united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

SMR1312/10

School on High-Dimensional Manifold Topology

(21 May - 8 June 2001)

Splitting the surgery map under a geometric assumption

F.T. Farrell

Binghamton University
Department of Mathematical Sciences
Binghamton, New York 13902-6000
U.S.A.

These are preliminary lecture notes, intended only for distribution to participants

Splitting the surgery map under a geometric assumption

F.T. Farrell[†]

[†] *Binghamton University, Department of Mathematical Sciences,
Binghamton, New York 13902-6000, USA*

Lecture given at the:
School on High Dimensional Manifold Topology, Trieste
21 May - 8 June 2001

LNS

Lecture 2

Throughout this lecture (unless otherwise stated) M (and N) will denote complete (connected) Riemannian manifolds. Furthermore Γ will denote the group of all deck transformations of the universal cover $\tilde{M} \to M$ and we identify Γ with $\pi_1(M)$. If v is a vector tangent to M (i.e. $v \in TM$ = tangent bundle of M) then

$$\alpha_v : \mathbb{R} \to M$$

denotes the unique geodesic such that $\dot{\alpha}_v(0) = v$.

Figure 1

The function $\mathbb{R} \times TM \to TM$ defined by

$$g^t(v) = \dot{\alpha}_v(t)$$

for $t \in \mathbb{R}$ and $v \in TM$, is a flow on TM; i.e. it is smooth and satisfies the equation

$$g^s(g^t(v)) = g^{s+t}(v)$$

for all $s, t \in \mathbb{R}$ and $v \in TM$. This flow leaves invariant SM = unit sphere bundle of M and its restriction to SM is called the *geodesic flow*. Closely related to the geodesic flow is the *exponential function* $\mathrm{Exp} : TM \to M$ defined by

$$\mathrm{Exp}(v) = \alpha_v(1).$$

It is also a smooth function. If we fix a base point $x_0 \in M$, then the restriction of Exp to $T_{x_0}M$ = tangent space to M at x_0 is also called the exponential function and denoted by

$$\exp_{x_0} : T_{x_0}M \to M.$$

(Or more simply by exp when no ambiguity is possible.) Note that the vector space $T_{x_0}M$ considered as a smooth manifold $N = T_{x_0}M$ has a natural complete Riemannian metric; namely, if $u \in TN$, then $|u| = \sqrt{U \cdot U}$ where U is the parallel translate of u to 0.

Figure 2

We say that M is *non-positively curved* (resp. *negatively curved*) if all its sectional curvatures are ≤ 0 (resp. < 0). And a negatively curved manifold is *pinched negatively curved* if its sectional curvatures are bounded away from 0 and $-\infty$. Note that a closed negatively curved manifold is pinched negatively curved.

Definition. A smooth map $f : M \to N$ is called (weakly) *expanding* if

$$|df(v)| \geq |v|$$

for all vectors $v \in TM$.

There is the following important result relative to these definitions.

Theorem. *(Cartan) Let M be non-positively curved and $x_0 \in M$ be a base point. Then* $\exp : T_{x_0}M \to M$ *is an expanding map. Furthermore it is a covering projection and hence a diffeomorphism when $\pi_1(M) = 0$.*

Because of Cartan's theorem a non-positively curved (Riemannian) manifold M^m is aspherical since its universal cover $\tilde{M}$ is diffeomorphic to $\mathbb{R}^m$. It also leads to the following useful alternate description of TM as the bundle with fiber $\tilde{M}$ associated to the principal Γ-bundle $\tilde{M} \to M$; namely

$$\tilde{M} \times_\Gamma \tilde{M} \to M.$$

In fact this bundle is indentified with $TM \to M$ as Diff($\mathbb{R}^m$)-bundles via the Γ-equivariant diffeomorphism

$$T\tilde{M} \to \tilde{M} \times \tilde{M}$$

which sends $v \in T\tilde{M}$ to $(\alpha_v(0), \alpha_v(1))$. The 0-section of TM corresponds (under this identification) with the image of the diagonal Δ of $\tilde{M} \times \tilde{M}$ in $\tilde{M} \times_\Gamma \tilde{M}$.

There is also a natural *geodesic ray compactification* $\bar{M}$ of $\tilde{M}$ due to Eberlein and O'Neill such that $(\bar{M}, \tilde{M})$ is homeomorphic to $(\mathbb{D}^m, \mathrm{Int}\ \mathbb{D}^m)$ where

$$\mathbb{D}^m = \{v \in \mathbb{R}^m \mid |v| \leq 1\}.$$

Let $M(\infty) = \bar{M} - \tilde{M}$ denote the points added; called *ideal points.* Each ideal point is an asymptoty class of geodesic rays in $\tilde{M}$. A *geodesic ray* is a subset of $\tilde{M}$ of the form

$$\{\alpha_v(t) \mid t \in [0, +\infty)\}$$

for some $v \in S\tilde{M}$. Two rays R_1 and R_2 are *asymptotic* if there exists a positive number b such that each point of R_1 is within distance b of some point of R_2 and vice-versa.

Figure 3

The deck transformation action of Γ on $\tilde{M}$ extends to an action on $\bar{M}$ since Γ acts via isometries on $\tilde{M}$ and isometries preserve both geodesic rays and the relation of being asymptotic.

W.C. Hsiang and I abstracted an additional key property possessed by the geodesic ray compactification in the following definition. (For the rest of this lecture M denotes a closed topological manifold and *not necessarily a Riemannian manifold.*)

Definition. A closed manifold M^m satisfies condition $(*)$ provided there exists an action of $\Gamma = \pi_1(M^m)$ on $\mathbb{D}^m$ with the following two properties.

1. The restriction of this action to $\text{Int}(\mathbb{D}^m)$ is equivalent via a Γ-equivariant homeomorphism to the action of Γ by deck transformations on the universal cover $\tilde{M}$ of M^m.
2. Given any compact subset K of $\text{Int}(\mathbb{D}^m)$ and any $\epsilon > 0$, there exists a real number $\delta > 0$ such that the following is true for every $\gamma \in \Gamma$. If the distance between γK and $S^{m-1} = \partial\mathbb{D}^m$ is less than δ, then the diameter of γK is less than ϵ.

Figure 4

The above picture illustrates property 2 of condition $(*)$.

Remark 1. Hsiang and I showed that every closed (connected) non-positively curved Riemannian manifold M satisfies condition $(*)$ by using its geodesic ray compactification.

Remark 2. Any manifold satisfying condition $(*)$ is obviously aspherical. It was conceivable 20 years ago, when this condition was formulated, that every closed aspherical manifold M^m satisfies condition $(*)$. But then Mike Davis constructed closed aspherical manifolds M^m where $\tilde{M} \neq \mathbb{R}^m$ contradicting property 1 of condition $(*)$.

On the other hand, $M^m \times S^1$ satisfies property 1 of condition $(*)$ whenever $\tilde{M} = \mathbb{R}^m$. This is seen as follows. Let $\mathbb{Z}$ denote the additive group of integers. Its natural action by translations on $\mathbb{R}$ extends to an action on $[-\infty, +\infty)$ where each group element fixes $-\infty$. We hence have a product action of $\pi_1(M \times S^1) = \pi_1(M) \times \mathbb{Z}$ on

$$\tilde{M} \times [-\infty, +\infty) = \mathbb{R}^m \times [0, +\infty)$$

which extends to its one point compactification $\mathbb{D}^{m+1}$. If we let this be the action posited in the above Definition, then it satisfies property 1 of condition $(*)$ but *not* property 2.

Figure 5

We also note that the universal cover X of $M^m \times S^1$ is $\mathbb{R}^{m+1}$ for any closed aspherical manifold M^m where $m \geq 5$ because X is contractible and simply connected at ∞. This is a result of Newman (1966).

Theorem. *(Farrell-Hsiang 1981) Let M^m be a closed manifold satisfying condition $(*)$. Then the map in the (simple) surgery sequence*

$$\mathcal{S}^s(M^m \times \mathbb{D}^n, \partial) \to [M^m \times \mathbb{D}^n, \partial; G/\mathrm{Top}]$$

is identically zero when $n \geq 1$ and $n + m \geq 6$.

So as not to obscure the argument, we sketch the proof of this Theorem under the extra assumptions that M is triangulable and $n = 1$. Set

$$E^{2m} = \tilde{M} \times_\Gamma \tilde{M}$$

and let $p : E^{2m} \to M$ denote the bundle projection. Then the following square commutes:

$$\begin{tikzcd}
\mathcal{S}^s(\mathbb{D}^1 \times M, \partial) \arrow[r] \arrow[d, "\alpha"'] & {[\mathbb{D}^1 \times M, \partial; G/\mathrm{Top}]} \arrow[d, "(\mathrm{id}\times p)^*"] \\
\mathcal{S}(\mathbb{D}^1 \times E, \partial) \arrow[r] & {[\mathbb{D}^1 \times E, \partial; G/\mathrm{Top}]}
\end{tikzcd}$$

where α is the transfer map defined as follows. Let the simple homotopy equivalence

$$h : (W, \partial W) \to (\mathbb{D}^1 \times M, \partial)$$

represent an element $b \in \mathcal{S}^s(\mathbb{D}^1 \times M, \partial)$. Then the proper homotopy equivalence

$$\hat{h} : (\mathcal{W}, \partial W) \to (\mathbb{D}^1 \times E, \partial)$$

represents $\alpha(b) \in \mathcal{S}(\mathbb{D}^1 \times E, \partial)$ where

$$\mathcal{W} = \{(x, y) \in W \times (\mathbb{D}^1 \times E) \mid h(x) = \mathrm{id} \times p(y)\}$$

and $\hat{h}(x, y) = y$. Since p is a homotopy equivalence, $(\mathrm{id} \times p)^*$ is an isomorphism. Hence the Theorem is a consequence of the following:

Assertion. *The map α is identically zero.*

We proceed to verify this. Note first that W is an s-cobordism and hence a cylinder because of the s-cobordism theorem. We may therefore assume that $W = [0, 1] \times M$ and that h is a homotopy between id_M and a self-homeomorphism $f : M \to M$. Furthermore, if f is pseudo-isotopic to id_M via a pseudo-isotopy homotopic to h rel ∂, then $b = 0$.

Let $\tilde{h}$ be the unique lift of h to $[0, 1] \times \tilde{M}$ such that $\tilde{h}$ is a proper homotopy between $\mathrm{id}_{\tilde{M}}$ and a self-homeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}$, which is a lift of f. Then $\tilde{h} \times \mathrm{id}_{\tilde{M}}$ determines a proper homotopy

$$k : [0, 1] \times E \to [0, 1] \times E$$

between id_E and a self-homeomorphism $g : E \to E$ (which is also determined by $\tilde{f} \times \mathrm{id}_{\tilde{M}}$). Since

$$\hat{h} : (\mathcal{W}, \partial W) \to (\mathbb{D}^1 \times E, \partial)$$

can be identified with

$$k : ([0,1] \times E, \partial) \to ([0,1] \times E, \partial),$$

the Assertion is an immediate consequence of the following.

Lemma. *g is pseudo-isotopic to id_E via a pseudo-isotopy which is properly homotopic to k rel ∂.*

We now use our assumption that M^m satisfies condition $(*)$ to prove this lemma. Identify $\tilde{M}$ with $\mathbb{D}^m$ and define a manifold $\bar{E}$ by

$$\bar{E} = \mathbb{D}^m \times_\Gamma \tilde{M}.$$

Then $E = \mathrm{Int}(\bar{E})$ and property 2 of condition $(*)$ implies that $\tilde{f}$ extends to a Γ-equivariant homeomorphism

$$\bar{f} : \mathbb{D}^m \to \mathbb{D}^m$$

by setting $\bar{f}|_{S^{m-1}} = \mathrm{id}_{S^{m-1}}$. Consequently $\bar{f} \times \mathrm{id}_{\tilde{M}}$ determines a self-homeomorphism

$$\bar{g} : \bar{E} \to \bar{E}$$

which extends $g : E \to E$ and satisfies $\bar{g}|_{\partial\bar{E}} = \mathrm{id}_{\partial\bar{E}}$. We proceed to construct a pseudo-isotopy

$$\phi : \bar{E} \times [0,1] \to \bar{E} \times [0,1]$$

satisfying

1. $\phi|_{\bar{E}\times 0} = \bar{g}$;
2. $\phi|_{\bar{E}\times 1} = \mathrm{id}_{\bar{E}\times 1}$;
3. $\phi|_{(\partial\bar{E})\times[0,1]} = \mathrm{id}_{(\partial\bar{E})\times[0,1]}$.

Properties (1-3) define ϕ on $\partial(\bar{E} \times [0,1])$. To construct ϕ over $\mathrm{Int}(\bar{E} \times [0,1])$ consider the natural fiber bundle

$$\bar{E} \times [0,1] \xrightarrow[q]{} M$$

with fiber $\mathbb{D}^m \times [0,1]$. And note the following. If Δ is an n-simplex in M, then $q^{-1}(\Delta)$ can be identified with $\mathbb{D}^{n+m+1}$.

The construction of ϕ proceeds by induction over the skeleta of M via a standard obstruction theory argument. And the obstructions encountered in extending ϕ from over the $(n-1)$-skeleton to over the n-skeleton are the problem of extending a self-homeomorphism of S^{n+m} to one of $\mathbb{D}^{n+m+1}$. But these obstructions all vanish because of the Alexander Trick. Recall that this Trick asserts that any self-homeomorphism η of S^n extends to a self-homeomorphism $\bar{\eta}$ of $\mathbb{D}^{n+1}$. In fact

$$\bar{\eta}(tx) = t\eta(x)$$

where $x \in S^n$ and $t \in [0,1]$ is an explicit extension.

Figure 6

Now $\psi = \phi|_{E\times[0,1]}$ is the pseudo-isotopy from g to id_E posited in the Lemma. And a similar argument, which we omit, shows that ψ is properly homotopic to k rel ∂. Q.E.D.

Remark 3. It follows from results of Davis and Januszkiewicz that PL non-positively curved closed manifolds also satisfy condition $(*)$. And Bizhong Hu showed that every non-positive curved finite complex K is a retract of such a manifold. Hu (1995) deduced from this, using Ranicki's algebraic formulation of surgery theory, that the assembly map is split monic for such a K. Ferry-Weinberger and Carlsson-Pedersen also obtained this in addition to many further results on the split injectivity of σ.

Corollary. *Let $f : N \to M$ be a homotopy equivalence between closed smooth manifolds such that M supports a non-positively curved Riemannian metric. Then N and M are stably homeomorphic; i.e.*

$$f \times \mathrm{id} : N \times \mathbb{R}^{m+4} \to M \times \mathbb{R}^{m+4}$$

is homotopic to a homeomorphism where $m = \dim(M)$.

Proof. Let $\phi : N \times S^1 \to M \times S^1 \times \mathbb{R}^{m+3}$ be an embedding homotopic to the composition

$$N \times S^1 \xrightarrow{f\times \mathrm{id}_{S^1}} M \times S^1 \times 0 \subseteq M \times S^1 \times \mathbb{R}^{m+3}.$$

Note that ϕ exists because of the Whitney Embedding Theorem. And let v denote the normal bundle to ϕ. We proceed to show that v is topologically trivial. Now Kwan and Szczarba showed that $f \times \mathrm{id}_{S^1}$ is a simple homotopy equivalence and hence represents an element in $\mathcal{S}^s(M \times S^1)$. This element maps to 0 in $[M \times S^1; G/\mathrm{Top}]$ because of the Theorem and the 4-fold (semi) periodicity of the topological surgery exact sequence. But v (equipped with a specific homotopy trivialization) is this image element; in particular, v is topologically trivial.

Since the region outside an open tubular neighborhood of image(ϕ) is a (half open) h-cobordism, we can use the h-cobordism theorem to show that the total space E of v is diffeomorphic to $M \times S^1 \times \mathbb{R}^{m+3}$. But E can also be topologically identified with $N \times S^1 \times \mathbb{R}^{m+3}$ since v is topologically trivial. Hence there is a homeomorphism

$$\psi : N \times S^1 \times \mathbb{R}^{m+3} \to M \times S^1 \times \mathbb{R}^{m+3}$$

such that $\psi_{\#}(\pi_1 N) = \pi_1(M)$. The homeomorphism posited to exist in the Corollary is obtained by lifting ψ to the infinite cyclic covering spaces corresponding to $\pi_1(N)$ and $\pi_1(M)$, respectively. $\square$

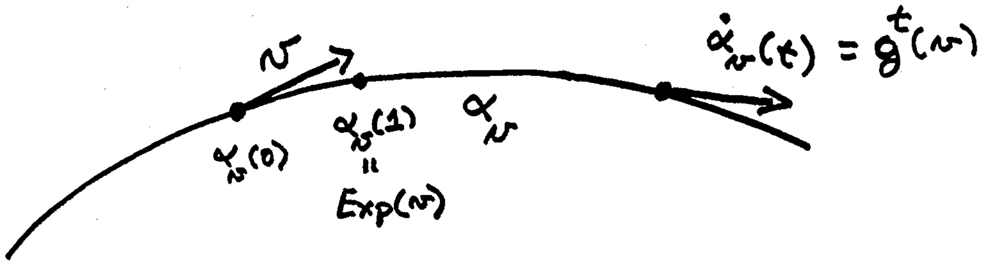


Figure 1.

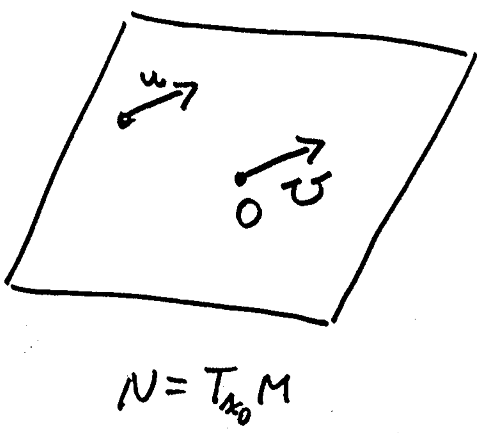


Figure 2.

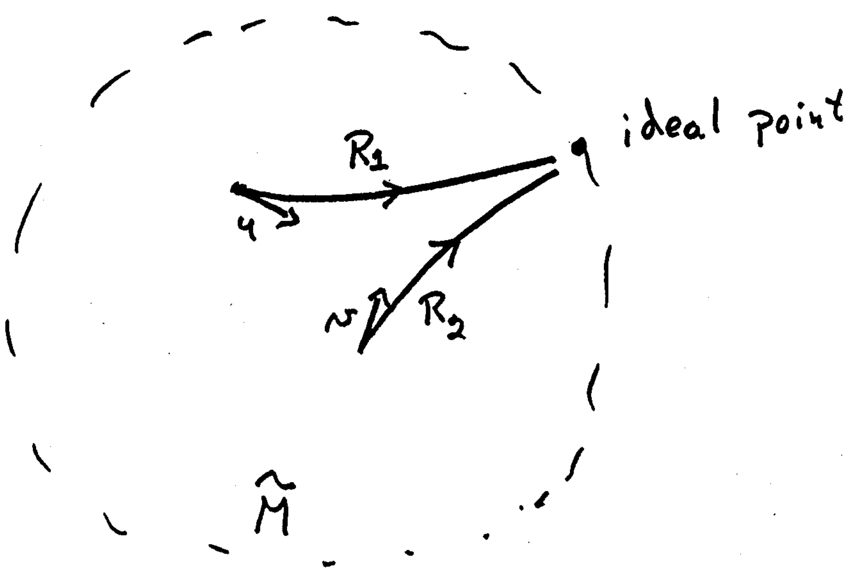


Figure 3.

Lecture 2 figures

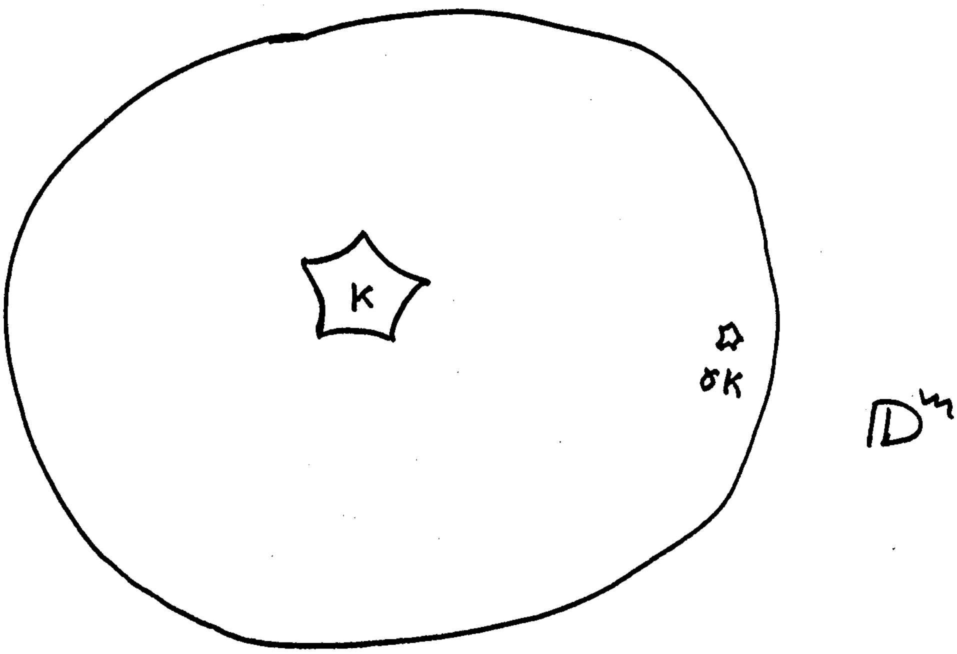


Figure 4

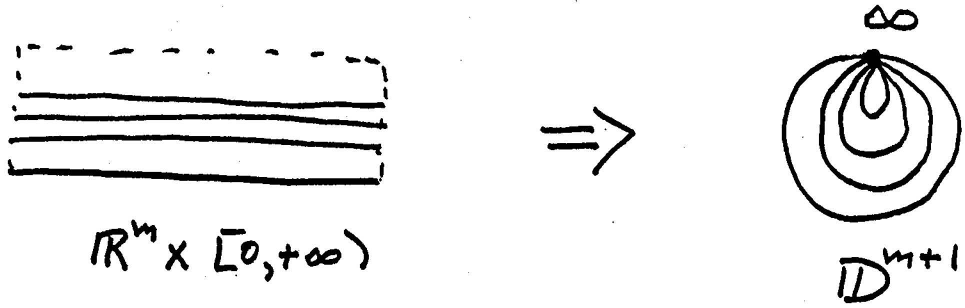
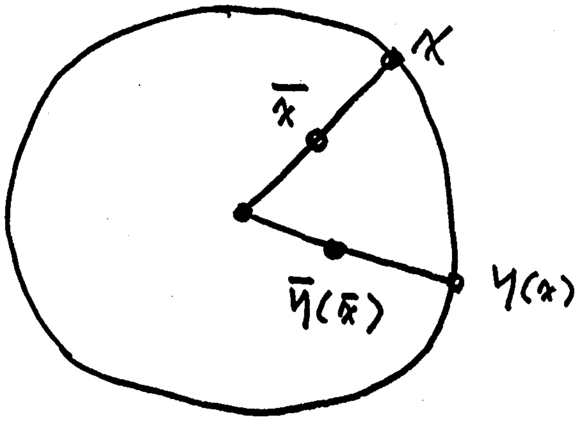


Figure 5.



$\bar{x} = t\,x$ $\quad \bar{y}(\bar{x}) = t\,y(x)$

Figure 6.