalence relation $\sim$ identifies $H_- \times \{-1\}$ with $H_- \times \{+1\}$. The boundary of $h(\Delta^n)$ consists of two copies of H_+ glued together, i.e., it can be identified with $h((\partial\Delta^n)')$. Another way to describe this procedure is to form the manifold $h((\partial\Delta^n)') \times S^1$ and then cut it open along $H_+ \times \{1\}$. It is clear that $h(\Delta^n)$ is an orientable manifold with boundary (assuming that $h((\partial\Delta^n)')$ is orientable). Assuming that $h((\partial\Delta^n)')$ is a cubical cell complex, we see that $h(\Delta^n)$ inherits the structure of a cubical cell complex (possibly after subdividing the [-1,1] factor). Moreover, the link of a vertex v in $h(\Delta^n)$ is the cone on $Lk(v, h((\partial\Delta^n)'))$. It follows that the link of any vertex in $h(\Delta^n)$ is a flag complex and hence, that $h(\Delta^n)$ is nonpositively curved. Finally, as indicated above, the hyperbolization of an arbitrary n-dimensional simplicial complex is defined by the fiber product construction.

Using the fact that $H_n(h(\Delta^n), \partial h(\Delta^n)) \cong \mathbb{Z}$, it is not hard to verify property 7). Property 8) follows from the observation that when Λ is a manifold, $h(\Lambda)$ is defined as a submanifold of $\Lambda \times \Delta^n$ with trivial normal bundle.

Remark. One consequence of 8) is that $c : h(\Lambda) \to \Lambda$ pulls back the stable characteristic classes of Λ (e.g., its Pontriagin classes and Stiefel-Whitney classes) to those of $h(\Lambda)$. Thus, the characteristic numbers $h(\Lambda)$ are the same as those of Λ. This shows that the condition of being aspherical does not impose any restrictions on the characteristic numbers of a manifold.

16. A nontriangulable aspherical 4-manifold. The E_8-form is a certain positive definite, unimodular, even symmetric bilinear form on $\mathbb{Z}^8$. (So, its signature is 8). The plumbing construction gives a smooth, simply connected 4-manifold with boundary, N^4, with intersection form the E_8-form. The boundary of N^4 is a homology 3-sphere; moreover, it is not simply connected. (In fact, ∂N^4 is Poincaré's homology 3-sphere). Let Λ^4 be a simplicial complex formed by triangulating N^4 and then attaching the cone on ∂N^4. Of course, Λ^4 is not a 4-manifold since there is no Euclidean neighborhood of the cone point. On the other hand, it is a polyhedral homology manifold in the sense that the link of each vertex is a homology sphere. It is called the E_8 *homology* 4*-manifold*. Stiefel-Whitney classes make sense for homology manifolds and it follows from the fact that N^4 is stably parallelizable that the Stiefel-Whitney classes of Λ^4 all vanish. Its signature is 8.

A famous theorem of Rohlin asserts that for any smooth or PL closed 4-manifold M^4 with $w_1(M^4) = 0 = w_2(M^4)$, the signature of its intersection form must be divisible by 16. It follows that Λ^4 is not homotopy equivalent to a smooth or PL 4-manifold. On the other hand, by Freedman's result (stated in Section 9), there is a contractible manifold N' with $\partial N' = \partial N$. Hence, Λ^4 is homotopy equivalent to the topological manifold $M^4 = N \cup N'$. (We have replaced the cone on a homology 3-sphere by the contractible manifold N'.) By Rohlin's Theorem M^4 cannot be homotopy equivalent to a PL 4-manifold.

After Freedman's result was proved, it still seemed possible that Λ^4 could be homotopy equivalent to triangulated manifold M^4. However, it follows from Casson's work on the Casson invariant that this is also not the case. Indeed, the link of any vertex in a triangulation of M^4 must be a homotopy 3-sphere (and at least one such link must be a fake 3-sphere since the triangulation cannot be PL). One can then arrange that the connected sum of all fake 3-spheres which arise as links bounds a PL submanifold of M^4 of signature 8. This implies that the Casson invariant of such a fake 3-sphere must be an odd integer. However, since the Casson invariant of a homology sphere depends only on its fundamental group, this integer is 0. This contradiction shows that M^4 is not triangulable.

The hyperbolization technique of the previous section allows us to promote this result to aspherical 4-manifolds. Consider $h(\Lambda^4)$, the result of applying Gromov's oriented hyperbolization procedure to Λ^4. It is a polyhedral homology manifold with only one non-manifold point (namely, the hyperbolization of the cone point). By property 8) of the previous section, its Stiefel-Whitney classes vanish. Since the complement of a regular neighborhood of the cone point is oriented cobordant rel ∂N to N^4, it follows that the signature of $h(\Lambda^4)$ is 8. Now let M^4 denote the result of replacing the regular neighborhood of the cone point in $h(\Lambda^4)$ by the contractible manifold N'. Since M^4 is homotopy equivalent to $h(\Lambda^4)$, it is aspherical. The argument of the previous paragraph now proves the following.

Theorem 16.1. *There is an aspherical 4-manifold M^4 which is not homotopy equivalent to any triangulable 4-manifold.*

In particular, M^4 is not homotopy equivalent to a PL manifold. Standard arguments show that this property is preserved when we take the produce with a k-torus; hence, we have proved the following.

Theorem 16.2. *For each $n \geq 4$, there is a closed aspherical n-manifold which is not homotopy equivalent to a PL manifold.*

17. Relative hyperbolization. As we have been learning in these lectures, Farrell and Jones have proved the Isomorphism Conjecture (cf. Section 10) for the fundamental group of any nonpositively curved, closed Riemannian manifold. It seems likely (or at least plausible) that the Farrell-Jones program can be adapted to prove the Isomorphism Conjecture for the fundamental group of any closed PL manifold equipped with a nonpositively curved, piecewise Euclidean metric (cf. Section 6). In this section we describe a variant of the reflection group trick (it is also a variant of hyperbolization) which can be used to prove the following.

Theorem 17.1. *Suppose that the Isomorphism Conjecture is true for the fundamental group of any closed PL manifold with a nonpositively curved, piecewise Euclidean metric. Then it is also true for the fundamental group of any finite polyhedron with a nonpositively curved, piecewise Euclidean metric.*

Remark. In the case of the Whitehead group, this program was carried out about ten years ago by B. Hu. He first proved the vanishing the Whitehead group of the fundamental group of any closed PL manifold with a nonpositively curved polyhedral metric. He then used a construction similar to the one described below to derive the same result for the fundamental group of any nonpositively curved polyhedron.

Suppose B is a cell complex equipped with a piecewise Euclidean metric. Subdividing if necessary, we may assume that B is a simplicial complex. Suppose further that B is a subcomplex of another simplicial complex X. Possibly after another subdivision, we may assume that B is a full subcomplex of X. This means that if a simplex σ of X has nonempty intersection with B, then the intersection is a simplex of B (and a face of σ). We note that while each simplex of B is given a Euclidean metric, no metric is assumed on the simplices of X which are not in B. In practice X will always be a PL manifold.

We will define a new cell complex $D(X, B)$ equipped with a polyhedral metric. We will also define a covering space $\tilde{D}(X, B)$. Each cell of $D(X, B)$ (or of $\tilde{D}(X, B)$) will have the form $\alpha\times$ $[-1,1]^k$, for some integer $k \geq 0$, where α is a simplex of B and where $\alpha\times[-1,1]^k$ is equipped with the product metric. (Usually we will use α for a simplex of B and σ for a simplex in X which is not in B.) Here are some properties of this construction.

1) For $n = \dim X$, there will be 2^n disjoint copies of B in $D(X, B)$.

2) For each such copy and for each vertex v in B, the link of v in $D(X, B)$ will be isomorphic to a subdivision of $Lk(v, X)$. In particular, if X is a manifold then $D(X, B)$ will be a manifold.

3) If the metric on B is nonpositively curved, then the metric on $D(X, B)$ will be nonpositively curved and each copy of B will be a totally geodesic subspace of $D(X, B)$.

4) The group $(\mathbb{Z}/2)^n$ will act as a reflection group on $D(X, B)$. A fundamental chamber for this action will be denoted by $K(X, B)$. It will be homeomorphic to a regular neighborhood of B in X. Thus, $K(X, B)$ will be a retract of $D(X, B)$ and B will be a deformation retract of $K(X, B)$.

In fact, the entire construction depends only on a regular neighborhood of B in X. More precisely, it depends only on the set of simplices of X which intersect B.

Let $\mathcal{P}$ denote the poset of simplices σ in X such that $\sigma \cap B \neq \emptyset$ and such that σ is not a simplex of B. For each simplex α of B, let $\mathcal{P}_{>\alpha}$ denote the subposet of $\mathcal{P}$ consisting of all σ which have α as a face. Let $\mathcal{F}$ =Flag($\mathcal{P}$) denote the poset of chains in $\mathcal{P}$ (an element of $\mathcal{F}$ is a nonempty, finite, totally ordered subset of $\mathcal{P}$).

Given a chain $f = \{\sigma_0 < \cdots < \sigma_k\} \in \mathcal{F}$, let σ_f denote its least element, i.e., $\sigma_f = \sigma_0$. Given a simplex α of B, let $\mathcal{F}_{>\{\alpha\}}$ denote the set of chains f with $\sigma_f > \alpha$.

We begin by defining the fundamental chamber K $(= K(X, B))$. Each cell of K will have the form $\alpha\times$ $[0,1]^f$, for some $f \in \mathcal{F}_{>\{\alpha\}}$. Thus, the number of interval factors of $\alpha\times[0,1]^f$ is the number of elements of f. If $f \leq f'$, then we identify $[0,1]^f$ with the face of $[0,1]^{f'}$ defined by setting the coordinates $x_\sigma = 1$, for all $\sigma \in f' - f$.

We define an incidence relation on the set of such cells as follows: $\alpha\times[0,1]^f \leq \alpha'\times[0,1]^{f'}$ if and only if $\alpha \leq \alpha'$ and $f \leq f'$. (Notice that if $\alpha \leq \alpha'$, then $\mathcal{F}_{>\{\alpha'\}} \subset \mathcal{F}_{>\{\alpha\}}$.) K is defined to be the cell complex formed from the disjoint union $\coprod \alpha\times[0,1]^f$ by gluing together two such cells whenever they are incident. It is clear that K is homeomorphic to a regular neighborhood of B in X.

Next we define the mirrors of K. For each $\sigma \in \mathcal{P}_{>\alpha}$ and for each chain f with $\sigma_f = \sigma$ define $\delta_\sigma(\alpha\times[0,1]^f)$ to be the face of $\alpha\times[0,1]^f$ defined by setting $x_\sigma = 0$, i.e.,

$$\delta_\sigma(\alpha \times [0,1]^f) = \alpha \times 0 \times [0,1]^{f-\{\sigma\}}.$$

The mirror $\delta_\sigma K$ is the subcomplex of K consisting of all such cells. In other words, $\delta_\sigma K = \alpha \times S_\sigma$, where $\alpha = B \cap \sigma$ and where S_σ denotes the star of the barycenter of σ in the simplicial complex $\mathcal{F}$ (=Flag($\mathcal{P}$)).

Next we apply the reflection group trick. Set $J = (\mathbb{Z}/2)^{\mathcal{P}}$. Define $\tilde{D} = D(X, B)$ by

$$\tilde{D} = (J \times K)/\sim$$

where the equivalence relation $\sim$ is defined as in Section 7.

Remark. Suppose X is the cone on a simplicial complex ∂X and that B is the cone point. Then $\tilde{D}(X, B)$ coincides with the cubical complex P_L defined in Section 3 (where L is the barycentric subdivision of ∂X).

The definition of the space $D = D(X, B)$ is similar to that of $\tilde{D}$ only one uses the smaller group $(\mathbb{Z}/2)^n$, $n = \dim X$, instead of J. If $\{r_1, \ldots, r_n\}$ is the standard set of generators for $(\mathbb{Z}/2)^n$, then we identify the points (gr_i, x) and (g, x) whenever x belongs to a mirror $\delta_\sigma K$ with $i = \dim \sigma$.

There is another definition of $D(X, B)$ which is similar to the definitions of the hyperbolization constructions in the previous two sections. Define $\dim(X, B)$ to be the maximum dimension of any simplex in X which is not in B. We shall define a space $D^{(k)}(X, B)$ for any pair (X, B) with $\dim(X, B) \leq k$. The definition is by induction on $\dim(X, B)$. First of all, $D^{(0)}(X, B)$ is defined to be B. Assume that $D^{(n-1)}$ has been defined and that $\dim(X, B) = n$. Set

$$D^{(n)}(X^{(n-1)} \cup B, B) = D^{(n-1)}(X^{(n-1)} \cup B, B) \times \{-1, 1\}.$$

If σ is an n-simplex such that $\sigma \cap B \neq \emptyset$ and σ is not in B, then define

$$D^{(n)}(\sigma, \sigma \cap B) = D^{(n-1)}(\partial\sigma, \partial\sigma \cap B) \times [-1, 1].$$

We note that the boundary of $D^{(n)}(\sigma, \sigma \cap B)$ is naturally a subcomplex of $D^{(n)}(X^{(n-1)} \cup B, B)$. Hence, we can glue in each hyperbolized simplex $D^{(n)}(\sigma, \sigma \cap B)$ to obtain $D^{(n)}(X, B) = D(X, B)$.

The advantage of this definition is that it makes it easier to prove property 3) (that if B is nonpositively curved, then so is $D(X, B)$.) The proof is based on a Gluing Lemma of Gromov. This lemma asserts that if we glue together two nonpositively curved spaces via an isometry of a common totally geodesic subspace, then the new metric space is nonpositively curved. The inductive hypothesis gives that the spaces $D^{(n-1)}(X^{(n-1)} \cup B, B)$ and $D^{(n-1)}(\partial\sigma, \partial\sigma \cap B)$ are nonpositively curved. Using the Gluing Lemma, we get that $D^{(n)}(X, B)$ is also nonpositively curved.

Remark. The construction of $D(X, B)$ was explained to me about eight years ago by L. Jones. It is a variation of the "cross with interval" hyperbolization procedure which had been described previously by T. Januszkiewicz and me. Relative versions of this were described by B. Hu and by R. Charney and me. In these earlier versions the 1-skeleton of X was not changed. Jones realized that the construction is nicer if, as in this section, we also hyperbolize the 1-simplices.