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SMR1312/5

School on High-Dimensional Manifold Topology

(21 May - 8 June 2001)

Algebraic *K*- and *L*-theory and applications to the topology of manifolds

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ALGEBRAIC K- AND L-THEORY AND APPLICATIONS TO THE TOPOLOGY OF MANIFOLDS

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ABSTRACT. Some lecture notes for the Summer School in High-Dimensional Topology, May 20 - June 8, 2001 at the Abdus Salam International Centre for Theoretical Physics.

CONTENTS

The development of geometric topology has led to the identification of specific algebraic structures of great richness and usefulness. A common theme in this area is the study of algebraic invariants of discrete groups or rings by topological methods. The resulting subject is now called algebraic K-theory.

The purpose of these lecture notes is to survey some of the main constructions and techniques in algebraic K-theory, together with an indication of the topological background and applications. More details about proofs and references will be given in the lectures, as time permits. The material is organized

Date: May 15, 2001.
Partially supported by an NSERC Research Grant.

into some introductory sections, concerning linear and unitary K-theory, followed by descriptions of four important geometric problems and their related algebraic methods. Good general sources for much of the preliminary material are the books of K. Brown [3], Curtis-Reiner [11], [10], Milnor [33], Milnor-Husemoller [34], and Swan-Evans [44]. Some of the material in Section 7 is based on my DMV lecture notes [45].

1. Homology of coverings

Let X be a CW-complex with fundamental group $\pi = \pi_1(X, x_0)$, and denote by $\Lambda := \mathbf{Z}\pi$ the integral group ring of π. The standard involution $\lambda \mapsto \bar{\lambda}$ on Λ is induced by the formula

$$\sum n_g g \mapsto \sum n_g g^{-1}$$

for $n_g \in \mathbf{Z}$ and $g \in \pi$. Notice that this gives an anti-automorphism of the group ring since $\overline{uv} = \bar{v}\bar{u}$ for all u, $v \in \Lambda$.

If $\widetilde{X}$ denotes the universal covering of X, then π acts cellularly on $\widetilde{X}$ by deck transformations, and the cellular chain complex $C_*(\widetilde{X}; \mathbf{Z})$ becomes a free Λ-chain complex of right Λ-modules. We define $C_*(X) := C_*(\widetilde{X}; \mathbf{Z})$ with this right Λ-module structure, and

$$H_*(X; \Lambda) := H(C_*(X)) \ . \tag{1.1}$$

These are just the homology groups of the universal covering together with the π-action. More generally, if M is any right Λ-module, we define

$$H_*(X; M) := H(C_*(X) \otimes_\Lambda M) \tag{1.2}$$

where to define the tensor product we convert M into a left Λ-module by the rule $\lambda m = m\bar{\lambda}$.

Lemma 1.3. *Let $\rho \lhd \pi$ and $X(\rho)$ be the orbit space of $\widetilde{X}$ under the action of ρ. Then $H_*(X; \mathbf{Z}[\pi/\rho]) = H_*(X(\rho); \mathbf{Z})$.*

For $\rho = \pi$ acting trivially on $\mathbf{Z}$ this agrees with the ordinary homology of $X = X(\pi)$. The homology of the coverings $X(\rho)$ are related to $H_*(X; \mathbf{Z})$ via the projection maps $p\colon X(\rho) \to X$ and the transfer $trf\colon \Sigma^\infty X_+ \to \Sigma^\infty X(\rho)_+$.

Proposition 1.4. *If $\pi = \pi_1(X, x_0)$ is a finite group, the composition $p_* \circ trf_*\colon H_i(X; \mathbf{Z}) \to H_i(X; \mathbf{Z})$ is multiplication by $|\pi|$.*

We can also define the cohomology of X with coefficients in a right Λ-module M by

$$H^*(X; M) := H(\mathrm{Hom}_\Lambda(C_*(X), M)) \ .$$

Lemma 1.5. *For X a finite CW-complex, the groups $H^*(X; \Lambda)$ are isomorphic to the cohomology groups $H^*_{cp}(\widetilde{X}; \mathbf{Z})$ of $\widetilde{X}$ with compact support.*

A finite connected CW-complex X which homologically resembles a manifold of dimension n is called a finite Poincaré complex of formal dimension n. More precisely, we start with a pair (X, w) where $w\colon \pi_1(X, x_0) \to \{\pm 1\}$ is a homomorphism (in the case of an actual manifold, this is the orientation data dual to the first Stiefel-Whitney class). If w is trivial we suppress it from the notation. For such a pair (X, w) we define a new involution on $\Lambda = \mathbf{Z}\pi_1(X, x_0)$ by the formula

$$\sum n_g g \mapsto \sum w(g) n_g g^{-1}$$

taking into account the values of the orientation homomorphism. Then w-twisted homology groups $H_*^w(X; M) = H(C_*(X) \otimes_\Lambda M)$ are defined as above, using the w-twisted involution to convert M from a right Λ-module into a left Λ-module. Poincaré duality is defined with respect to a fundamental class $[X] \in H_n^w(X; \mathbf{Z})$. Let $\xi \in C_n(X) \otimes_\Lambda \mathbf{Z}$ be a representative cycle for $[X]$, so the transfer $trf\, \xi \in C_n(X)$ is a locally finite chain on $\widetilde{X}$. Then (X, w) is a Poincaré complex of formal dimension n and orientation class w if the chain map

$$\xi \cap \colon C^*(X) \to C_*(X)$$

defined by the cap product with $trf\, \xi$ is a chain homotopy equivalence. Since the choice of representative ξ for $[X]$ is unique up to chain homotopy, the Poincaré duality condition just says that the cap product induces isomorphisms

$$[X] \cap \colon H^r(X; M) \to H_{n-r}^w(X; M)$$

for all r, and any coefficient module M. In the special case where $M = \Lambda$ this is the usual duality between homology and cohomology with compact supports on $\widetilde{X}$.

2. Homology of groups

Let G be a discrete group and recall that $K(G, 1)$ denotes any CW-complex X with $\pi_1(X, x_0) = G$ and $\pi_i(X) = 0$ for $i > 1$. Such a space is uniquely determined up to homotopy equivalence by G. The homology of the group G with coefficients in a right G-module (i.e. a right $\mathbf{Z}G$-module) is defined to be

$$H_*(G; M) := H_*(K(G, 1); M)\ . \tag{2.1}$$

Similarly, we can define the cohomology of G with coefficients in a G-module as the cohomology of $K(G, 1)$.

Example 2.2. For $G = \mathbf{Z}/2$, the space $X = K(\mathbf{Z}/2, 1) = RP^\infty$ is the union of all the real projective spaces RP^n as $n \to \infty$. Then $\widetilde{X} = S^\infty$ and

$$\Lambda = \mathbf{Z}[\mathbf{Z}/2] = \mathbf{Z}[T]/(T^2 - 1) \cong \mathbf{Z} + \mathbf{Z}T$$

where T denotes the deck transformation given by the antipodal map on S^∞. The chain complex $C_*(RP^\infty; \Lambda)$ in this case is

$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r, "\partial"] \arrow[d, equal] & C_2 \arrow[r, "\partial"] \arrow[d, equal] & C_1 \arrow[r, "\partial"] \arrow[d, equal] & C_0 \arrow[r, "\epsilon"] \arrow[d, equal] & \mathbf{Z} \arrow[d, equal] \\
\cdots \arrow[r] & \Lambda \arrow[r, "1-T"] & \Lambda \arrow[r, "1+T"] & \Lambda \arrow[r, "1-T"] & \Lambda \arrow[r, "\epsilon"] & \mathbf{Z}
\end{tikzcd}$$

where $\epsilon\colon \Lambda \to \mathbf{Z}$ is the augmentation map $\sum n_g g \mapsto \sum n_g$. Since S^∞ is contractible (this always holds for the universal covering of a $K(G,1)$), the sequence above is exact and we get a resolution of $\mathbf{Z}$ by free Λ-modules. In order to compute the homology groups $H_*(\mathbf{Z}/2; M)$ for a $\mathbf{Z}/2$-module M, we tensor the complex above with M to obtain

$$\cdots \longrightarrow M \xrightarrow{1-T} M \xrightarrow{1+T} M \xrightarrow{1-T} M \xrightarrow{1\otimes\epsilon} M$$

The differentials $\Lambda \otimes M \to \Lambda \otimes M$ are given by $\lambda \otimes m \mapsto \partial\lambda \otimes m$, which is just multiplication by $1 \pm T$. Therefore

$$H_{2k}(\mathbf{Z}/2; M) = \{m \in M \mid m = -Tm\}/\{m - Tm \mid m \in M\}$$

and

$$H_{2k+1}(\mathbf{Z}/2; M) = \{m \in M \mid m = Tm\}/\{m + Tm \mid m \in M\}$$

for $k > 0$, and these groups are all 2-torsion. □

Proposition 2.3. *For G a finite group, the homology $H_*(G; M)$, $* > 0$, is torsion of exponent $|G|$ for any G-module M.*

If $M = \mathbf{Z}$ with trivial G-action, we write $H_*(G) = H_*(G; \mathbf{Z})$. Here are some useful properties:

(i) $H_1(G) = G/[G,G]$.

(ii) If $G = F/R$ where F is a free group and R is a normal subgroup of F, then $H_2(G) = R \cap [F,F]/[F,R]$ (Hopf's formula).

(iii) If $G = B_1 *_A B_2$ is the amalgamated free product of B_1 and B_2 over a common subgroup A, then there is an exact sequence

$$\cdots \to H_i(A) \to H_i(B_1) \oplus H_i(B_2) \to H_i(G) \to H_{i-1}(A) \to \ldots$$

(iv) $H_i(G \times \mathbf{Z}) = H_i(G) \oplus H_{i-1}(G)$.

(v) If $f\colon G_1 \to G_2$ is a group homomorphism, there is a long exact sequence

$$\ldots H_i(G_1) \to H_i(G_2) \to H_i(f) \to H_{i-1}(G_1) \ldots$$

where $H_*(f)$ is the homology of the mapping cylinder of the induced map $f\colon K(G_1,1) \to K(G_2,1)$.

Sometimes the space $K(G,1)$ is homotopy equivalent to a finite Poincaré complex, or even a closed manifold. These groups are of great interest in geometric topology (see [14] for a comprehensive survey of progess on the Novikov and Borel conjectures concerning the topology of aspherical manifolds). The basic example is $G = \mathbf{Z}^k$, where $K(G,1) = T^k$ is a k-dimensional torus.

Lemma 2.4. *Suppose that $K(G,1)$ is homotopy equivalent to a finite complex. Then G contains no elements of finite order except the identity.*

This is the first necessary condition for G to be the fundamental group of an aspherical manifold.

3. Projective Modules

Let R be a ring with unit element. An R-module P is *projective* if it is a direct summand of a free R-module. The projective class group $K_0(R)$ is the Grothendieck group of the category $\mathcal{P}(R)$ of *finitely-generated* projective R-modules. More explicitly, the generators of $K_0(R)$ are isomorphism classes $[P]$, for each $P \in \mathcal{P}(R)$, and relations $[P \oplus Q] = [P] + [Q]$ for all P, Q in $\mathcal{P}(R)$. Then $K_0(R)$ is an abelian group, and $[P] = [Q]$ in $K_0(R)$ if and only if $P \oplus R^k \cong Q \oplus R^k$ for some integer $k \geq 0$. This relation is called *stable* isomorphism. In many cases (e.g. for R a field or skew field), stable isomorphism implies isomorphism. We will discuss this "cancellation" problem more below.

Proposition 3.1. *If R is a field, skew field, local ring, or a principal ideal domain, then $K_0(R) \cong \mathbf{Z}$, where the isomorphism is given by the rank.*

For R a Dedekind domain, such as the ring of integers in an algebraic number field, the group $K_0(R)$ is difficult to calculate since it involves the ideal class group of R. If K denotes the field of fractions of R, then a fractional R-ideal is a finitely-generated R-submodule of K. The product J_1J_2 of two fractional ideals is the R-ideal consisting of all finite sums $\sum x_iy_i$ with $x_i \in J_1$ and $y_i \in J_2$. The inverse ideal $J^{-1} = \{x \in K | xJ \subseteq R\}$, and the product $JJ^{-1} = R$. Two fractional ideals are called equivalent if they are isomorphic as R-modules, and the equivalence class of J is denoted $[J]$. The ideals equivalent to R are called principal ideals, and the abelian group (under multiplication) of ideal classes modulo principal ideals is the ideal class group $Cl(R)$.

Theorem 3.2. *Let R be a Dedekind domain whose quotient field is an algebraic number field or a function field. Then the ideal class group $Cl(R)$ is finite.*

Even for $R = \mathbf{Z}[e^{2\pi i/p}]$, where p is an odd prime, the structure of the ideal class group $Cl(R)$ is generally unknown (see [33, p. 30]), although there is an explicit formula for its order, called the *ideal class number*.

Theorem 3.3. *Let R be a Dedekind domain. Then $K_0(R) = \mathbf{Z} \oplus Cl(R)$.*

The ideal class number is often non-trivial (e.g. for $p = 23$, 29), so these rings have projective modules which are not stably isomorphic to free modules.

The projective class group respects products of rings

$$K_0(R \times S) \cong K_0(R) \oplus K_0(S)$$

and if $f\colon R \to S$ is a ring homomorphism, there is an induced map

$$f_*\colon K_0(R) \to K_0(S)$$

induced by $f_*(P) = P \otimes_R S$, with the usual functorial properties. In addition, K_0 is Morita invariant so that

$$K_0(M_n(R)) \cong K_0(R) .$$

We define $\widetilde{K}_0(R)$ to be the quotient of $K_0(R)$ by the subgroup generated by the free modules $[R^k]$. In other words, if $i\colon \mathbf{Z} \to R$ maps $1 \in \mathbf{Z}$ to the unit element of R, then

$$\widetilde{K}_0(R) := K_0(R)/ \operatorname{Im} i_*$$

The finiteness result above for the class group of Dedekind domains has a striking generalization due to R. Swan. A module is called *locally free* if it becomes free after tensoring with $\mathbf{Z}_{(p)}$ for all primes p.

Theorem 3.4 (Swan). *Let R be a Dedekind domain of characteristic 0, and G be a finite group such that no rational prime dividing the order of G is invertible in R. Then every finitely generated projective RG module is locally free. Moreover, $K_0(RG) = \mathbf{Z} \oplus \widetilde{K}_0(RG)$ and $\widetilde{K}_0(RG)$ is finite.*

If R can be embedded into a (skew) field F, then we define $\operatorname{rank}(P) = \dim_F(P \otimes_R F)$. It follows that $\widetilde{K}_0(R) \cong \ker\{r\colon K_0(R) \to K_0(F)\}$, where $r([P]) = \operatorname{rank} P$.

Lemma 3.5. *If R can be embedded in a field or skew field, then $K_0(R) \cong \mathbf{Z} \oplus \widetilde{K}_0(R)$.*

This direct sum splitting doesn't always hold. For example, if $R = M_n(F)$ is the ring of $n \times n$ matrices over a field F, then $K_0(R) \cong \mathbf{Z}$ generated by the simple module, but $\widetilde{K}_0(R) = \mathbf{Z}/n$.

The primary methods for computing $K_0(R)$ are localization sequences and the Mayer-Vietoris type exact sequences arising from fibre squares of rings. The general idea is to reduce the study of projective R-modules to the same problem over simpler rings, such as full matrix rings over (skew) fields, or complete rings. Recall that R with a 2-sided ideal I is a complete in the I-adic topology if $R = \varprojlim R/I^k$.

Lemma 3.6. *If R be a complete in the I-adic topology, then $K_0(R) = K_0(R/I)$ via the projection map.*

For example, this holds if R is either a left artinian ring, or a finitely generated algebra over a complete noetherian local ring, and I is contained in $\operatorname{Rad} R$. We will be particularly interested in group rings RG, where G is a finite group and $R = \widehat{\mathbf{Z}}_p$. In this case, $K_0(\widehat{\mathbf{Z}}_pG) \cong K_0(\mathbf{F}_pG) \cong K_0(\widehat{\mathbf{Z}}_pG/Rad)$.

To recontruct projectives over R out of projectives over simpler rings, we need the technique of fibre squares. Suppose that a commutative square of rings

(3.7)
$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "k"] & S_1 \arrow[d, "l"] \\ S_2 \arrow[r, "j"] & T \end{tikzcd}$$

has the following properties

(i) The maps i, j, k, and l are ring homomorphisms.
(ii) R is the fibre product of S_1 and S_2 over T.
(iii) At least one of j and l is surjective.

Given left modules P_1 and P_2 over S_1 and S_2 respectively, together with an isomorphism

$$h\colon j_*(P_1) = P_1 \otimes_{S_1} T \cong P_2 \otimes_{S_2} T = l_*(P_2)$$

let $M(P_1, P_2, h) := \{(p_1, p_2) \in P_1 \times P_2 \mid h(j_*(p_1)) = l_*(p_2)\}$. This has an R-module structure by the formula $r \cdot (p_1, p_2) := (i(r)p_1, k(r)p_2)$.

Proposition 3.8 (Milnor). *If P_1 and P_2 are finitely generated or projective, then so is $M(P_1, P_2, h)$. Every projective R-module is isomorphic to some $M(P_1, P_2, h)$ for suitable choices of P_1, P_2 and h.*

The choice of isomorphism h does change the isomorphism class of $M(P_1, P_2, h)$ in general, but the information so far tells us that the sequence

$$K_0(R) \to K_0(S_1) \oplus K_0(S_2) \to K_0(T)$$

is exact. Extending this Mayer-Vietoris type sequence to the left or right will involve the definition of new K-theory functors.

Example 3.9. Let $G = \mathbf{Z}/p$ for p a prime, and let $R = \mathbf{Z}G$. Then there is a fibre square of the kind just considered

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "k"] & \mathbf{Z} \arrow[d, "l"] \\ \mathbf{Z}[\zeta_p] \arrow[r, "j"] & \mathbf{F}_p \end{tikzcd}$$

where $\zeta_p = e^{2\pi i/p}$. The sequence above shows that the new ingredient in the calculation of $\widetilde{K}_0(\mathbf{Z}G)$ is the kernel

$$D(\mathbf{Z}G) := \ker\{K_0(\mathbf{Z}G) \to K_0(\mathbf{Z}[\zeta_p] \oplus K_0(\mathbf{Z})\}\ .$$

By analysing this fibre square one can show:

Theorem 3.10 (Reiner). *Let $G = \mathbf{Z}/p$ where p is a prime. Then $D(\mathbf{Z}G) = 0$.*

Using this result, Reiner was able to completely classify the integral representation of $\mathbf{Z}/p$, or in other words, the finitely-generated modules over $\mathbf{Z}[\mathbf{Z}/p]$ which are torsion-free as abelian groups. For most finite groups such a classification is not available. □

For G any finite group, there exists a maximal $\mathbf{Z}$-order $\mathcal{M} \subset \mathbf{Q}G$ containing $\mathbf{Z}G$. In particular, $\mathcal{M}$ is a subring of $\mathbf{Q}G$ which is finitely generated as a $\mathbf{Z}$-module, and such that $\mathcal{M} \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}G$. In the example above, $\mathcal{M} = \mathbf{Z}[\zeta_p] \oplus \mathbf{Z}$. For any finite group G, we define

$$D(\mathbf{Z}G) := \ker\{K_0(\mathbf{Z}G) \to K_0(\mathcal{M})\} \subseteq \widetilde{K}_0(\mathbf{Z}G) .$$

The calculation of this group has been a major research goal in the algebraic K-theory of finite groups (see [35] or [10] for references). Note that $D(\mathbf{Z}G)$ has finite order by Swan's theorem.

4. Finiteness obstructions

For X a finite CW-complex and $\Lambda = \pi_1(X, x_0)$, the chain complex $C_*(X; \Lambda)$ is a complex of finitely generated free Λ-modules. We say that a CW-complex X is *finitely dominated* if there exists a finite CW-complex Y and continuous maps $r\colon Y \to X$ and $i\colon X \to Y$ such that $r \circ i \simeq id_X$. Here is a nice result of Mather and Ferry:

Theorem 4.1. *Let X be a finitely dominated CW-complex. Then the product space $X \times S^1$ has a canonical finite CW-structure (independent of the finite domination).*

The chain complex $C_*(X; \Lambda)$ of a finitely dominated space is a finite length complex of finitely generated projective Λ-modules. In this situation, C. T. C. Wall defined the finiteness obstruction

$$\theta_W(X) = \sum (-1)^i [C_i(\widetilde{X}; \mathbf{Z})] \in \widetilde{K}_0(\mathbf{Z}\pi_1(X, x_0))$$

and proved:

Theorem 4.2 (Wall). *If X is a finitely dominated CW-complex, then $\theta_W(X)$ is a homotopy invariant. Moreover $\theta_W(X) = 0$ if and only if X is homotopy equivalent to a finite complex.*

Wall also proved that any element of $\widetilde{K}_0(\mathbf{Z}G)$ could arise as the finiteness obstruction of some finitely dominated complex. In some interesting cases there are restrictions on the allowable finiteness obstructions.

Theorem 4.3 (Mislin-Varadarajan). *Suppose that X is a finitely dominated nilpotent space with finite fundamental group G. Then $\theta_W(X) \in D(\mathbf{Z}G)$.*

One example of a nilpotent space is the quotient of sphere $X = S^n/G$, where G is a nilpotent group acting freely.

Another assumption which restricts the possible finiteness obstructions is Poincaré duality. If (X, w) is a finitely dominated Poincaré complex, the w-twisted involution $\lambda \mapsto \bar{\lambda}$ on Λ induces an $\mathbf{Z}/2$-module structure on $\widetilde{K}_0(\mathbf{Z}G)$ by the formula $[P] \mapsto -[P^*]$, where P is a projective right Λ-module and $P^* = \mathrm{Hom}_\Lambda(P, \Lambda)$ is converted from a left to a right Λ-module by the involution.

Lemma 4.4. *Let (X, w) be a finitely dominated Poincaré complex with fundamental group G and with formal dimension n. Then $\overline{\theta_W(X)} = (-1)^{n+1}\theta_W(X)$*

This shows that the finiteness obstruction of a Poincaré n-complex gives a well-defined element in $H^{n+1}(\mathbf{Z}/2; \widetilde{K}_0(\mathbf{Z}G))$.

5. Whitehead torsion

In this section we consider automorphisms of finitely generated free R-modules. If the free module has rank k, the group of all automorphisms is $GL_k(R)$. Identifying each $A \in GL_k(R)$ with the matrix

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \in GL_{k+1}(R)$$

we obtain the inclusions

$$GL_1(R) \subset GL_2(R) \subset GL_3(R) \dots$$

and the union is the infinite general linear group $GL(R)$. A matrix is called elementary if its entries coincide with those of the identity matrix except for one off-diagonal entry.

Lemma 5.1 (Whitehead). *The subgroup $E(R) \subset GL(R)$ generated by all elementary matrices is just the commutator subgroup of $GL(R)$.*

Proof. Let aE_{ij} denote the matrix which has at most one non-zero entry a in the (i, j) position. Then the relation

$$(I + aE_{ij})(I + E_{jk})(I - aE_{ij})(IE_{jk}) = (I + aE_{ik})$$

for i, j, k all distinct, shows that every elementary matrix of $GL_n(R)$ is a commutator for $n \geqq 3$. On the other hand, Whitehead's identities below show that any commutator $ABA^{-1}B^{-1}$ in $GL_n(R)$ can be written as a product of elementary matrices in $GL_{2n}(R)$.

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I - A^{-1} & I \end{pmatrix} \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I - A & I \end{pmatrix}$$

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \prod_{i=1}^{n} \prod_{j=n+1}^{n} (I + x_{ij}E_{ij}) \ .$$

□

The abelian quotient group

$$K_1(R) = GL(R)/E(R)$$

was defined by J. H. C. Whitehead in order to compare homotopy equivalent complexes. Notice that a ring homomorphism $f\colon R \to S$ induces a map $f_*\colon K_1(R) \to K_1(S)$ with the usual functorial properties. In addition the functor K_1 respects products of rings

$$K_1(R \times S) \cong K_1(R) \oplus K_1(S)$$

and is Morita invariant

$$K_1(M_n(R)) \cong K_1(R) \ .$$

For calculations in the case where R is commutative we have a homomorphism

$$\det\colon K_1(R) \to R^\times$$

given by the determinant. If R is a field, or $R = \mathbf{Z}$ then $K_1(R) \cong R^\times$. For R a skew field, then $K_1(R) = R^\times/[R^\times, R^\times]$ via the "non-commutative determinant". If R is a $\mathbf{Z}$-order in a semi-simple $\mathbf{Q}$-algebra S, then $S \otimes_{\mathbf{Q}} \mathbb{C}$ is a product of full matrix rings over the complex numbers by Wedderburn's theorem. Composing the inclusion $R \subset S$ with projection onto one of these factors $M_n(\mathbb{C})$ gives a homomorphism

$$\mathrm{nr}\colon K_1(R) \to K_1(S) \to K_1(M_n(\mathbb{C})) \xrightarrow{\det} \mathbb{C}^\times$$

detecting the torsion-free part of $K_1(R)$. The image of this homomorphism lies in the ring of integers of the centre field of the associated factor of S.

Theorem 5.2 (Bass). *Let R be a $\mathbf{Z}$-order in a semisimple $\mathbf{Q}$-algebra S. Then $K_1(R)$ is a finitely generated abelian group of rank $r - q$, where q denotes the number of simple factors in S and r denotes the number of simple factors in $S \otimes_{\mathbf{Q}} \mathbb{R}$.*

For more precise calculations, the method of pull-back squares is available. If R is the pullback of S_1 and S_2 over T as in (3.7), then we have a six term exact sequence

$$K_1(R) \to K_1(S_1) \oplus K_1(S_2) \to K_1(T) \xrightarrow{\partial} K_0(R) \to K_0(S_1) \oplus K_0(S_2) \to K_0(T) \ .$$

For example, this sequence explains the role of the isomorphism h in the pullback construction $M(P_1, P_2, h)$ of projectives over R from projectives over S_1 and S_2.

Remark 5.3. There is an interesting connection between the following three questions:

(i) given $a \in K_1(R)$, what is the minimum n such that $a = [A]$ for some matrix $A \in GL_n(R)$?
(ii) given an ideal $I \subset R$, what is the minimum number of generators $I = \langle r_1, \ldots r_k \rangle$ among all generating sets for I ?
(iii) given an isomorphism $M \oplus N \cong M' \oplus N$, does it follow that $M \cong M'$?

The last question is the cancellation problem for modules over R. The unifying idea linking these three questions is transitivity of elementary matrices on the set of unimodular elements (i.e. those generating a free direct summand) in a given R-module M.

Theorem 5.4 (Bass). *Suppose that R is a ring with Krull dimension d, and M, M' and N are right R-modules, with N projective, such that $M \oplus N \cong M' \oplus N$. If N contains a free direct summand R^k of rank $k \geq d+2$, then $M \cong M'$.*

Proof. We may assume that $N = R^k$. Under the given stability condition, the elementary linear automorphisms of $M \oplus R^k$ act transitively on the set of unimodular elements. Therefore, any isomorphism $M' \oplus R^k \cong M \oplus R^k$ can be composed with an elementary automorphism to ensure that the standard basis of R^k is mapped by the identity. It follows that $M \cong M'$. □

A similar method gives stability bounds for the other two questions.

We turn now to the original geometric motivation for introducing the K_1 functor. If X is a finite CW-complex, its fundamental group $\pi := \pi_1(X, x_0)$ acts on the cells of $\widetilde{X}$ to give $C_*(X)$ the structure of a free Λ-module chain complex. To obtain a basis for this chain complex, order the cells of X (of a given dimension r), orient each one, and then choose a lifting of each cell to an r-cell of $\widetilde{X}$. This gives a free Λ-base for $C_r(\widetilde{X})$, unique up to order, sign, and multiplication on the right by elements of $\pi_1(X, x_0)$. Now if $f\colon X \to Y$ is a homotopy equivalence of finite CW-complexes, we have a short exact exact sequence of chain complexes

$$0 \to C_*(X) \to C_*(Y) \to C_*(f) \to 0$$

where the chain complex of the mapping cylinder of f has chain groups $C_i(f) := C_{i-1}(X) \oplus C_i(Y)$ and its differential is given by the formula

$$\partial_i := \begin{pmatrix} -\partial^X_{i-1} & 0 \\ f & \partial^Y_i \end{pmatrix} .$$

The chosen bases of $C_*(X)$ and $C_*(Y)$ induce a basis for $C_*(f)$, so the above is an exact sequence of free, based, Λ-module chain complexes. Moreover, since f is asssumed to be a homotopy equivalence, the homology $H(C_*(f))$ of the mapping cylinder is zero. In this situation, one can define a K_1-invariant called the Whitehead torsion of f.

To explain the process, we will consider any acyclic (i.e. zero homology) chain complex

$$C_n \to C_{n-1} \to \cdots \to C_2 \to C_1 \to C_0$$

of free Λ-modules, and assume that each group C_i has a given Λ-basis $\{c_i\}$. Let B_i denote the image of $\partial_{i+1}\colon C_{i+1} \to C_i$, and note that we have exact sequences

$$0 \to B_i \to C_i \to B_{i-1} \to 0$$

for each i, together with the equality $B_0 = C_0$. Inductively we see that all of these sequences split, so the modules B_i are all stably free. By taking the direct sum of the complex with elementary based complexes of the form

$$0 \to \Lambda^r \to \Lambda^r \oplus \Lambda^s \to \Lambda^s \to 0$$

we may assume that all the modules B_i are free to begin with. Choose a basis $\{b_i\}$ for each B_i, and notice that we now have two different bases, namely c_i and $\{b_i, b_{i-1}\}$ for each C_i. Let $[c_i/b_ib_{i-1}] \in K_1(\Lambda)$ denote the element given by the change of basis isomorphism on C_i.

Lemma 5.5. *The element*

$$\tau(C) = \sum(-1)^i[c_i/b_ib_{i-1}] \in K_1(\Lambda)$$

is independent of the choice of bases $\{b_i\}$ for the B_i.

Now in the geometric situation, we have made choices of the bases for $C_*(X)$ and $C_*(Y)$. To allow for the effect of these choices, we define the Whitehead group

$$\mathrm{Wh}(\mathbf{Z}G) := K_1(\mathbf{Z}G)/\{\pm g \mid g \in G\}$$

for any group G. Then we have

Theorem 5.6 (Whitehead). *Let $f\colon X \to Y$ be a homotopy equivalence of finite CW-complexes with fundamental group π. Then the element $\tau(f) := \tau(C_*(f)) \in \mathrm{Wh}(\mathbf{Z}\pi)$ is a homotopy invariant.*

Whitehead went on to show that $\tau(f) = 0$ if and only if X and Y were related by a sequence of cellular operations called "elementary expansions and collapses". In addition, if G is a finitely presented group, Whitehead proved that any element of $\mathrm{Wh}(\mathbf{Z}G)$ can be realized by some homotopy equivalence of finite CW-complexes with fundamental group G.

A homotopy equivalence $f\colon X \to Y$ with $\tau(f) = 0$ is called a *simple* homotopy equivalence. The study of simple homotopy types is now an important subject within homotopy theory. Here is another result of Whitehead which opened up an active research area. Let $X \vee rS^2$ denote the wedge of X with r copies of S^2.

Theorem 5.7 (Whitehead). *Let X and Y be finite 2-complexes with the same Euler characteristic, and let $\alpha\colon \pi_1(X, x_0) \cong \pi_1(Y, y_0)$ be an isomorphism between their fundamental groups. Then there is a simple homotopy equivalence*

$f\colon X \vee rS^2 \simeq Y \vee rS^2$ realizing the given isomorphism α on fundamental groups.

There is also a geometric analogue of the cancellation problem for modules, namely to remove as many S^2 wedge summands as possible from a stable homotopy equivalence. For complexes with finite fundamental groups, we can remove all but one S^2.

Theorem 5.8 (Hambleton-Kreck). *Let X and Y be finite 2-complexes with the same Euler characteristic and finite fundamental group. Let $\alpha\colon \pi_1(X, x_0) \cong \pi_1(Y, y_0)$ be a given isomorphism and suppose that $X \simeq X_0 \vee S^2$. Then there is a simple homotopy equivalence $f\colon X \to Y$ inducing α on the fundamental groups.*

If (X, w) is a finite Poincaré complex of formal dimension n, then the mapping cylinder of the duality map $[X]\cap\colon C^* \to C_*$ is acylic. We call $\tau(X, w) := \tau([X]\cap) \in \mathrm{Wh}(\mathbf{Z}\pi_1(X, x_0)$ the *torsion* of (X, w) and say that (X, w) is a *simple* Poincaré complex if $\tau(X, w) = 0$. The w-twisted involution on Λ induces an involution $A \mapsto \overline{(A^t)}$ on $GL(\Lambda)$ and hence an involution on $\mathrm{Wh}(\Lambda)$. Any closed manifold is a simple Poincaré complex.

Theorem 5.9. *Let (X, w) be a finite Poincaré complex of formal dimension n. Then $\overline{\tau(X, w)} = (-1)^n\tau(X, w)$. If (X, w) is homotopy equivalent to a closed n-manifold with orientation class w, then $\tau(X, w) = 0$.*

One of the most famous results about Whitehead torsion is that $\tau(f)$ is an obstruction for f to be homotopic to a homemorphism.

Theorem 5.10 (Chapman). *Let $f\colon X \to Y$ be a homeomorphism of finite CW-complexes. Then $\tau(f) = 0$.*

For the geometric applications of Whitehead torsion we must develop methods to compute $\mathrm{Wh}(\mathbf{Z}G)$ for finitely presented groups G. In this problem, there are two sharply different approaches depending on whether G is finite or infinite. If G is infinite and torsion-free, then the main conjecture is that $\mathrm{Wh}(\mathbf{Z}G) = 0$ and the methods are geometric (see [14]). On the other hand, if G is finite the Whitehead group is generally non-trivial and there are extensive calculations available using algebraic methods (see [35]). Of course this summary leaves open what to do about infinite groups which have non-trivial elements of finite order, for example $\mathbf{Z} \times G$ where G is finite. More generally it would clearly be useful to have some idea how the Whitehead groups change under Laurent polynomial extension (i.e. direct product with $\mathbf{Z}$), amalgamated free products and HNN extension. We mention only the result on polynomial extensions, involving new K-theory functors $\widetilde{Nil}(\mathbf{Z}G)$ based on nilpotent matrices over the group ring.

Theorem 5.11 (Bass-Heller-Swan). *For any group G,*

$$\mathrm{Wh}(\mathbf{Z}[\mathbf{Z} \times G]) \cong \mathrm{Wh}(\mathbf{Z}G) \oplus \widetilde{K}_0(\mathbf{Z}G) \oplus \widetilde{Nil}(\mathbf{Z}G) \oplus \widetilde{Nil}(\mathbf{Z}G)\ .$$

For G a finite group, we define the *arithmetic square*

$$\begin{tikzcd} \mathbf{Z}G \arrow[r] \arrow[d] & \mathbf{Q}G \arrow[d] \\ \widehat{\mathbf{Z}}G \arrow[r] & \widehat{\mathbf{Q}}G \end{tikzcd}$$

where $\widehat{\mathbf{Z}}$ is the direct product of all the rings $\widehat{\mathbf{Z}}_p$ and $\widehat{\mathbf{Q}}$ is the restricted product of the rings $\widehat{\mathbf{Q}}_p$. An element of the direct product $\prod \widehat{\mathbf{Q}}_p$ is in the restricted product if all but finitely many of its entries are in $\widehat{\mathbf{Z}}_p$. Although the arithmetic square is not a pullback in the sense of (3.7), the strong approximation theorem in algebraic number theory gives and exact sequence

$$K_1(\mathbf{Z}G) \to K_1(\mathbf{Q}G) \oplus K_1(\widehat{\mathbf{Z}}G) \to K_1(\widehat{\mathbf{Q}}G) \xrightarrow{\partial} K_0(\mathbf{Z}G) \to \\ \to K_0(\mathbf{Q}G) \oplus K_0(\widehat{\mathbf{Z}}G) \to K_0(\widehat{\mathbf{Q}}G)$$

which is very effective for calculations. For example, Swan's theorem show that the map $K_1(\widehat{\mathbf{Q}}G) \xrightarrow{\partial} \widetilde{K}_0(\mathbf{Z}G)$ is surjective, and this suggests that the finiteness obstruction $\theta_W(X)$ for a finitely dominated space X should have a lifting to $K_1(\widehat{\mathbf{Q}}G)$. This is indeed the case: after choosing bases $\mathbf{h}$ for the homology of $C_*(X) \otimes_{\mathbf{Z}} \widehat{\mathbf{Q}}$, one can define the idelic Reidemeister torsion $\hat{\Delta}(X, \mathbf{h}) \in K_1(\widehat{\mathbf{Q}}G)$ so that $\partial\hat{\Delta}(X, \mathbf{h}) = \theta_W(X)$. This invariant plays an important role in the solution of the spherical space form problem.

Let

$$SK_1(\mathbf{Z}G) := \ker\{K_1(\mathbf{Z}G) \to K_1(\mathbf{Q}G)\} .$$

Then

Theorem 5.12 (Wall). *For G a finite group, the torsion subgroup of $K_1(\mathbf{Z}G)$ is just $\{\pm G^{ab}\} \oplus SK_1(\mathbf{Z}G)$. The standard oriented involution induces the identity on the torsion-free quotient* $\mathrm{Wh}'(\mathbf{Z}G) := \mathrm{Wh}(\mathbf{Z}G)/SK_1(\mathbf{Z}G)$.

For G finite cyclic, $SK_1(\mathbf{Z}G) = 0$ and the Whitehead group is torsion-free (the rank was given above). In general however, the groups $SK_1(\mathbf{Z}G)$ are highly non-trivial. A homotopy equivalence $f\colon X \to Y$ with $\tau(f) \in SK_1(\mathbf{Z}G) \oplus \{\pm G^{ab}\}$ is called a *weakly simple homotopy equivalence* . Similarly, a Poincaré complex X is weakly simple if its duality map has zero torsion in $\mathrm{Wh}'(\mathbf{Z}\pi_1(X, x_0))$.

Corollary 5.13 (Wall). *An orientable finite Poincaré complex of odd formal dimension, with finite fundamental group, is weakly simple. A homotopy equivalence between (weakly) simple oriented Poincaré complexes of even formal dimension, with finite fundamental group, is weakly simple.*

6. Hermitian forms

An involution $\alpha(r) = \bar{r}$ on a ring R with unit has the properties

(i) $\alpha(r+s) = \alpha r + \alpha s$ for all r, $s \in R$.
(ii) $\alpha(rs) = \alpha s \alpha r$ for all r, $s \in R$.
(iii) $\alpha^2(r) = r$ for all $r \in R$.
(iv) $\alpha(1) = 1$.

Let R be a ring with involution, and let $\epsilon \in R^\times$ be a unit of R with $\epsilon\bar{\epsilon} = 1$. An ϵ-hermitian form on a right R-module M is a map $h\colon M \times M \to R$ such that

(i) $h(x+y,z) = h(x,z) + h(y,z)$ for all $x, y, z \in M$.
(ii) $h(x, yr) = h(x,y)r$ for all $x, y \in M$ and all $r \in R$.
(iii) $h(y,x) = \epsilon\overline{h(x,y)}$ for all $x, y \in M$.

The adjoint map $ad(h)\colon M \to M^* = \mathrm{Hom}_R(M,R)$ is defined by $ad(h)(x)(y) = h(x,y)$ for all $x, y \in M$. The form (M,h) is non-degenerate if the adjoint map $ad(h)$ is injective, and non-singular if $ad(h)$ is an isomorphism of right R-modules. As usual, we convert M^* from a left R-module to a right R-module by using the involution. The form can be described either by h or by $ad(h)$, whichever is most convenient. We usually take $\epsilon = \pm 1$. Two ϵ-hermitian forms (M,h) and (N,k) are isometric if there exists an R-module isomorphism $\varphi\colon M \to N$ such that $h(x,y) = k(\varphi(x), \varphi(y))$ for all $x, y \in M$. There is an obvious notion of orthogonal direct sum $(M,h) \perp (N,k) = (M \oplus N, h \perp k)$, where $h \perp k = \begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix}$. We can then define $K_0(\mathcal{H}(R,\alpha))$ to be the Grothendieck group of the category of hermitian forms on finitely generated projective R-modules. This is a hermitian version of $K_0(R)$ and we have a forgetful map

$$K_0(\mathcal{H}(R,\alpha)) \to K_0(R),$$

taking a form (M,h) to its underlying module M.

Example 6.1. Let $M = N \oplus N^*$ and $h(x,f),(y,g)) = f(y) + \epsilon\overline{g(x)}$, for all $x, yinN$ and $f, g \in N^*$. This defines the hyperbolic form $\mathbf{H}(N)$ on $N \oplus N^*$. Applying this to projective modules N, we get a homomorphism

$$\mathbf{H}\colon K_0(R) \to K_0(\mathcal{H}(R,\alpha))$$

called the hyperbolic map.

If $M \cong R^n$ is a free R-module and $\{e_1, \dots, e_n\}$ is a basis, then a non-singular hermitian form (M,h) has a K_1-valued "determinant" invariant $d(M,h) = [ad(h)] \in K_1(R)$ where M^* is given the dual basis. This gives an invariant of forms on free based modules which is additive under orthogonal direct sums.

Notice that $\overline{d(M,h)} = \epsilon d(M,h)$. Changing the basis of M changes the matrix for h by the usual formula $A \mapsto \overline{P}^t AP$, so the K_1-invariant changes by an element of the form $u + \bar{u}$. We get a well-defined invariant $[d(M,h)] \in H^k(\mathbf{Z}/(2); K_1(R))$, additive under orthogonal direct sums.

If $R = K$ is a field with fixed field F under the involution, this invariant is just the usual determinant of the hermitian form taking values in $F^\times$ with indeterminancy from choice of basis in the image of the norm map $N_{K/F}(K^\times)$.

The hyperbolic form on the standard based free module $M = R^n \oplus R^n$ of rank n has $d(M,h) = (-1)^n$. For hermitian forms on based free modules of rank $2n$ there is a refinement of the K_1-valued determinant

$$\operatorname{disc}(M,h) = (-1)^n d(M,h)$$

called the discriminant of (M,h). The discriminant vanishes on hyperbolic forms. □

Example 6.2. Let $R = \mathbf{Z}$, $\epsilon = \pm 1$, and $M = \mathbf{Z}^n$ be a free abelian group of rank n. Then a non-singular ϵ-hermitian form on M is just a symmetric or skew-symmetric unimodular form on $\mathbf{Z}^n$. In the symmetric case, we say that a form h is *even* if $h(x,x) \equiv 0 \bmod 2$ for all $x \in M$, and otherwise h is *odd* . A form is called *definite* if $h(x,x) \neq 0$ whenever $x \neq 0$, and otherwise indefinite.

Theorem 6.3. *Indefinite unimodular symmetric forms are classified by the rank, type (odd or even), and the signature.*

The classification of definite symmetric unimodular forms over $\mathbf{Z}$ is a fascinating subject (see [34]). The number of distinct isometry classes grows rapidly with the rank n. In contrast, the classification of skew-symmetric unimodular forms over $\mathbf{Z}$ is trivial: there is just one such form (the hyperbolic form) for each even rank. Similarly, we often encounter (skew) symmetric forms on vector spaces over fields with trivial involution (e.g. $\mathbf{Q}$, bR, or finite fields $\mathbf{F}_p$. If the characteristic of the field is not 2, every symmetric form can be diagonalized and every non-singular skew-symmetric form is hyperbolic. For symmetric forms (M,h) over $\mathbf{Q}$ or $\mathbb{R}$, the *signature* $\sigma(M,h) \in \mathbf{Z}$ is defined to be the number of positive entries minus the number of negative entries in any diagonalization of (M,h). This integer is well-defined, and together the rank, determinant, and signature classify the forms over $\mathbf{Q}$ or $\mathbb{R}$ up to isometry. Over $R = \mathbf{F}_p$, for p odd, the rank and determinant classify the forms.

An interesting contrast is the case $R = \mathbf{F}_2$, or any finite field with characteristic 2. In this case there is no difference between symmetric and skew-symmetric forms, but there are non-isometric forms, for example $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, with the same rank and determinant. Any non-singular form over $\mathbf{F}_2$ is isometric to orthogonal direct sums of these with the rank 1 form $\langle 1 \rangle$. □

Example 6.4. Let $R = \mathbb{C}$ with the involution given by complex conjugation, and $M = \mathbb{C}^n$. Then the form $h(z,w) = \sum \bar{z}_i w_i$ is a non-singular hermitian form. □

7. Normal maps and surgery obstructions

We now describe a geometrical setting for the algebra of hermitian forms. This is the Browder-Novikov-Sullivan-Wall theory of surgery, which has had such a decisive impact on geometric topology.

Suppose that W^{n+1} is a smooth compact manifold with two boundary components M_0 and M_1. Let $f\colon W \to [0,1]$ denote a Morse function, namely a smooth function with $f(M_0) = 0$, $f(M_1) = 1$, non-degenerate critical points and distinct critical values $0 < c_1 < c_2 < ... < c_r < 1$. By the Morse lemma, in a neighborhood U of critical point $p_0 \in W$ with $f(p_0) = c$, there exists a co-ordinate system $x_i = x_i(p)$, $1 \leq i \leq n+1$, so that

$$f(p) = f(p_0) - x_1^2 \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n+1}^2$$

for all $p \in U$. The integer k, $0 \leq k \leq n+1$ is the index of the critical point. If $\epsilon > 0$ is so small that $W = f^{-1}(c-\epsilon, c+\epsilon)$ has no critical points other than p_0, then $M_{c+e} = f^{-1}(c+\epsilon)$ is obtained from $M_{c-\epsilon}$ by an elementary surgery of type $(k, n-k)$:

$$M_{c+\epsilon} = (M_{c-\epsilon} - \varphi(S^{k-1} \times D^{n-k+1})) \cup_\varphi (D^k \times S^{n-k})$$

where $\varphi\colon S^{k-1} \times D^{n-k+1} \to M_{c-\epsilon}$ is an embedding. The manifold $W = (M_{c-\epsilon} \times I) \cup D^k \times D^{n-k+1}$ is usually called the trace of the surgery. This is the basic construction in surgery.

The discussion above shows that the equivalence relation cobordism of manifolds is generated by elementary surgeries. To reverse this point of view, and produce a scheme for the classification of manifolds requires a way to keep track of the effect of elementary surgeries. First we define, for any space X, the n-dimensional structure set $\mathcal{S}_n(X)$. This is the set of equivalence classes of pairs (M^n, f), where M^n is a closed n-manifold and $f\colon M \to X$ is a homotopy equivalence. Two such pairs (M_0, f_0), (M_1, f_1) are equivalent if there is a diffeomorphism $g\colon M_0 \to M_1$ such that $f_1 \circ g \simeq f_0$. One can now ask for a "computation" of $\mathcal{S}_n(X)$ given X. Of course it would be reasonable to start with X a closed n-manifold, or at least a finite Poincaré complex of formal dimension n, and then $\mathcal{S}_n(X)$ would measure the manifolds in the same homotopy type.

A Poincaré space (X, w) resembles a manifold in another way. Let $X \to \mathbb{R}^{n+k}$ be an embedding (for k large) and N a regular neighborhood. Then it turns out that the composite $i\colon \partial N \to N \to X$ is (up to homotopy) a spherical fibration, with each fibre homotopy equivalent to S^{k-1}. If k is sufficiently large, this fibration ν_X is unique up to fibre homotopy equivalence and is called the Spivak normal fibre space of X. By construction, the collapse map

$$c\colon S^{n+k} \to \mathbb{R}^{n+k}/\mathbb{R}^{n+k} - N := T(\nu)$$

together with the Thom isomorphism Φ induces a degree 1 map

$$H_{n+k}(S^{n+k};\mathbf{Z}) \xrightarrow{c_*} H_{n+k}(T(\nu);\mathbf{Z}) \xleftarrow[\approx]{\Phi} H_n^w(X;\mathbf{Z})$$

taking a generator $[S^{n+k}]$ onto $[X]$. Conversely, the Spivak normal fibre space is characterized, up to stable fibre homotopy equivalence, as a spherical fibration ν over X such that $\pi_{n+k}(T(\nu))$ contains a map of degree 1.

We now define a degree 1 normal map with target (X, w). This consists of a degree 1 map $f\colon M^n \to X$ where M is a closed n-manifold and $f^*w = w_1(M)$, together with a bundle map $b\colon \nu_M \to \xi$ covering f, for some vector bundle ξ over X. Two normal maps (M_i, f_i, b_i), $i = 0, 1$ are normally cobordant if there is a cobordism W^{n+1} from M_0 to M_1 and maps $F\colon W \to X \subset I$, $B\colon \nu_N \to \xi \oplus 1$ extending (f_i, b_i). The set of normal maps with target (X, w) is denoted $T(X, w)$. Note that from the discussion above, each bundle ξ occurring in a degree 1 normal map must be fibre homotopy equivalent to ν_x (such a ξ is called a vector bundle reduction of ν_X). The elements of $T(X, w)$ are in bijection with the union of all elements of degree 1 in $\pi_{n+k}(T(\xi))$ as ξ varies over all vector bundle reductions of ν_x.

A primary obstruction for the existence of any manifold homotopy equivalent to X is therefore the existence of some reduction of ν_x. For arbitrary Poincaré complexes X, these need not exist. Assuming that $T(X, w)$ is non-empty, we seek a procedure for determining when a normal map is normally cobordant to a homotopy equivalence.

We first notice that the set $T(X, w)$ provides a good way to keep track of the effect of surgeries. If $f\colon M \to X$ is a degree 1 map, the main observation is that the diagram

$$\begin{tikzcd} H^{n-k}(M) \arrow[d, "\cap[X]"] & H^{n-k}(X) \arrow[l, "f^*"'] \arrow[d, "\cap[X]"] \\ H_k^w(M) \arrow[r, "f_*"] & H_k^w(X) \end{tikzcd}$$

commutes. Therefore, in each dimension, f_* is split surjective and f^* is split injective. Let $K_i(f)$, (respectively $K^i(f)$) denote the i-dimensional kernel (respectively cokernel) of f_* (respectively f^*). Then $[M]\cap$ induces an isomorphism of $K^{n-i}(f)$ onto $K_i(f)$ for all $i \geq 0$. Now f is a homotopy equivalence if and only if it induces an isomorphism on π_1 and $K_i(f) = 0$ for all $i \geq 0$.

Furthermore, if $b\colon \nu_M \to \xi$ is a bundle map covering f and $\phi\colon S^i \to M$ is an embedding of a sphere in M with $f \circ \phi \simeq *$, then $\phi^*\nu_M = \phi^* f^*(\xi)$ is a trivial bundle. Since the tangent bundle of a sphere is trivial after stabilizing once, we see that $\phi(S^i)$ has trivial normal bundle in M if $i < [n/2]$. Therefore, starting with a degree 1 normal map, we can simplify it by elementary surgeries, to obtain:

Proposition 7.1. *A degree 1 normal map $(f, b)\colon M^n \to X$ is normally cobordant to an $[n/2]$-connected normal map.*

Proof. By elementary surgeries on 0 and 1 spheres we can assume that f induces an isomorphism on π_0 and π_1. By induction we assume that f is i-connected for $i+1 \leq [n/2]$. Then $\pi_{i+1}(f) \cong K_i(f)$ and any element is represented by an embedded i-sphere with trivial normal bundle. We perform an elementary surgery on this class. Since we have used the normal bundle trivialization arising from an extension of $f \circ \phi$ over D^{i+1}, the bundle map b extends over the trace of the surgery. □

When we do surgery on an i-sphere, the homology class in $K_i(f)$ carried by this sphere is eliminated, but a dual class in dimension $(n-i-1)$ is introduced. If $i < [n/2]$ the new class is in dimension $\geq [n/2]$, so progress can be made easily. It remains to discuss the middle dimensions.

Note that if $n = 2k$ and we do surgery on a trivial $S^{k-1} \times D^{i+1}$(*i.e.* contained in a 2k-disk in M), the result is to replace M by $M \# S^k \times S^k$. Similarly, if $n = 2k+1$ and we surger $S^k \times D^{k+1} \subset D^{2k+1}$, we get $M \# S^k \times S^{k+1}$.

If $n = 2k$, it is no longer true that every class in $K_k(f)$ is represented by an embedded sphere with trivial normal bundle. Since $L = K_k(f)$ is the single non-trivial homology group of the chain complex $C_*(f)$ of the mapping case, it follows that L is a stably-free finitely generated Λ-module. By surgering on some trivial $(k-1)$-spheres, we may assume L is a free n-module. So is $K^k(f) \cong Hom_\Lambda(K_k(f), \Lambda)$, where the isomorphism is given by Poincaré duality. This gives a $(-1)^k$-hermitian pairing

$$\lambda\colon L \times L \to \Lambda$$

induced by intersection numbers, which will now be described more geometrically following [47, Chap. 5]. From the discussion, a new algebraic structure emerges - the notion of a quadratic refinement for the intersection pairing.

According to a theorem of Haefliger, regular homotopy classes of immersions $\phi\colon S^k \to M^{2k}$ correspond bijectively (by the tangent map) to stable homotopy classes of stable bundle monomorphisms $\tau_{S^k} \to \phi^*\tau_M$. We represent elements of $K_k(f)$ by immersions equipped with a path in M joining a fixed base point $x_0 \in M$ to $\phi(p_0)$, where $p_0 \in S^k$ is a base point. These immersions may be chosen so that the Euler class of the normal bundle is trivial. Note that $\pi_1(M, x_0)$ acts on such an immersed sphere by composing the path with a loop at x_0.

Suppose that S_1 and S_2 are two immersed k-spheres in M, meeting transversely in a finite set of points p. To each point P we assign a fundamental group element g_P and an orientation $\epsilon_P = \pm 1$. The Λ-valued intersection form is defined by

$$\lambda(S_1, S_2) = \sum_p \epsilon_P g_P.$$

This is related to the ordinary intersection form $\lambda_0\colon L\times L\to \mathbf{Z}$ by the formula

$$\lambda(x,y)=\sum_{g\in\pi_1}\lambda_0(x,yg^{-1})g$$

The same procedure can be used to define the self-intersection of an immersed sphere S_1 (in general position). At each intersection point P, after an order of the branches is chosen, the quantities ϵ_P and g_P are defined as before. If the order is interchanged, $\epsilon_P g_P$ becomes $(-1)^k w(g_P)\epsilon_P g_P^{-1} = (-1)^k\epsilon_P\bar{g}_P$ (using the notation introduced before for the anti-involution). Therefore, the self-intersection defines a map

$$\mu\colon L\to \Lambda/I_k\ .$$

where $I_k := \{\nu-(-1)^k\bar{\nu}\,|\,\nu\in\Lambda\}$.

Theorem 7.2. *The properties of the quadratic form (L,λ,μ) are given by:*

(i) *For $x\in L$ fixed, $y\to\lambda(x,y)$ is a Λ-homomorphism $L\to\Lambda$.*
(ii) $\lambda(y,x)=(-1)^k\overline{\lambda(x,y)}$, *for $x,y\in L$.*
(iii) $\lambda(x,x)=\mu(x)+(-1)^k\overline{\mu(x)}$, *for $x\in L$.*
(iv) $\mu(x+y)-\mu(x))-\mu(y)=\lambda(x,y)$, *for $x,y\in L$.*
(v) $\mu(xa)=\bar{a}\mu(x)a$, *for $x\in L$, $a\in\Lambda$.*
(vi) *If $k\geq 3$, the class x is represented by an embedding if and only if $\mu(x)=0$.*

The assumption the $k\geq 3$ in the last property is critical for the whole theory. M. Freedman's celebrated Field's Medal work on the Disk Theorem for topological 4-manifolds [15], [16] deals with the case $k=2$ for special fundamental groups (including finite fundamental groups). Notice that the first two properties just say that (L,λ) is a $(-1)^k$-hermitian form. The new algebraic ingredient is the quadratic refinement μ. By property (iv) the quadratic form determines the associated hermitian form. Note that μ takes values in an abelian group Λ/I_k, and that the action $a\mapsto\bar{a}\mu a$ is well-defined for $a\in\Lambda$, $\mu\in\Lambda/I_k$ (independent of the choice of lift for μ. Also the map $\Lambda/I_k\to\Lambda$ as in (iii) given by $\mu\mapsto\mu+(-1)^k\bar{\mu}$ is computed by taking any lift of μ to Λ.

The definition of quadratic form (L,λ,μ) makes sense for modules over arbitrary rings R with involution. We say that two quadratic forms (L,λ,μ) and (L',λ',μ') are isomorpic if there is an isometry $f\colon L\to L'$ of the hermitian forms λ, λ' such that $\mu'\circ f=\mu$. A hyperbolic form in this setting is one that is isomorphic to $\mathbf{H}(\Lambda^n)=\Lambda^n\oplus\Lambda^n$ with μ vanishing on the direct summands $\Lambda^n\oplus 0$ and $\oplus\Lambda^n$.

Note that if R is a field of characteristic not 2, with trivial involution, the relation $\lambda(x,x)=\mu(x)+(-1)^k\overline{\mu(x)}$ shows that a $\frac{1}{2}\lambda(x,x)=\mu(x)$. Therefore, in the symmetric case the quadratic form is determined by the associated hermitian form. On the other hand, if R has characteristic 2, there is a difference between quadratic and hermitian forms.

Example 7.3. Let $R = \mathbf{F}_2$ and (L, λ, μ) a non-singular quadratic form on a free $\mathbf{F}$-modules L. The associated hermitian form λ is always hyperbolic: let $\{e_1, \ldots, e_n; f_1, \ldots, f_n\}$ be a hyperbolic basis for L with $\lambda(e_i, e_j) = \lambda(f_i, f_j) = 0$ and $\lambda(e_i, f_j) = \delta_{ij}$. The Arf invariant $c(L, \lambda, \mu) \in \mathbf{Z}/2$ is defined by the formula

$$c(L, \lambda, \mu) = \sum \mu(e_i)\mu(f_i)$$

Theorem 7.4 (Arf). *The Arf invariant is an isometry invariant of the quadratic forms over $\mathbf{F}_2$, and additive under orthogonal direct sums. A form (L, λ, μ) is hyperbolic if and only if $c(L, \lambda, \mu) = 0$.*

□

To relate the algebra of quadratic forms to the problem of eliminating $K_k(f)$, we make the following two geometric observations.

(i) If $(f, b)\colon M \to X$ is normally cobordant to a homotopy equivalence, then (L, λ, μ) contains a free-direct summand L_0 such that $L_0 = L_0^{\perp}$ and $\mu(L_0) = 0$. This is called a subkernel. An easy algebraic argument implies that a quadratic form contains a subkernel if and only if it is isomorphic to an orthogonal direct sum of hyperbolic planes (these are free Λ-modules of rank 2 with base $\{x, y\}$, $\mu(x) = \mu(y) = 0$ and $\lambda(x, y) = 1$).

(ii) A hyperbolic plane can be removed from (L, λ, μ) by surgery on one of the basis elements if $k \geq 3$. The picture to keep in mind here is the "plumbing" of two copies of $S^k \times D^k$, which just $S^k \times S^k - D^{2k}$, and has boundary S^{2k-1}.

These points motivate the definition of the even-dimensional surgery obstruction group $L_{2k}(\mathbf{Z}[\pi_1 X], w)$: the stable isomorphism classes of $(-1)^k$-quadratic forms (L, λ, μ) on free Λ-modules L, modulo hyperbolic forms. Here "stable isomorphism" means that the forms become isomorphic after adding hyperbolics. Similarly we can define $L_{2k}(R, \alpha)$ for any ring R with involution α.

The odd-dimensional case leads to a more complicated situation. Suppose that $(f, b)\colon M^{2k+1} \to X$ is a degree 1 normal map with $K_i(f) = 0$ for $i < k$. Choose a set of generators $\phi_j\colon S^k \times D^{k+1} \to M$ (each joined by a path to the base-point) for $K_k(f)$ as a Λ-module. These may be assumed to have disjoint images in M, so let U be the union of the images and $M_0 = M - U$. We assume further that $f(U) = * \in X$, and $X = X_0 \cup D^{2k+1}$ where $(X_0, \partial X_0)$ is a finite Poincaré pair. We can then obtain a map of triads

$$f\colon (M, M_0, U) \to (X, X_0, D^{2k+1}),$$

leading to the diagram (see [47, Chap. 6]):

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr, bend left] \arrow[dr] & & K_{k+1}(U,\partial U) \arrow[rr, bend left] \arrow[dr] & & K_k(M_0) \arrow[rr, bend left] \arrow[dr] & & 0 \\
& K_{k+1}(M) \arrow[ur] \arrow[dr] & & K_k(\partial U) \arrow[ur] \arrow[dr] & & K_k(M) \arrow[ur] \arrow[dr] & \\
0 \arrow[rr, bend right] \arrow[ur] & & K_{k+1}(M_0,\partial U) \arrow[rr, bend right, "\tau"'] \arrow[ur] & & K_k(U) \arrow[rr, bend right] \arrow[ur] & & 0
\end{tikzcd}$$

Now $\partial U \approx \#\,(S^k \times S^k)_i$, so the term $K_k(\partial U)$ supports a hyperbolic form with two standard subkernels $K_{k+1}(U, \partial U)$ and $K_k(U)$. Furthermore $K_{k+1}(M_0, \partial U)$ is also a subkernel in $K_k(\partial U)$. The main observation is that $K_k(M)$ and $K_{k+1}(M)$ are trivial if and only if $K_{k+1}(M_0, \partial U)$ is a complementary subkernel to $K_{k+1}(U, \partial U)$ for some choice of the $\{\phi_i\}$. In the diagram above, this is equivalent to τ being an isomorphism.

The discussion so far suggests that the relevant data is $(\mathbf{H}(\Lambda^r), L_0, L_1)$ where $\mathbf{H}(\Lambda^r)$ is the hyperbolic form $(\Lambda^r \oplus \Lambda^t, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ and L_0, L_1 are two subkernels. This is correct and the precise definition of this "formation" structure are due to Ranicki, following earlier work of Mischenko. For our purposes, the original definition of Wall for $L_{2k+1}(\mathbf{Z}[\pi_1 X], w)$ is more convenient. It rests on an algebraic fact:

Lemma 7.5. *If L_0, L_1 are subkernels in a quadratic form (L, λ, μ), then any Λ-module isomorphism $\theta\colon L_0 \to L_1$ extends to an isometry of (L, λ, μ).*

Let $SU_r(\Lambda)$ denote the group of isometries of the standard hyperbolic form $\mathbf{H}(\Lambda^r)$, and $TU_r(\Lambda)$ the subgroup leaving the subkernel $\Lambda^r \oplus 0$ invariant. A detailed analysis of the construction above, shows that there is a well-defined invariant after allowing for

(i) stabilization: $SU_r(\Lambda) \subset SU_{r+1}(\Lambda) \subset \cdots \subset SU(\Lambda)$.
(ii) the action of $TU_r(\Lambda) \subset TU_{r+1}(\Lambda) \subset \cdots \subset TU(\Lambda)$.
(iii) interchanging $\Lambda^r \oplus 0$ and $0 \oplus \Lambda^r$.

Let $\sigma = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix} \in SU_1(\Lambda)$ and let $RU(\Lambda)$ be the subgroup of $SU(\Lambda)$ generated by σ and $TU(\Lambda)$. Then surgery to a homotopy equivalence is possible, if and only if the automorphism relating $K_{k+1}(U, \partial U)$ to $K_{k+1}(M_0, \partial U)$ is equivalent to $\sigma \oplus \sigma \oplus \cdots \oplus \sigma$ (the automorphism in (iii) above) under the 2-sided action of $RU(\Lambda)$. Wall finally proves (with the aid of a remarkable identity) that $RU(\Lambda) \supset [SU(\Lambda), SU(\Lambda)]$ and so

$$L_{2k+1}(\mathbf{Z}[\pi_1 X], w) := SU(\Lambda)/RU(\Lambda)$$

is an abelian group.

The main outcome of this analysis is the surgery exact sequence:

Theorem 7.6 (Browder-Novikov-Sullivan-Wall). *If (X, w) is a finite Poincaré complex of formal dimension $n \geq 5$, there is an exact sequence (of groups and pointed sets)*

$$L_{n+1}(\mathbf{Z}[\pi_1 X], w) \to \mathcal{S}_n(X) \xrightarrow{\eta} T(X, w) \xrightarrow{\lambda} L_n(\mathbf{Z}[\pi_1 X], w).$$

In the further development of geometric surgery, one shows that $L_n \cong L_{n+4}$ (geometrically this is just crossing a surgery problem with CP^2 in domain and range) and studies the maps in the surgery exact sequence. The set $T(X, w)$ is related to classifying spaces for topological bundles and spherical fibrations. The "assembly map" description of the surgery obstruction map λ will be described in the next section.

One variation of the whole setup which is important for the applications is to take account of Whitehead torsion. This idea is due to S. Cappell. The definition for $\mathcal{S}_n(X)$ is given in terms of homotopy equivalences $f\colon M \to X$. Since a homotopy equivalence has a torsion $\tau(f) \in Wh(\mathbf{Z}[\pi_1 X])$, we could define $\mathcal{S}_n^U(X)$ for subgroups $U \subseteq Wh(\mathbf{Z}[\pi_1 X])$ by requiring that all torsions lie in U. Notice that Poincaré duality imposes the condition $\tau(f) = (-1)^n \overline{\tau(f)}$ so it is natural to suppose that U is an involution-invariant subgroup. If two homotopy equivalences f_0, f_1 are normally cobordant, then $\tau(f_0) - \tau(f_1) = v + (-1)^n \bar{v}$, for some $v \in Wh(\mathbf{Z}[\pi_1(x)]$.

The definition of the surgery obstruction group must be modified by choosing bases for our free modules, and then requiring that any isomorphisms which occur have torsions in U. The special choices $U = \{0\}$ and $U = Wh(\mathbf{Z}\pi)$ are denoted L^s and L^h respectively. If $U \subseteq V$ are involution-invariant subgroups of $Wh(\mathbf{Z}[\pi])$ then there is a long exact sequence

$$\cdots \to H^{n+1}(\mathbf{Z}/2, V/U) \to L_n^U(\mathbf{Z}\pi, w) \to L_n^U(\mathbf{Z}\pi, w) \to H^n(\mathbf{Z}/2; V/U) \to$$

8. Computation of L-groups

In order to compute the surgery obstruction groups $L_*(\mathbf{Z}G, w)$ for finite groups G, we want to take advantage of the fact that the L-groups are algebraically defined, so we have groups $L_*(R, \alpha)$ for any ring with involution. To use this generality effectively, we would first like to establish the methods already described for K-theory, namely reducing to more tractable rings via exact sequences arising from pullback squares or the arithmetic square. However, operations such as change of rings, which are natural algebraically have no geometric analogue, so it isn't clear that any purely algebraic calculation can give usable geometric information. The algebraic theory of surgery developed by Ranicki, based on the work of Wall and Mischenko, answers both of these objectives.

The algebraic theory of surgery starts from the notion of an algebraic Poincaré complex. This is a chain complex (C, d) of finitely-generated projective modules over a ring (R, α) with involution

$$C_n \xrightarrow{d} C_{n-1} \to \cdots \to C_1 \xrightarrow{d} C_0,$$

together with a collection of R-module maps

$$\varphi_s \colon C^{n-r+s} \to C_r \ (s \geq 0)$$

such that

$$d\varphi_s + (-1)^r \varphi_s d^* + (-1)^{n+s-1}(\varphi_{s-1} + (-1)^s T\varphi_{s-1}) = 0$$

and such that the chain map

$$\varphi_0 \colon C^{n-*} \to C_*$$

is a chain equivalence. Here C^{n-*} is the dual complex (shifted by n) and T is the duality involution

$$\begin{array}{rccc} T\colon & \mathrm{Hom}_R(C^p, C_q) & \to & \mathrm{Hom}_R(C^q, C_p) \\ & \varphi & \mapsto & (-1)^{pq}\varphi^* \end{array}$$

The map φ_0 induces the Poincaré duality isomorphisms $H^{n-r}(C) \to H_r(C)$, φ_1 is a chain homotopy between φ_0 and $T\varphi_0$, and so on.

If $(f, b) \colon M \to X$ is a degree 1 normal map, then the kernel complex $C(f)$ has the structure of an algebraic Poincaré complex. Furthermore, the bundle map b gives in a natural way, a quadratic refinement of this structure (a "quadratic Poincaré complex") which determines the surgery obstruction. These definitions generalize those of forms and formations. For example, an algebraic Poincaré complex of dimension zero is just a non-singular hermitian form on the projective module C_0, and a quadratic Poincaré complex of dimension zero is just a non-singular quadratic form on C_0.

One of the main results of the algebraic theory is the description of $L_n(R, \alpha)$ as the cobordism group of algebraic n-dimensional quadratic Poincaré complexes. There is no difficulty in replacing projective R-module chain complexes by free chain complexes, but we apparently lose the possibility of Whitehead torsion variant L-groups since the Whitehead group is only defined for group rings. However if $\widetilde{U} \subseteq \widetilde{K}_1(R) := K_1(R)/\{\pm 1\}$ is an involution-invariant subgroup, the groups $L_*^{\widetilde{U}}(R, \alpha)$ are defined as the cobordism groups of complexes with $\tau(\varphi_0) \in \widetilde{U}$. This is consistent with our previous definitions for group rings. For example, if $R = \mathbf{Z}G$ and $\widetilde{U} = \{\pm G^{ab}\}$, then

$$L_n^s(\mathbf{Z}G) = L_n^{\widetilde{U}}(R)$$

since $Wh(\mathbf{Z}G) = K_1(\mathbf{Z}G)/\{\pm G^{ab}\}$.

If we add to our chain complexes the requirement that the Euler characteristic $\chi(C) = 0$, then we can define variant L-groups $L_n^U(R,\alpha)$ based on involution-invariant subgroups $U \subseteq K_1(R)$. The extreme cases $U = \{0\}$ and $U = K_1(R)$ are denoted L^S and L^K respectively. These L-groups are well-behaved under products and Morita equivalence.

They are related to the previous groups by an exact sequence,

$$0 \to L_{2k}^U(R,\alpha) \to L_{2k}^{\tilde{U}}(R,\alpha) \to \mathbf{Z}/2 \to L_{2k-1}^U(R,\alpha) \to L_{2k-1}^{\tilde{U}}(R,\alpha) \to 0$$

When $R = \mathbf{Z}G$ and $U = K_1(R)$, $L_{2k}^K(\mathbf{Z}G) \cong L_{2k}^h(\mathbf{Z}G)$ and

$$L_{2k+1}^h(\mathbf{Z}G) = L_{2k+1}^K(\mathbf{Z}G)/ < \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix} > .$$

In terms of our original discussion of L^h this means: define L_n^K using forms of even rank if $n = 2k$, and let $L_{2k+1}^K(\mathbf{Z}G) = SU(\Lambda)/TU(\Lambda)$.

The cobordism description provides a uniform way to derive exact sequences, which can then be used for calculations. For example, if $R \to S$ is a map of rings with involution, there is a long exact sequence

$$\cdots \to L_n(R) \to L_n(B) \to L_n(R \to B) \to L_{n-1}(R) \to \ldots$$

The most important of these is the "Main Exact Sequence" of Wall, which is obtained from the arithmetic square.

Theorem 8.1 (Wall). *Let G be a finite group and $X = SK_1(\mathbf{Z}G) \subset K_1(\mathbf{Z}G)$ or its image in $K_1(\widehat{\mathbf{Z}}G)$. Then there is a long exact sequence*

$$\cdots \to L_{n+1}^S(\widehat{\mathbf{Q}}G) \to L_n^X(\mathbf{Z}G) \to L_n^X(\widehat{\mathbf{Z}}G) \oplus L_n^S(\mathbf{Q}G) \to L_n^S(\widehat{\mathbf{Q}}G) \ldots$$

For geometric surgery problems, we must have the freedom to change our Λ-bases for $C_i(f)$ by elements $g \in G$. This means that the smallest geometrically relevant torsion decoration containing $X = SK_1(\mathbf{Z}G)$ is

$$Y = SK_1(\mathbf{Z}G) \oplus \{\pm G^{ab}\} .$$

Then there are natural maps,

$$L_n^s(\mathbf{Z}G) \to L_n^{\tilde{Y}}(\mathbf{Z}G) \to L_n^h(\mathbf{Z}G),$$

so that $L_n^{\tilde{Y}}(\mathbf{Z}G)$ is "intermediate", between the two L-groups of most geometric significance.

It is worth remarking that the L-groups $L_n^p(\mathbf{Z}G)$ based on projective Λ-module chain complexes also have some geometric use. In fact, if $(f,b)\colon M^n \to X$ is a degree 1 normal map and X is a finitely dominated (but not necessarily finite) Poincaré complex, then a surgery obstruction $\lambda(f,b)$ is defined in

$L^p_n(\mathbf{Z}[\pi_1 X])$. Moreover when $n \geq 5$, $\lambda(f,b) = 0$ if and only if the product normal map $(f,b) \times 1\colon M \times S^1 \to X \times S^1$ is normally cobordant to a homotopy equivalence. In addition, the projective L-groups (and their generalizations) are the natural obstruction groups for surgery on non-compact manifolds. The version of this setting which incorporates bounded or controlled surgery problems has been particularly useful (see [37], [14]).

The projective L-groups can also be studied by an arithmetic sequence. If $L^P_n(\mathbf{Z}G)$ denotes the L-groups with the added condition $\chi = 0$, then

Let G be a finite group. Then there is an exact squence

$$\cdots \to L^K_{n+1}(\widehat{\mathbf{Q}}G) \to L^P_n(\mathbf{Z}G) \to L^K_n(\widehat{\mathbf{Z}}G) \oplus L^K_n(\mathbf{Q}G) \to L^K_n(\widehat{\mathbf{Q}}G) \to \dots$$

The arithmetic exact sequences relate the computation of surgery obstruction groups to the L-theory of rings with much better algebraic properties. For example, $\mathbf{Q}G = \prod M_{n_i}(D_i)$ where the D_i are skew fields (Wedderburn's theorem) and

$$L^K_n(\mathbf{Q}G) = \prod L^K_n(D_i, \alpha_i),$$

by invariance under products and Morita equivalence. The terms $L^K_n(D_i, \alpha_i)$ must be interpreted with some care: our involution α on $\mathbf{Q}G$, induces an involution n the centre of each invariant factor $A_i = M_{n_i}(D_i)$, however in the transition from forms over A_i to forms over D_i a change of symmetry can occur. Nevertheless the product decomposition formula suggests that we should use the rational representation theory of G in a systematic way to organize and simplify the calculation.

The basic building blocks for character theory are the p-hyperelementary groups: extensions

$$1 \to C \to G \to P \to 1$$

where C is cyclic of order prime to p and P is a p-group.

Theorem 8.2 (Dress Induction). *Let G be a finite group and $U \subseteq K_1(\mathbf{Z}G)$ an involution-invariant subgroup. Then $L^U_n(\mathbf{Z}G)$ can be computed in terms of $\{L^U_n(\mathbf{Z}H) \mid H \subseteq G$ is 2-hyperelementary$\}$.*

This result means in particular that the sum of all the restricton maps $L^U_n(\mathbf{Z}G) \to L^U_n(\mathbf{Z}H)$ to the 2-hyperelementary subgroups is an injection. Therefore to decide whether a surgery obstruction is zero it is sufficient to restrict to these groups. Notice that for a normal map, restriction to a proper subgroup is given geometrically by taking a finite covering of the normal map. Dress induction exploits the Mackey functor structure on the L-groups modelled on classical induction and restriction of representations. This additional structure is a powerful tool for calculations.

Even with the help of character theory, the group $L^K_n(\mathbf{Q}G)$ is not easy to study. For example, $L^K_0(\mathbf{Q}G)$ is not finitely-generated! A classical remedy is the "local-global" comparison or "Hasse principle". This can be incorporated into our formulation by setting

$$CL_n^U(\mathbf{Q}G) = L_n^U(\mathbf{Q}G \to \widehat{\mathbf{Q}}G \oplus \mathbb{R}G)$$

for $U \subseteq K_1(\mathbf{Q}G)$, and rewriting the arithmetic sequences for example as

$$\cdots \to CL_{n+1}^K(\mathbf{Q}G) \to L_n^P(\mathbf{Z}G) \to L_n^K(\widehat{\mathbf{Z}}G) \oplus L_n^K(\mathbb{R}G) \to CL_n^K(\mathbf{Q}G) \dots$$

$$\cdots \to CL_{n+1}^S(\mathbf{Q}G) \to L_n^X(\mathbf{Z}G) \to L_n^X(\widehat{\mathbf{Z}}G) \oplus L_n^S(\mathbb{R}G) \to CL_n^S(\mathbf{Q}G) \dots$$

The computation of the $CL_n^U(D)$ for D a division algebra (with involution) is the deepest part of the theory, and involves methods from Galois cohomology (see Kneser's Tata Institute notes).

Let us consider now the other terms in the arithmetic sequence. For $L_n^K(\mathbb{R}G)$ we have an immediate expression (via character theory) in terms of the most classical calculations in quadratic forms, namely forms over $\mathbb{R}$, $\mathbb{C}$ and (the quaternions) $\mathbf{H}$. For these cases, the signature, discriminant (and Pfaffian for L^S) give a complete list of invariants.

The term $L_n^K(\widehat{\mathbf{Z}}_pG)$ also reduces to quadratic forms over fields, since the L^K-groups have the property that

$$L_n^K(\widehat{\mathbf{Z}}_pG) = L_n^K(\widehat{\mathbf{Z}}_pG/J_pG)$$

where $J_pG \subseteq \widehat{\mathbf{Z}}_pG$ is the Jacobson radical. The quotient ring is finite and semi-simple, so we reduce via Morita equivalence to the L^K-groups of finite fields. In odd characteristic, the discriminant and Pfaffian are sufficient invariants; in characteristic 2 we must add in the Arf invariant. We remark that for finite fields with non-trivial involution, the L^K-groups are zero in characteristic 2 and the L^S-groups are all zero.

The corresponding term $L_n^X(\widehat{\mathbf{Z}}_pG)$ in is also easy when $p \neq 2$. For p odd,

$$L_n^X(\widehat{\mathbf{Z}}_pG) \cong L_n^S(\widehat{\mathbf{Z}}_pG) \xrightarrow{\approx} L_n^S(\widehat{\mathbf{Z}}_pG/J_pG)$$

and we have L^S-groups of finite fields. If $p = 2$, there is an exact sequence

$$\to H^{n+1}(K_1(\widehat{\mathbf{Z}}_2G)/X) \to L_n^X(\widehat{\mathbf{Z}}_2G) \to L_n^K(\widehat{\mathbf{Z}}_2G) \to \dots$$

so the new difficulty is the left-hand term and determining the maps in the exact sequence (see [23] for more information).

Even if we completely understand the terms $L_n^K(\hat{\mathbf{Z}}G)$ or $L_n^X(\hat{\mathbf{Z}}G)$, a problem still remains. If p divides $|G|$, the map

$$L_n^K(\widehat{\mathbf{Z}}_pG) \to L_n^K(\widehat{\mathbf{Q}}_pG)$$

in the arithmetic sequence is badly behaved, since $\widehat{\mathbf{Q}}_pG$ splits into more factors than $\widehat{\mathbf{Z}}_pG$ and the image spreads over these factors in a complicated way. To control this problem, we introduce our final improvement in the arihmetic sequences. Let $G = \mathbf{Z}/m \rtimes \sigma$ be a 2-hyperelementary group, where m is odd and σ is a 2-group. The extension is given by a homomorphism $t\colon \sigma \to Aut(\mathbf{Z}/m)$. For each $d \mid m$, let

$$R(d) = \mathbf{Z}[\zeta_d]^t\sigma$$

and $S(d) = R(d) \oplus \mathbf{Q}$, $T(d) = R(d) \otimes \mathbb{R}$.

Theorem 8.3 (Hambleton-Madsen). *There is a natural direct sum splitting*

$$L^p_n(\mathbf{Z}G) = \bigoplus_{d|m} L^p_n(\mathbf{Z}G)(d)$$

such that

(i) *$L^p_i(\mathbf{Z}G)(d)$ is mapped isomorphically to $L^p_i(\mathbf{Z}[\mathbf{Z}/d \rtimes \sigma])(d)$ by the restriction map, and $L^p_n(\mathbf{Z}G)(d) = L^P_n(\mathbf{Z}G)(d)$ for $d > 1$.*

(ii) *There is an exact sequence for each $d \mid m$*

$$\to CL^K_{n+1}(S(d)) \to L^p_n(\mathbf{Z}G)(d) \to \prod_{p\nmid d} L^K_n(\hat{R}_p(d)) \oplus L^K_i(T(d)) \to CL^K_i(S(d)) \to \dots$$

The improvement that has been made here is in the local term. Now if $p \nmid 2d$ and $G = \mathbf{Z}/d \rtimes \sigma$, the map

$$L^k_n(\hat{R}_p(d)) \to L^K_n(\hat{S}_p(d))$$

splits according to the rational representations of G which are faithful on $\mathbf{Z}/d$. The remaining problem occurs for $p = 2$, in determining the map

$$L^k_n(\widehat{\mathbf{Z}}_2 \otimes \mathbf{Z}[\zeta_d]^t\sigma) \to L^k_n(\widehat{\mathbf{Q}}_2 \otimes \mathbf{Z}[\zeta_d]^t\sigma).$$

but we refer to [23] for further details.

There is also an analogue of this splitting theorem for $L^X_n(\mathbf{Z}G)(d)$, and again the only remaining "spreading" occurs at $p = 2$.

This concludes our brief outline of the techniques for the calculation of $L_*(\mathbf{Z}G)$, developed (for the most part) by C. T. C. Wall over a 10 year period. The answers for specific groups G are likely to be complicated. Here are two nice cases.

Example 8.4. Kervaire and Milnor calculated the L-groups of the trivial group

$$L^s_n(\mathbf{Z}) = 8\mathbf{Z}, 0, \mathbf{Z}/2, 0 \quad \text{for } n = 0, 1, 2, 3 \bmod 4\ ,$$

where the non-zero groups are detected by the signature or Arf invariant, and the notation $8\mathbf{Z}$ means that the signature can take on any value $\equiv 0 \bmod 8$. More geometrically, the generator in dimension $4k \geq 8$ is represented by the Milnor manifold surgery problem $(f, b)\colon M^{4k} \to S^{4k}$. Here $M^{4k} = W^{4k}(E_8) \cup D^{4k}$ is the closed topological manifold obtained by adjoining a disk D^{4k} to the boundary of the smooth plumbing manifold $W^{4k}(E_8)$ (see [24]). The boundary $\partial W^{4k}(E_8)$ is a homotopy sphere of dimension $4k - 1$, and is homeomorphic to S^{4k-1} by the generalized Poincaré conjecture (proved by S. Smale). Alternately, $W^{4k}(E_8)$ is the Brieskorn variety given by the points $z = (z_0, z_1, \ldots, z_{2k})$ in $\mathbb{C}^{2k}$ satisfying the two equations

$$z_0^3 + z_1^5 + z_2^2 + \cdots + z_{2k}^2 = \epsilon$$

for a fixed small $\epsilon > 0$, and

$$|z_0|^2 + |z_1|^2 + \cdots + |z_{2k}|^2 = 1 .$$

The generator of $L_2(\mathbf{Z}) = \mathbf{Z}/2$ in dimension $4k + 2 \geq 6$ is represented by the Kervaire manifold surgery problem $(f, b)\colon K^{4k+2} \to S^{4k+2}$, where $K^{4k+2} = W^{4k+2}(A_2) \cup D^{4k+2}$ and $W^{4k+2}(A_2)$ also has a plumbing description, or as the Brieskorn variety

$$z_0^2 + z_1^2 + z_2^2 + \cdots + z_{2k+1}^2 = \epsilon$$

for a fixed small $\epsilon > 0$, and

$$|z_0|^2 + |z_1|^2 + \cdots + |z_{2k+1}|^2 = 1 .$$

The simply connected surgery obstruction in $L_{4k}(\mathbf{Z}) \cong 8\mathbf{Z}$ is defined for any degree 1 normal map $(f, b)\colon M^{4k} \to X^{4k}$, by the formula

$$\mathrm{Index}(f) = \mathrm{Index}(M) - \mathrm{Index}(X),$$

where $\mathrm{Index}(M)$ is the signature of the intersection form on $H_{2k}(M; \mathbf{Z})$. □

A. Bak made extensive computations of L-groups. One result which has been very useful in topological applications is:

Theorem 8.5 (Bak). *Let G be a finite group of odd order. Then $L^?_{2k+1}(\mathbf{Z}G) = 0$ for $? = s,\ h$ or p.*

A much more complete survey and more results in particular cases can be found in [23].

9. TOPOLOGICAL 4-MANIFOLDS WITH FINITE FUNDAMENTAL GROUP

M. Freedman [15] proved that the surgery exact sequence is valid for topological 4-manifolds with finite fundamental groups (and more generally for polycyclic-by-finite fundamental groups). In particular, Freedman proved the 5-dimensional s-cobordism theorem in this setting. At the same time, S. Donaldson showed how the Yang-Mills gauge theory could give new information about smooth 4-manifolds, and demonstrated that smooth 5-dimensional s-cobordisms need not be products. The combination of these two dramatic developments led to an exciting period of discovery, in which many of the long-standing open problems in the topology of 4-manifolds were settled. In this section, we will stick to applications of surgery theory and topological 4-manifolds.

Theorem 9.1 (Freedman). *Let X be a closed, simply connected, topological 4-manifold. Then X is classified up to homeomorphism by the intersection form on $H_2(X;\mathbf{Z})$ and the Kirby-Siebenmann invariant.*

The Kirby-Siebenmann invariant (in $\mathbf{Z}/2$) is the obstruction to finding a PL-structure on a closed topological 4-manifold. Freedman's classification can be generalized by geometric "cancellation" techniques, based on the unitary analogue of cancellation for modules, for 4-manifolds with finite fundamental groups.

We say that two closed topological 4-manifolds X and $Y$are *stably homeomorphic* if there exists a homeomorphism

$$h\colon X \sharp r(S^2 \times S^2) \approx Y \sharp r(S^2 \times S^2)$$

for some integer r. The cancellation problem is to remove copies of $S^2 \times S^2$, or in other words to determine the minimum r for which the two sides are homeomorphic. There is also a version of this problem for smooth manifolds (about which almost nothing is known !).

Theorem 9.2 (Hambleton-Kreck). *Let X and Y be closed, oriented topological 4-manifolds with finite fundamental group. Suppose that the connected sum $X \sharp r(S^2 \times S^2)$ is homeomorphic to $Y \sharp r(S^2 \times S^2)$. If $X = X_0 \sharp (S^2 \times S^2)$, then X is homeomorphic to Y.*

Note that the assumption that X splits off one $S^2 \times S^2$ cannot be omitted in general. There are, for example, even simply–connected closed topological 4-manifolds which are stably homeomorphic but not homeomorphic because they have non–isometric intersection forms.

There is now a fairly clear two part strategy for classifying 4-manifolds. First we try to classify up to stable homeomphism, and then we apply cancellation. For the first part, there are fairly explicit results, but complete answers for the second part are available only for special fundamental groups (e.g. π_1 cyclic).

The following notation is useful for keeping track of the second Stiefel-Whitney data of a manifold X. We say that X has *w_2--type*:

(I) if $w_2(\widetilde{X}) \neq 0$.
(II) if $w_2(X) = 0$.
(III) if $w_2(\widetilde{X}) = 0$ and $w_2(X) \neq 0$.

Theorem 9.3 (Hambleton-Kreck). *Let X be a closed, oriented 4-manifold with finite cyclic fundamental group. Then X is classified up to homeomorphism by the fundamental group, the intersection form on $H_2(X;\mathbf{Z})/Tors$, the w_2-type, and the Kirby-Siebenmann invariant. Moreover, any isometry of the intersection form can be realized by a homeomorphism. The invariants can all be realized independently, except in the case of w_2-type II, where the Kirby-Siebenmann invariant is determined by the intersection form.*

Note that we do not assume any stability condition here, so the proof requires a sharper version of the cancellation theorem. In the remainder of the section, we will describe M. Kreck's approach to the stable classification. Further details about proofs of the above results can be found in [22].

There is a close analogy between the stable classification of homotopy types of 2-complexes (as discussed in Section 5) and stable homeomorphism types of 4-manifolds. Consider the thickening functor from finite 2-complexes to closed 4-manifolds, obtained by embedding a 2-complex K as polyhedron in $\mathbb{R}^5$ and taking the boundary of a smooth regular neighborhood. If two 2-complexes are simply homotopy equivalent the corresponding 4-manifolds are s-cobordant (implying homeomorphic, if the fundamental groups are poly-(finite or cyclic) [16]) and we denote the corresponding s-cobordism class by $M(K)$. If we replace the 2-complex by its 1-point union with S^2, the corresponding 4-manifold changes by connected sum with $S^2 \times S^2$. This indicates the analogy of stable equivalence classes of 2-complexes with the following notation for 4-manifolds.

Since the smooth stable s-cobordism theorem (implying that two s-cobordant 4-manifolds are stably diffeomorphic) holds, the stable diffeomorphism class of $M(K)$ is determined by the stable simple homotopy class of K and so, (see §1) by $\pi_1(K)$.

However, the stable classification of 4-manifolds (in contrast to the situation for 2-complexes) needs more invariants than the fundamental group and the Euler characteristic. For example, we must include the orientation, signature, and existence of a spin-structure. To obtain a complete answer we express the stable classification as a bordism problem and compute the bordism groups by the Atiyah-Hirzebruch spectral sequence.

Let $c\colon X \to K(\pi,1)$ be the classifying map of the universal covering $\widetilde{X}$, where $\pi := \pi_1(X, x_0)$. The Kirby-Siebenmann invariant of X will be denoted $KS(X)$. There is an isomorphism $c^*\colon H^1(\pi;\mathbf{Z}/2) \to H^1(X;\mathbf{Z}/2)$ and an exact sequence

$$0 \to H^2(\pi;\mathbf{Z}/2) \xrightarrow{c^*} H^2(X;\mathbf{Z}/2) \to H^2(\tilde{X};\mathbf{Z}/2)\ .$$

Thus we can always pull back $w_1(X)$ by c from a class denoted $w_1 \in H^1(\pi;\mathbf{Z}/2)$, and we can pull back $w_2(X)$ from a class denoted $w_2 \in H^2(\pi;\mathbf{Z}/2)$, if $w_2(\tilde{X}) = 0$.

For a smooth 4-manifold X, the normal 1-type of X is a fibration $p\colon B(\pi, w_1, w_2) \to BO$. If $w_2(\widetilde{X}) \neq 0$, then $B(\pi, w_1, w_2) = K(\pi,1) \times BSO$ and p is given by the composition

$$p\colon K(\pi,1) \times BSO \xrightarrow{E\times i} BO \times BO \xrightarrow{\oplus} BO,$$

wherer $E\colon K(\pi,1) \to BO$ is the classifying map of the stable line bundle given by w_1, $i\colon BSO \to BO$ is the inclusion, and and $\oplus$ is the H-space structure on BO given by the Whitney sum.

If $w_2 \neq \infty$ we define the normal 1-type as the fibration $p : B(\pi, w_1, w_2) \longrightarrow BO$ given by the following pullback square

$$\begin{tikzcd}
B(\pi, w_1, w_2) \arrow[r] \arrow[d, "p"] & K(\pi, 1) \arrow[d, "w_1 \times w_2"] \\
BO \arrow[r, "\hat{w}_1 \times \hat{w}_2"] & K(\mathbf{Z}/2, 1) \times K(\mathbf{Z}/2, 2)
\end{tikzcd}$$

where $\hat{w}_i := w_i(EO)$ are the Stiefel-Whitney classes of the universal bundle and we interpret w_i as maps to $K(\mathbf{Z}/2, i)$.

If $w_1 = 0$, $B(\pi, 0, w_2)$ factorizes over BSO and we choose one of the possible lifts. To deal with the oriented case ($w_1 = 0$) and the non-oriented case simultaneously we write $p\colon B(\pi, w_1, w_2) \longrightarrow B(S)O$.

For topological manifolds one can make the obvious changes (replace the linear normal bundle by the topological normal bundle given by a map $\nu\colon X \to B(S)Top$) the normal 1-type $p\colon B(\pi, w_1, w_2) \longrightarrow B(S)Top$.

Given any fibration $B \to B(S)O$, abbreviated for short as B, we consider the B-bordism group $\Omega_n(B)$ consisting of bordism classes of closed smooth n-manifolds, which are oriented if the fibration is over BSO, together with a lift $\bar{\nu}$ over B of the classifying map $\nu\colon X \to B(S)O$ for the stable normal bundle of X. Such a lift is called a *normal 1-smoothing* if $\bar{\nu}$ is a 2-equivalence. By construction, X admits a normal 1-smoothing in $B(\pi, w_1, w_2)$. Similarly for topological manifolds one starts with a fibration $B \to B(S)Top$, and introduces the analogous bordism group of topological manifolds denoted $\Omega_n^{Top}(B)$.

Theorem 9.4 (Kreck). *Two smooth (topological) 4-manifolds X_0 and X_1 with the same normal 1-type $B(\pi, w_1, w_2)$ are stably diffeomorphic (homeomorphic) if and only if:*

(i) *they have the same Euler characteristic, and*
(ii) *they admit normal 1-smoothings $\bar{\nu}_0$ and $\bar{\nu}_1$ such that $(X_0, \bar{\nu}_0)$ and $(X_1, \bar{\nu}_1)$ represent the same bordism class in $\Omega_4(B(\pi, w_1, w_2))$ (in $\Omega_4(B^{Top}(\pi, w_1, w_2))$).*

A computation of the bordism groups now gives:

Theorem 9.5 (Kreck). *Two oriented smooth (topological) 4-manifolds X_0 and X_1 with the same fundamental group and with $w_2(\widetilde{X}_i) \neq 0$ are stably diffeomorphic (homeomorphic), if and only if they have the same Euler characteristic and signature, if $c_*[X_0] = c_*[X_1] \in H_4(K(\pi, 1); \mathbf{Z})/Out(\pi)$ and, in the topological case, $KS(X_0) = KS(X_1)$.*

10. Surgery obstruction on closed manifolds

The surgery exact sequence

$$L_{n+1}(\mathbf{Z}[\pi_1 X], w) \to \mathcal{S}_n(X) \xrightarrow{\eta} T(X, w) \xrightarrow{\lambda} L_n(\mathbf{Z}[\pi_1 X], w)$$

provides a good framework for classifying manifolds, but to obtain concrete results in particular cases we must know how to compute the maps in the sequence. In this section we will consider only the *oriented* case ($w \equiv 1$) and suppose that X is a closed oriented topological manifold of dimension $n \geq 5$, with *finite* fundamental group $\pi := \pi_1(X, x_0)$. Then $\mathcal{S}_n(X)$ is just the set of manifolds homotopy equivalent to X modulo h-cobordism.

Suppose that $(f, b)\colon M \to N$ is a degree 1 normal map of closed manifolds, or in other words, a closed manifold surgery problem. It turns out that the closed manifold surgery obstructions are very restricted, related to the low dimensional group homology $H_*(\mathbf{Z}\pi; \mathbf{Z}/2)$, while $L_n^h(\mathbf{Z}\pi)$ is usually large. We will actually obtain results about the *weakly simple* surgery obstructions in $L'_n(\mathbf{Z}\pi) := L_n^U(\mathbf{Z}\pi)$, where $U = SK_1(\mathbf{Z}\pi) \oplus \{\pm\pi^{ab}\}$. The natural map $L_n^U(\mathbf{Z}\pi) \to L_n^h(\mathbf{Z}\pi)$ shows that these results also hold for L^h.

One way to obtain closed manifold surgery problems form the cartesian product of a simply connected surgery problem with a closed manifold P in domain and range. The standard simply connected connected surgery problems are the Milnor problem and the Kervaire problem (8.4).

Theorem 10.1 ([17]). *Let P^k be a closed, oriented, topological manifold with $\pi_1(P)$ finite, and let $(f, b)\colon M^n \to N^n$ be a simply connected closed manifold surgery problem, with $n + k \geq 5$. Then the product normal map*

$$(f \times id, b \times id)\colon M \times P \to N \times P$$

is normally cobordant to a weakly simple homotopy equivalence either

(i) *for $n \equiv 2 (\mathrm{mod}\, 4)$ and $k \equiv 0 (\mathrm{mod}\, 4)$, if the Euler characteristic of P is even, or*
(ii) *for $n \equiv 0 (\mathrm{mod}\, 4)$ if $\mathit{Index}(P) = 0$.*

The most complete result is for odd dimensional surgery problems. Recall that if $\bar{\pi}$ is a subquotient of π (that is, $\bar{\pi} = \rho/\rho_0$ where $\rho_0 \triangleleft \rho \subseteq \pi$) there is a "transfer-projection" homomorphism $L'_n(\mathbf{Z}\pi) \to L'_n(\mathbf{Z}\bar{\pi})$ induced geometrically by surgery on a covering normal map. Let $C(2)$ denote the cyclic group of order 2, and $Q(2^k)$ the generalized quaternion group of order 2^k.

Theorem 10.2 ([17]). *Let N^n be a closed oriented topological manifold with $\pi_1(N)$ finite and $n \geq 5$ odd. Then a closed manifold surgery problem $(f, b)\colon M \to N$ is normally cobordant to a weakly simple homotopy equivalence if and only if:*

(i) *$n \equiv 1 (\mathrm{mod}\, 4)$ and $\lambda(f, b)$ maps to zero under transfer-projection to all quaternionic subquotients $Q(2^k)$ of $\pi_1(N)$, or*
(ii) *$n \equiv 3 (\mathrm{mod}\, 4)$ and $\lambda(f, b)$ maps to zero under transfer-projection to all $C(2)$ quotients of $\pi_1(N)$.*

A closed manifold X provides a base point for the homotopy theoretic description $T(X) \cong [X, G/TOP]$, due to Sullivan and Kirby-Siebenmann, for the set of degree 1 normal maps. The right-hand side has an abelian group structure (since G/TOP is an H-space), and the surgery obstruction gives a *homomorphism*

$$\sigma_X \colon [X, G/TOP] \to L_n^U(\mathbf{Z}\pi)$$

which can be understood in terms of a "universal" family of homomorphisms

$$\kappa_j^U \colon H_j(\pi, \mathbf{Z}/2) \to L_{j+2}^U(\mathbf{Z}\pi)_{(2)}$$

depending only on the fundamental group.

The definition of the $\{\kappa_j^U\}$ depends on the 2-local splitting $G/TOP_{(2)} = \prod_{k>0} K(\mathbf{Z}_{(2)}, 4k) \times K(\mathbf{Z}/2, 4k-2)$ given by the cohomology classes $\ell = \{\ell_{4*}\} \in H^{4*}(G/TOP, \mathbf{Z}_{(2)})$ and $k = \{k_{4*+2} \in H^{4*+2}(G/TOP, \mathbf{Z}/2)$ of Morgan-Sullivan, Rourke-Sullivan, and Milgram.

The next major ingredient is the fact that a closed topological n-manifold satisfies an enriched form of Poincaré duality, namely,

$$\cap [X]_{\mathbb{L}_0} \colon [X, G/TOP] \cong H^0(X; \mathbb{L}_0) \xrightarrow{\approx} H_n(X; \mathbb{L}_0)$$

where $\mathbb{L}_0$ is the Quinn-Ranicki connective L-spectrum with 0-th space G/TOP (see [39]). Note that the generalized homology functor on the right-hand side can be applied to any space, not just to n-manifolds. Let $c \colon X \to K(\pi, 1)$ be the classifying map of the universal covering space $\widetilde{X}$.

Theorem 10.3 (Quinn, Ranicki). *For any group π, and any integer n, there exists an assembly map*

$$A_\pi \colon H_n(\pi; \mathbb{L}_0) \to L_n^U(\mathbf{Z}\pi),$$

functorial in π. Furthermore, if X is a closed, oriented topological n-manifold, the surgery obstruction homomorphism $\sigma_X(f) = A_\pi \circ c_*(\sigma_X(f) \cap [X]_{\mathbb{L}_0})$, for all maps $f \colon X \to G/TOP$.

The L-spectrum has a 2-local splitting as above into Eilenberg-Maclane spectra, so that

$$H_n(\pi; \mathbb{L}_0) = \bigoplus_{k>0} H_{n-4k}(\pi; \mathbf{Z}_{(2)}) \times H_{n-4k-2}(\pi; \mathbf{Z}/2)$$

and the corresponding splitting of the assembly map A_π restricted to one of the $\mathbf{Z}/2$-homology summands gives κ_j^U for $j = n - 4k - 2$. It isn't obvious (but true) that another pair (n', k') with the same value $j = n' - 4k' - 2$ leads to the same homomorphism κ_j^U, after identifying $L_n \cong L_{n+4(k-k')}$ by the periodicity isomorphism (note that $n' = n + 4(k - k')$).

Let V_X denote the total Wu class of the stable normal bundle ν_X, and for any map $f \colon X \to G/TOP$ let

$$\mathrm{ARF}_j(f) = \{(V_X^2 \cup f^*(k)) \cap [X]\} \in H_j(X; \mathbf{Z}/2)$$

be the j-dimensional component of the indicated homology class. We let $\mathrm{ARF}(f)$ and $\mathrm{Index}(f)$ denote the ordinary (simply-connected) Arf invariant and index of the surgery problem given by f (considered as elements in $L_*(\mathbf{Z})$). Finally, let

$$s_r \colon H_{2^r+2}(X;\mathbf{Z}/2) \to H_4(X;\mathbf{Z}/2)$$

for $r \geq 0$ be the Hom-dual of the iterated squaring maps in cohomology.

Theorem 10.4 ([17]). *Let X be a closed, oriented topological n-manifold with finite fundamental group π. Let $U \subset \mathrm{Wh}(\mathbf{Z}\pi)$ be an involution invariant subgroup containing $Im(SK_1(\mathbf{Z}\rho) \to SK_1(\mathbf{Z}\pi))$, where $\rho \subseteq \pi$ is a 2-Sylow subgroup. For any surgery problem $f \colon X \to G/TOP$ of closed manifolds, the surgery obstruction $\sigma_X(f) \in L^U_n(\mathbf{Z}\pi)$ is equal to:*

(i) *$Index(f) + \kappa^U_2\{c_*(ARF_2(f))\}$ for $n \equiv 0(\mathrm{mod}\, 4)$.*
(ii) *$\kappa^U_3\{c_*(ARF_3(f))\}$ for $n \equiv 1(\mathrm{mod}\, 4)$.*
(iii) *$ARF(f) + \kappa^U_4\{c_*(\sum_{r\geq 0} s_r(ARF_{2^r+2}(f)))\}$ for $n \equiv 2(\mathrm{mod}\, 4)$.*
(iv) *$\kappa^U_1\{c_*(ARF_1(f))\}$ for $n \equiv 3(\mathrm{mod}\, 4)$.*

This result and the applications above are proved by factoring the κ-homomorphisms through a more computable form of L-theory, and then using the arithmetic square techniques (see [17] for more details).

11. The spherical space form problem

The classification of orthogonal spherical space forms up to isometry [48] was first proposed by Killing in 1891, and the problem attracted the attention of famous mathematicians of the time, such as Clifford, Hopf, Klein, and Poincaré. In 1925, H. Hopf's proved [25]:

Theorem 11.1 (Hopf). *The following is a list of all finite fixed-point free subgroups of $SO(4)$:*

(a) *The cyclic group $C(n)$, the generalized quaternion group $Q(4n)$, the binary tetrahedral group $T^*(24)$, the binary octahedral group $O^*(48)$, and the binary icosahedral group $I^*(120)$.*
(b) *The semidirect product $C(2n+1) \rtimes C(2^k)$ of an odd order cyclic group with a cyclic 2-group. More explicitly $C(2n+1) \rtimes C(2^k)$ is given by the presentation $\{A, B : A^{2^k} = B^{2n+1} = 1, ABA^{-1} = B^{-1}\}$ where $k \geq 2, n \geq 1$.*
(c) *A semidirect product $Q(8) \rtimes C(3^k)$ of the quaternion group $Q(8)$ with a cyclic 3-group. More explicitly, $Q(8) \rtimes C(3^k)$ is given by the presentation $\{P, Q, X : P^2 = (PQ)^2 = Q^2, X^{3^k} = 1, XPX^{-1} = Q, XQX^{-1} = PQ\}$ where $k \geq 1$. For $k = 1$, this is the binary tetrahedral group $T^*(24)$.*
(d) *The product of any of the above groups with a cyclic group of coprime order.*

At first glance, the above list may appear to be random. In the forties and fifties, efforts were made to interpret Hopf's list using group cohomology [9] and it was discovered that all these groups have periodic Tate cohomology of period four. In general, a finite group has periodic cohomology if and only if it satisfies the p^2-conditions ("any subgroup of order p^2 is cyclic") for all primes p. From the viewpoint of group theory, this condition means that the odd Sylow subgroup is cyclic and the 2-Sylow subgroup is cyclic or generalized quaternion. If the cohomology has period four then, in addition, the pq-conditions hold ("every subgroup of order pq is cyclic") for p and q distinct odd primes.

The necessity of the $2q$-conditions was established by J. Milnor [32] in 1957, when he showed that the dihedral group of order $2q$ cannot operate freely on any $\mathbf{Z}/2$-homology sphere despite the fact that it has periodic cohomology of period 4. In [32] Milnor also compiled the following list of all finite groups, not in Hopf's list (11.1.a)-(11.1.d), but satisfying the restrictions known at the time on fundamental groups of 3-manifolds.

Theorem 11.2 (Milnor). *The following are the finite groups with periodic cohomology of period 4, containing no dihedral subgroups.*

(a) *The semidirect product $Q(8n,k,l)$ of the odd cyclic group $C(kl)$ with the generalized quaternion group $Q(8n)$. More explicitly, $Q(8n,k,l)$ has the presentation: $\{X,Y,Z : X^2 = Y^{2n} = (XY)^2,\ Z^{kl} = 1,\ XZX^{-1} = Z^r,\ YZY = Z^{-1}\}$. Here n,k,l are all odd integers and relatively prime to each other, $n > k > l \geq 1$, and r satisfies $r \equiv -1 \pmod k$, $r \equiv 1 \pmod l$. If $l = 1$, we set $Q(8n,k) \equiv Q(n,k,1)$.*

(b) *The group $Q(8n,k,l)$ with the same presentation as (1.5), but with n even.*

(c) *An extension $O(48;3^{k-1},l)$ of the odd order cyclic group $C(3^{k-1}l)$, $3 \nmid l$, by the binary octahedral group $O^*(48)$. More precisely, $O(48;3^{k-1},l)$ has five generators X,P,Q,R,A and the following relations:*

$$X^{3^k} = P^4 = A^l = 1,\ P^2 = Q^2 = R^2,\ PQP^{-1} = Q^{-1}$$
$$XPX^{-1} = Q,\ XQX^{-1} = PQ,\ RXR^{-1} = X^{-1},\ RPR^{-1} = QP$$
$$RQR^{-1} = Q^{-1},\ AP = PA,\ AQ = QA,\ RAR^{-1} = A^{-1}.$$

(d) *The product of any of the above groups with a cyclic group of coprime order.*

Thus Hopf's problem is to prove that groups in the above list (11.2.a)-(11.2.d) do not act freely on homotopy 3-spheres.

In the late sixties, C. T. C. Wall asked whether Milnor's result could be interpreted using the new theory of nonsimply connected surgery. Ronnie Lee [27] answered this question in 1973 by defining a "semicharacteristic" obstruction for the problem. As well as recovering the previous result of Milnor, the semicharacteristic rules out the family of groups $Q(8n,k,l)$, n even, in (11.2.b). Later C. B. Thomas observed that this also eliminates the family of groups $O(48,3^{k-1},l)$ in (11.2.c) because groups of this type always contain a subgroup isomorphic to $Q(16,3^{k-1},1)$. These results leave undecided only the groups $Q(8n,k,l)$, n odd, in (11.2.a) and their products with cyclic groups of coprime order in (11.2.d) from Milnor's original list.

The remaining part of Hopf's problem is to prove that for any distinct odd primes p, q, the group $Q(8p, q)$ does not operate freely on any homotopy 3-sphere. Notice that a group $Q(8n, k, l)$ in the family (11.2.a) always contains a subgroup of the form $Q(8p, q)$. Hence ruling out the groups $Q(8p, q)$ would also eliminate the family (11.2.c) in Milnor's list and the corresponding products in (11.2.d).

In contrast with the 3-dimensional case, the analogous spherical space form problem in higher dimensions has been almost completely resolved. The goal is the topological (smooth) classification of finite group actions (Σ^{2n-1}, G) on (homotopy) spheres Σ^{2n-1} of dimension $2n-1$, $n \geq 3$. This problem was both a motivation and an important test case for the techniques of algebraic and geometric topology developed in the period 1960–1985. P. A. Smith had already shown in 1944 that the p^2 conditions were necessary for a G-action on any homology sphere. Conversely, Swan [43] proved:

Theorem 11.3 ([43]). *Every group with periodic cohomology acts freely and simplicially on a CW complex homotopy equivalent to a sphere.*

Given a group G with periodic cohomology of period $2d$, Swan's contruction produces finitely dominated Poincaré complexes X with $\pi_1(X, x_0) = G$, and $\widetilde{X} \simeq S^{2n-1}$, for some multiple n of d. We call these Swan complexes for short. The chain complex $C_*(X)$ gives an exact sequence (or periodic resolution) of the form

$$0 \to \mathbf{Z} \to P_{2n-1} \to \cdots \to P_1 \to P_0 \to \mathbf{Z} \to 0$$

where the C_i are finitely generated projective $\mathbf{Z}G$-modules. Two such sequences C_* and C'_* are isomorphic if there is a chain map $f\colon C \to C'$ inducing the identity on the homology groups $H_0 = H_{2n-1} = \mathbf{Z}$, and the homotopy types of X are in bijective correspondence with the isomorphism classes of the periodic resolutions. The Wall finiteness obstruction is just

$$\theta_W(X) = \sum(-1)^i[P_i] \in \widetilde{K}_0(\mathbf{Z}G)$$

and Swan discovered a beautiful formula for the difference $\theta_W(X) - \theta_W(X')$ if X, X' are two Swan complexes of the same dimension.

Any two periodic resolutions can be compared by a chain map:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z} \arrow[r] \arrow[d, "r"] & P_{2n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{Z} \arrow[r] & P'_{2n-1} \arrow[r] & \cdots \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r] & \mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

inducing a map of degree r. Equivalently, if X and Y are Swan complexes of the same dimension, there is a map $f\colon X \to Y$ of degree r between them. We let $\langle r, N\rangle \subset \mathbf{Z}G$ denote the ideal generated by the integer r and the group ring element $N = \sum\{g \mid g \in G\}$

Theorem 11.4 ([43]). *Let X and Y be Swan complexes for G of the same dimension. Then $\theta_W(Y) = \theta_W(X) + [\langle r, N\rangle] \in \widetilde{K}_0(\mathbf{Z}G)$.*

The existence and classification of Swan complexes opened the way for a systematic attack on the problem using surgery theory. Throughout the 1970's remarkable progress was made on the higher dimensional space form problem, culminating in the paper of Madsen, Thomas and Wall [29].

Theorem 11.5 ([29]). *Any finite group G satisfying the p^2 and $2p$ conditions (for all primes p) acts freely and smoothly on a homotopy sphere of some odd dimension $2n - 1 > 3$.*

The precise dimensional bounds were not determined, although for G of period $2d$ they show that $n = 2d$ is always realizable ($n = d$ is best possible).

The next big step forward was the explicit calculation by Milgram [31] in 1979 of the finiteness obstruction for some of the period 4 groups $G = Q(8p, q)$, following the method of [46]. Tensoring a periodic resolution for G with the adele ring $\widehat{\mathbf{Q}}$ allows one to define an "idelic" Reidemeister torsion invariant

$$\widehat{\Delta}(X) \in K_1(\widehat{\mathbf{Q}}G)$$

whose image under the boundary map $\partial\colon K_1(\widehat{\mathbf{Q}}G) \to \widetilde{K}_0(\mathbf{Z}G)$ gives the formula

$$\partial\widehat{\Delta}(X) = \theta_W(X)$$

Now the arithmetic square techniques can be applied to compute the finiteness obstruction in terms of units in algebraic number fields. In particular, Milgram showed that some of the groups in Milnor's list are not fundamental groups of spherical space forms in any dimension (including dimension 3).

After this followed a sequence of papers by Milgram (see the survey in [12]), and independently by Madsen [30], aiming at the calculation of the relevant surgery obstruction. Here the problem is to determine which of the groups $Q(8p, q)$ act freely on Σ^{8k+3}, for $k > 0$, since they act linearly on S^{8k+7} for all $k \geq 0$. It turned out that the answer is computable in principle, but depends sporadically on the number theory of the primes p, q. Note that the vanishing of the high-dimensional obstruction is equivalent to the existence of a free action of the corresponding group $Q(8p, q)$ on an integral homology 3-sphere.

12. Bounded K and L-theory

The next two sections give an introduction to "bounded" topology, and generalize algebraic K-theory to this setting. This algebra has many applications in topology, including the problem discussed in Section 14 of these notes.

Let M be a metric space. Assume there is a group G acting on M by eventual Lipschitz maps [36]. Recall an eventual Lipschitz map $g : M \longrightarrow M$ is a map so there exists $k, l \in \mathbb{R}_+$ so that $d(gx, gy) \leq k \cdot d(x, y) + l$. We want k and l to be independent of g.

Example 12.1. Let M be a finitely generated group exhibited with the word metric, and $G \subseteq M$ a subgroup. Then the action of G on M by conjugation is by eventual Lipschitz equivalences. Specifically if $g \in G$ has length l then $d(gxg^{-1}, gyg^{-1}) = d(gxy^{-1}g^{-1}, e) \leq 2l + d(x,y)$

Example 12.2. Let (V,G) be an orthogonal representation. Then G acts by isometries on V hence clearly by eventual Lipschitz maps.

Given M and G as above, and a commutative ring with unit R, we define a category $\mathcal{G}_{M,G}(R)$ as follows:

Definition 12.3. An object A is a right RG-module together with a map $f : A \to F(M)$, where $F(M)$ is the set of finite subsets of M, satisfying

(i) f is G-equivariant.
(ii) $A_x = \{a \in A | f(a) \subseteq \{x\}\}$ is a finitely generated RG_x-module, free as an R-module.
(iii) As an R-module $A = \bigoplus_{x \in M} A_x$.
(iv) $f(a+b) \subseteq f(a) \cup f(b)$.
(v) For each ball $B \subset M$, the subset $\{x \in B | A_x \neq 0\}$ is finite, and relatively G-compact.

A morphism $\phi : A \to B$ is a morphism of RG-modules, satisfying the following condition: there exists k so that the components $\phi_n^m : A_m \to B_n$ (which are R-module morphisms) are zero when $d(m,n) > k$. The category $\mathcal{G}_{M,G}(R)$ is an additive category in an obvious way.

Remark 12.4. When M has more than one point it follows from these conditions that $f(a) = \emptyset$ if and only if $a = 0$. When M is precisely one point, this has to be added as an extra assumption. It follows easily from the conditions that f measures exactly where an element has components. In other words, if $x_1, \ldots x_n \in M$ are different points and $a_i \in A_{x_i}$, $a_i \neq 0$ then $f(a_1 + \ldots + a_n) = \{x_1, \ldots, x_n\}$.

Given an object A, an R-module homomorphism $\phi : A \to R$ is said to be locally finite if the set of $x \in M$ for which $\phi(A_x) \neq 0$ is finite. Define $A^* = \mathrm{Hom}_R^{l.f.}(A, R)$, as the set of locally finite R-homomorphisms. We want to make $*$ a functor from $\mathcal{G}_{M,G}(R)$ to itself to make $\mathcal{G}_{M,G}(R)$ a category with involution. We define $f^* : A^* \to FM$ by $f^*(\phi) = \{x | \phi(A_x) \neq 0\}$ which is finite by assumption. The dual module A^* has an obvious left action of G, turning it into a left RG module via the formula $\phi g(a) = \phi(ga)$, and f^* is equivariant with respect to the left action on M given by $xg = g^{-1}x$. To make $*$ an endofunctor of $\mathcal{G}_{M,G}(R)$ we need to replace the left action by a right action. As is usual in surgery theory, this may be done in various ways, the standard one being to let g act on the right by letting g^{-1} act on the left. However given a homomorphism $w : G \to \{\pm 1\}$, we may let g act on the right of A^* by $w(g) \cdot g^{-1}$ on the left.

Proposition 12.5. *$(\mathcal{G}_{M,G}(R), *)$ is an additive category with involution.*

For many purposes we are more interested in the subcategory of $\mathcal{G}_{M,G}(R)$ for which all objects are free RG modules.

Definition 12.6. The subcategory of $\mathcal{G}_{M,G}(R)$ where the modules are required to be free RG modules is denoted by $\mathcal{C}_{M,G}(R)$.

It is easy to see that $*$ induces a functor on $\mathcal{C}_{M,G}(R)$, so that $\mathcal{C}_{M,G}(R)$ is a subcategory with involution.

Example 12.7. If G acts trivially on M and G is finite, then $\mathcal{C}_{M,G}(R)$ is naturally equivalent to $\mathcal{C}_M(RG)$ where RG is the category of free finitely generated based RG modules. We can also give a nice expression for the lower algebraic K-theory functors defined by Bass: $K_{-i}(\mathbf{Z}G) = K_1(\mathcal{C}_{\mathbb{R}^{i+1}}(\mathbf{Z}G))$ for $i > 0$.

Example 12.8. If G is finitely generated and $|G|$ denotes the metric space with the same underlying set as G, and the word metric, then $\mathcal{C}_{|G|,G}(R)$ is naturally equivalent to $\mathcal{C}_{pt}(RG)$ (as categories with involution). Notice it does not matter which generating set we choose for G since 2 different generating sets will give eventual Lipschitz equivalent metrics. In case G is finite, this means $\mathcal{C}_{|G|,G}(R)$ is equivalent to $\mathcal{C}_{pt,G}(R)$ which is equivalent to $\mathcal{C}_{pt}(RG)$.

Using the algebraic L-theory of additive categories with involution, as developed by Ranicki, we immediately have defined functors $L^K_n(\mathcal{C}_{M,G}(R))$ where K is some $*$ invariant subgroup of $\widetilde{K}_i(\mathcal{C}_{M,G}(R))$, $i = 0, 1$. Here $\widetilde{K}_1(\mathcal{C}_{M,G}(R)) = K_1(\mathcal{C}_{M,G}(R))/\{\pm 1\}$ and $\widetilde{K}_0(\mathcal{C}_{M,G}(R)) = (K_0(\mathcal{C}_{M,G}(R))^\wedge)/K_0(\mathcal{C}_{M,G}(R))$, where $^\wedge$ denotes idempotent completion.

Let N be a sub metric space of the metric space M. In the equivariant case, we suppose that N is an invariant subspace.

Definition 12.9. The category $\mathcal{C}^{>N}_{M,G}(R)$ of germs away from N, has the same objects as $\mathcal{C}_{M,G}(R)$, and morphisms are germs of morphism away from N: two morphisms are identified if there exists k so that they only differ in a k-neighborhood of N.

Consider the metric space $M \times \mathbb{R}$ where G acts trivially on the $\mathbb{R}$-factor. Inside we have the metric space $M \cup N \times [0, \infty)$. It follows immediately from the methods of [36], see also [1] for a more formalized description, that the natural functor

$$\mathcal{C}_{M\cup N\times[0,\infty),G}(R) \longrightarrow \mathcal{C}^{>N}_{M,G}(R)$$

induces an isomorphism on K-theory, and it follows from the proofs of [40], that it induces an isomorphism in L-theory (Eilenberg swindle is allowed in L-theory).

Theorem 12.10 ([36]). *There is a long exact sequence*

$$\dots K_*(\mathcal{C}_{N,G}(R)) \longrightarrow K_*(\mathcal{C}_{M,G}(R)) \longrightarrow K_*(\mathcal{C}^{>N}_{M,G}(R)) \longrightarrow K_{*-1}(\mathcal{C}_{N,G}(R)) \dots$$

Here it should be noted that we are using the non-connective deloopings of [36] to define K-theory in negative dimensions.

Theorem 12.11. *There is a 4-periodic long exact sequence*

$$\dots L_n^h(\mathcal{C}_{(N,G)}(R)) \to L_n^h(\mathcal{C}_{(M,G)}(R)) \to L_n^K(\mathcal{C}_{(M,G)}^{>N}(R)) \to L_{n-1}^h(\mathcal{C}_{(N,G)}(R)) \dots$$

Where $K = \operatorname{Im}(\widetilde{K}_1(\mathcal{C}_{(M,G)}(R)) \longrightarrow \widetilde{K}_1(\mathcal{C}_{(M,G)}^{>N}(R)))$.

The formulation in [40] is using $\mathcal{C}_{M\cup N\times[0,\infty)}$ instead of $\mathcal{C}_M^{>N}$. We saw in example 12.7 that trivial group action corresponds to RG coefficients. This is part of a more general phenomenon motivating the following definition

Definition 12.12. Suppose G is acting on the metric space M with invariant subspace N. We say that the set of subgroups $\{H_\alpha\}$ of G is the effective fundamental group for (M, G) away from N if the following is satisfied: For every $k > 0$ the set $\{x \in M \mid \operatorname{diam}(H_\alpha \cdot x) < k\}$ is not contained in a bounded neighborhood of N.

Example 12.13. let (V, G) be a representation. Then the effective fundamental group away from 0 is the set of isotropy subgroups of the representation.

On the geometric side we need the following result from [13]. A map $X \longrightarrow M$ from a space to a metric space is eventually continuous if there exist a covering $\{U_\alpha\}$ of X so that $\operatorname{diam}(pU_\alpha)$ is uniformly bounded, and the inverse image of a bounded set is precompact. When the metric space is a cone, an eventually continuous map may always be replaced by a continuous map which is only a bounded distance away.

Theorem 12.14. *Let X be a free $G-CW$ complex together with a G-equivariant, eventually continuous map $X \to M$ such that $X \to M$ is boundedly simply connected, and X satisfies Poincaré duality with respect to some homomorphism $w\colon G \to \mathbf{Z}/2$, in the category $\mathcal{C}_{M,G}(\mathbb{Z})$, dim$(X) \geq 5$. Let $W \to X$ be a degree one normal map. Then W is normally cobordant to a bounded homotopy equivalence if and only if an invariant in $L_n(\mathcal{C}_{M,G}(\mathbb{Z}))$ vanishes.*

The concept boundedly simply connected is defined in [13, 2.7] . As in standard surgery theory, normal invariants corresponds to lifts of the Spivak normal fibre space $X \to BF$ to $BTOP$. If we fix a lift (defining a basepoint) then [13]:

Theorem 12.15. *There is a long exact sequence of surgery*

$$\dots \to L_{n+1}^h\mathcal{C}_{M,G}(\mathbb{Z})) \to \mathcal{S}^b\begin{pmatrix} X/G \\ \downarrow \\ M/G \end{pmatrix} \to [X/G, F/TOP] \to L_n^h(\mathcal{C}_{M,G}(\mathbb{Z}))$$

Tensor product defines a pairing

$$\mathcal{C}_{|G|,G}(R) \times \mathcal{G}_{M,G}(R) \longrightarrow \mathcal{C}_{|G|\times M,G}(R)$$

whenever R is a commutative ring with unit. When $|G|$ is finite, this means we may replace $\mathcal{C}_{|G|,G}(R)$ by $\mathcal{C}_{pt}(RG)$ and $\mathcal{C}_{|G|\times M,G}(R)$ by $\mathcal{C}_{M,G}(R)$, so for finite G we have a pairing

$$\mathcal{C}_{pt}(RG) \times \mathcal{G}_{M,G}(R) \longrightarrow \mathcal{C}_{M,G}(R)$$

Using the fact that $(A \otimes B)^* = A^* \otimes B^*$ for finitely generated R-modules, it follows that this commutes with the pairings, so it follows from [40] that there is a pairing

$$L_n(RG) \otimes L^k(\mathcal{G}_{M,G}(R)) \longrightarrow L_{n+k}(\mathcal{C}_{M,G}(R))$$

geometrically corresponding to the twisted product.

13. Mackey properties

Let M be a metric space and G a finite group acting on M by eventual Lipschitz maps, R a commutative ring with unit. Consider the category $\mathcal{C}_{M,G}(R)$. Given two subgroups $G_1 \subset G_2 \subset G$ we have G_1 and G_2 acting on M by restriction and there are restriction functors $\mathcal{C}_{M,G_2}(R) \to \mathcal{C}_{M,G_1}(R)$ and induction functors $\mathcal{C}_{M,G_1}(R) \to \mathcal{C}_{M,G_2}(R)$. The restriction functor is obtained just by restriction of the group action, and the induction functor sends an object A to $RG_2 \otimes_{RG_1} A$. The required map from $RG_2 \otimes_{RG_1} A$ to the finite subsets of M is extended from the map of A to the finite subsets of M by equivariance: let $f(g \otimes a) = g \cdot f(a)$. Clearly restriction and induction are functors. We need

Lemma 13.1. *Restriction and induction are functors of categories with involution.*

Proof. The involution is given by $A^* = \mathrm{Hom}^{l.f.}(A, R)$ turned into a right RG-module as described above, and it does not matter whether we restrict before or after applying $\mathrm{Hom}^{l.f.}$. Also

$$\begin{aligned}\mathrm{Hom}^{l.f.}(RG_2 \otimes_{RG_1} A, R) &= \mathrm{Hom}(RG_2, \mathrm{Hom}^{l.f.}(A, R)) \\ &= RG_2 \otimes_{RG_1} \mathrm{Hom}^{l.f.}(A, R)\end{aligned}$$

□

Given two functors between additive categories with involution, we may form a new functor, the direct sum of the two functors. It is easy to see that

Lemma 13.2. *A functor between additive categories with involution induces a map of L-groups. The sum of two functors induces the sum of the two maps.*

Proof. Direct from the definitions since L-groups are defined as a bordism theory where direct sum is turned into addition [38]. □

Consider the category $A(G)$ defined as follows. The objects are the subgroups of G, and the $\mathrm{Hom}(H_1, H_2)$ is the Grothendieck construction applied to the collection of finite "free bi-sets" (these are just finite sets Z with free left H_2 action and free right H_1-action) where the addition is disjoint union. The balanced product

$$({}_{H_3}Z_{H_2}) \times_{H_2} ({}_{H_2}Y_{H_1})$$

is a free biset and can be easily shown to induce a composition $\operatorname{Hom}(H_1, H_2) \times \operatorname{Hom}(H_2, H_3) \to \operatorname{Hom}(H_1, H_3)$ which is bilinear. The set H as an $H - H$ biset is the identity element for $\operatorname{Hom}(H, H)$.
There is a functor $Gr(G) \to A(G)$ from the category of subgroups of G (morphisms are $\operatorname{Maps}(H_1, H_2) = \{g \in G | gH_1g^{-1} \subset H_2\}$). It is the identity on the objects and sends $g \in \operatorname{Maps}(H_1, H_2)$ to the equivalence class of H_2 considered as a left H_2 set in the obvious manner, and $h_2h_1 = h_2gh_1g^{-1}$ for all $h_1 \in H_1$, and all $h_2 \in H_2$. As noted in [21, 4.1], this is a Mackey functor and any functor out of $A(G)$ to an additive category yields a Mackey functor by composition. It follows that

Theorem 13.3. *Given a finite group G and a metric space M as above, then $\mathcal{C}_{M,?}(R)$ is a Mackey functor, and hence $L_n(\mathcal{C}_{M,?}(R))$ is a Mackey functor.*

Remark 13.4. We suppress the upper index in the L-groups in the above statement. The point is that the upper index has to be a subgroup of a K-theoretic group which is in itself a Mackey functor e.g. the whole group or the trivial subgroup, but also naturally defined image groups will work.

Proof. Given an $H_1 - H_2$ biset Z then sending A to $RZ \otimes_{RH_2} A$ and extending the reference map by equivariance defines a functor from $\mathcal{C}_{X,H_2}(R)$ to $\mathcal{C}_{X,H_1}(R)$. □

14. Non-linear similarity

Let G be a finite group and V, V' finite dimensional real orthogonal representations of G. Then V is said to be *topologically equivalent* to V' (denoted $V \sim_t V'$) if there exists a homeomorphism $h\colon V \to V'$ which is G-equivariant. If V, V' are topologically equivalent, but not linearly isomorphic, then such a homeomorphism is called a non-linear similarity. These notions were introduced and studied by de Rham [41], [42], and developed extensively in [4], [5], [26], [28], and [8].

Recently, Erik Pedersen and I have completed de Rham's program by showing that Reidemeister torsion invariants and number theory determine non-linear similarity for finite cyclic groups. I will describe some of our results in this section. The new ingredient is the use of "bounded surgery" techniques.

A G-representation is called *free* if each element $1 \neq g \in G$ fixes only the zero vector. Every representation of a finite cyclic group has a unique maximal free subrepresentation.

Theorem 14.1 (Hambleton-Pedersen). *Let G be a finite cyclic group and V_1, V_2 be free G–representations. For any G–representation W, the existence of a* *non-linear similarity $V_1 \oplus W \sim_t V_2 \oplus W$ is entirely determined by explicit congruences in the weights of the free summands V_1, V_2, and the ratio $\Delta(V_1)/\Delta(V_2)$ of their Reidemeister torsions, up to an algebraically described indeterminacy.*

This is just a general formulation, intended to give an overview of the answer. Precise statements of our results are given in [20]. For example, for cyclic groups of 2-power order, we obtain a complete classification of non-linear similarities.

Two fundamental result on the problem were proved in the 1980's by Cappell-Shaneson [4], Hsiang-Pardon [26], and Madsen-Rothenberg [28].

Theorem 14.2 (Cappell-Shaneson). *Non-linear similarities $V \sim_t V'$ exist for cyclic groups $G = C(4q)$ of every order $4q \geqq 8$.*

Theorem 14.3 (Hsiang-Pardon, Madsen-Rothenberg). *If $G = C(q)$ or $G = C(2q)$, for q odd, topological equivalence of G-representations implies linear equivalence.*

This is called the Odd Order Theorem (the missing case $G = C(4)$ is trivial). Since linear G-equivalence for general finite groups G is detected by restriction to cyclic subgroups, it is reasonable to study this case first. For the rest of this section, unless otherwise mentioned, G denotes a finite cyclic group.

Further positive results can be obtained by imposing assumptions on the isotropy subgroups allowed in V and V'. For example, de Rham [41] proved in 1935 that piecewise linear similarity implies linear equivalence for free G-representations, by using Reidemeister torsion and the Franz Independence Lemma. Topological invariance of Whitehead torsion shows that his method also rules out non-linear similarity in this case. In [19, Thm.A] we studied "first-time" similarities, where $\operatorname{Res}_K V \cong \operatorname{Res}_K V'$ for all proper subgroups $K \subsetneqq G$, and showed that topological equivalence implies linear equivalence if V, V' have no isotropy subgroup of index 2. This result is an application of bounded surgery theory (see [18], [19, §4]), and provides a more conceptual proof of the Odd Order Theorem. These techniques are extended in [20] to provide a necessary and sufficient condition for non-linear similarity in terms of the vanishing of a bounded transfer map. This gives a new approach to de Rham's problem. The main work of [20] is to establish methods for effective calculation of the bounded transfer in the presence of isotropy groups of arbitrary index.

An interesting question in non-linear similarity concerns the minimum possible dimension for examples. It is easy to see that the existence of a non-linear similarity $V \sim_t V'$ implies $\dim V = \dim V' \geqq 5$. Cappell, Shaneson, Steinberger and West proved:

Theorem 14.4 ([8]). *Non-linear similarity starts in dimension 6 for $G = C(2^r)$, with $r \geqq 4$.*

A 1981 Cappell-Shaneson preprint (now published [7]) shows that 5–dimensional similarities do not exist for any finite group.

In [5], Cappell and Shaneson initiated the study of *stable* topological equivalence for G–representations. We say that V_1 and V_2 are stably topologically similar ($V_1 \approx_t V_2$) if there exists a G–representation W such that $V_1 \oplus W \sim_t V_2 \oplus W$. Let $R_{\text{Top}}(G) = R(G)/R_t(G)$ denote the quotient group of the real representation ring of G by the subgroup $R_t(G) = \{[V_1] - [V_2] \mid V_1 \approx_t V_2\}$. In [5], $R_{\text{Top}}(G) \otimes \mathbf{Z}[1/2]$ was computed, and the torsion subgroup was shown to be 2–primary. As an application of our general results, we determine the structure of the torsion in $R_{\text{Top}}(G)$, for G any cyclic group. In Theorem 14.11 we give the calculation of $R_{\text{Top}}(G)$ for $G = C(2^r)$. This is the first complete calculation of $R_{\text{Top}}(G)$ for any group that admits non–linear similarities.

In order to state a sample of the results from [20] precisely, we need some notation. Let $G = C(4q)$, where $q > 1$, and let $H = C(2q)$ denote the subgroup of index 2 in G. The maximal odd order subgroup of G is denoted G_{odd}. We fix a generator $G = \langle t \rangle$ and a primitive $4q^{th}$-root of unity $\zeta = \exp 2\pi i/4q$. The group G has both a trivial 1-dimensional real representation, denoted $\mathbf{R}_+$, and a non-trivial 1-dimensional real representation, denoted $\mathbf{R}_-$.

A free G–representation is a sum of faithful 1-dimensional complex representations. Let t^a, $a \in \mathbf{Z}$, denote the complex numbers $\mathbf{C}$ with action $t \cdot z = \zeta^a z$ for all $z \in \mathbf{C}$. This representation is free if and only if $(a, 4q) = 1$, and the coefficient a is well–defined only modulo $4q$. Since $t^a \cong t^{-a}$ as real G–representations, we can always choose the weights $a \equiv 1 \bmod 4$. This will be assumed unless otherwise mentioned.

Now suppose that $V_1 = t^{a_1} + \cdots + t^{a_k}$ is a free G–representation. The Reidemeister torsion invariant of V_1 is defined as

$$\Delta(V_1) = \prod_{i=1}^{k} (t^{a_i} - 1) \in \mathbf{Z}[t]/\{\pm t^m\} .$$

Let $V_2 = t^{b_1} + \cdots + t^{b_k}$ be another free representation, such that $S(V_1)$ and $S(V_2)$ are G-homotopy equivalent. This just means that the products of the weights $\prod a_i \equiv \prod b_i \bmod 4q$. Then the Whitehead torsion of any G-homotopy equivalence is determined by the element

$$\Delta(V_1)/\Delta(V_2) = \frac{\prod(t^{a_i} - 1)}{\prod(t^{b_i} - 1)}$$

since $\mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\mathbf{Q}G)$ is monic [35, p.14]. When there exists a G-homotopy equivalence $f\colon S(V_2) \to S(V_1)$ which is normally cobordant to the identity map on $S(V_1)$, we say that $S(V_1)$ and $S(V_2)$ are *normally cobordant.* More generally, we say that $S(V_1)$ and $S(V_2)$ are *s–normally cobordant* if $S(V_1 \oplus U)$ and $S(V_2 \oplus U)$ are normally cobordant for all free G–representations U. This is a necessary condition for non-linear similarity, which can be decided by explicit congruences in the weights.

This quantity, $\Delta(V_1)/\Delta(V_2)$ is the basic invariant determining non-linear similarity. It represents a unit in the group ring $\mathbf{Z}G$, explicitly described for $G = C(2^r)$ by Cappell and Shaneson in [6, §1] using a pull-back square of rings. To state concrete results we need to evaluate this invariant modulo suitable indeterminacy.

The involution $t \mapsto t^{-1}$ induces the identity on $\mathrm{Wh}(\mathbf{Z}G)$, so we get an element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^0(\mathrm{Wh}(\mathbf{Z}G))$$

where we use $H^i(A)$ to denote the Tate cohomology $H^i(\mathbf{Z}/2; A)$ of $\mathbf{Z}/2$ with coefficients in A.

Let $\mathrm{Wh}(\mathbf{Z}G^-)$ denote the Whitehead group $\mathrm{Wh}(\mathbf{Z}G)$ together with the involution induced by $t \mapsto -t^{-1}$. Then for $\tau(t) = \frac{\prod(t^{a_i}-1)}{\prod(t^{b_i}-1)}$, we compute

$$\tau(t)\tau(-t) = \frac{\prod(t^{a_i}-1)\prod((-t)^{a_i}-1)}{\prod(t^{b_i}-1)\prod((-t)^{b_i}-1)} = \prod \frac{(t^2)^{a_i}-1}{((t^2)^{b_i}-1)}$$

which is clearly induced from $\mathrm{Wh}(\mathbf{Z}H)$. Hence we also get a well defined element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H)) \ .$$

This calculation takes place over the ring $\Lambda_{2q} = \mathbf{Z}[t]/(1+t^2+\cdots+t^{4q-2})$, but the result holds over $\mathbf{Z}G$ via the involution–invariant pull-back square

$$\begin{array}{ccc} \mathbf{Z}G & \to & \Lambda_{2q} \\ \downarrow & & \downarrow \\ \mathbf{Z}[\mathbf{Z}/2] & \to & \mathbf{Z}/2q[\mathbf{Z}/2] \end{array}$$

Consider the exact sequence of modules with involution:

$$K_1(\mathbf{Z}H) \to K_1(\mathbf{Z}G) \to K_1(\mathbf{Z}H \to \mathbf{Z}G) \to \widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G) \tag{14.5}$$

and define $\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) = K_1(\mathbf{Z}H \to \mathbf{Z}G)/\{\pm G\}$. We then have a short exact sequence

$$0 \to \mathrm{Wh}(\mathbf{Z}G)/\mathrm{Wh}(\mathbf{Z}H) \to \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) \to \mathbf{k} \to 0$$

where $\mathbf{k} = \ker(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$. Such an exact sequence of $\mathbf{Z}/2$-modules induces a long exact sequence in Tate cohomology. In particular, we have a coboundary map

$$\delta \colon H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H)) \ .$$

Our first result deals with isotropy groups of index 2, as is the case for the non-linear similarities constructed in [4].

Theorem 14.6 ([20, Thm. A]). *Let $V_1 = t^{a_1}+\cdots+t^{a_k}$ and $V_2 = t^{b_1}+\cdots+t^{b_k}$ be free G-representations, with $a_i \equiv b_i \equiv 1 \bmod 4$. There exists a topological similarity $V_1 \oplus \mathbf{R}_- \sim_t V_2 \oplus \mathbf{R}_-$ if and only if*

(i) $\prod a_i \equiv \prod b_i \bmod 4q$,

(ii) $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$, *and*

(iii) *the element* $\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$ *is in the image of the coboundary* $\delta\colon H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$.

Remark 14.7. More general isotropy is handled in the other results of [20]. Theorem 14.6 should be compared with [4, Cor.1], where more explicit conditions are given for "first-time" similarities of this kind under the assumption that q is odd, or a 2-power, or $4q$ is a "tempered" number.

The case $\dim V_1 = \dim V_2 = 4$ gives a reduction to number theory for the existence of 5-dimensional similarities.

We turn now to results on the structure of $R_{\mathrm{Top}}(G)$. There is a filtration

$$R_t(G) \subseteq R_n(G) \subseteq R_h(G) \subseteq R(G) \tag{14.8}$$

on the real representation ring $R(G)$, inducing a filtration on

$$R_{\mathrm{Top}}(G) = R(G)/R_t(G)\ .$$

Here $R_h(G)$ consists of those virtual elements with no homotopy obstruction to similarity, and $R_n(G)$ the virtual elements with no normal invariant obstruction to similarity Note that $R(G)$ has the nice basis $\{t^i, \delta, \epsilon \mid 1 \leqq i \leqq 2q-1\}$, where $\delta = [\mathbf{R}_-]$ and $\epsilon = [\mathbf{R}_+]$.

Let $R^{free}(G) = \{t^a \mid (a, 4q) = 1\} \subset R(G)$ be the subgroup generated by the free representations. To complete the definition, we let $R^{free}(C(2)) = \{\mathbf{R}_-\}$ and $R^{free}(e) = \{\mathbf{R}_+\}$. Then

$$R(G) = \bigoplus_{K \subseteq G} R^{free}(G/K)$$

and this direct sum splitting can be intersected with the filtration above to define $R_h^{free}(G)$, $R_n^{free}(G)$ and $R_t^{free}(G)$. In addition, we can divide out $R_t^{free}(G)$ and obtain subgroups $R_{h,\mathrm{Top}}^{free}(G)$ and $R_{n,\mathrm{Top}}^{free}(G)$ of $R_{\mathrm{Top}}^{free}(G) = R^{free}(G)/R_t^{free}(G)$. By induction on the order of G, we see that it suffices to study the summand $R_{\mathrm{Top}}^{free}(G)$.

Let $\widetilde{R}^{free}(G) = \ker(\mathrm{Res}\colon R^{free}(G) \to R^{free}(G_{odd}))$, and then project into $R_{\mathrm{Top}}(G)$ to define

$$\widetilde{R}_{\mathrm{Top}}^{free}(G) = \widetilde{R}^{free}(G)/R_t^{free}(G)\ .$$

Theorem 14.9 ([20]). *The torsion subgroup of* $R_{\mathrm{Top}}^{free}(G)$ *is precisely* $\widetilde{R}_{\mathrm{Top}}^{free}(G)$, *and the subquotient* $\widetilde{R}_{n,\mathrm{Top}}^{free}(G) = \widetilde{R}_n^{free}(G)/R_t^{free}(G)$ *always has exponent two.*

Here is a specific computation.

Theorem 14.10 ([20, Thm. D]). *Let* $G = C(4q)$, *with* $q > 1$ *odd, and suppose that the fields* $\mathbf{Q}(\zeta_d)$ *have odd class number for all* $d \mid 4q$. *Then* $\widetilde{R}_{\mathrm{Top}}^{free}(G) = \mathbf{Z}/4$ *generated by* $(t - t^{1+2q})$.

For any cyclic group G, both $R^{free}(G)/R_h^{free}(G)$ and $R_h^{free}(G)/R_n^{free}(G)$ are torsion groups which can be explicitly determined by congruences in the weights.

We conclude this list of sample results with a calculation of $R_{\mathrm{Top}}(G)$ for cyclic 2-groups.

Theorem 14.11 ([20, Thm. E]). *Let $G = C(2^r)$, with $r \geqq 4$. Then*

$$\widetilde{R}_{\mathrm{Top}}^{free}(G) = \langle \alpha_1, \alpha_2, \ldots, \alpha_{r-2}, \beta_1, \beta_2, \ldots, \beta_{r-3} \rangle$$

subject to the relations $2^s\alpha_s = 0$ for $1 \leqq s \leqq r-2$, and $2^{s-1}(\alpha_s + \beta_s) = 0$ for $2 \leqq s \leqq r-3$, together with $2(\alpha_1 + \beta_1) = 0$.

The generators for $r \geqq 4$ are given by the elements

$$\alpha_s = t - t^{5^{2^{r-s-2}}} \quad \text{and} \quad \beta_s = t^5 - t^{5^{2^{r-s-2}}+1}.$$

We remark that $\widetilde{R}_{\mathrm{Top}}^{free}(C(8)) = \mathbf{Z}/4$ generated by $t - t^5$.

Our approach to the non-linear similarity problem starts with an elementary observation about topological equivalences for cyclic groups.

Lemma 14.12. *If $V_1 \oplus W \sim_t V_2 \oplus W'$, where V_1, V_2 are free G-representations, and W and W' have no free summands, then there is a G-homeomorphism $h\colon V_1 \oplus W \to V_2 \oplus W$ such that*

$$h| \bigcup_{1 \neq H \leq G} W^H$$

is the identity.

Proof. Let h be the homeomorphism. We will successively change h, stratum by stratum. For every subgroup K of G, consider the homeomorphism of K-fixed sets

$$f^K\colon W^K \to W'^K.$$

This is a homeomorphism of G/K, hence of G–representations. As G–representations we can split

$$V_2 \oplus W' = U \oplus W'^K \sim_t U \oplus W^K = V_2 \oplus W''$$

where the similarity uses the product of the identity and $(f^K)^{-1}$. Notice that the composition of f with this similarity is the *identity* on the K-fixed set. Rename W'' as W' and repeat this successively for all subgroups. We end up with $W = W'$ and a homeomorphism as claimed. □

One consequence is

Lemma 14.13. *If $V_1 \oplus W \sim_t V_2 \oplus W$ then there exists a G-homotopy equivalence $S(V_2) \to S(V_1)$.*

Proof. If we 1-point compactify h we obtain a G-homeomorphism

$$h^+ \colon S(V_1 \oplus W \oplus \mathbf{R}) \to S(V_2 \oplus W \oplus \mathbf{R}).$$

After an isotopy, the image of the free G-sphere $S(V_1)$ may be assumed to lie in the complement $S(V_2 \oplus W \oplus \mathbf{R}) - S(W \oplus \mathbf{R})$ of $S(W \oplus \mathbf{R})$ which is G-homotopy equivalent to $S(V_2)$. □

Any homotopy equivalence $f \colon S(V_2)/G \to S(V_1)/G$ defines an element $[f]$ in the structure set $\mathcal{S}^h(S(V_1)/G)$. We may assume that $\dim V_i \geqq 4$. This element must be non-trivial: otherwise $S(V_2)/G$ would be topologically h-cobordant to $S(V_1)/G$, and Stallings infinite repetition of h-cobordisms trick would produce a homeomorphism $V_1 \to V_2$ contradicting [2], since V_1 and V_2 are free representations. More precisely, we use Wall's extension of the Atiyah–Singer equivariant index formula to the topological locally linear case [47]. If $\dim V_i = 4$, we can cross with $\mathbf{CP}^2$ to avoid low–dimensional difficulties. Crossing with W and parameterising by projection on W defines a map from the classical surgery sequence to the bounded surgery exact sequence

$$(14.14)\qquad \begin{array}{ccccc} L_n^h(\mathbf{Z}G) & \longrightarrow & \mathcal{S}^h(S(V_1)/G) & \longrightarrow & [S(V_1)/G, \mathrm{F/Top}] \\ \downarrow & & \downarrow & & \downarrow \\ L_{n+k}^h(\mathcal{C}_{W,G}(\mathbf{Z})) & \longrightarrow & \mathcal{S}_b^h\begin{pmatrix} S(V_1)\times W/G \\ \downarrow \\ W/G \end{pmatrix} & \longrightarrow & [S(V_1) \times_G W, \mathrm{F/Top}] \end{array}$$

The L–groups in the upper row are the ordinary surgery obstruction groups for oriented manifolds and surgery up to homotopy equivalence. In the lower row, we have bounded L–groups corresponding to an orthogonal action $\rho_W \colon G \to O(W)$, with orientation character given by $\det(\rho_W)$. Our main criterion for non–linear similarities is:

Theorem 14.15. *Let V_1 and V_2 be free G–representations with $\dim V_i \geqq 2$, and suppose that $f \colon S(V_2) \to S(V_1)$ is a G-homotopy equivalence. Then, there is a topological equivalence $V_1 \oplus W \sim_t V_2 \oplus W$ if and only if the element $[f] \in \mathcal{S}^h(S(V_1)/G)$ is in the kernel of the bounded transfer map*

$$trf_W \colon \mathcal{S}^h(S(V_1)/G) \to \mathcal{S}_b^h\begin{pmatrix} S(V_1)\times_G W \\ \downarrow \\ W/G \end{pmatrix}.$$

Proof. For necessity, we refer the reader to [19] where this is proved using a version of equivariant engulfing. For sufficiency, we notice that crossing with $\mathbf{R}$ gives an isomorphism of the bounded surgery exact sequences parameterized by W to simple bounded surgery exact sequence parameterized by $W \times \mathbf{R}$. By the bounded s-cobordism theorem, this means that the vanishing of the bounded transfer implies that

$$\begin{array}{ccc} S(V_2) \times W \times \mathbf{R} & \xrightarrow{f\times 1} & S(V_1) \times W \times \mathbf{R} \\ & & \downarrow \\ & & W \times \mathbf{R} \end{array}$$

is within a bounded distance of an equivariant homeomorphism h, where distances are measured in $W \times \mathbf{R}$. We can obviously complete $f \times 1$ to the map

$$f * 1 \colon S(V_2) * S(W \times \mathbf{R}) \to S(V_1) * S(W \times \mathbf{R})$$

and since bounded in $W \times \mathbf{R}$ means small near the subset

$$S(W \times \mathbf{R}) \subset S(V_i) * S(W \times \mathbf{R}) = S(V_i \oplus W \oplus \mathbf{R}),$$

we can complete h by the identity to get a homeomorphism

$$S(V_2 \oplus W \oplus \mathbf{R}) \to S(V_1 \oplus W \oplus \mathbf{R})$$

and taking a point out we have a homeomorphisms $V_2 \times W \to V_2 \times W$ □

By comparing the ordinary and bounded surgery exact sequences (14.14), and noting that the bounded transfer induces the identity on the normal invariant term, we see that a necessary condition for the existence of any stable similarity $f \colon V_2 \approx_t V_1$ is that $f \colon S(V_2) \to S(V_1)$ has s–normal invariant zero. Assuming this, under the natural map

$$L_n^h(\mathbf{Z}G) \to \mathcal{S}^h(S(V_1)/G),$$

where $n = \dim V_1$, the element $[f]$ is the image of $\sigma(f) \in L_n^h(\mathbf{Z}G)$, obtained as the surgery obstruction (relative to the boundary) of a normal cobordism from f to the identity. The element $\sigma(f)$ is well-defined in $\tilde{L}_n^h(\mathbf{Z}G) = \operatorname{Coker}(L_n^h(\mathbf{Z}) \to L_n^h(\mathbf{Z}G))$. Since the image of the normal invariants

$$[S(V_1)/G \times I, S(V_1)/G \times \partial I, \mathrm{F/Top}] \to L_n^h(\mathbf{Z}G)$$

factors through $L_n^h(\mathbf{Z})$ (see [17, Thm.A, 7.4] for the image of the assembly map), we may apply the criterion of 14.15 to any lift $\sigma(f)$ of $[f]$. This reduces the evaluation of the bounded transfer on structure sets to a bounded L–theory calculation.

Theorem 14.16. *Let V_1 and V_2 be free G–representations with $\dim V_i \geqq 2$, and suppose that $f \colon S(V_2) \to S(V_1)$ is a G–homotopy equivalence which is G–normally cobordant to the identity. Then, there is a topological equivalence $V_1 \oplus W \sim_t V_2 \oplus W$ if and only if $trf_W(\sigma(f)) = 0$, where $trf_W \colon L_n^h(\mathbf{Z}G) \to L_{n+k}^h(\mathcal{C}_{W,G}(\mathbf{Z}))$ is the bounded transfer.*

Acknowledgments

The author is greatly indebted to Wu-Chung Hsiang for introducing him to this wonderful area of mathematics, and to Ronnie Lee for his inspiring lectures on this subject at Yale University in Fall, 1970. I would also like to thank my colleagues and co-authors whose work is reported on in these notes, particularly Matthias Kreck, Ronnie Lee, Ib Madsen, Jim Milgram, Erik Pedersen, Andrew Ranicki, Larry Taylor, and Bruce Williams. Needless to say, they are not responsible for the mistakes in my exposition.

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