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Foliated control theory and its applications

L. Jones
Department of Mathematics
State University of New York at Stony Brook
Stony Brook, NY 11794
U.S.A.

FOLIATED CONTROL THEORY AND ITS APPLICATIONS

LOWELL E. JONES

ABSTRACT. The control theorems and fibered control theorems due to Chapman, Ferry and Quinn, concerning controlled h-cobordisms and controlled homotopy equivalences, are reviewed. Some foliated control theorems, due to Farrell and Jones, are formulated and deduced from from the fibered control theorems. The role that foliated control theory plays in proving the Borel conjecture for closed Riemannian manifolds having non-positive sectional curvature, and in calculating Whitehead groups for the fundamental group of such manifolds, is described. The importance of the geometric collapsing theory of Cheeger-Fukaya-Gromov in extending the preceeding results is explained.

1. INTRODUCTION

Recall that the **Borel Conjecture** states that any homotopy equivalence $f : M \longrightarrow N$ from the closed aspherical manifold M to the closed aspherical manifold N must be homotopic to a homeomorphism. There is a related conjecture, which in these notes will be referred to as the **Whitehead Group Conjecture**, which states that the Whitehead group $Wh(\pi)$ of any torsion free group π vanishes. In particular the Whitehead Conjecture claims that $Wh(\pi) = 0$ for π the fundamental group of any aspherical manifold.

Any closed Riemannian manifold N whose sectional curvature values satisfy $K \leq 0$ everywhere must be an aspherical manifold; infact, by a theorem due to Hadamard (cf. reference [1]), the universal covering space of N must be diffeomorphic to Euclidean space. Thus the following two theorems, due to Farrell and Jones, represent a partial verification of the Borel Conjecture and the Whitehead Group Conjecture (cf. references [13] and [14]).

Theorem 1.1. *Let N denote a smooth closed Riemannian manifold whose sectional curvature values satisfy $K \leq 0$ everywhere. Also suppose that $dim(M) \geq 4$. Then any homotopy equivalence*

$$M \xrightarrow{f} N$$

from the closed aspherical manifold M is homotopic to a homeomorphism.

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Theorem 1.2. *Let N denote a smooth closed Riemannian manifold whose sectional curvature values satisfy $K \leq 0$ everywhere; and let π denote the fundamental group of N. Then $Wh(\pi) = 0$.*

Strategy for proving 1.2.

An **h-cobordism** of the compact Riemannian manifold pair $(X, \partial X)$ consists of a cobordism (W, W_∂) from $(X, \partial X) = \partial_-(W, W_\partial)$ to $\partial_+(W, W_\partial)$, such that (W, W_∂) may be equipped with deformation retracts of the pair (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial)$; these deformation retracts are denoted by

$$r_t^- : (W, W_\partial) \longrightarrow (W, W_\partial) \quad \text{and} \quad r_t^+ : (W, W_\partial) \longrightarrow (W, W_\partial),$$

for $t \in [0,1]$. If dimX ≥ 5 then any elemenent $\omega \in Wh(\pi_1(X))$ can be represented by an h-cobordism (W, W_∂) of $(X, \partial X)$; and α=0 iff W is diffeomrophic to $X \times [0,1]$. Thus one approach to proving 1.2 would be to show that any h-cobordism of the manifold N in 1.2 is a product cobordism. To show this we need some tools which allow us to conclude that under certain circumstances an h-cobordism is a product cobordism; the tools that suffice are the controlled h-cobordism theorems of Ferry, Chapmann and Quinn, and their foliated versions.

Control theory for h-cobordisms.

An h-cobordism W of the closed Riemannian manifold X is said to be **ε-controlled** if there are deformation retracts r_t^- and r_t^+ of W onto $\partial_- W$ and $\partial_+ W$ respectively, such that for each $w \in W$ both of the paths

$$r_1^- \circ r_t^-(w) \text{ and } r_1^- \circ r_t^+(w), t \in [0,1]$$

have diameter less than ε in X. S. Ferry has proven the following control theorem (cf. reference [16]).

Theorem 1.3. *Suppose that dimX ≥ 5. There is an $\varepsilon > 0$ which depends only on the compact Riemannian manifold X. Any ε-controlled h-cobordism W of X is diffieomorphic to $X \times [0,1]$.*

Fibered control theory for h-cobordisms.

Fibered versions of 1.3 have also been proven. In these versions we need a smooth fiber bundle projection

$$\rho : (X, \partial X) \longrightarrow (Y, \partial Y)$$

onto the compact Riemannian manifold pair $(Y, \partial Y)$. An h-cobordism (W, W_∂) of $(X, \partial X)$ is said to be **$(\varepsilon; \rho)$-controlled** if there are deformation retracts of r_t^- and r_t^+ of (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial$ respectively, such that for each $w \in W$ both of the paths

$$\rho \circ r_1^- \circ r_t^-(w) \text{ and } \rho \circ r_1^- \circ r_t^+(w), t \in [0,1]$$

have diameter less than ε in Y. The follow theorem was proven by F. Quinn and T.A. Chapman (cf. references [3], [17] and [18]).

Theorem 1.4. *Suppose that $dimX \geq 6$. Suppose that the fiber of*

$$\rho : X \longrightarrow Y,$$

denoted by F, satisfies $Wh(\pi_1(F) \oplus G)$=0 for any finitely generated free abelian group G. Then there is an $\varepsilon > 0$ which depends only on the compact Riemannian manifold Y. Any $(\varepsilon; \rho)$-controlled h-cobordism (W, W_∂) of $(X, \partial X)$ is diffeomorphic to the product $(X, \partial X)$.

Remark 1.5. If $\partial X = \emptyset$ in 1.4 then we may weaken the dimension hypothesis of 1.4 to $dimX \geq 5$.

Remark 1.6. In the course of this paper we will be using that the hypothesis of 1.4, that $Wh(\pi_1(F) \oplus G)$=0 for any finitely generated free abelian group G, is known to be satisfied for the following types of manifolds F.

(a) F equal an n-dimensional torus T^n (cf. reference [2]).

(b) F equal an aspherical manifold whose fundamental group is a poly-$\mathbb{Z}$ group (cf. reference [6]).

Remark 1.7. Note that it follows from 1.2 that this hypothesis of 1.4 is also satisfied for F equal a closed Riemannian manifold having non-positive sectional curvature values everywhere. Infact the product $F \times T^n$, when equipped with the product of the given Riemannian structure on F with the standard flat Riemannian structure on the n-dimensional torus T^n, is a closed Riemannian manifold having non-positive sectional curvature values everywhere. Moreover $\pi_1(F \times T^n) = \pi_1(F) \oplus G$, where G is a free abelian group of rank n. So by applying 1.2 to $F \times T^n$ for all $n = 0, 1, 2, ...$ we get that F satisfies the hypothesis of 1.4.

Foliated control theory for h-cobordisms.

Let Ξ denote a smooth one-dimensional foliation for the compact Riemannian which also foliates the boundary ∂X. A path $p : [0,1] \longrightarrow N$ in N is said to have **Ξ-diameter less than** (α, ε) if the following hold: there is a connected subset A of a leaf of Ξ which has length ; the length less than α; any point in image(p) is within a distance less than ε to A. An h-cobordism (W, W_∂) of $(X, \partial X)$ is said to be $(\alpha, \varepsilon; \Xi)$-**controlled** if there are deformation retracts r_t^+ and r_t^- of (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial)$ respectively, such that for any fixed $w \in W$ both of the paths

$$r_1^- \circ r_t^-(w) \quad \text{and} \quad r_1^- \circ r_t^+(w), t \in [0,1]$$

have Ξ-diameter less than (α, ε) . The following theorem has been proven by Farrell and Jones (cf. references [7] and [8]).

Theorem 1.8. *Suppose that* $dimX \geq 6$. *Given any number* $\alpha > 0$ *there is a number* $\varepsilon > 0$ *which depends only on* α *and* Ξ. *Any h-cobordism* (W, W_∂) *of* $(X, \partial X)$ *which is* $(\alpha, \varepsilon; \Xi)$*-controlled is diffeomorphic to* $(X, \partial X) \times [0,1]$.

Remark 1.9. If $\partial X = \emptyset$ in 1.8 then the dimension hypothesis may be weakened to $dimX \geq 5$.

Note that 1.8 generalizes 1.4 in the special case that Ξ of 1.4 is of dimension one: and 1.8 is an immediate corollary of 1.4 in the special case that the the leaves of Ξ are equal to the fibers of a smooth fiber bundle projection $\rho : X \longrightarrow Y$ onto a closed Riemannian manifold Y. A complete proof for 1.8, which is carried out in section 5 below, is based upon some "local, relative " versions of 1.4. These "local, relative" versions of 1.4 are formlated in sections 2 and 3 below.

There is also a fibered versions of 1.8. This fibered version requires a smooth fiber bundle projection $\rho : X \longrightarrow Y$ onto a compact Riemannian manifold Y. Now we let Ξ denote a one-dimensional smooth foliation for Y which also foliates ∂Y. An h-cobordism (W, W_∂) of $(X, \partial X)$ is said to be $(\alpha, \varepsilon; \Xi, \rho)$-**controlled** if there are deformation retracts r_t^+ and r_t^- of (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial)$ respectively, such that for any fixed $w \in W$ both of the paths

$$\rho \circ r_1^- \circ r_t^-(w) \quad \text{and} \quad \rho \circ r_1^- \circ r_t^+(w), t \in [0,1]$$

have Ξ-diameter less (α, ε). The following theorem is also due to Farrell and Jones (cf. references [7] and [8]).

Theorem 1.10. *Suppose* $dimX \geq 6$. *Suppose also that* $Wh(\pi_1(E) \oplus G) = 0$ *for any finitely generated free abelian group* G, *and for any space* E *of the form* $\rho^{-1}(L)$ or $\rho^{-1}(y)$ *where* L *denotes any leaf of* Ξ *and* y *denotes any point in* Y. *For any number* $\alpha > 0$ *there is a number* $\varepsilon > 0$ *which depends only on* α *and* Ξ. *Any h-cobordism* (W, W_∂) *of* $(X, \partial X)$ *which is* $(\alpha, \varepsilon; \Xi, \rho)$*-controlled is diffeomorphic to* $(X, \partial X) \times [0,1]$.

Remark 1.11. If $\partial X = \emptyset$ in 1.10 then we may weaken the dimension hypothesis of 1.10 to $dimX \geq 5$.

Completion of the proof of Theorem 1.2.

Let N denote closed Riemannian manifold of 1.2, and let $N \times S^1$ denote the product of N with the circle S^1 equipped with the product metric. Note that $N \times S^1$ also has non-positive sectional curvature every where. Let $S(N \times S^1)$ denote the unit sphere bundle for $N \times S^1$, and let $u : N \times S^1 \longrightarrow S(N \times S^1)$ denote a unit vector field which is always tangent to the second factor of $N \times S^1$. Let X denote the subset of all $v \in S(N \times S^1)$ such that $\langle v, u(p) \rangle \geq 0$, where $p \in N \times S^1$ is the base point of v; note that X is the total space of a smooth disc bundle over $N \times S^1$. Thus X is a compact smooth manifold with boundary ∂X. Choose any Riemannian metric for X. Denote by

$$g_t : S(N \times S^1) \longrightarrow S(N \times S^1), \quad t \in [0,1]$$

the geodesic flow for $N \times S^1$; note that this flow leaves invariant the subsets X and ∂X. Thus there is the one-dimensional smooth foliation Ξ for X whose leaves are just the orbits of the geodesic flow which intersect X; note that Ξ also foliates ∂X. The following theorem, due to Farrell and Jones, is the key result which allows one to apply foliated control theory to prove 1.2.

Theorem 1.12. *Suppose $dimN \geq 2$. Given $\omega \in Wh(\pi_1(N \times S^1))$ there is an $\alpha > 0$. For each $\varepsilon > 0$, if the number $s > 0$ is choosen sufficiently large, there is an $(\alpha, \varepsilon; \Xi, g_s)$-controlled h-cobordism (W, W_∂) of $(X, \partial X)$ which represents α in the following sense: $\alpha = 0$ iff W is diffeomorphic to $X \times [0,1]$.*

Remark 1.13. The h-cobordism (W, W_∂) in 1.12 is gotten as follows. Choose an h-cobordism V of $N \times S^1$ which represents $\omega \in Wh(\pi_1(N \times S^1))$. Choose deformation retracts s_t^- and s_t^+ for V onto $\partial_- V = N \times S^1$ and onto $\partial_+ V$ respectively. Define W to be the total space of the fiber bundle obtained by pulling back the disc bundle $X \longrightarrow N \times S^1$ along the map $s_1^- : V \longrightarrow N \times S^1$. Define the subset $W_\partial \subset W$ to be the pull back along the map $s_1^- : V \longrightarrow N \times S^1$ of the subbundle of $S(N \times S^1)$ consisting of all unit vectors tangent to the first factor of $N \times S^1$. The deformation retracts s_t^- and s_t^+ for V can be "lifted" to deformation retracts r_t^- and r_t^+ for (W, W_∂) by using the homotopy lifting property for fiber bundles. There are many such liftings, most of which are of no use in verifying the control property claimed for (W, W_∂) in 1.12. Liftings which are useful in this regard are the "focal transfers" which are described in detail in Tom Farrell's lecture notes.

Now we will complete the proof of 1.2. First note that, since group $\pi_1(N)$ is a retract of the group $\pi_1(N \times S^1)$, the group $Wh(\pi_1(N))$ is also a retract of the group $Wh(\pi_1(N \times S^1))$. So to complete the proof of 1.2 it will suffice to show that $Wh(\pi_1(N \times S^1)) = 0$. This last equality follows by applying 1.10 and 1.12. In more detail we represent any $\omega \in Wh(\pi_1(N \times S^1))$ by an h-cobordism (W, W_∂) as in 1.12. We choose the bundle projection map $\rho : X \longrightarrow Y$ of 1.10 be equal to the map $g_s : X \longrightarrow X$ of 1.12, where s is choosen sufficiently large so that (W, W_∂) is $(\alpha, \varepsilon; \Xi, g_s)$-contolled. Now we just apply 1.10 to see that (W, W_∂) is diffeomophic to $(X, \partial X) \times [0,1]$; hence, by 1.12, $\alpha = 0$ as desired. If $dim(N) = 1$ then 1.2 is a consequence of 1.6.

This completes the proof of 1.2.

Outline of sections 2-5.

In section 2 we discuss a "local-relative" version of the fibered control Theorem 1.4. The word "local" here refers to the fact that the fiber bundles involved all project to subsets of Euclidean space; and the word "relative" here refers to the fact that fibered control result discussed yield product structures which extend previously existing product structures.

In section 3 we formulate a very general "Foliated Control Conjecture" for h-cobordisms which would imply 1.4 and 1.10 as corollaries. Some supporting evidence is given for this conjecture (in addition to 1.4 and 1.10). A "local-relative" version of 1.10 is discussed.

In section 4 we consider "long and thin" pieces of a closed Riemannian manifold Y which comes equipped with a smooth one-dimensional foliation Ξ. By a "long and thin" piece of Y we simply mean a small neighborhood in Y of either a closed leaf of Ξ or a small neighborhood of a long arc within a non-closed leaf of Ξ. The main result of section 4 states that there is is a covering of Y by "long and thin pieces" which satisfies a number of useful properties. For example each piece in the covering looks like either a long thin rectangle (i.e like $[0,a] \times [0,b]^n$, where a is a large number and the b is a smaller number), or it looks like the total space of a vector bundle over the product $S^1 \times B$, where B is a ball of small radius in Euclidean space.

In section 5 we complete the proof of 1.10. This is done by "restricting" the h-cobordism (W, W_∂) of 1.10 to that part of X which lies over each of the "long and thin" pieces of X discussed in section 4. We then apply to each of these pieces of (W, W_∂) the "local-relative" versions of 1.10 which was are proven in section 2 and 3 below.

2. Fibered control theory for h-cobordisms.

In this section we formulate "relative" and "local-relative" versions of 1.4. The word "local" here means that instead of considering the entire h-cobordism of the manifold X which is given in 1.4 we consider only a piece of it which lies over a subset of the manifold Y of 1.4 ; these pieces are the "partial h-cobordisms" which are defined below. The word "relative" here means that the hypothesis of 1.4 is now strengthened to include a product structure for part of the the "partial h-cobordism" , and the conclusion of 1.4 is now strenthened to extending most of the given product structure to a product structure for most of the "partial h-cobordism".

Relative fibered control theory for h-cobordisms.

Let F denote a smooth closed manifold and let $\rho : X \longrightarrow Y$ denote a smooth fiber bundle projection from the Riemannian manifold X to the Riemannian manifold Y having F for fiber; X and Y need not be compact, but are assumed to have empty manifold boundaries. S denotes a compact subset of Y.

A **partial h-cobordism of** X **over** S consists of the following objects: two open subsets A, B of Y which satisfy $S \subset B \subset A$; a smooth cobordism pair (W, V) of the pair $\rho^{-1}(A, B)$, having boundary components $(\partial_- W, \partial_- V) = \rho^{-1}(A, B)$ and $(\partial_+ W, \partial_+ V)$; a retraction map

$$H : W \longrightarrow \rho^{-1}(A),$$

and maps

$$h_t^- : V \longrightarrow W \quad \text{and} \quad h_t^+ : V \longrightarrow W, \quad t \in [0,1],$$

which depend continuously on the variable t. The cobordism V satisfies $H^{-1}(\rho^{-1}(S)) \subset V$. In addition the mappings H, h_t^- and h_t^- must satisfy the following properties: $h_1^- = H \mid V$ and $h_1^+(V) \subset \partial_+ W$; h_0^- and h_0^+ are both equal to the inclusion map $V \subset W$; $h_t^- \mid (\partial_- V)$ and $h_t^- \mid (\partial_- V)$ are equal to the inclusion maps

$$\partial_- V \subset W \quad \text{and} \quad \partial_+ V \subset W$$

respectively for all $t \in [0,1]$.

Let T denote any subset of S. We say that the partial h-cobordism (W, V, H, h_t^-, h_t^+) of X over S is $(\varepsilon; \rho)$-**controlled over** T if for any fixed $z \in H^{-1}(\rho^{-1}(T))$ both of the paths

$$\rho \circ H \circ h_t^-(z) \quad \text{and} \quad \rho \circ H \circ h_t^-(z), \quad t \in [0,1],$$

have diameter less than ε in Y.

A **product structure over** T for the partial h-cobordism (W, V, H, h_t^-, h_t^+) of X over S consists of a smooth embedding

$$P : \rho^{-1}(T) \times [0,1] \longrightarrow V$$

which satisfies $P(z,0) = (z, f)$ and $P(z,1) \in \partial_+ V$ for all $z \in \rho^{-1}(T)$. This product structure is said to be $(\varepsilon; \rho)$-**controlled over** U, for some subset $U \subset T$, if for each $z \in \rho^{-1}(U)$ the path

$$p(t) = \rho \circ H \circ P(z,t), \quad t \in [0,1]$$

has diameter less than ε in Y.

We will need the following notation in this next theorem. For any subset Z of Y and any number $\delta > 0$ we let Z^δ denote the subset of all points in Y which are a distance less or equal to δ from Z, and we let $Z^{-\delta}$ denote the subset of all points in Z which are a distance greater than or equal to δ from $Y \setminus Z$. This next theorem, which generalizes 1.4, is also due to T.A. Chapman and F. Quinn (cf. references [3] and [17]).

Theorem 2.1. *Suppose that dimX ≥ 5. Suppose also that*

$$Wh(\pi_1(F) \oplus G) = 0$$

for any finitely generated free abelian group G. Given compact subsets T and S of Y, with $T \subset S$, there is a number $\kappa > 0$ which depends only on the isometry type of the triple (Y, S, T); ε will denote any number in $(0, \kappa)$. Let (W, V, H, h_t^-, h_t^+) denote a partial h-cobordism of X over S which is $(\varepsilon; \rho)$-controlled over S; and let

$$P : \rho^{-1}(T) \times [0, 1] \longrightarrow V$$

denote a product structure for this partial h-cobordism over the subset $T \subset Y$ which is $(\varepsilon; \rho)$-controlled over T.

(a) *There is $\varepsilon' > \varepsilon$, which depends only on ε and on the isometry type of (Y, S, T), and which satisfies*

$$\lim_{\varepsilon \to 0} \varepsilon' = 0\,.$$

(b) *There is another product structure*

$$P' : \rho^{-1}(S^{-\varepsilon'} \times [0, 1] \longrightarrow V$$

(c) *$P\prime$ is $(\varepsilon'; \rho)$-controlled over $S^{-\varepsilon'}$.*

(d) *P and P' are equal over $T^{-\varepsilon}$.*

Local-relative fibered control theory for h-cobordisms.

When Y of 2.1 is equal to Euclidean space $\mathbb{R}^n$ then we can strengthen the conclusions of 2.1 as follows.

Theorem 2.2. *If Y is equal an open subset of $\mathbb{R}^n$ in 2.1 then there is a number $\lambda > 0$, which depends only on n, such that ε and $\varepsilon\prime$ of 2.1 are related by $\varepsilon\prime = \lambda\epsilon$. Moreover the number κ of 2.1 may be taken to be any positive number.*

It is not difficult to deduce 2.2 from 2.1. To do so we will need the following lemma. We will first use this lemma to complete the proof of 2.2, and then we will deduce the lemma from 2.1.

It is also true that 2.1 can be deduced from 2.2. This is the subject of homework problem ?? below. It would be a good idea for the reader to work thru this homework problem as preparation to reading the proof for 1.10 in section 5 below.

Before stating the next lemma we need some notation. Let C denote a cell structure for $\mathbb{R}^n$ defined as follows. Let J denote the standard unit n-cube in $\mathbb{R}^n$, i.e. $(x_1, x_2, ..., x_n) \in J$ iff $0 \leq x_i \leq 1$ for all i. The n-dimensional cells of C are just the translates of J by vectors which have all integer valued coordinates; the lower dimensional cells of C are the translates of all the faces of J by these same vectors. Let K and L denote subcomplexes of C

with $L \subset K$; let (W, V, H, h_t^-, h_t^+) denote a partial h-cobordism over $K^{1/3}$; and let P denote a product structure for (W, V, h_t^-, h_t^+) over $L^{1/3}$

Lemma 2.3. *There is a number $\delta \in (0, 1/3)$ which depends only on $n = dimX$. Suppose (W, V, h_t^-, h_t^+) is $(\delta; \rho)$-controlled over $K^{1/3}$ and if P is $(\delta; \rho)$-controlled over $L^{1/3}$. Then there is a another product structure Q for (W, V, h_t^-, h_t^+) over K which is $(1/3; \rho)$-controlled over K and is equal to P over L.*

Proof of Theorem 2.2

We will prove this for $Y = \mathbb{R}^n$; the proof for the general situation is handled in the same way.

Define a number $\alpha > 0$ by

2.2.1 $\alpha = \delta\varepsilon^{-1}$,

and define a map

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

to be multiplication by α. Let K and L denote the maximal subcomplexes of C which satisfy $K^{1/3} \subset f(S)$ and $L^{1/3} \subset f(T)$ respectively.

We now apply 2.3 to the partial h-cobordism (W, V, H, h_t^-, h_t^+) and to the product structure P of 2.1 and 2.2; in this application we use the bundle projection map $\rho' = f \circ \rho$ instead of the projection map ρ. Note that the control hypothesis of 2.2 and 2.2.1 together imply that (W, V, H, h_t^-, h_t^+) and P satisfy the control hypothesis of 2.3 with respect to the projection ρ'. Thus 2.3 applies to yield a another product structure

$$Q : \rho\prime^{-1}(K) \times [0, 1] \longrightarrow W$$

which is equal to P on $(\rho')^{-1}(L) \times [0, 1]$. Now we define the desired product structure P' and number $\lambda > 0$ of 2.2 by

2.2.2.

(a) $\lambda = 3n\delta^{-1}$.

(b) $P' = Q$ on $\rho^{-1}(S^{-\varepsilon'}) \times [0, 1]$.

Now the conclusions of 2.2 follow from 2.2.1, 2.2.2 and from the conclusions of 2.3. This completes the proof of 2.2.

Proof of Lemma 2.3

Let $c_i, i \in I$, denote all the cells of K. Note that there is a sequence

$$I_1 \subset I_2 \subset I_3 \subset .... \subset I_{\mu_n}$$

of subsets of I which satisfy the following properties.

2.3.1.

(a) μ_n depends only on the number n.

(b) For any r and any $i, j \in I_{r+1} \setminus I_r$ we have that $c_i \cap c_j = \emptyset$.

(c) For any r and any $i \in I_{r+1}$ and any $j \in I_r$ we have $dim(c_i) \leq dim(c_j)$.

For each $r = 1, 2, ..., \mu_n$ we set

$$K_r = \bigcup_{i \in I_r} c_i.$$

Note that each K_r is a subcomplex of K, and that $K_r \subset K_{r+1}$. The proof of 2.3 will be carried out by induction over the increasing sequence

$$K_1 \subset K_2 \subset .... \subset K_{\mu_n} = K$$

of subsets of K. We shall assume thruought this induction argument that the number δ of 2.3 is a variable; its final value will be fixed at the end of this proof.

Induction Hypothesis 2.3.2.(r) Set $\sigma_r = 1/3r$. If δ is choosen sufficiently small, where how small is sufficient depends only on the numbers r and n, then there will be a product structure Q_r for (W, V, H, h_t^-, h_t^+) over $(L \bigcup K_r)^{\sigma_r}$, and there will be a number $\delta_r > 0$, all of which satisfy the following.

(a) δ_r depends only on the numbers r and n.
(b) $\lim_{\delta \to 0} \delta_r = 0$.
(c) Q_r is equal to P over L^{σ_r}.

Now we will carry out the induction step

$$2.3.2(r) \to 2.3.2(r+1).$$

For each $i \in I_{r+1} \setminus I_r$ define S_i and T_i by

2.3.3.

(a) $S_i = (c_i)^{\sigma_r}$
(b) $T_i = (c_i)^{\sigma_r} \bigcap (L \bigcup K_r)^{\sigma_r})$.

We may regard (W, V, h_t^-, h_t^+) as a partial h-cobordism over S_i; define a product structure P_i for (W, V, H, h_t^-, h_t^+) over T_i by letting P_i equal the restriction of Q_r to $\rho^{-1}(T_i) \times [0,1]$. Note that (W, V, H, h_t^-, h_t^+) is $(\delta_r; \rho)$-controlled over S_i, and P_i is $(\delta_r; \rho)$-controlled over T_i. Thus we may apply 2.1 to this situation to conclude that there is a number $\delta_{r,i} > \delta_r$, and that there is another product structure P_i' for (W, V, h_t^-, h_t^+) over $(S_i)^{-\delta_{r,i}}$, which satisfy the following properties.

2.3.4

(a) $\delta_{r,i}$ depends only on δ_r and on the isometry typeof the triple (Y, S_i, T_i).
(b) $\lim_{\delta_r \to 0} \delta_{r,i} = 0$.
(c) P_i is equal to Q_r over $T_i^{-\delta_{r,i}} \setminus (S_i \setminus T_i)^{\delta_{r,i}}$
(d) P_i is $(\delta_{r,i}; \rho)$-controlled over $(S_i)^{-\delta_{r,i}}$.

Note that there is an upper bounds, which depends only on the numbers r and n, to the number of distinct isometry types of the triples (Y, S_i, T_i), $i \in I_{r+1} \setminus I_r$. So, by 2.3.4(a) above, there is the same upper bound for the number of distinct numbers $\delta_{r,i}$, $i \in I_{r+1} \setminus I_r$. Thus we may define δ_{r+1} by

bf 2.3.4.

(e)

$$\delta_{r+1} = max\{\delta_{r,i} : i \in I_{r+1} \setminus I_r\}.$$

The following properties are then deduced immediately from 2.3.4.

2.3.5.

(a) δ_{r+1} depends only on δ_r and on the numbers r and n.
(b) $limit_{\delta_r \to 0} \delta_{r+1} = 0$; and $\delta_{r+1} > \delta_r$.
(c) P_i is equal to Q_r over $T_i^{-\delta_{r+1}}$.
(d) P_i is $(\delta_{r+1}; \rho)$-controlled over $(S_i)^{-\delta_{r+1}}$.

Now we can complete the induction step as follows. Note that it follows from 2.3.2(b), 2.3.3(a)(b) and 2.3.5(b) that if δ is choosen sufficiently small then we will have

2.3.6.

(a) $(c_i)^{\sigma_{r+1}} \subset (S_i)^{-\delta_{r+1}}$,
(b) $(c_i)^{\sigma_{r+1}} \bigcap (L \cup K_r)^{\sigma_{r+1}} \quad \subset \quad T_i^{\delta_{r+1}}$;

and it follows from 2.3.2(a) and 2.3.5(a) that how small is "sufficient" here depends only on the numbers r and n. Note also that it follows from 2.3.1(b) that

2.3.6.

(c) $(c_i)^{\sigma_{r+1}} \bigcap (c_j)^{\sigma_{r+1}} = \emptyset$ for any $i, j \in I_{r+1} \setminus I_r$ with $i \neq j$.

Now it follows from 2.3.6 and 2.3.5(c) that for sufficiently small δ, where how small is sufficient depends only on the numbers r and n, the product structure Q_{r+1} of 2.3.1(r+1) is well defined by

2.3.7.

(a) $Q_{r+1} = Q_r$ over $(L \cup K_r)^{\sigma_{r+1}}$,
(b) Q_{r+1} is equal to P_i over $(c_i)^{\sigma_{r+1}}$ for each $i \in I_{r+1} \setminus I_r$.

This completes the proof for 2.3.

Stratified fibered version of Theorem 2.2.

In proving the foliated control Theorem 1.10 in section 5 below we shall also need a generalization of 2.2 for a stratified fiber bundle projection map

$$\rho' : X\prime \longrightarrow Y'$$

described as follows. Let $\rho : X \longrightarrow Y$ be as in 2.2 and we let $\pi : X' \longrightarrow X$ denote bundle projection for a vector bundle over X with total space X'; we assume that this vector bundle is equipped with an innerproduct which gives rise to the norm $\| \quad \|$. Set $Y' = Y \times [0, \infty)$ equipped it with the product metric, and define the projectin ρ' by

$$\rho'(v) = (\rho \circ \pi(v), \|v\|)$$

for each $v \in X'$. We can define as before the following notions: a partial h-cobordism of X' over the subset S' of Y'; an h-cobordism over S' being $(\varepsilon; \rho')$-controlled over S'; a product structure over a subset $T' \subset S'$ for a partial h-cobordism; a product structure being $(\varepsilon; \rho')$-controlled over T'.

There is a "stratified fibered" version of Theorem 2.2 in which we simply replace the projection map $\rho : X \longrightarrow Y$ of 2.2 by the projection map $\rho' : X' \longrightarrow Y'$ just described. Unfortunately this version is not sufficiently general for the applications we have in mind in section 5 below. What is needed in section 5 is a type of "stratified version" of 2.2 where the control is of a bit more exotic nature then the "$(\varepsilon; \rho')$-control" just discussed.

A partial h-cobordism (W, V, H, h_t^-, h_t^+) of X' over the susubset $S' \subset Y'$ is said to be $(\varepsilon; \lambda; \rho')$**-controlled over** S' if it is $(\varepsilon + \lambda s; \rho')$-controlled over $S' \cap (Y \times [0, s])$ for all $s > 0$. A product structure P' for (W, V, H, h_t^-, h_t^+) over a subset $T' \subset S'$ is said to be $(\varepsilon; \lambda; \rho')$**-controlled over** T' if P' is $(\varepsilon + \lambda s; \rho')$-controlled over $T' \cap (Y \times [0, s])$ for all $s > 0$

Theorem 2.4. *Given any positive number λ_1 there is another positive number $\lambda_2 \in (\lambda_1, \infty)$ which depends only on λ_1 and $n = \dim(Y)$. Let a, ε be positive numbers which satisfy $\lambda_2 \varepsilon < a$; let T, S be compact subsets of Y with $T \subset S$; let(W, V, h_t^-, h_t^+) denote a partial h-cobordism of X' over*

$$S' = S \times [0, a]$$

which is $(\varepsilon; \lambda; \rho')$-contolled over S'; and let P' denote a product structure for this partial h-cobordism defined over the subset

$$T' = (T \times [0, a]) \cup (S \times [\varepsilon, a])$$

and which is $(\varepsilon; \lambda; \rho')$-controlled over T'.

(a) *There is another product structure P' for (W, V, H, h_t^-, h_t^+) over $S^{-\lambda_2 \varepsilon} \times [0, a]$.*
(b) *P' is $(\lambda_2 \varepsilon; \lambda_2; \rho')$-controlled over $S^{-\lambda_2 \varepsilon} \times [0, a]$.*
(c) *P' is equal to P over $(T^{-\lambda_2 \varepsilon} \times [0, a]) \bigcup (S^{-\lambda_2 \varepsilon} \times [\lambda_2 \varepsilon, a])$.*

The proof of Theorem 2.4 is outlined in homeworks 2.5, 2.6 and 2.7 below.

Homework for section 2.

Homework 2.5. There is a more general version of Theorem 2.1 (cf. [17]) for fiber bundle projections $\rho : X \longrightarrow Y$ between Riemannian manifolds X and Y where the fiber of ρ is smooth and compact but not necessarily closed, and where Y has no boundary. Note that if $\partial F \neq \emptyset$ then this forces X to have a boundary ∂X; infact the restricted projection $\rho : \partial X \longrightarrow Y$ is a smooth fiber bundle projection having the smooth closed manifold ∂F for fiber. In this version of 2.1 the partial h-cobordism (W, V, H, h_t^-, h_t^+) of 2.1 must contain a subcobordism (W_∂, V_∂) of the pair $(A \cap \partial X, B \cap \partial X)$;and V_∂ comes equipped with a product structure

$$P_\partial : B \times [0,1] \longrightarrow V_\partial$$

which satisfies

$$h_t^- \circ P_\partial(v,s) = (v,(1-t)s) \quad \text{and} \quad h_t^+ \circ P_\partial(v,s) = (v,(1-t)s+t)$$

for all $v \in V_\partial$ and for all $t \in [0,1]$. Also in this version of 2.1 both of the product structures P and $P\prime$ must be extensions of the product structure P_∂.

In this homework the reader is asked to give a precise formulation of the generalization of 2.1 that has just been sketched. Then deduce this generalization of 2.1 from 2.1.

Homework 2.6. There is also a a more general version of Theorem 2.2 where the fiber F of the projection map $\rho : X \longrightarrow Y$ in 2.2 is a compact smooth manifold but not necessarily closed. Give a precise formulation of this version of 2.2 and use the more general version of Theorem 2.1 discussed in the the preceeding homework to prove this more general version of 2.2.

Homework 2.7. Use the general version of Theorem 2.2 discussed in the preceeding homework to prove Theorem 2.4. Here is a rough idea of how to procede with this homework.

Let

$$\pi : X\prime \longrightarrow X$$

denote the vector bundle over X introduced just prior to the statement of Theorem 2.4; and let $X'(\varepsilon)$ denote the subset of all vectors $v \in X\prime$ satisfying $\|v\| \leq \varepsilon$. Note that the restricted map

$$\rho \circ \pi : X\prime(\varepsilon) \longrightarrow Y$$

is a fiber bundle projection having for fiber a compact (but not closed) smooth manifold. We obtain a partial h-cobordism $(W(\varepsilon), V(\varepsilon), G, g_t^-, g_t^+)$ of $X'(\varepsilon)$ as follows: set $W(\varepsilon)$ equal to the closure in W of its subset $W \setminus \text{image}(P)$, where P is the product structure given in 2.4; and set $V(\varepsilon)$ equal to the closure in W of its subset. Note that $(W(\varepsilon), V(\varepsilon))$ contains a subcobordism pair $(W(\varepsilon)_\partial, V(\varepsilon)_\partial)$ defined to be the interesetion of $(W(\varepsilon), V(\varepsilon))$ with image(P), and that P restricts to give a product structure for $(W(\varepsilon)_\partial, V(\varepsilon)_\partial)$.To get the deformation retracts (g_t^-, g_t^+) we must slightly modify the deformation retractions (h_t^-, h_t^+), because the (h_t^-, h_t^+) do not at present necessarily map $V(\varepsilon)$ into $W(\varepsilon)$, nor do are they at present necessarily linked with the product structure for $(W(\varepsilon)_\partial, V(\varepsilon)_\partial)$ as described in 2.5. To complete this homework apply the version of 2.2 described in 2.6 to the partail h-cobordism $(W(\varepsilon), V(\varepsilon), G, g_t^-, g_t^+)$; note that this partial h-cobordism is well controlled with respect to the projection map $\rho \circ \pi : X\prime(\varepsilon) \longrightarrow Y$.

Homework 2.8. Use Theorem 2.2 to prove Theorem 1.4 in the special case that $\partial Y = \emptyset$ in 1.4. Carrying out this homework, which is outlined below, will prepare the reader for the proof of the foliated control Theorem 1.10 given in section 5 below. The first step towards completing this homework is to choose a sufficiently small number $\epsilon > 0$ and a finite number of smooth charts

$$g_i : \mathbb{B}_\epsilon^n \longrightarrow Y, i = 1, 2, ..., m,$$

for Y, from the open ball of radius ϵ centered at the origin of $\mathbb{R}^n$, such that the following properties hold.

(a) The images $g_i(\mathbb{B}^n_{\epsilon/2})$, i=1,2,3,...,m, cover all of Y.

(b) The derivatives dg_i for the g_i satisfy

$$||2v|| \geq ||dg_i(v)|| \geq ||v/2||$$

holds for all vectors v which are tangent to $\mathbb{B}_\epsilon^n$.

(c) There is an increasing sequence

$$J^1 \subset J^2 \subset ..... \subset J^{\pi(n)} = 1, 2, ..., m,$$

where $\pi(n)$ depends only on n . If $k, j \in J^i \setminus J^{i-1}$ for some integers i, j, k, then $g_j(\mathbb{B}_\epsilon^n)$ and $g_k(\mathbb{B}_\epsilon^n)$ are disjoint.

To complete this homework problem one proceeds by induction over the sequence of subsets $J^1 \subset J^2 \subset ... \subset J^{\pi(n)}$. Choose a decreasing sequence of small numbers $3/4 > \epsilon_1 > \epsilon_2 > .... > \epsilon_{\pi(n)} > 1/2$. Our induction hypothesis is that a product structure P_r for the h-cobordism of 1.4 has been constructed over the subset $\bigcup_{i\in J(r)} g_i((B^n_{\epsilon_r})$ with good control over this same subset. To construct P_{r+1} one applies 2.2 to the "pieces" of the h-cobordism which "lie over" each subset $g_i(\mathbb{B}_\epsilon^n)$ with $i \in (J^{r+1} \setminus J^r)$. Note (by (c) above) that 2.2 may be applied independently over each subset; the relevant projections for these applications of 2.2 are maps

$$\rho_i : X_i \longrightarrow \mathbb{B}_\epsilon^n,$$

where $X_i = \rho^{-1}(g_i((B^n))$ and $\rho_i = g_i^{-1} \circ \rho$.

Homework 2.9. In Theorem 1.4 the base space Y of the fiber bundle projection is allowed to have a manifold boundary ∂Y, but all the fibered control theorems of this section we have assumed that Y has no boundary. However each result of this section is true even when Y does have a boundary. Deduce from 2.1 a more general version of 2.1 in which Y may have a non-empty boundary. Do the same for 2.2 and 2.4.

3. Foliated control theory for h-cobordisms.

In this section we formulate the general notion of foliated control theory, formulate a general conjecture concerning foliated control, and prove a theorem which will be used in the proof of Theorem 1.10 in section 5 below.

Let X denote a compact Riemannian manifold with boundary ∂X, and let Ξ denote a smooth foliation for X which also foliates ∂X. A path $p : [0,1] \longrightarrow X$ is said to have **Ξ-diameter less than** (α, ε) **in** (X, Ξ) if there is a connected subset A in some leaf L of Ξ such that the following hold: the diameter of A in L, with respect to the Riemannian structure that L inherits from X, is less than α; any point $p(t)$, $t \in [0,1]$, is a distance in X less than ε from the subset A.

An h-cobordism (W, W_∂) of $(X, \partial X) = \partial_-(W, W_\partial)$ is said to be $(\alpha, \varepsilon; \Xi)$-**controlled** if there are deformation retracts

$$r_t^- : (W, W_\partial) \longrightarrow (W, W_\partial), \quad t \in [0,1]$$

and

$$r_t^+ : (W, W_\partial) \longrightarrow (W, W_\partial), \quad t \in [0,1]$$

for (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial)$ respectively, such that for each $w \in W$ both of the paths

$$r_1^- \circ r_t^-(w) \quad \text{and} \quad r_1^- \circ r_t^-(w), \quad t \in [0,1]$$

have Ξ-diameters less than (α, ε) in (X, Ξ). (We say that the deformation retracts r_t^- and r_t^+ are $(\alpha, \varepsilon; \Xi)$-**controlled** if they satisfy the preceeding properties.)

The following conjecture is an attempt to generalize to the foliated setting the fibered control h-cobordism Theorem 1.4.

Conjecture 3.1. *Suppose each leaf L of the foliation Ξ satisfies*

$$Wh(\pi_1(L) \oplus G) = 0$$

for every free abelian group G. Suppose also that $dim(X) \geq 6$. Then for any number $\alpha > 0$ there is a number $\varepsilon > 0$. If (W, W_∂) is an h-cobordism of X which is $(\alpha, \varepsilon; \Xi)$-controlled, then $(W, W_\partial$ is a product cobordism.

Special cases of Conjecture 3.1.

In the next few paragraphs we remark upon some special cases of this conjecture.

Case I: dim(Ξ)=1. In this case Theorem 1.8 above states that Conjecture 3.1 is true.

Case II: dim(Ξ)=2. Whether this special case of Conjecture 3.1 is true is not known. However in this case the hypothesis of 3.1, that

$$Wh(\pi_1(L) \oplus G) = 0$$

for every finitely generated free abelian group G, may be dropped from 3.1 because this equality holds automatically for any leaf L of a smooth two-dimensional foliation Ξ. To see this we first note that any such leaf L is a connected 2-dimensional manifold without boundary. Next recall that each such manifold is homeomorphic to one of the following four types of surfaces (cf. reference [19]).

(a) L is homeomorphic to the 2-sphere.
(b) L is homeomorphic to the 2-dimensional real projective space.
(c) L is homeomorphic to a closed manifold with constant sectional curvature equal 0.
(d) L is homeomorphic to a complete Riemannian manifold with constant sectional curvature equal -1.

Note that if L is of type (a) then $Wh(\pi_1(L)) = 0$ is follows from 1.6; if L is of type (b) then $Wh(\pi_1(L)) = 0$ is proven in reference [15]; if L is of type (c) or (d) then $Wh(\pi_1(L)) = 0$ is proven in reference [11].

Case III: the leaves of Ξ have bounded diameter. The diameter refered to here is the diameter of each leaf L of Ξ computed with respect to the Riemannian metric inherited by L from X. Our assumption is that there is a finite upper bound for the diameters of all the leaves of Ξ. Note, that since X is compact, this implies that each leaf of Ξ is compact. Let Y denote the quotient space obtained from X by collapsing each leaf of Ξ to a point and let

$$\rho : X \longrightarrow Y$$

denote the quotient map. Note that Y can be equipped with a finite triangulation (infact it inherits a PL structure from the smooth structure on X); and the map $\rho : X \longrightarrow Y$, although not in general a fiber bundle projection, is the projection map of a "stratified fibration" in the sense of Quinn (cf. reference [18]). Quinn has proven in reference [18] a more general "stratified" version of Theorem 1.4 in which the fiber bundle projection of 1.4 may be replaced by the projection map for any "stratified fibration". In our present context, Quinn's "stratified fibration" version of 1.4 allows us to apply the conclusions of 1.4 to the quotont map $\rho : X \longrightarrow Y$. Note that if W is an h-cobordism of X which is $(\alpha, \varepsilon; \Xi)$-controlled, then W is also $(\varepsilon'; \rho)$-controlled with respect to some metric on Y, where

$$\lim_{\varepsilon \to 0} \varepsilon' = 0.$$

Thus Quinn's "stratified fibration" version of Theorem 1.4 may be applied to conclude that W is a product. Thus we have proven (or more precisely Frank Quinn has proven) the following theorem.

Theorem 3.2. *If the leaves of Xi have bounded diameter then Conjecture 3.1 is true.*

Case IV: each leaf of Ξ has non-postive sectional curvature. Each leaf L inherits a Riemannian structure from X; in this case we require that L has non-positive sectional curvature values everywhere with respect to this structure. In this case Conjecture 3.1 is known to be true, provided we replace the word "controlled" in **??** by the phrase "simply-controlled", which we define now.

Let X^{cov} denote the universal covering space for X and let Ξ^{cov} denote the lifting to X^{cov} of the foliation Ξ; X^{cov} is equipped with a Riemannian structure lifted from that on X. Note that any h-cobordism W of X lifts to an h-cobordism W^{cov} of X^{cov}. An h-cobordism W of X is said to be $(\alpha, \varepsilon; \Xi)$-**simply-controlled** if there are deformation retracts

$$r_t^- : (W, W_\partial) \longrightarrow (W, W_\partial), \quad t \in [0,1],$$

and

$$r_t^+ : (W, W_\partial) \longrightarrow (W, W_\partial), \quad t \in [0,1]$$

for (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial)$ respectively, which lift to $(\alpha, \varepsilon; \Xi^{cov})$-controlled deformation retracts retracts for $(W, W_\partial)^{cov}$ onto $\partial_-(W, W_\partial)^{cov}$ and $\partial_+(W, W_\partial)^{cov}$ respectively.

Theorem 3.3. *Suppose tha leaves of Ξ have non-positive sectional curvature values everywhere. For any number $\alpha > 0$ there is a number $\varepsilon > 0$ which depends only on α and Ξ. Any $(\alpha, \varepsilon; \Xi)$-controlled h-cobordism (W, W_∂) of $(X, \partial X)$ is a product cobordism.*

Note that this theorem follows from Theorem 1.2 and the s-cobordism theorem in the special case that the there is only the one leaf X in the foliation Ξ. This theorem is proven in [8] for the special case that each leaf L of Ξ has constant sectional curvature -1. To prove Theorem 3.2 in general one should follow the ideas in the proof of [8], replacing the "asymptotic transfer" constructions used in that proof by the "focal transfer" constructions used in references [13] and [14]. (As of yet no one has worked out the details of this proof.) For more information about the "asymptotic and focal transfer" constructions the reader is referred to Tom Farrell's lecture notes.

Foliated-control over Euclidean space.

In this subsection we deduce a very simple foliated control result from the the results of section 2. Let

$$\rho : X \longrightarrow Y$$

denote the fiber bundle projection of Theorem 2.2 above; thus Y is an open subset of Euclidean space $\mathbb{R}^n$. We let Γ denote a foliation of $\mathbb{R}^n$ whose leaves are gotten by choosing a plane in $\mathbb{R}^n$ and translating this plane to every point of $\mathbb{R}^n$. A foliation Ξ for Y is gotten by just restricting Γ to Y.

For any subset $S \subset \mathbb{R}^n$ and any positive numbers α and ε we denote by $S^{\alpha,\varepsilon}$ the set of all points $y \in \mathbb{R}^n$ for which there is a path

$$p : [0,1] \longrightarrow \mathbb{R}^n$$

such that $p(0) \in S$, the distance from p(1) to y is less than ε, the image of the path lies in a subset $A \subset L$ of a leaf L of Γ, and A has diamter less than α. We denote by $S^{-\alpha,-\varepsilon}$ the difference subset $S \setminus (\mathbb{R}^n \setminus S)^{\alpha,\varepsilon}$.

In the next theorem we consider a partial h-cobordism $(W, V, , H, h_t^-, h_t^+)$ over the compact subset $S \subset Y$ (as defined in section 2), and a product structure

$$P : \rho^{-1}(T) \times [0,1] \longrightarrow W$$

for W over the compact subset $T \subset S$ (also as described in section 2). We say that this partial h-cobordism is $(\alpha, \varepsilon; \Xi, \rho)$-**controlled** over S if for each $w \in W$ both of the paths

$$\rho \circ H \circ h_t^-(w) \quad \text{and} \quad \rho \circ H \circ h_t^+(w), \quad t \in [0,1],$$

have Ξ-diameter less than (α, ε) in (Y, Ξ). The product structure P is said to be $(\alpha, \varepsilon; \Xi, \rho)$-**controlled** over T if for each $x \in \rho^{-1}(T)$ the path

$$\rho \circ H \circ P(x,t) \quad , \quad t \in [0,1],$$

has Ξ-diameter less than (α, ε) in (Y, Ξ).

Theorem 3.4. *Suppose that each fiber F of $\rho : X \longrightarrow Y$ satisfies*

$$Wh(\pi_1(F) \oplus G) = 0$$

for all finitely generated free abelian groups. Suppose also that

$$dimension(X) \geq 5.$$

There is positive number λ which depends only on the integer n. Supose that partial h-cobordism $(W, V, , H, h_t^-, h_t^+)$ is $(\alpha, \varepsilon; \Xi, \rho)$-controlled over S, and that the product structure P is also $(\alpha, \varepsilon; \Xi, \rho)$-controlled over T, for some positive numbers α and ε. Then there is another product structure P' which satisfies the following properties.

(a) *There is another product structure P' for the partial h-cobordism defined over $S^{-\lambda\alpha,-\lambda\varepsilon}$.*
(b) *P' is equal to P over $T{-}\lambda\alpha, -\lambda\varepsilon$.*
(c) *P' is $(\lambda\alpha, \lambda\varepsilon; \Xi, \rho)$-controlled over $S^{-\lambda\alpha,-\lambda\varepsilon}$.*

Proof of Theorem 3.4

This is really an immediate corollary of Theorem 2.2 in section 2. To see this we let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ denote the linear map that multiplies a vector v by the number $\alpha^{-1}\varepsilon$ if v is tangent to the foliation Γ, and which multiplies v by 1 if v is perpendicular to the foliation Γ. Note that the partial h-cobordism $(W, V, , H, h_t^-, h_t^+)$ is $(2\varepsilon; f \circ \rho)$-controlled over $f(S)$, and that P is $(2\varepsilon; f \circ \rho)$-controlled over $f(T)$. Thus we may apply Theorem 2.2 to conclude that there there is another product structure P' over $f(S)^{-\lambda(2\varepsilon)}$ which is $(\lambda(2\varepsilon); f \circ \rho)$-controlled over this same set, where λ is the number in Theorem 2.2. This translates into the desired conclusions for Theorem 3.4 provided the λ of 3.4 is chosen to be equal to four times the value of λ in Theorem ??.

This completes the proof of Theorem 3.4.

4. Long and thin charts for a one-dimensional foliation.

In this section we consider the smooth one-dimensional foliation Ξ of a closed Riemannian manifold Y. We formulate two lemmas which describe a nice covering of Y by "long and thin" open subsets of Y (long in the foliation direction and thin in the direction perpendicular to the foliation). In section 5 we will complete the proof of Theorem 1.10 by applying fiber control results of sections 2 an 3 to the pieces of the h-cobordism in 1.10 which lie over each long and thin set in the open covering of Y

Let γ denote a given positive number and let Ξ^γ denote all the closed leaves of Ξ which have length less than or equal to γ. We define a sequence of subsets

4.1.

$$\emptyset = \Xi_0^\gamma \subset \Xi_1^\gamma \subset \Xi_2^\gamma \subset \ldots \subset \Xi_z^\gamma = \Xi^\gamma$$

as follows: each Ξ_i^γ is a union of closed leaves; a leaf L in Ξ_i^γ is also a leaf of Ξ_{i-1}^γ iff there is a tubular neighborhood A for L in X, with projection $p_A : A \longrightarrow L$, and a sequence of leaves $L_j, j = 1, 2, 3, \ldots$, in Ξ_i^γ which converge to L, such that each composite map $L_j \subset A \longrightarrow L$ is an n_j-fold covering map for $n_j > 1$. The following lemma is easy to verify.

Lemma 4.1. *Let ν denote the length of the shortest closed leaf of Ξ. Then the number z in 4.1 depends only on ν and γ.*

The next lemma, which is more difficult to verify, is proven in sections 7 and 8 of reference [8]. We shall need some notation and a definition before stating this next lemma.

Let n denote the dimension of Y, and let U denote an open subset of Euclidean space $\mathbb{R}^{n-1}$, and let $(-r, r)$ denote the interval of length $2r$ centered at 0 in the real numbers. A smooth map

$$g : (-r, r) \times U \longrightarrow Y$$

is a called a **rectangular foliation chart** for Ξ if the following properties hold: g is a one-one immersion; for each $u \in U$ the path $g(t,u)$ has unit speed is and contained in a leaf of Ξ.

Lemma 4.2. *Let C denote a compact subset of $Y \setminus \Xi^\gamma$. There are numbers $\sigma_1, \sigma_1, \epsilon \in (0,1)$; the σ_i depend only on $n = dim(Y)$, and ϵ depends on C, γ and on Ξ. There is a finite collection $g_i : (-r_i, r_i) \times U_i, i \in I$, of rectangular foliation charts for Ξ such that the following properties hold.*

(a) *We have that $\sigma_1\gamma < r_i$ for all $i \in I$.*
(b) *The images $g_i((-\sigma_2 r_i, \sigma_2 r_i) \times U_i^{-\epsilon})$, $i \in I$, cover C.*
(c) *For each $i \in I$ and for each vector v which is tangent to $(-r_i, r_i) \times U_i$, we have that $\epsilon\|v\| \leq \|dg_i(v)\| \leq \epsilon^{-1}\|v\|$.*
(d) *The index set I is a disjoint union $I = I_1 \cup I_2 \cup ... \cup I_{n+1}$ of subsets. For any integer k satisfying $1 \leq k \leq n+1$, and for any $i, j \in I_k$, the images of g_i and of g_j are disjoint.*

Remark 4.3. Note it follows from 4.2 that there are numbers $\lambda, \mu > 0$; with λ depending only on $n = \dim(Y)$; and with μ depending on C, γandΞ. For any of the rectangular charts g_i of 4.2, and for any path $g : [0,1] \longrightarrow \text{image}(g_i)$ the following is true. If the path g has Ξ-diameter less than (α, ε) in (Y, Ξ) then the path $g_i^{-1} \circ g$ has Ω-diameter less then $(\lambda\alpha, \mu\varepsilon)$ in $(-r_i, r_i) \times U_i$, where Ω denotes for foliation of $(-r_i, r_i) \times U_i$ by the lines $(-r_i, r_i) \times x$ with $x \in U_i$. If the path $g_i^{-1} \circ g$ has Ω-diameter less than (α, ε) in $(-r_i, r_i) \times U_i$ then the path g has Ξ-diameter less then $(\lambda\alpha, \mu\varepsilon)$ in (Y, Ξ).

We will need some more notation and definitions before stating the final lemma of this section.

Let $\mathbb{R}^n \times S^1$ denote the product of Euclidean space $\mathbb{R}^n$ with the unit circle S^1. Let

$$p : \tau \longrightarrow \mathbb{R}^n \times S^1$$

denote the projection map for a vector bundle over $\mathbb{R}^n \times S^1$ which is equipped with an inner product structue: for each $v \in \tau$ let $\|v\|$ denote the length of v. We denote the first factor map of p by

$$p_1 : \tau \longrightarrow \mathbb{R}^n$$

Define another map

$$q : \tau \longrightarrow \mathbb{R}^n \times [0, \infty)$$

by $q(v) = (p_1(v), \|v\|)$. For any numbers $r, x > 0$ set

$$\tau(r,s) = q^{-1}(\mathbb{B}^n_r \times [0,s)),$$

where $\mathbb{B}^n_r$ denotes the open ball of radius r centered at the origin of $\mathbb{R}^n_r$, and set

$$\tau(r) = \tau(r,r).$$

A mapping

$$f : \tau(r) \longrightarrow Y$$

is called a **circular chart for** Y if the following properties hold: $\dim(\tau) = \dim(Y)$; f is a one-one smooth immersion.

A circular chart $f : \tau(r) \longrightarrow Y$ is called an $(\alpha, \beta; \mu)$-bf approximation to Ξ if the following properties hold for any path $g : [0,1] \longrightarrow f(\tau(r))$.

(a) If g has Ξ-diameter less than (α, ε) in (Y, Ξ), for any $\varepsilon \in (\beta, \infty)$, then the path $q \circ f^{-1} \circ g$ must have diameter less than $\mu\|f^{-1}(g(0))\| + \mu\varepsilon$ in $\mathbb{R}^n \times [0, \infty)$.

(b) If the path $q \circ f^{-1} \circ g$ has diameter less than ε in $\mathbb{R}^n \times [0, \infty)$ then the path g must have Ξ-diameter less than $(\alpha, \mu\|f^{-1}(g(0))\| + \mu\varepsilon)$ in (Y, Ξ).

A proof for the next lemma can be found in sections 1 and 2 of reference [8].

Lemma 4.4. *There is a small postive number μ, and a continuous function $h : [0, \infty) \longrightarrow [0, \infty)$ satisfying $h(0) = 0$, which depend only on the number γ and on the foliation Ξ. Let the number z be as in 4.1, let $j \in \{1, 2, 3, ..., z\}$ and let $C(j)$ denote any compact subset of $\Xi_j^\gamma \setminus \Xi_{j-1}^\gamma$. There is a number $\delta_j > 0$ which depends on $C(j)$, γ and on Ω. For each $\epsilon_j \in (0, \delta_j)$ there is a finite collection of circular charts $f_{j,i} : \tau_{j,i}(\epsilon_j) \longrightarrow Y$, $i \in I_j$, for Y, all of which satisfy the following properties.*

(a) *Each $f_{j,i}$ is an $(\gamma, h(\epsilon_j)\epsilon_j; \mu)$-approximation to Ξ.*

(b) *Each set $f_{j,i}(\tau_{j,i}^{\epsilon_j} \setminus f_{j,i}(\tau_{j,i}^{h(\epsilon_j)\epsilon_j})$ is disjoint from Ξ_j^γ.*

(c) *The sets $f_{j,i}(\tau^{h(\epsilon)\epsilon, \epsilon/2)})$, for $i \in I_j$, cover $C(j)$.*

(d) *The index set I_j is a disjoint union $I_{j,1} \bigcup I_{j,2} \bigcup ... \bigcup I_{j,s}$, where s is some postive integer satisfying $s < \mu^{-1}$. For each i and each $k, l \in I_{j,i}$, we have that the images of $f_{j,k}$ and of $f_{j,l}$ are disjoint.*

5. Proof of the foliated control theorem for h-cobordisms.

Let $\rho : X \longrightarrow Y$ denote the smooth fiber bundle map of Theorem 1.101.5, and let

$$\Xi_1^\gamma \subset \Xi_2^\gamma \subset .... \subset \Xi_z^\gamma$$

be the increasing sequence of subsets of Y described in 4.1 above. The proof of Theorem 1.10 proceedes by induction over the increasing sequence of subspaces

$$Y \setminus \Xi_z^\gamma \subset Y \setminus \Xi_{z-1}^\gamma \subset Y \setminus \Xi_{z-2}^\gamma \subset .... \subset Y \setminus \Xi_1^\gamma \subset Y$$

. The first step in our induction argument is carried out in Lemma 5.5 below. The induction hypothesis is the same as the hypothesis of Lemma 5.7 below; the induction step is carried out in Lemma 5.7.

For any subset $C \subset Y$ and for any numbers $\alpha, \varepsilon > 0$ we denote by $C^{\alpha,\varepsilon}$ the subset of all $y \in Y$ for which there is a path $g : [0,1] \longrightarrow Y$ which begins at y and has Ξ-diameter less than (α, ε) in (Y, Ξ). We define the subset $C^{-\alpha,-\varepsilon}$ by

$$C^{-\alpha,-\varepsilon} = C \setminus (Y \setminus)^{\alpha,\varepsilon}.$$

Lemma 5.5. *There is number $1 > \lambda > 0$, which depends only on the number $n = dimY$; there is a number $\omega > 1$, which depends on the the numbers n and γ and on the foliation Ξ. Given an arbitrarily large compact subset C of $Y \setminus \Xi^\gamma$, there is a number $\delta \in (0,1)$, which depends on C,$n = \dim Y$,γ and Ξ. For any numbers $\alpha \in (0, \lambda\gamma)$ and $\varepsilon \in (0, \delta)$, and for any h-cobordism W of X which is (α, ε)-controlled over Y, there is a product structure P for W defined over $C^{-\gamma,-\omega\varepsilon}$, which is $(\gamma, \omega\varepsilon)$-controlled over $C^{-\gamma,-\omega\varepsilon}$.*

Remark 5.6. In preceeding lemma if a pair of deformation retractions $r_t^\pm$ for the h-cobordism are given with respect to which the control data for W is as stated in the lemma, then the product structure P may be choosen so that its control data with respect to the same deformation retractions $r_t^\pm$ is also as stated in 5.5.

Lemma 5.7. *Choose a number j satisfing $1 \leq j \leq z$ and let C_j denote an arbitrarily large compact subset of $Y \setminus \Xi_j^\gamma$. There is a number $1 > \omega - j > 0$ and a number $\delta_j \in (0,1)$ which depend on n,γ and on the foliation Ξ. For any number $\varepsilon_j \in (0, \delta_j)$ let W denote an h-cobordism of X which is (γ, ε_j)-controlled over Y, and let P_j denote a product structure for W defined over C_j which is (γ, ε_j)-controlled over C_j. Then there is an arbitrarily large compact subset C_{j-1} of $Y \setminus \Xi_{j-1}^\gamma$, and a product strucuture P_{j-1} for W over C_{j-1}, which satisfy the following properties.*

(a) $C_j^{-\gamma,-\omega_j\varepsilon_j} \subset C_{j-1}$.

(b) P_{j-1} is equal to P_j over the subset $C_j^{-\gamma,-\omega_j\varepsilon_j}$ of Y.

(c) P_{j-1} is $(\gamma, \omega_j\varepsilon_j)$-controlled over the subset C_{j-1} of Y.

Remark 5.8. In preceeding lemma if a pair of deformation retractions $r_t^\pm$ for the h-cobordism are given with respect to which the control data for W and P_j are as stated in the lemma, then the product structure P_{j-1} may be choosen so that its control data with respect to the same deformation retractions $r_t^\pm$ is also as stated in 5.7.

Proof of Lemma 5.1.

We cover the set C by a finite number of foliation charts $\{g_i : i \in I\}$ as provided in Lemma 4.2. Let

$$I = I_1 \cup I_2 \cup ... \cup I_{n+1}$$

denote the disjoint union given in Lemma 4.2(d). The proof proceedes by induction over the sequence $I_1, I_2, ..., I_{n+1}$. The induction hypothesis consists of assuming that the product structure has already been constructed over a large compact subset of the union

$$\bigcup_{j \in I(r)} \text{image}(g_i),$$

where

$$I(r) = I_1 \cup I_2 \cup ... \cup I_r.$$

The induction step $r \longrightarrow r+1$ consists of applying Theorem 3.4 "over" each each of the subsets image(g_i) with $i \in I_{r+1}$. Note that 4.3 is relavant to this application of 3.4.

Proof of Lemma 5.2.

Let $C(j)$ denote an arbitrarly large compact subset of $\Xi_j^\gamma \setminus \Xi_{j-1}^\gamma$ .We cover the set $C(j)$ by a finite number of circular charts $\{fj, i : i \in I_j\}$ for (Y) as provided in Lemma 4.4. Let

$$I_j = I_{j,1} \cup I_{j,2} \cup ... \cup I_{j,n+1}$$

denote the disjoint union given in Lemma 4.4(d). The proof proceedes by induction over the sequence $I_{j,1}, I_{j,2}, ..., I_{j,n+1}$. The induction hypothesis consists of assuming that the product structure has already been constructed over a large compact subset of the union

$$\bigcup_{j \in I_j(r)} \text{image}(g_i),$$

where

$$I_j(r) = I_{j,1} \cup I_{j,2} \cup ... \cup I_{j,r}.$$

The induction step $r \longrightarrow r+1$ consists of applying Theorem 2.4 "over" each each of the subsets image($f_{j,i}$) with $i \in I_{j,r+1}$.

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY AT STONY BROOK, STONY BROOK, NY 11794

E-mail address: lejones@math.sunysb.edu