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INTRODUCTION TO NONCOMMUTATIVE FIELD THEORY

Lectures I & II

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Please note: These are preliminary notes intended for internal distribution only.

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INTRODUCTION TO NONCOMMUTATIVE FIELD THEORY

TRIESTE June-July 2001

JLF Barbón (CERN)

NONCOMMUTATIVE FIELD THEORY (NCFT) IS A FIELD THEORY DEFINED ON A SPACE-TIME ENDOWED WITH A NONCOMMUTATIVE GEOMETRY **(NCG)**

NCG IS A GENERALIZATION OF ORDINARY GEOMETRY SO THAT, ROUGHLY-SPEAKING, COORDINATES ARE REPLACED BY NONCOMMUTING OPERATORS SATISFYING SOME OPERATOR ALGEBRA

$$\boxed{[\hat{x}^i, \hat{x}^j] = f_{ij}(\hat{x}^k)}$$

PHYSICAL MOTIVATION

MOSTLY THEORETICAL.

* HAVING A BUILT-IN QUANTUM STRUCTURE IN SPACE-TIME LOOKS LIKE A GOOD INGREDIENT FOR A THEORY OF QUANTUM GRAVITY

* SO, IF THE NCG STRUCTURE CONSTANTS $f_{ij}(\hat{X}^k)$ HAVE DIMENSIONAL PARAMETERS, THEY SHOULD BE SET BY THE Planck SCALE.

* Powerful math in your arsenal to try achieving this goal (Connes '80)

MATH EXAMPLE: Fuzzy S^2

FIND THREE NONCOMMUTING OPERATORS $\hat{X}_1, \hat{X}_2, \hat{X}_3$ SATISFYING $\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = R^2$

AN OBVIOUS CHOICE IS ANY REPRESENTATION OF $SU(2)$. with ANGULAR MOMENTA J_a

$$\boxed{\begin{gathered}[J_a, J_b] = i\,\varepsilon_{abc}\,J_c \\ J_1^2 + J_2^2 + J_3^2 = j(j+1)\,\mathbb{1}_{d_j \times d_j}\end{gathered}}$$

IN THE SPIN-j REPRESENTATION

$d_j = 2j+1$

THUS, WE CAN TAKE

$$\boxed{\hat{X}_a = \frac{R}{\sqrt{j(j+1)}}\,J_a}$$

$$\boxed{\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = R^2\,\mathbb{1}_{d_j \times d_j}}$$

* THE RESULTING ALGEBRA

$$[\hat{X}_a, \hat{X}_b] = i \frac{R}{\sqrt{j(j+1)}} \varepsilon_{abc} \hat{X}_c$$

* RESPECTS THE $SO(3)$ SYMMETRY THAT CHARACTERIZES S^2

* POSITIONS OF "POINTS" ARE FUZZY BECAUSE COORDINATES DO NOT COMMUTE

* DIMENSION $2j+1$ IS FINITE : A KIND OF LATTICE APPROXIMATION TO S^2

* THIS CAN BE MADE MORE PRECISE BY NOTICING THAT IN THE $j \to \infty$ LIMIT AT FIXED R WE RECOVER THE ALGEBRA OF COMMUTATIVE FUNCTIONS ON S^2

* NCG MAY ARISE IN PHYSICAL SYSTEMS WHEN SOME EFFECTIVE POSITION OPERATOR BECOMES NC AS A RESULT OF INTERACTIONS

$$\left[\hat{X}^{\mu}_{eff}, \hat{X}^{\nu}_{eff}\right] \neq 0$$

(typically 1^{st} quantized non-relativistic systems)

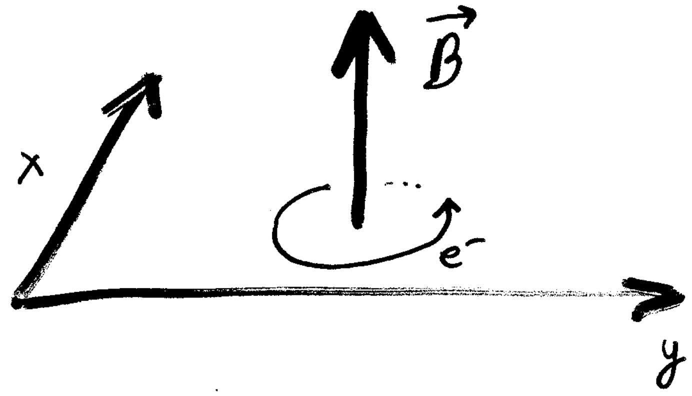
* IN THE MICROSCOPIC THEORY, $\hat{X}^{\mu}_{micro}$ MAY OR MAY NOT BE COMMUTING

TWO EXAMPLES

- electrons in strong magnetic field
- D-branes

N C G IN A PHYSICAL SYSTEM

Electrons in lowest Landau level



$$A_i = -\frac{1}{2} B_{ij} x^j$$

$$\boxed{H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2}$$

NOTATIONAL DEVICE

$$z \equiv \sqrt{\frac{eB}{2}}\,(x + iy)$$

$$a = \partial_{\bar{z}} + \frac{z}{2}$$

$$a^+ = -\partial_z + \frac{\bar{z}}{2}$$

GET A HARMONIC OSCILLATOR !

$$[a, a] = [a^+, a^+] = 0 \qquad [a, a^+] = 1$$

$$\omega = \frac{eB}{m} = \text{Larmor}$$

$$H = \hbar\omega\left(a^+ a + \frac{1}{2}\right)$$

LANDAU LEVELS

$$H\, u_{\ell,m} = E_\ell\, u_{\ell,m}$$

$m \to$ degenerate quantum number

$$\boxed{E_\ell = \hbar\omega\left(\ell + 1/2\right) \qquad \ell \in \mathbb{Z}}$$

Wave functions:

$$u_{\ell,m} = (a^+)^\ell\, u_{0,m}$$

$$0 = a\, u_{0,m} = \left(\partial_{\bar{z}} + \frac{z}{2}\right) u_{0,m}$$

solve

$$\boxed{u_{0,m} = \frac{z^m}{\sqrt{m!}}\, e^{-z\bar{z}/2}}$$

∞ degeracy on the plane

LOOK AT LOWEST LANDAU LEVEL (LLL)

$\ell = 0$ $\qquad \vartheta_m \equiv e^{z\bar{z}/2} u_{m,0}$

THE ϑ_m ARE ANALYTIC AND LIVE IN BARGMAN'S HILBERT SPACE:

$$(\vartheta_m | \vartheta_n) \equiv \langle u_{0m} | u_{0n} \rangle = \int d^2z \, e^{-z\bar{z}} \, \bar{\vartheta}_m \vartheta_n$$

$$\equiv \int d\mu(z) \, \bar{\vartheta}_m(\bar{z}) \, \vartheta_n(z)$$

Integrating by parts:

$$(f | \partial_z | g) = \int d^2z \, e^{-z\bar{z}} \, \bar{f} \, \partial g =$$

$$= \int d^2z \, e^{-z\bar{z}} \, \bar{f} \, \bar{z} \, g = (f | \bar{z} | g)$$

SO, RESTRICTING TO LLL:

$$\boxed{P_{LLL} \, \partial_z \, P_{LLL} = P_{LLL} \, \bar{z} \, P_{LLL}}$$

DENOTE $\mathcal{H}_{LLL}$ THE HILBERT SPACE OF ANALYTIC FUNCTIONS WITH INNER PRODUCT $(,)$

On this space:

$$\bar{z}\Big|_{\mathcal{H}_{LLL}} = \partial_z\Big|_{\mathcal{H}_{LLL}}$$

HENCE

$$\boxed{[\bar{z}, z]\Big|_{\mathcal{H}_{LLL}} = 1}$$

BACK TO ORIGINAL VARIABLES:

$$\boxed{[x, y] = i\,\theta_B \quad , \quad \theta_B = \frac{1}{eB}}$$

NCG DOES PLAY A ROLE IN THE QUANTUM HALL EFFECT!

HEURISTIC DERIVATION FROM LAGRANGIAN

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{e}{2} B_{ij} x^i \dot{x}^j$$

If: $$|m \dot{x}^i| \ll |B_{ij} x^j|$$

Just neglect the kinetic term:

$$\mathcal{L} \simeq -\frac{e}{2} B_{ij} x^i \dot{x}^j$$

Canonical quantization:

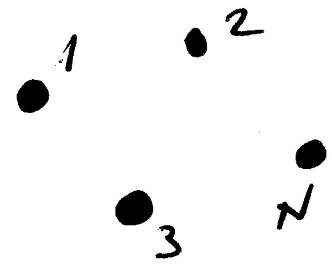
$$\pi_j = \frac{dL}{d\dot{x}^j} = -e B_{jk} x^k$$

$$[\pi_j, x^l] = \frac{1}{i} \delta_j^{\ l} = -e B_{jk} [x^k, x^l]$$

$$\boxed{[x^k, x^l] = i \left(\frac{1}{eB}\right)^{kl}}$$

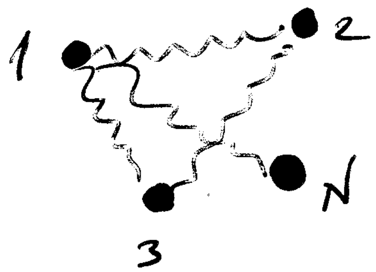
D-BRANES

N WIDELY SEPARATED D-PARTICLES HAVE POSITIONS $(X_1, X_2, X_3, \dots X_N)$



HOWEVER, FOR $|X_i - X_j| \lesssim l_s$

STRETCHING STRINGS CAN BE EXCITED



THE QUANTUM MECHANICS OF COLLECTIVE COORDINATES IS BETTER APPROXIMATED BY MATRIX QUANTUM MECHANICS OF $N \times N$ HERMITEAN MATRICES

$$X^{+} = X = (X_{ij}) \qquad i,j = 1 \dots N$$

$$\mathcal{L} = \frac{1}{2} \operatorname{tr} \dot{X}^2 + \cdots$$

DEFINE POSITION OF i-th PARTICLE

$$X_{ii} \equiv \langle i | X | i \rangle$$

FUZZINESS OF i-th PARTICLE

$$\Delta X_i^2 = \langle X^2 \rangle_i - \left(\langle X \rangle_i \right)^2$$

$$= \sum_j X_{ij} X_{ji} - X_{ii}^2 =$$

$$= \sum_{j \neq i} | X_{ij} |^2 > 0$$

if the other particles "interfere"

POSITIONS ARE SHARP NOTIONS ONLY FOR DIAGONAL MATRICES

$$X_{diag} = (X_i \delta_{ij}) \longrightarrow \Delta X_i^2 = 0$$

* FOR IDENTICAL PARTICLES, THE STATISTICAL PERMUTATION GROUP $G_{Bose} = S_N$ GETS NATURALLY PROMOTED TO THE FULL $\mathcal{U}(N)$

$$X \to \mathcal{U}^{+} X \mathcal{U}$$

* IF PHYSICS IS $\mathcal{U}(N)$-INVARIANT WE CAN DIAGONALIZE X TO HAVE SHARP POSITIONS, EXCEPT THAT, WITH MORE THAN ONE DIMENSION WE CANNOT DIAGONALIZE ALL X^{μ} SIMULTANEOUSLY.

THIS IS POSSIBLE IF THE DYNAMICS ENFORCES $[X^{\mu}, X^{\nu}] = 0$ AS SOME GROUND-STATE CONDITION.

FOR EXAMPLE

$$V_{eff} = - \operatorname{tr} [X^{\mu}, X^{\nu}][X_{\mu}, X_{\nu}]$$

is minimized by sharp D0-brane locations, up to global $\mathcal{U}(N)$ transformations

IN NCFT WE GO ONE STEP FURTHER. ASSUME SPACE-TIME IS NONCOMMUTATIVE PRIOR TO QUATIZATION

$$[\hat{X}_i, \hat{X}_j] = f_{ij}(\hat{X}^k) \neq 0$$

Even for $\hbar = 0$

* THE HILBERT SPACE WHERE $\hat{X}_i$ ACT IS AUXILIARY, NOT RELATED TO THE QUANTUM HILBERT SPACE ARISING FOR $\hbar \neq 0$

↳ TRY TO BUILD CLASSICAL FIELD THEORY ON THIS SPACE, LATER INTRODUCE $\hbar$

* SO WE INVENT A NAME FOR THE OPERATOR PRODUCT OF THE $\hat{X}^i$ IN THIS AUXILIARY HILBERT SPACE: THE "~~DOT~~ PRODUCT"

$$[\hat{X}_i, \hat{X}_j]_\bullet = \hat{X}_i \bullet \hat{X}_j - \hat{X}_j \bullet \hat{X}_i$$

THE CLASSICAL FIELD CONFIGURATIONS ARE FUNCTIONS $\varphi(\hat{X}_i)$ DEFINED WITH THIS PRODUCT

$$\varphi(\hat{X}_1, \hat{X}_2) = \sum' C_{n_1, n_2} \, \hat{X}_1^{n_1} \bullet \hat{X}_2^{n_2}$$

$$X_i^{n_i} = \underbrace{\hat{X}_i \bullet \hat{X}_i \bullet \hat{X}_i \cdots \bullet \hat{X}_i}_{n_i}$$

TO SPECIFY THE COEFFICIENTS $C_{n_1 n_2 \cdots}$ NEED AN ORDERING CONVENTION

→ A "TRIVIAL" EXAMPLE IS ORDINARY GAUGE THEORY. THIS IS NCFT WITH ALGEBRA

$$C(\mathbb{R}^d) \otimes \mathrm{MAT}_{N \times N}$$

Continuous functions

matrices of rank N

$$A_\mu = A_\mu^a(x) \, T^a_{ij}$$

THE NC ALGEBRA IS FINITE DIMENSIONAL

$$[T^a, T^b] = i f^{abc} \, T^c$$

* ANOTHER "TRIVIAL" EXAMPLE is SOME SORT OF FIELD THEORY ON THE FUZZY SPHERE. FIELDS ARE FUNCTIONS OF SU(2) MATRICES IN THE SPIN-j REPRESENTATION

THIS IS LIKE A LATTICE MODEL WITH FINITE NUMBER OF D.O.F.

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* SIMPLEST NON-TRIVIAL GENERALIZATION OF FIELD THEORY IS BASED ON AN ALGEBRA

$$f_{ij}(\hat{X}^k) = (\text{constant})_{ij}$$

and $\hat{X}^k$ with continuous spectrum

NC GEOMETRY IS A "QUANTUM" DEFORMATION OF CLASSICAL GEOM.

$$[\hat{x}^i, \hat{x}^j] = i\,\theta^{ij} \qquad \mathbb{R}^n$$

Connes (80's) Snyder (47)

INTRODUCE NON-LOCAL STRUCTURE AT LENGTH SCALE $\boxed{\ell_{NC} \sim \sqrt{\theta}}$

ANALOGOUS TO CLASSICAL VERSUS QUANTUM PHASE SPACE

WE CAN BUILD A NON-LOCAL GENERALIZATION OF QUANTUM FIELD THEORY ON SUCH SPACES

A PLANCKIAN FRAMEWORK ??

NCFT

* $\boxed{\text{NONLOCAL}}$ QFT WITH A NOTION OF "RENORMALIZABILITY" AND OTHER INTERESTING STRUCTURES

* HALF WAY BETWEEN LOCAL QFT AND STRINGS (UV/IR)

* NCYM ACTUALLY APPEARS AS A SPECIFIC LOW ENERGY LIMIT OF STRING THEORY

Connes, Douglas & Schwarz
Douglas & Hull
Seiberg & Witten

NCFT : applications?

* NCFT IS RELATED TO THE LARGE-N LIMIT OF 'tHooft IN ORDINARY GAUGE THEORIES

* NOT EXCLUDED THAT VIOLATIONS OF LORENTZ SYMMETRY IN NATURE COULD BE PARAMETRIZED IN THIS WAY

* A SOLID APPLICATION TO EFFECTIVE THEORIES OF THE QHE

REVIEW and references Douglas & Nekrasov hepth/0106048

MORE FORMAL REVIEWS

* Gracia-Bondía & Varilly & Figueroa
  "Elements of noncommutative geometry"
  Birkhauser, Boston (2001)

* Connes "Noncommutative geometry"
  Academic press (1994)

* Connes math.qa/0011193
  hep-th/0003006

CONSTRUCTING NCFT

$$[\hat{x}^\mu, \hat{x}^\nu] = i\,\theta^{\mu\nu}$$

Realize $\mathbb{R}^d$ as a phase space with effective Plank's constants $\sim \theta_{\mu\nu}$.

NOTICE THAT STILL $\hbar = 0$ AT THIS POINT!

START WITH A PLANE:

$$[\hat{x}, \hat{y}] = i\,\theta \quad = \quad \hat{x} \bullet \hat{y} - \hat{y} \bullet \hat{x}$$

$\hat{y} \to -i\,\theta\,\partial_x$ is a solution

BY COMMUTING OPERATORS:

$$\boxed{e^{ip\hat{y}} \bullet f(\hat{x}) = f(\hat{x} - p\,\theta) \bullet e^{ip\hat{y}}}$$

BY USING JUST THE COMMUTATOR,

$[\hat{x}^\alpha, \hat{x}^\beta] = i\theta^{\alpha\beta}$ ONE CAN RE-ORDER

OPERATORS USING BAKER-CAMPBELL-HAUSDORF

THAT

$$\boxed{e^{i\hat{p}_\mu \hat{x}^\mu} \cdot e^{iq_\mu \hat{x}^\mu} = e^{-\frac{i}{2} p \times q}\, e^{i(p+q)_\mu \hat{x}^\mu}}$$

AS AN OPERATOR STATEMENT, WITH

$$\boxed{p \times q \equiv p_\mu \theta^{\mu\nu} q_\nu}$$

FOR PLANE WAVES:

$$\boxed{e^{i p_\mu x^\mu} * e^{i q_\nu x^\nu} = \underbrace{e^{-\frac{i}{2} p \times q}}_{\text{Moyal phase}} \, e^{i (p+q)_\alpha x^\alpha}}$$

$$\boxed{p \times q \equiv p_\mu \, \theta^{\mu\nu} \, p_\nu}$$

Superposing this, we obtain the MOYAL PRODUCT for any pair of functions:

$$\boxed{f(x) * g(x) = f(x) \, e^{\frac{i}{2} \overleftarrow{\partial}_\alpha \theta^{\alpha\beta} \overrightarrow{\partial}_\beta} \, g(x)}$$

NONLOCAL, NONCOMMUTATIVE, ASSOCIATIVE

$$f(x) * g(x) = f(x) g(x) + \frac{i}{2} \underbrace{\{f, g\}_\theta}_{\text{POISSON}} + \cdots$$

A CONVENIENT WAY OF MANIPULATING THE OPERATOR ALGEBRA IS TO MAP IT TO SOME FUNCTION ALGEBRA

WEYL MAP:

$$\hat{O}(\hat{x}^\mu) \xmapsto{\mathbf{W}} W[\hat{O}] = f_{\hat{O}}(x^\mu)$$

classical coordinate

$f_{\hat{O}}(x^\mu)$ ARE THE "COMPONENTS" OF $\hat{O}$ IN SOME BASIS OF THE SPACE OF OPERATORS.

THE FINITE-DIMENSIONAL ANALOG OF THIS IS WHEN YOU WRITE

$$A_\mu = \sum_{a=1}^{\dim G} A^a T^a$$

analog of $f_{\hat{O}}(x_\mu)$

x_μ is just a label for the basis, as "a"

NEXT WE DEFINE THE MOYAL PRODUCT BY :

$$\boxed{W[\hat{O}\hat{O}'] = W[\hat{O}] * W[\hat{O}']}$$

$$\Downarrow$$

$$\boxed{f_{\hat{O}\cdot\hat{O}'}(x) = f_{\hat{O}}(x) * f_{\hat{O}'}(x)}$$

FINITE DIM. ANALOG:

GIVEN A MATRIX ALGEBRA $(Gl(d,\mathbb{R}))$.

$$T^a T^b = \sum_c C^{ab}_{\ \ c} T^c$$

Define two operators $A = A_a T^a$, $B = B_b T^b$

So that $W[A] = A_a$ $W[B] = B_b$.

$$A \cdot B = \sum A^a B^b \sum_c C^{ab}_{\ \ c} T^c = (A\cdot B)^c T^c$$

$$\boxed{(A\cdot B)_c = \Big(W[A] * W[B]\Big)_c = \sum_{ab} C^{ab}_{\ \ c} A_a B_b}$$

PRESENTED IN THIS WAY, NCG IS EQUIVALENT TO JUST $\boxed{\text{DEFORMING}}$ THE ALGEBRA OF ORDINARY FUNCTIONS ON $\mathbb{R}^d$ BY CHANGING THE PRODUCT

One nice choice is Weyl ordering

$$f_{\hat{O}}(x) = \int \frac{d^d k}{(2\pi)^d} \, \mathrm{Tr} \, e^{ik(x-\hat{x})} \, \hat{O}(\hat{x})$$

This defines a $*$ product (Moyal) such that

$$\boxed{\underbrace{e^{ip\cdot x}}_{\text{ORDINARY FUNCTION}} * e^{iq\cdot x} = e^{-\frac{i}{2} p \times q} \, e^{i(p+q)x}}$$

But there are as many as there are ordering prescriptions

VERY IMPORTANT PROPERTY

$$\int d^d x \; f(x) * g(x) =$$

$$= \int d^d p \int d^d q \;\; \delta(p+q) \; \hat{f}(p) \; \hat{g}(q) \; e^{-\frac{i}{2} p \times q}$$

$$= \int d^d x \;\; f(x) \, g(x)$$

CAN ALWAYS DROP ONE $*$ PRODUCT INSIDE INTEGRALS

$$\int f * g * h = \int (f*g) \, h = \int h \, (f*g)$$

$$= \int h * f * g = \int g * h * f .$$

SO, INTEGRALS BEHAVE LIKE MATRIX TRACES. "BOSE SYMMETRY" BREAKS DOWN TO JUST CYCLIC SYMMETRY.

CLASSICAL NCFT

A NONLOCAL FIELD THEORY WITH A CLASSICAL ACTION OBTAINED FROM YOUR FAVORITE ORDINARY FIELD THEORY BY REPLACING ORDINARY PRODUCTS BY MOYAL PRODUCTS.

(A CORRESPONDENCE PRINCIPLE AT THE CLASSICAL LEVEL)

$$S = \frac{1}{2} \int \partial\varphi * \partial\varphi + m^2 \varphi * \varphi$$

$$+ \sum_n \frac{\lambda_n}{n!} \int \overbrace{\varphi * \varphi * \varphi * \cdots * \varphi}^{n}$$

AS USUAL WITH CORRESPONDENCE PRINCIPLES, MUST EXERCISE SOME CARE WITH ORDERING AMBIGUITIES.

EXAMPLE, IF THE FIELD φ^i CARRIES AN INTERNAL-SYM VECTOR INDEX:

$$\lambda_1 \int \varphi_i * \varphi^i * \varphi_j * \varphi^j$$

$$\lambda_2 \int \varphi_i * \varphi_j * \varphi^i * \varphi^j$$

ARE INEQUIVALENT COUPLINGS AS SOON AS $\theta \neq 0$

GAUGE THEORIES

$$S = \frac{1}{4g^2} \int \mathrm{tr} \left| F_{\mu\nu} \right|^2 + \int \bar{\psi} * \gamma^\mu D_\mu \psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i A_\mu * A_\nu - i A_\nu * A_\mu$$

$$D_\mu \psi = \partial_\mu \psi + i A_\mu * \psi$$

$$\delta \psi = i \, \varepsilon * \psi$$

$$\delta A_\mu = - \partial_\mu \varepsilon - i A_\mu * \varepsilon + i \varepsilon * A_\mu$$

TYPICALLY NEED **U(N)** GAUGE GROUP

$$[A_\mu, A_\nu]_* = \frac{1}{2} [T^a, T^b] \{ A_\mu^a * A_\nu^b \} + \frac{1}{2} \{ T^a, T^b \} [A_\mu^a * A_\nu^b]$$

$\boxed{\text{Out of } SU(N) \text{ Lie algebra}}$

One can implement $SO(N)$, $Sp(N)$ WITH SOME EFFORT BONORA, Sheikh-Jabbari, Schnabl & Tomasiello 01

A PECULIARITY

$\mathrm{tr}\, F^2$ IS <u>NOT</u> GAUGE INVARIANT

$$\mathrm{tr}\, F^2 \to \mathrm{tr}\, g^{-1} * F^2_* g$$

CYCLIC PERMUTATION TO CANCEL THE GAUGE TRANSFORMATION REQUIRES A SPATIAL INTEGRAL FOR THE $*$ PRODUCT

$$\int \mathrm{tr}\, g^{-1} * F^2 * g = \int \mathrm{tr}\, g * g^{-1} * F^2 =$$

$$= \int \mathrm{tr}\, F^2$$

<u>GAUGE INVARIANT OPERATORS ARE</u>

<u>NONLOCAL</u>, LIKE INTEGRATED OPS.

$\Rightarrow$ ONE CAN DO SLIGHTLY BETTER AND DEFINE QUASI-LOCAL OPS, SUCH AS OPEN WILSON LINES

$\boxed{\text{ANOTHER ONE}}$ DIFFICULT TO MAKE ELECTRONS AND QUARKS NONCOMMUTATIVE AT THE SAME TIME

Normalize the NCQED Lagrangian

$$\mathcal{L}_{NCQED} = \frac{1}{4e^2} \int F_{\mu\nu} * F^{\mu\nu}$$

$$+ \quad i \int \overline{\Psi} * \not{D} * \Psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i \; A_\mu * A_\nu - A_\nu * A_\mu$$

$$D_\mu \psi = \partial_\mu \psi + i e' A_\mu * \psi$$

$$\delta \psi = i e' \varepsilon * \psi$$

$$\delta A_\mu = -\partial_\mu \varepsilon - i A_\mu * \varepsilon + i \varepsilon * A_\mu$$

normalization of δA_μ fixed by photon lagrangian

IS THE COVARIANT DERIVATIVE REALLY COVARIANT?

$$\delta\left(D_\mu \psi\right) = \delta\left(\partial_\mu \psi + i e' A_\mu * \psi\right) =$$

$$= \partial_\mu \delta\psi + i e' \delta A_\mu * \psi + i e' A_\mu * \delta\psi =$$

$$= i e' \partial_\mu \varepsilon * \psi + i e' \varepsilon * \partial_\mu \psi +$$

$$+ i e' \left(- \partial_\mu \varepsilon - i A_\mu * \varepsilon + i \varepsilon * A_\mu \right) * \psi$$

$$+ i e' A_\mu * \left(i e' \varepsilon * \psi \right) =$$

$$= i e' \varepsilon * \left(\partial_\mu \psi + i A_\mu * \psi \right)$$

$$+ \left(e' - e'^2 \right) A_\mu * \varepsilon * \psi$$

HENCE, WE NEED $\mathbf{e' = 1}$

(in units of e) TO HAVE A GOOD MINIMAL COUPLING.

CHARGE CONJUGATES CAN BE COUPLED VIA

$$D^{(-)}_\mu \psi = \partial_\mu \psi - i \psi * A_\mu$$

NEUTRAL MAYORANA FERMIONS (PHOTINO)

CAN BE COUPLED VIA.

$$D^{adj}_\mu \psi = \partial_\mu \psi + i A_\mu * \psi - i \psi * A_\mu$$

NOTE THAT THIS COUPLING IS TRIVIAL IN THE $\theta \to 0$ LIMIT

<u>QUARKS WITH FRACTIONAL CHARGE</u>

PROBLEMATIC

IN PHENOMENOLOGICAL GAMES,

WE RESTRICT TO NC QED OR

NC SUSY QED

THIS TALK $\boxed{\text{IS NOT}}$ ABOUT THE NON CONMUTATIVE STANDARD MODEL OF CONNES ET. AL.

$$\mathbb{R}^4 \times \{\cdot \;\; \cdot\}$$

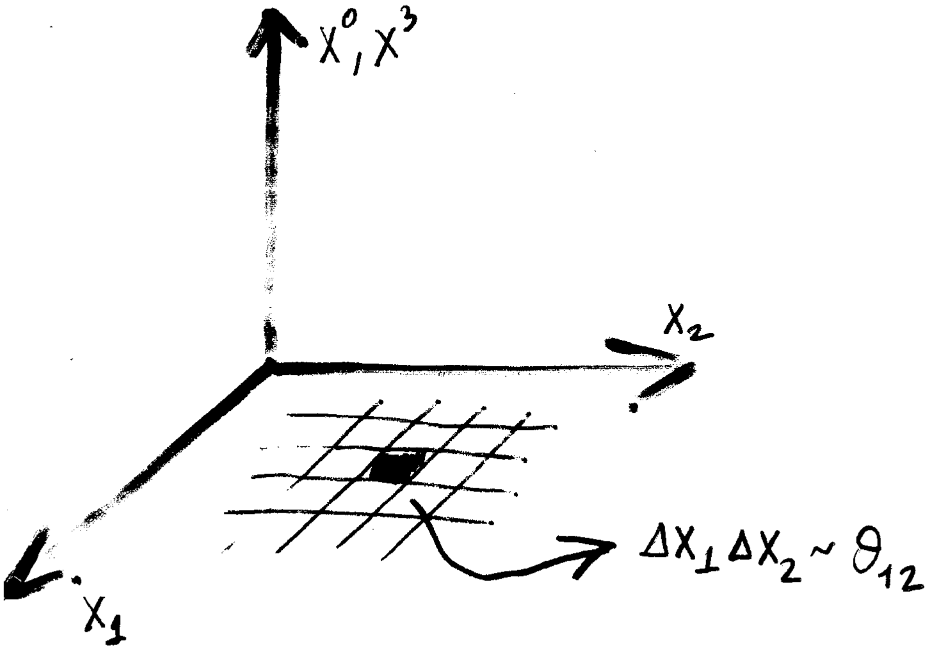
Standard Lorentz inv

"N.C"

SIMPLIFIED MODELS HAVE COUPLING / MASS RELATIONS NOT STABLE UNDER R.G.

(E. Alvarez, Gracia-Bondia & C.P. Martin (93)

WE WORK IN N.C. GENERALIZATION OF $\mathbb{R}^4$ (T^4)



NOT LORENTZ INVARIANT ON SCALES

$$\ell_{NC} \sim \sqrt{\theta}$$

PHYSICAL INTERPRETATION OF [*]

CONSIDER A CHARGED MAYORANA FERMION

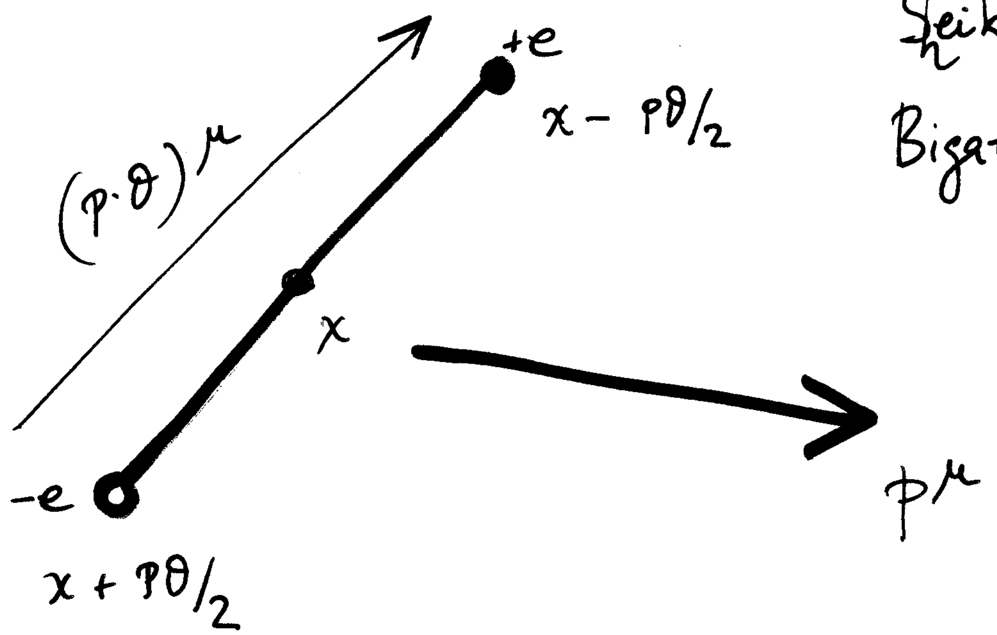
$$\mathcal{L}_\psi = \bar\psi \not\partial \psi + ie\,\bar\psi \not{A} * \psi - e\bar\psi\, \psi * \not{A}$$

FOR A PLANE WAVE $\psi \sim e^{ip\cdot x}$

$$\not{A}(x) * e^{ipx} = \not{A}\left(x^\mu + \frac{(p\theta)^\mu}{2}\right) e^{ipx}$$

$$e^{ipx} * A(x) = \not{A}\left(x^\mu - \frac{1}{2}(p\theta)^\mu\right) e^{ipx}$$

PHYSICAL PICTURE: A "DIPOLE" OF LENGTH $|p\theta|$

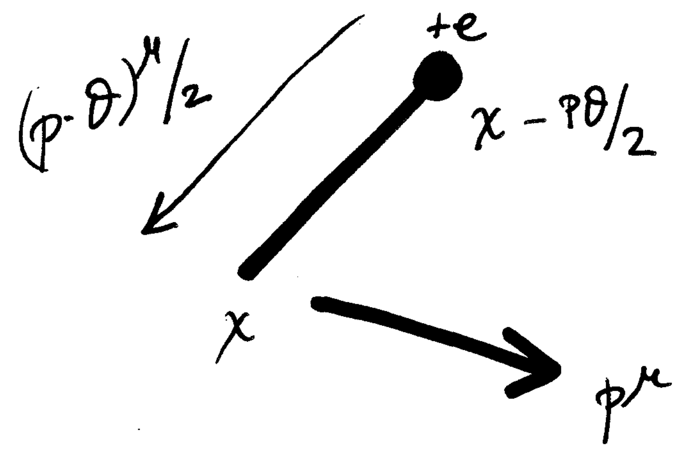


Sheikh-Jabbari 99

Bigatti & Susskind 00

DIRAC ELECTRON

$$D\!\!\!/_{*}\psi = \partial_\mu \psi + ie\, A_\mu * \psi$$

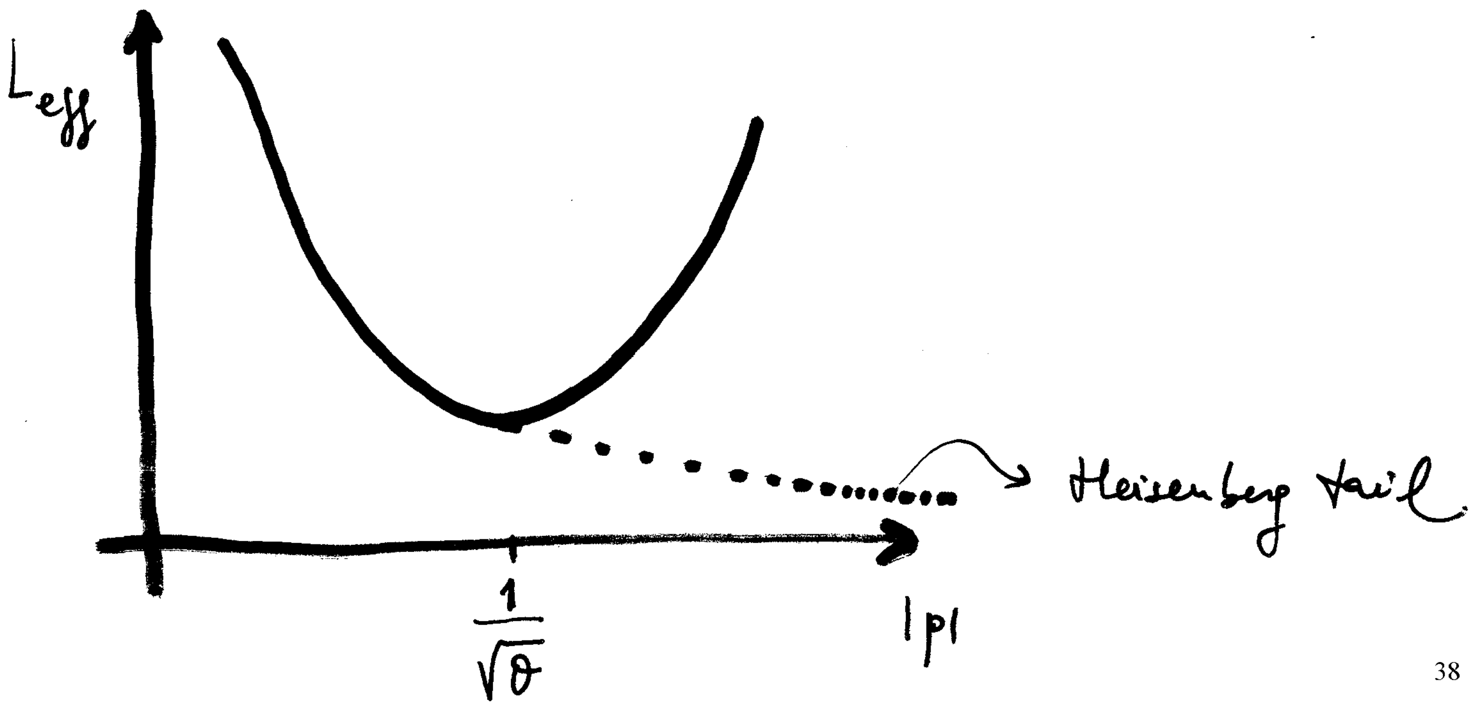


HALF A DIPOLE !

EFFECTIVE SIZE OF PARTICLE :

$$L_{eff} = \max\left(\lambda_{\text{Compton}},\ \ell_{\text{dipole}}\right) \sim$$

$$\sim \max\left(\frac{1}{|p|},\ |p\theta|\right)$$



NC FT "PARTICLES" ARE EFFECTIVELY EXTENDED "DIPOLES". RESEMBLE RIGID OPEN STRINGS.

VIBRATIONAL MODES OF OPEN STRINGS

$M^2_{osc} \sim \text{Tension} \sim 1/\alpha'$

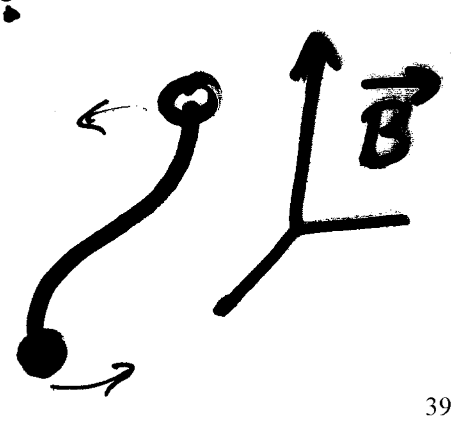
$\alpha' \to 0$ removes excited states

open string rigid only massless excitations survive.

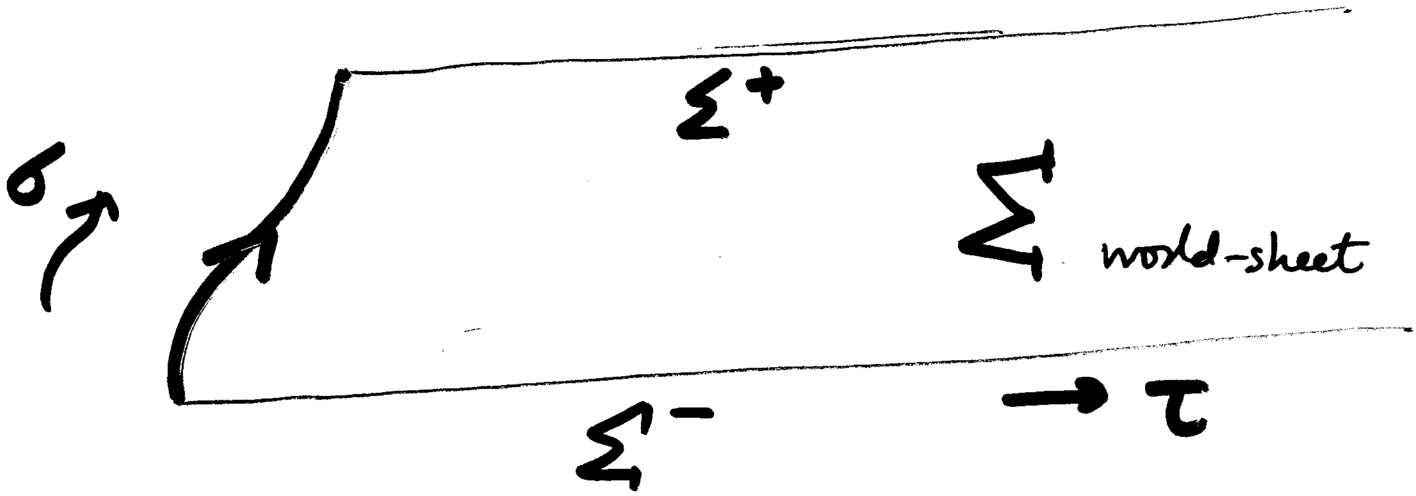
NORMALLY $\alpha' \to 0$ GIVES POINTLIKE OBJECTS.

HOW CAN WE KEEP THE STRING EXTENDED BUT RIGID ?

MAGNETIC FIELD



ORIENTED OPEN STRINGS ARE NATURAL DIPOLES



Electromagnetic coupling : Wilson line on endpoints

$$S_{em} = \int_{\Sigma^+} A_\mu dx^\mu - \int_{\Sigma^-} A_\mu dx^\mu =$$

$$= \int_{\Sigma^+ - \Sigma^-} A_\mu dx^\mu = \int_{\partial\Sigma} A_\mu dx^\mu$$

$$\underset{\text{(Stokes)}}{=} \frac{1}{2} \int_{\Sigma} F_{\mu\nu} \, dx^\mu \wedge dx^\nu$$

COMPLETE σ-MODEL ACTION

$$\boxed{S = \frac{1}{4\pi\alpha'} \int_\Sigma g_{\mu\nu}\, dx^\mu dx^\nu + \frac{1}{2}\int B_{\mu\nu}\, dx^\mu \wedge dx^\nu}$$

TAKE CONSTANT TARGET-SPACE METRIC AND CONSTANT MAGNETIC FIELD $B_{\mu\nu} = B_{ij} = \text{const}$

$$\boxed{\text{Tension} = \frac{1}{2\pi\alpha'}}$$

IMAGINE A REGIME $\boxed{|g_{ij}| << |\alpha' B_{ij}|}$

CAN APPROXIMATE $S \approx \frac{1}{2}\int B_{ij}\, dx^i \wedge dx^j =$

$$= \frac{1}{2} B_{ij} \int_{\partial\Sigma} \left(x^i \partial_\tau x^j - x^j \partial_\tau x^i \right)$$

endpoints behave as e^- in LLL !!

SAME CANONICAL QUANTIZATION :

$$\boxed{[x^j, x^k]\Big|_{\partial\Sigma} = i\left(\frac{1}{B}\right)^{jk}} \qquad \boxed{\theta^{jk} = \left(\frac{1}{B}\right)^{jk}}$$

IN ORDER TO GET NCYM TAKE $\alpha' \to 0$ TO PROJECT ON MASSLESS STATES BUT AT THE SAME TIME WE NEED $B_{ij} \sim \left(\frac{1}{\theta}\right)_{ij} \sim$ CONSTANT

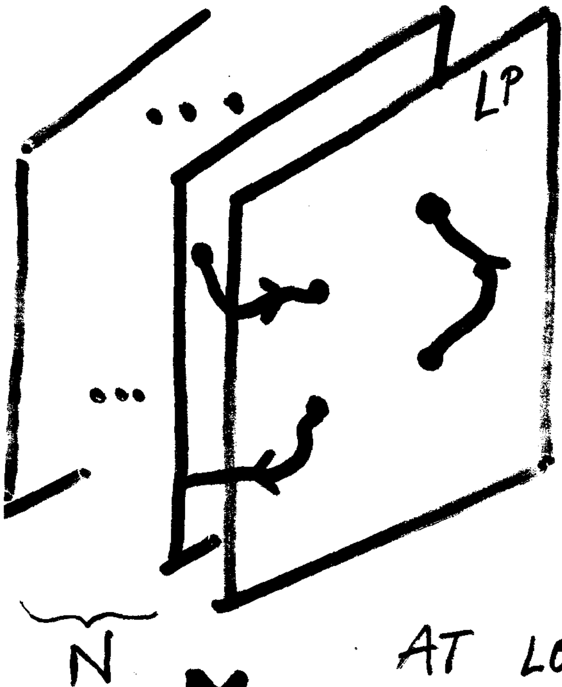
AND $|g_{ij}| << |\alpha' B_{ij}|$

THERE IS ONE SCALING LIMIT (SW) THAT GIVES A LOW ENERGY INTERACTING NCYM THEORY, AT LEAST CLASSICALLY

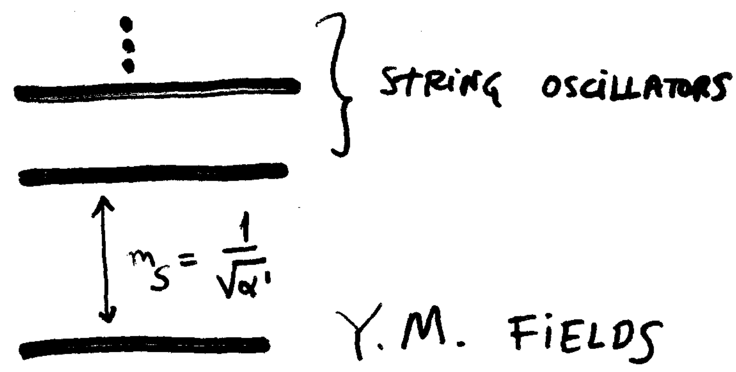
$$\boxed{\begin{gathered} g_{ij} \sim (\alpha')^2 B_{ij} B^{ij} \to 0 \\ \text{at fixed} \quad B_{ij} = (1/\theta)_{ij} \end{gathered}}$$

Seiberg & Witten 99

YANG-MILLS FROM Dp-BRANES



MASS SPECTRUM
OF OPEN STRINGS



AT LOW ENERGIES $\sqrt{\alpha'}\, E \ll 1$

$$\boxed{\begin{gathered} U(N)\quad \mathcal{N}=4 \quad SYM_{p+1} \\ g^2_{YM} = g_s\,(\alpha')^{\frac{p-3}{2}} \\ \frac{\Delta X_{\text{transverse}}}{\alpha'} \sim \langle \text{Higgs} \rangle \end{gathered}}$$

THIS DESCRIPTION IS PERTURBATIVE

$$\sim (g_s N)^{\#\,\text{holes}}$$

$$\boxed{\text{GOOD IF } g_s N \ll 1}$$

SMALL 't HOOFT COUPLING!

TURN ON NS B-FIELD

$$S_{\text{open}} = \frac{1}{4\pi\alpha'} \int_{\Sigma_2} \underbrace{g_{ij}}_{\text{closed string metric}} dx^i \wedge * dx^j + \frac{i}{4\pi} \int_{\Sigma_2} \underbrace{B_{ij}}_{\text{NS B-field}} dx^i \wedge dx^j$$

* OPEN STRINGS "SEE" A DIFFERENT METRIC G_{ij} AND "MAGNETIC" FIELD θ_{ij}

* TREE AMPLITUDES FOLLOW FROM WORLD-SHEET GREEN FUNCTION

$$\langle X^i(\tau)\, X^j(\tau') \rangle = -\alpha' G^{ij} \log(\tau-\tau')^2 + \frac{i}{2}\theta^{ij} \underbrace{\varepsilon(\tau-\tau')}_{\pm 1}$$

$$\boxed{G^{ij} = \left(\frac{1}{g + 2\pi\alpha' B}\right)^{ij}_{S} \qquad \theta^{ij} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B}\right)^{ij}_{A}}$$

LOW ENERGY LIMIT OF S-W

SEIBERG & WITTEN

$$\boxed{\mathrm{NCYM} = \lim_{\alpha' \to 0} \left(\text{D-BRANE}\right) \Bigg| \; G, \theta, g_{YM} \text{ fixed}}$$

NEED $\varepsilon \to 0$ WITH

$$g_{ij} \sim \varepsilon$$

$$\alpha' \sim \sqrt{\varepsilon}$$

$$g_s \sim \varepsilon^{\mathrm{rank}(B)}$$

$$B \sim \text{constant}$$

THEN: $\theta \to 1/B$ and:

$$\boxed{\mathcal{L}_{eff} \to \frac{1}{4 g_{YM}^2} \int \sqrt{\det G}\, G^{ij} G^{i'j'} \operatorname{Tr} \hat{F}_{ii'} * \hat{F}_{jj'}}$$

NCYM AND STRING THEORY

* WITTEN (86)
* CONNES, DOUGLAS & SCHWARZ (97)

$$\text{NCYM} = \text{M-THEORY} \big/ \text{LIGHT-LIKE CIRCLE}$$

with $\langle C_{-ij} \rangle \sim \theta_{ij} \neq 0$

* DOUGLAS & HULL (97)

$$\text{NCYM} = \lim_{g\, B_{NS} \to \infty} \left[\text{D-BRANE} \right]$$

- CHEUNG & KROGH (98), LI (98)
- CHU & HO (98)
- ARDALAN, ARFAEI & SHEIKH-JABBARI (98)
- SCHOMERUS (99)
- ASTASHKEVICH, NEKRASOV & SCHWARZ (98)
- BRACE, MORARIU, ZUMINO (98)
- HOFMAN, VERLINDE, ZWART (98)
- PIOLINE, SCHWARZ (99), KONECHNY, SCHWARZ (98)
- KAWANO, OKUYAMA (98), KATO KUROKI (99)
- LANDI, LIZZI, SZABO (98), CASALBUONI (98)
- SEIBERG & WITTEN (99)

WHAT ABOUT BACKGROUND ELECTRIC FIELDS

$$F^{0i} \neq 0 \longrightarrow \theta^{0i} \neq 0$$

NONCOMMUTATIVE TIME!

EXPECT PROBLEMS:

$$\ell^{0}_{eff} = \theta^{0i} p_i$$

NC PARTICLES ARE EXTENDED IN TIME

Advanced effects in scattering

$\downarrow$

Problems with causality?

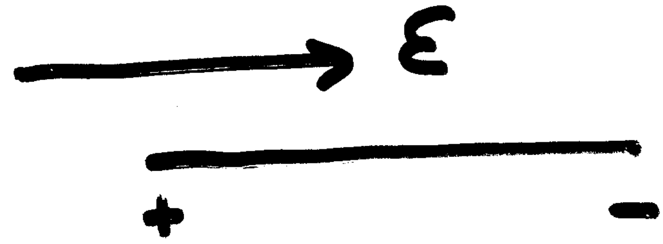
no obvious hamiltonian formalism

$\downarrow$

Problems with unitarity?

OPEN STRINGS IN SMALL ELECTRIC FIELD ARE OK, BUT:

CONSIDER A STRAIGHT OPEN STRING AT REST IN A BACKGROUND $\mathcal{E}$-FIELD



$$\text{Energy} = \text{Mass} + (\text{Potential Energy}) =$$

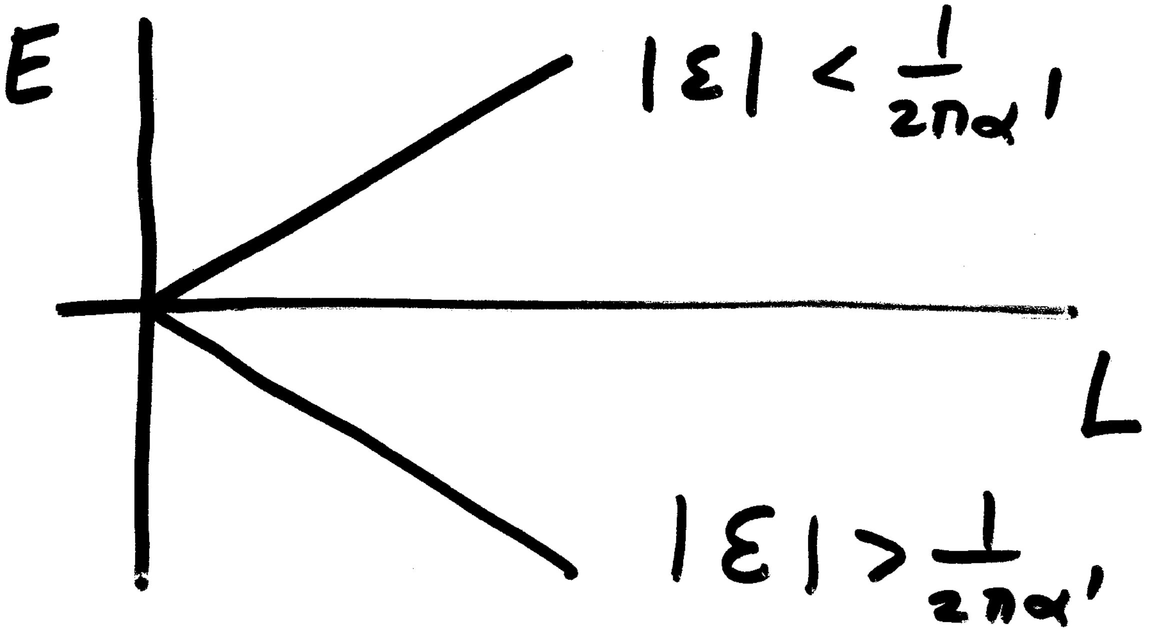
$$= (\text{Tension}) \cdot L - \mathcal{E} \cdot L =$$

$$= \left(\frac{1}{2\pi\alpha'} - \mathcal{E}\right) L$$

Effective tension: $\boxed{T_{eff} = \left(\frac{1}{2\pi\alpha'} - \mathcal{E}\right)}$

$\Rightarrow$ $\mathcal{E} = \mathcal{E}_{critical} = 2\pi\alpha' \longrightarrow$ Tensionless

$\Rightarrow$ $\mathcal{E} > \mathcal{E}_{crit} \longrightarrow$ Tachyonic



→ IF ELECTRIC FIELD IS TOO LARGE IN STRING UNITS THE OPEN STRING IS UNSTABLE TO INFINITE STRETCHING.

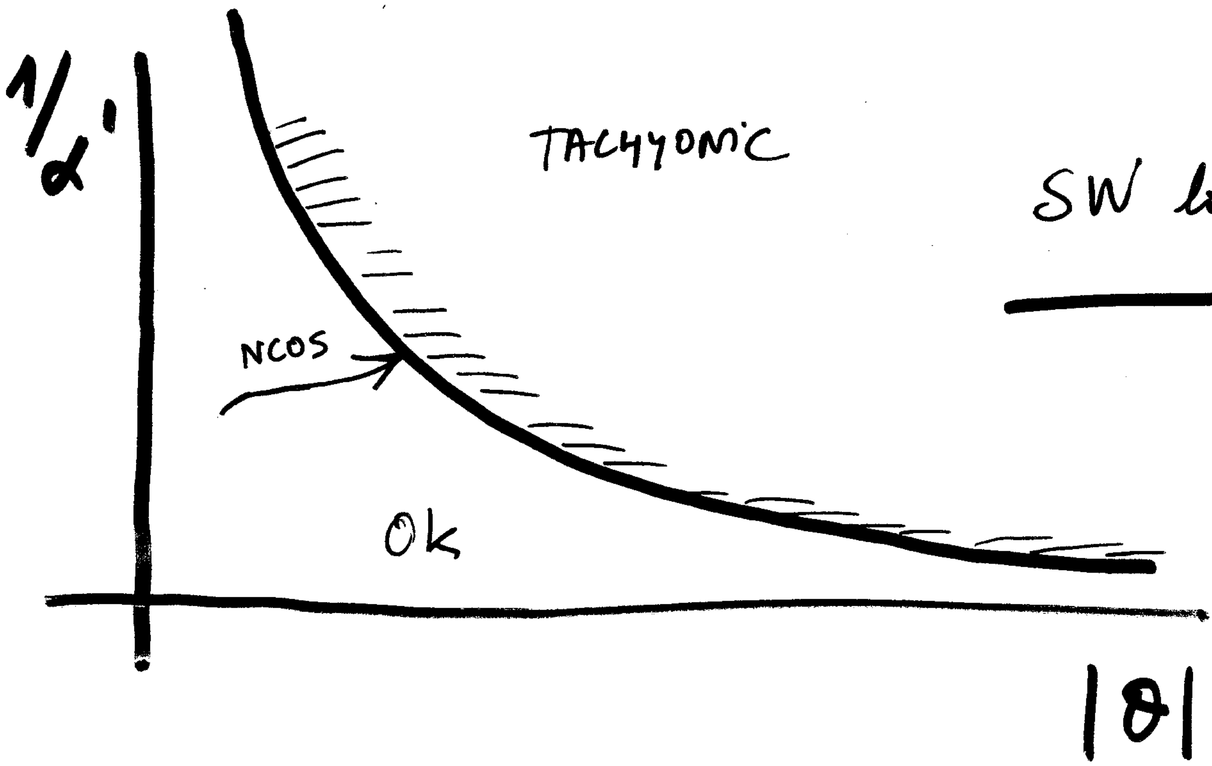
→ OPEN STRINGS CAN NUCLEATE AND CAUSE DECAY OF THE BACKGROUND $\mathcal{E}$-FIELD. LIKE SCHWINGER MECHANISM BUT CLASSICAL AND CATASTROPHIC!

(Burgess 85)

THUS, IN STRING THEORY:

$$|\alpha' \mathcal{E}| \lesssim 1 \longrightarrow |\theta^{0i}| \lesssim \alpha'$$

NONCOMMUTATIVITY IS "CENSORED" BY NORMAL STRING FUZZINESS



SW low energy lim

* Cannot achive SW limit
no string regularization of
TIME-NCFT BARBON & Rabinovici

* Can define a fully fledged interacting
string theory by scaling at
critical boundary

Gopakumar, Minwalla, Maldacena, Strominger → Seiberg, Susskind, Toumbas

$|\theta^{0i}| \sim \alpha'$

decoupled closed strings!

Mimics duality Web