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SUMMER SCHOOL ON PARTICLE PHYSICS

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PHENOMENOLOGY OF SUPERSYMMETRY

Lecture I

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Please note: These are preliminary notes intended for internal distribution only.

I. WHY IS IT NECESSARY TO GO BEYOND THE STANDARD MODEL ?

The Standard Model of particle physics is a superb description of fundamental particles and their interactions... [see lectures by John Ellis]

... with two notable footnotes:

(i) neutrinos are not exactly massless

suggestive, perhaps, of a new high-energy scale much larger than 1 TeV. There are simple ways to account for massive neutrinos, and we will occasionally return to this point.

(ii) The Higgs boson has not yet been discovered

or has it? [more on this by John Ellis]

Still, all indications point to the existence of a weakly-coupled Higgs boson with a mass of $\mathcal{O}(100\text{ GeV})$, and we will assume that this is the case in these lectures.

Precision electroweak data is consistent with $m_h \lesssim 200$ GeV. To avoid this conclusion, or to replace the weakly-coupled Higgs boson with another (say, strongly-coupled) mechanism for generating electroweak symmetry breaking, one would need additional new physics to conspire to produce the same virtual effects as the weakly-coupled Standard Model Higgs boson.

Nevertheless, the Standard Model (including the Higgs boson) cannot be considered to be a truly fundamental theory of particle physics, valid to arbitrarily high energy scales.

At best, it is an effective field theory, which is valid up to some scale Λ.

At energies above Λ, new physics enters and the Standard Model is no longer adequate for describing fundamental physics.

Below Λ, the Standard Model is a very good approximation to observable physics. Deviation are suppressed by at least a factor of E/Λ at an energy scale E.

<u>example</u>: neutrino masses may be non-zero as a consequence of physics at the scale Λ.

What's missing?

So far, gravity is not yet included. Quantum gravitational effects are relevant only at a very high energy scale, called the Planck scale

$$M_{\rm PL} = (c\hbar/G_N)^{1/2} \simeq 10^{19}\ \text{GeV}\,,$$

which arises as follows. The gravitational potential energy of a particle of mass M, $G_N M^2/r$ (where G_N is Newton's gravitational constant), evaluated at its Compton wavelength, $r = \hbar/Mc$, is of order the rest mass, Mc^2, when

$$G_N M^2 \left(\frac{Mc}{\hbar}\right) \sim Mc^2\,,$$

which implies that $M^2 \sim c\hbar/G_N$. When this happens, the gravitational energy is large enough to induce pair production, which means that quantum gravitational effects can no longer be neglected. Thus, the Planck scale, $M_{\rm PL} = (c\hbar/G_N)^{1/2}$, represents the energy scale at which gravity and all other forces of elementary particles must be incorporated into the same theory.

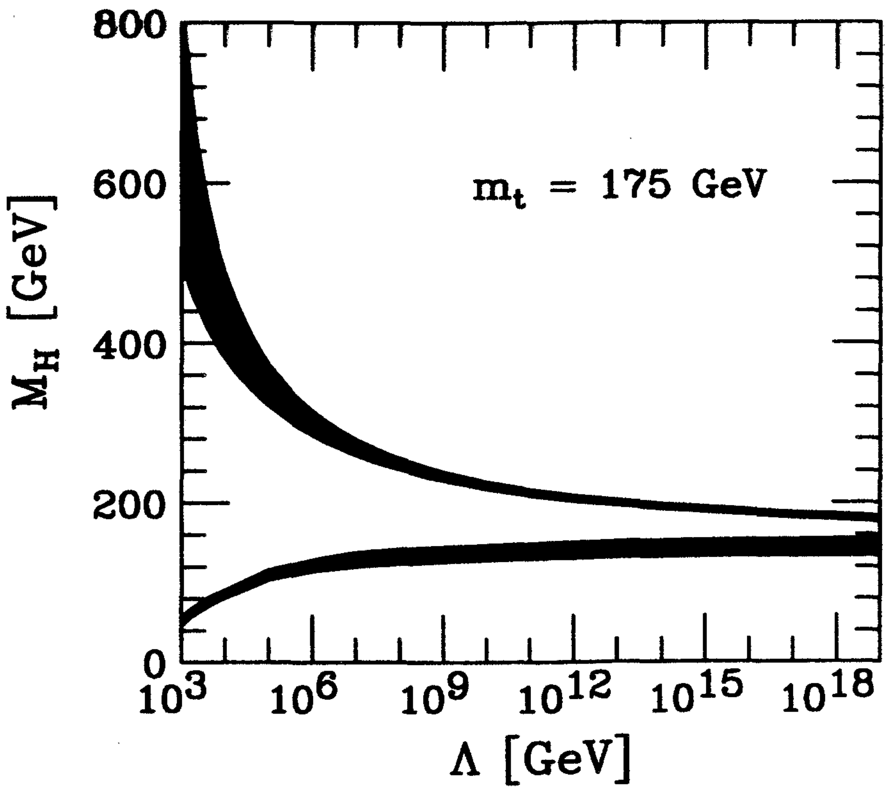
Where does the Standard Model Break Down?

The Standard Model (SM) describes quite accurately physics near the electroweak symmetry breaking scale [$v = 246$ GeV]. But, the SM is only a "low-energy" approximation to a more fundamental theory.

- The Standard Model cannot be valid at energies above the Planck scale, $M_{\rm PL} \simeq 10^{19}$ GeV, where gravity can no longer be ignored.
- Neutrinos are exactly massless in the Standard Model. But, recent experimental observations of neutrino mixing imply that neutrinos have very small masses ($m_\nu / m_e \lesssim 10^{-7}$). Neutrino masses can be incorporated in a theory whose fundamental scale is $M \gg v$. Neutrino masses of order v^2/M are generated, which suggest that $M \sim 10^{15}$ GeV.

- When radiative corrections are evaluated, one finds:
  - The Higgs potential is unstable at large values of the Higgs field ($|\Phi| > \Lambda$) if the Higgs mass is too small.
  - The value of the Higgs self-coupling runs off to infinity at an energy scale above Λ if the Higgs mass is too large.

This is evidence that the Standard Model must break down at energies above Λ.



Problems with Elementary Scalar Fields

In 1939, Weisskopf computed the self-energy of a Dirac fermion and compared it to that of an elementary scalar. The fermion self-energy diverged logarithmically, while the scalar self-energy diverged quadratically. If the infinities are cut-off at a scale Λ, then Weisskopf argued that for the particle mass to be of order the self-energy,

- For the e^-, $\Lambda \sim me^{\alpha^{-1}} \gg M_{\rm PL}$ [where $\alpha \equiv e^2/(4\pi\hbar c) \simeq 1/137$];
- For an elementary boson, $\Lambda \sim m/g$, where g is the coupling of the boson to gauge fields.

In modern times, this is called the hierarchy and naturalness problem. Namely, how can one understand the large hierarchy of energy scales from v to $M_{\rm PL}$ in the context of the SM? If the SM is superseded by a more fundamental theory at an energy scale Λ, one expects

$$m_H^2 = (m_H^2)_0 + Kg^2\Lambda^2$$

$(m_H^2)_0$ is a parameter of the fundamental theory, $K \sim \mathcal{O}(1)$ is determined by low-energy physics. The *natural* value for the scalar squared-mass is $g^2\Lambda^2$. Thus,

$$\Lambda \simeq m_H/g \sim \mathcal{O}(1\ \mathrm{TeV})$$

What new physics is lurking at the TeV scale?

On the Self-Energy and the Electromagnetic Field of the Electron

V. F. Weisskopf
University of Rochester, Rochester, New York
(Received April 12, 1939)

The charge distribution, the electromagnetic field and the self-energy of an electron are investigated. It is found that, as a result of Dirac's positron theory, the charge and the magnetic dipole of the electron are extended over a finite region; the contributions of the spin and of the fluctuations of the radiation field to the self-energy are analyzed, and the reasons that the self-energy is only logarithmically infinite in positron theory are given. It is proved that the latter result holds to every approximation in an expansion of the self-energy in powers of e^2/hc. The self-energy of charged particles obeying Bose statistics is found to be quadratically divergent. Some evidence is given that the "critical length" of positron theory is as small as $h/(mc)\cdot\exp(-hc/e^2)$.

I. Introduction and Discussions of Results

THE self-energy of the electron is its total energy in free space when isolated from other particles or light quanta. It is given by the expression

$$W = T + (1/8\pi)\int (H^2+E^2)d\mathbf{r}. \tag{1}$$

Here T is the kinetic energy of the electron; H and E are the magnetic and electric field strengths. In classical electrodynamics the self-energy of an electron of radius a at rest and without spin is given by $W \sim mc^2 + e^2/a$ and consists solely of the energy of the rest mass and of its electrostatic field. This expression diverges linearly for an infinitely small radius. If the electron is in motion, other terms appear representing the energy produced by the magnetic field of the moving electron. These terms, of course, can be obtained by a Lorentz transformation of the former expression.

The quantum theory of the electron has put the problem of the self-energy in a critical state. There are three reasons for this:

(a) Quantum kinematics shows that the radius of the electron must be assumed to be zero. It is easily proved that the product of the charge densities at two different points, $\rho(\mathbf{r}-\xi/2) \times \rho(\mathbf{r}+\xi/2)$, is a delta-function $e^2\delta(\xi)$. In other words: if one electron alone is present, the probability of finding a charge density simultaneously at two different points is zero for every finite distance between the points. Thus the energy of the electrostatic field is infinite as

$$W_{st} = \lim_{(a\to 0)} e^2/a.$$

(b) The quantum theory of the relativistic electron attributes a magnetic moment to the electron, so that an electron at rest is surrounded[4] by a magnetic field. The energy

$$U_{mag} = (1/8\pi)\int H^2 d\mathbf{r}$$

of this field is computed in Section III and the result is

$$U_{mag} = e^2h^2/(6\pi m^2c^2a^3).$$

This corresponds to the field energy of a magnetic dipole of the moment $eh/2mc$ which is spread over a volume of the dimensions a. The spin, however, does not only produce a magnetic field, it also gives rise to an alternating electric field. The closer analysis of the Dirac wave equation has shown[1] that the magnetic moment of the spin is produced by an irregular circular fluctuation movement (Zitterbewegung) of the electron which is superimposed to the translatory motion. The instantaneous value of the velocity is always found to be c. It must be expected that this motion will also create an alternating electric field. The existence of this field is demonstrated in Section III by the computation of the expression

$$U_{el} = (1/8\pi)\int E_s^2 d\mathbf{r}.$$

There E_s is the solenoidal part (div. $E_s = 0$) of the electric field strength created by the electron. The fact that the above expression does not vanish for an electron at rest proves the existence

[1] E. Schroedinger, Berl. Ber. 1930, 418 (1930).

zero in the one-electron theory, is negative and quadratically divergent in the positron theory. This is because of the negative contribution of the magnetic field and the interference effect of the electric field of the vacuum electrons.

(c) The energy W_{fluct} of forced vibrations under the influence of the zero-point fluctuations of the radiation field. The energies (b) and (c) compensate each other to a logarithmic term.

It is interesting to apply similar considerations to the scalar theory of particles obeying the Bose statistics, as has been developed by Pauli and the author.[7] Here the probability of finding two equal particles closer than their wave-lengths is *larger* than at longer distances. The effect on the self-energy is therefore just the opposite. The influence of the particle on the vacuum causes a higher singularity in the charge distribution instead of the hole which balanced the original charge in the previous considerations. It is shown in Section V that this gives rise to a quadratically divergent energy of the Coulomb field of the particle. Thus the situation here is even worse than in the classical theory. The spin term obviously does not appear and the energy W_{fluct} is exactly equal to its value for a Fermi particle.

A few remarks might be added about the possible significance of the logarithmic divergence of the self-energy for the theory of the electron. It is proved in Section VI that every term in the expansion of the self-energy in powers of e^2/hc

$$W=\sum_n W^{(n)} \tag{3}$$

diverges logarithmically with infinitely small electron radius and is approximately given by

$$W^{(n)}\sim z_n mc^2(e^2/hc)^n[\lg(h/mca)]^t, \quad t\leqq n.$$

Here the z_n are dimensionless constants which cannot easily be computed. It is therefore not sure, whether the series (3) converges even for finite a, but it is highly probable that it converges if $\delta=e^2/(hc)\cdot\lg(h/mca)<1$. One then would get $W=mc^2O(\delta)$ where $O(\delta)=1$ for a value of $\delta<1$. We then can define an electron radius in the same way as the classical radius e^2/mc^2 is defined, by putting the self-energy equal to mc^2. One obtains then roughly a value $a\sim h/(mc)\cdot\exp(-hc/e^2)$ which is about 10^{-58} times smaller than the classical electron radius. The "critical length" of the positron theory is thus infinitely smaller than usually assumed.

The situation is, however, entirely different for a particle with Bose statistics. Even the Coulombian part of the self-energy diverges to a first approximation as $W_{st}\sim e^2h/(mca^2)$ and requires a much larger critical length that is $a=(hc/e^2)^{-\frac{1}{2}}\cdot h/(mc)$, to keep it of the order of magnitude of mc^2. This may indicate that a theory of particles obeying Bose statistics must involve new features at this critical length, or at energies corresponding to this length; whereas a theory of particles obeying the exclusion principle is probably consistent down to much smaller lengths or up to much higher energies.

[7] W. Pauli and V. Weisskopf, Helv. Phys. Acta 7, 709 (1934).

II. The Charge Distribution of the Electron

The charge distribution in the neighborhood of an electron can be determined from the expression

$$G(\xi)=\int\rho(\mathbf{r}-\xi/2)\rho(\mathbf{r}+\xi/2)d\mathbf{r}; \tag{4}$$

here $\rho(\mathbf{r})$ is the charge density at the point $\mathbf{r}$. $G(\xi)$ is the probability of finding charge simultaneously at two points in a distance ξ. If applied to a situation in which one electron alone is present, direct information can be drawn from this expression concerning the charge distribution in the electron itself. The charge density is given by

$$\rho(\mathbf{r})=e\{\psi^*(\mathbf{r})\psi(\mathbf{r})\}-\sigma, \tag{5}$$

where $\psi(\mathbf{r})$, the wave function, is a spinor with four components ψ_μ, $\mu=1, 2, 3, 4$. We write

$$\{\psi^*\psi\}=\sum_{\mu=1}^{4}\psi_\mu{}^*\psi_\mu$$

for the scalar product of two spinors. σ is the charge density of the unperturbed electrons in the negative energy states which is to be subtracted in the positron theory. In the one-electron theory σ is zero. The wave function ψ can be expanded in wave functions φ_q of the

A lesson from history

The electron self-energy in classical electromagnetism goes like e^2/a $(a \to 0)$, *i.e.*, it is linearly divergent. In quantum theory, fluctuations of the electromagnetic fields (in the "single electron theory") generate a quadratic divergence. If these divergences are not canceled, one would expect that QED should break down at an energy of order m_e/e far below the Planck scale (a severe hierarchy problem).

The linear and quadratic divergences will cancel exactly if one makes a bold hypothesis: the existence of the positron (with a mass equal to that of the electron but of opposite charge).

Weisskopf was the first to demonstrate this cancellation in 1934... well, actually he initially got it wrong, but thanks to Furry, the correct result was presented in an erratum.

The self-energy of the electron

V. WEISSKOPF

Zeitschrift für Physik, 89: 27–39 (1934). Received 13 March 1934.

The self-energy of the electron is derived in a closer formal connection with classical radiation theory, and the self-energy of an electron is calculated when the negative energy states are occupied, corresponding to the conception of positive and negative electrons in the Dirac 'hole' theory. As expected, the self-energy also diverges in this theory, and specifically to the same extent as in ordinary single-electron theory.

1 Problem definition

The self-energy of the electron is the energy of the electromagnetic field which is generated by the electron in addition to the energy of the interaction of the electron with this field. Waller,[1] Oppenheimer,[2] and Rosenfeld[3] calculated the self-energy of the free electron by means of the Dirac relativistic wave equation of the electron and the Dirac theory of the interaction between matter and light. They here used an approximation method which represents the self-energy in powers of the charge e. They found that the first term, which is proportional to e^2, already becomes infinitely large. The essential reason for this is that the theory of the interaction of the electron with the electromagnetic field is built on the classical equations of motion of a point-shaped electron whose self-energy, as is well known, also becomes infinite in classical theory.[4]

In the present note, the expressions for the self-energy shall be derived without direct application of quantum electrodynamics, but by means of the Heisenberg radiation theory,[5] which is linked much more closely to classical electrodynamics. The radiation field is calculated classically from the current and charge densities of the atom; however, the amplitudes of the electromagnetic potentials are regarded as non-commuting in the final result. Just as was shown in a corresponding paper by Casimir[6] concerning the natural linewidth, this method yields the same result as explicit quantum

[1] I. Waller, *ZS. f. Phys.* 62, 673, 1930.

[2] R. Oppenheimer, *Phys. Rev.* 35, 461, 1930.

[3] L. Rosenfeld, *ZS. f. Phys.* 70, 454, 1931.

[4] Recently, G. Wentzel (*ZS. f. Phys.* 86, 479, 635, 1933) has shown that one can circumvent the divergence of the self-energy in classical electron theory by suitable limiting processes. The transfer of these methods to quantum theory has failed, however, since, according to Waller, the degree of infinity in quantum theory is higher than in classical theory. The hope expressed there that the degree of infinity will become smaller in the Dirac formalism of the 'hole' theory, does indeed hold for the electrostatic part but not for the electrodynamic part, so that the Wentzel method must fail here too.

[5] W. Heisenberg, *Ann. d. Phys.* 338, 1931; see also W. Pauli's article in Geiger-Scheel, *Handb. d. Phys.* XXIC/1, 2nd edn., pp. 201–10.

[6] H. Casimir, *ZS. f. Phys.* 81, 496, 1933.

Correction to the paper: The self-energy of the electron

Zeitschrift für Physik, 90: 817–18 (1934). Received 20 July 1934.

On [p. 166] of the paper cited above, there is a computational error which has seriously garbled the results of the calculation for the electrodynamic self-energy of the electron according to the Dirac hole theory. I am greatly indebted to Mr Furry (University of California, Berkeley) for kindly pointing this out to me.

The degree of divergence of the self-energy in the hole theory is *not*, as asserted in [the preceding paper], just as great as in the Dirac one-electron theory, but the divergence is only logarithmic. The expression for the electrostatic and electrodynamic parts of the self-energy E of an electron with momentum p now correctly reads, in the notations used in [the preceding paper]:

$$E = E^{\mathrm{S}} + E^{\mathrm{D}},$$

$$E^{\mathrm{S}} = \frac{e^2}{h(m^2c^2 + p^2)^{1/2}}(2m^2c^2 + p^2)\int_{k_0}^{\infty}\frac{\mathrm{d}k}{k} + \text{finite terms},$$

$$E^{\mathrm{D}} = \frac{e^2}{h(m^2c^2 + p^2)^{1/2}}(m^2c^2 - \tfrac{4}{3}p^2)\int_{k_0}^{\infty}\frac{\mathrm{d}k}{k} + \text{finite terms}.$$

For comparison, we cite the expressions obtained on the basis of the single-electron theory:

$$E^{\mathrm{S}} = \frac{c^2}{h}\int_0^{\infty}\mathrm{d}k + \text{finite terms},$$

$$E^{\mathrm{D}} = \frac{e^2}{h}\left[\frac{m^2c^2}{p(m^2c^2 + p^2)^{1/2}}\log\frac{(m^2c^2 + p^2)^{1/2} + p}{(m^2c^2 + p^2)^{1/2} - p} - 2\right]\int_0^{\infty}\mathrm{d}k + \frac{2e^2}{h(m^2c^2 + p^2)^{1/2}}\int_0^{\infty}k\,\mathrm{d}k.$$

The computational error arose in the transformation of the electrodynamic portion E^{D} for the case of the hole theory:

$$E^{\mathrm{D}} = J_+^k(\vec{p}) - J_-^{\prime k}(\vec{p}), \quad k = 1 \text{ or } 2,$$

where $J_+^k(\vec{p})$ is defined on [p. 166] whereas

$$J_-^{\prime k}(\vec{p}) = -\frac{e^2}{2\pi h}\int\frac{\mathrm{d}\vec{k}}{k}\,\frac{PP_+ + \frac{1}{k^2}(\vec{k}\vec{p})^2 + (\vec{k}\vec{p}) + m^2c^2}{PP_+(P + P_+ + k)}$$

and is not equal to the quantity $J_-^k(\vec{p})$, from which it differs only by a sign. Likewise, one must set

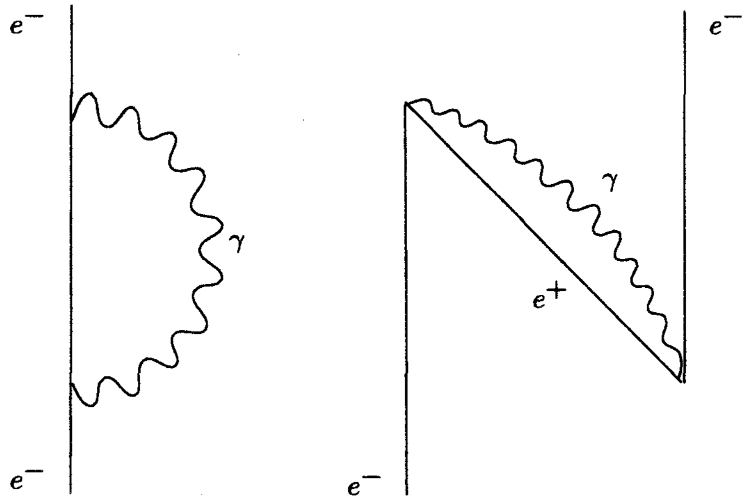
$$E_{\mathrm{vac}}^{\mathrm{D}} = \sum_{k=1,2}\int J_-^{\prime k}(\vec{p})\,\mathrm{d}\vec{p}$$

for the self-energy of the vacuum.

As a consequence of the new result, the question raised in note 4 of the paper requires a new examination, whether the Wentzel method,[15] to avoid the infinite self-energy by suitable limiting processes, might not still lead to the objective in the hole theory.

[15] G. Wentzel, *ZS. f. Phys*, **86**, 479, 635, 1933.

A remarkable result:



(a) (b)

> The linear and quadratic divergences of a quantum theory of elementary fermions are precisely canceled if one doubles the particle spectrum—for every fermion, introduce an anti-fermion partner of the same mass and opposite charge.

In the process, we have introduced a new CPT-symmetry that associates a fermion with its anti-particle and guarantees the equality of their masses.

Low-Energy Supersymmetry

Will history repeat itself? Let's try it again. Take the Standard Model and double the particle spectrum. Introduce a new symmetry—supersymmetry—that relates fermions to bosons: for every fermion, there is a boson of equal mass and vice versa. Now, compute the self-energy of an elementary scalar. Supersymmetry relates it to the self-energy of a fermion, which is only logarithmically divergent [or logarithmically sensitive to the fundamental high energy scale]. Conclusion: quadratic divergences cancel! The hierarchy problem is resolved.

No analogy can be precise. In this case, a serious flaw arises. No super-partners have ever been seen. (There is no scalar-electron degenerate in mass with the electron.) Supersymmetry, if it exists in nature, must be a broken symmetry. Previous arguments imply that:

> The scale of supersymmetry-breaking must be of order 1 TeV or less, if supersymmetry is associated with the scale of electroweak symmetry breaking.

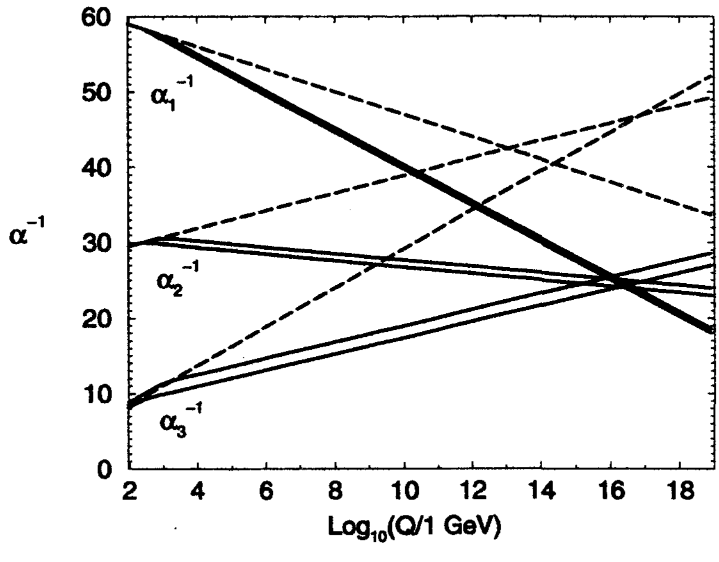
Still to be understood—the origin of supersymmetry breaking. Nevertheless, TeV-scale physics could provide our first glimpse of the Planck scale regime.

Benefits of Low-Energy Supersymmetry

- In low-energy SUSY theories, quadratic sensitivity to Λ is replaced by quadratic sensitivity to the SUSY-breaking scale.

- Provides a framework for the hierarchy of energy scales between the scale of electroweak symmetry breaking and the Planck scale ($M_{\rm PL} \simeq 10^{19}$ GeV), which characterizes the fundamental scale of gravity.

- Unification of the three gauge couplings at $\sim 10^{16}$ GeV.



- A candidate for cold dark matter

  (see lectures by Keith Olive)

II. Fermions in Quantum Field Theory

Some Technical Details

Supersymmetry is a bose-fermi symmetry. It will be very useful to understand and manipulate fermion fields at their most basic level.

We begin with some well known facts about the Poincaré algebra, which is the underlying space-time symmetry of relativistic quantum field theory.

$$[P^\mu, P^\nu] = 0$$

$$[J^{\mu\nu}, P^\lambda] = i(g^{\nu\lambda}P^\mu - g^{\mu\lambda}P^\nu)$$

$$[J^{\mu\nu}, J^{\lambda\rho}] = i(g^{\nu\lambda}J^{\mu\rho} - g^{\mu\lambda}J^{\nu\rho} + g^{\mu\rho}J^{\nu\lambda} - g^{\nu\rho}J^{\mu\lambda})$$

where P^μ generates space-time translations

$J^{\mu\nu}$ generates rotations and Lorentz boosts $\xrightarrow{\text{rot.}} J^i = \frac{1}{2}\epsilon^{ijk}J^k$, $\xrightarrow[\text{boosts}]{} K^i = J^{0i}$

The $J^{\mu\nu}$ satisfy an $SO(3,1) \simeq SL(2,C)$ Lie algebra.

In my conventions,

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \varepsilon_{0123} = +1.$$

The Pauli-Lubanski vector:

$$W_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma \qquad W^0 = \vec{J}\cdot\vec{P}, \quad \vec{W} = P^0\vec{J} + \vec{K}\times\vec{P}$$

Note that:

(i) $W_\mu P^\mu = 0$

(ii) $[W_\mu, P_\nu] = 0$

(iii) $[W_\mu, W_\nu] = -i\varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma$

Casimir operators of the Poincaré algebra:

(i) $P^2 = P_\mu P^\mu$

(ii) $W^2 = W_\mu W^\mu$

i.e. $[P^2, P^\mu] = [P^2, J^{\mu\nu}] = [W^2, P^\mu] = [W^2, J^{\mu\nu}] = 0$

Eigenvalues of P^2

$$P^2 = m^2$$

Eigenvalues of W^2

case 1: $m^2 > 0$. Go to rest frame where $P^\mu = (m; \vec{0})$
$\vec{J} = \vec{S}$

Then, $\vec{w} = m\vec{S}$

$w^2 = -m^2 \vec{S}^2$ with eigenvalues $-m^2 s(s+1)$, $s = 0, \frac{1}{2}, 1, \ldots$

case 2: $m^2 = 0$

Then $w^2 = 0$ [slight cheat here. But $w^2 = 0$ corresponds to the case of interest.]

$$w^2 = p^2 = w^\mu p_\mu = 0 \implies w^\mu = \lambda p^\mu.$$

From $\vec{w} = p^0 \vec{J} + \vec{K} \times \vec{P}$,

$$\vec{w} \cdot \vec{P} = p^0 \vec{J} \cdot \vec{P}$$

For $p^2 = 0$, we have $p^0 = |\vec{P}|$. Thus, inserting $\vec{w} = \lambda \vec{P}$,

$$\lambda = \frac{\vec{J} \cdot \vec{P}}{|\vec{P}|} \qquad \text{helicity} \qquad |\lambda| = 0, \tfrac{1}{2}, 1, \ldots$$

Conclusion

1. Massive states of QFT characterized by $|m, s\rangle$
2. Massless states of QFT characterized by $|\lambda\rangle$

Since λ changes sign under parity, when we add the antiparticle (the CPT-conjugate) the states are doubled: $|\lambda\rangle \oplus |-\lambda\rangle$. [example: the photon with $\lambda = \pm 1$.]

Fermions in QFT

Under a Lorentz transformation, $x'_\mu = L_\mu{}^\nu x_\nu$,

$$\psi'_\alpha(x') = \exp\left(-\tfrac{i}{2}\theta^{\mu\nu}S_{\mu\nu}\right)_\alpha{}^\beta \psi_\beta(x)$$

where the $S_{\mu\nu}$ are finite-dimensional matrices that satisfy the Lorentz algebra (same commutation relations as the $J_{\mu\nu}$). Different choices for the $S_{\mu\nu}$ (irreducible representations) correspond to different spin. Clearly $S_{\mu\nu}=0$ corresponds to spin 0.

Define:
$$J^i = \tfrac{1}{2}\varepsilon^{ijk}S_{jk} \qquad\qquad [J^i, J^j] = i\varepsilon^{ijk}J^k$$
$$K^i = S^{0i} \qquad \Rightarrow \qquad [J^i, K^j] = i\varepsilon^{ijk}K^k$$
$$[K^i, K^j] = -i\varepsilon^{ijk}J^k$$

and

$$\vec{J}_+ = \tfrac{1}{2}(\vec{J} + i\vec{K})$$
$$\vec{J}_- = \tfrac{1}{2}(\vec{J} - i\vec{K})$$

satisfy

$$[J_+^i, J_+^j] = i\varepsilon^{ijk}J_+^k$$
$$[J_-^i, J_-^j] = i\varepsilon^{ijk}J_-^k$$
$$[J_+^i, J_-^j] = 0$$

Thus, the irreducible representations of the Lorentz group correspond to (j_+, j_-), where the eigenvalues of $J_\pm^2$ are $j_\pm(j_\pm+1)$, respectively. The dimension of (j_+, j_-) is $(2j_+ + 1)(2j_- + 1)$.

example: $(0,0)$ is a scalar

Infinitesimally,

$$\exp\left(-\tfrac{i}{2}\theta_{\mu\nu}S^{\mu\nu}\right) \simeq I - i\vec{\theta}\cdot\vec{J} - i\vec{\beta}\cdot\vec{K}$$

$$\theta^i = \tfrac{1}{2}\epsilon^{ijk}\theta_{jk}$$
$$\beta^i = \theta^{0i}$$

Two-dimensional (spin-½) representations

$(\tfrac{1}{2},0)$
$$\left.\begin{aligned}\vec{J}_+ &= \tfrac{1}{2}(\vec{J}+i\vec{K}) = \frac{\vec{\sigma}}{2}\\ \vec{J}_- &= \tfrac{1}{2}(\vec{J}-i\vec{K}) = 0\end{aligned}\right\} \Longrightarrow \quad \begin{aligned}\vec{J} &= \frac{\vec{\sigma}}{2}\\ \vec{K} &= -\frac{i\vec{\sigma}}{2}\end{aligned}$$

$(0,\tfrac{1}{2})$
$$\left.\begin{aligned}\vec{J}_+ &= \tfrac{1}{2}(\vec{J}+i\vec{K}) = 0\\ \vec{J}_- &= \tfrac{1}{2}(\vec{J}-i\vec{K}) = \frac{\vec{\sigma}}{2}\end{aligned}\right\} \Longrightarrow \quad \begin{aligned}\vec{J} &= \frac{\vec{\sigma}}{2}\\ \vec{K} &= \frac{i\vec{\sigma}}{2}\end{aligned}$$

For the $(\tfrac{1}{2},0)$ representation, introduce the two-component field ξ_α ($\alpha=1,2$) which transforms under Lorentz transformations as

$$\xi_\alpha \longrightarrow \xi'_\alpha = M_\alpha{}^\beta \xi_\beta$$

where $M \simeq I - \frac{i\vec{\theta}\cdot\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}$ is a two-dimensional representation of $SL(2,C)$. In QFT, ξ_α is an anti-commuting two-component fermion field.

Aside: If M is a matrix representation of $SL(n,C)$, then M^*, $(M^{-1})^T$ and $(M^{-1})^\dagger$ are also representations, i.e. they preserve the group multiplication law. For $n>2$, all four representations are inequivalent. For $SL(2,C)$ only two of the four are distinct matrix representations for a given dimension, corresponding to (j_1,j_2) and (j_2,j_1).

For the $SL(2,C)$ matrices M, it is simple to check that:

$$(M^{-1})^T = i\sigma^2 M (i\sigma^2)^T \qquad i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which follows from:

$$\sigma^2 \vec{\sigma} \sigma^{2T} = \vec{\sigma}^T$$

Introduce the <u>contragradient</u> representation $(M^{-1})^T$:

$$\xi^\alpha \longrightarrow \xi'^\alpha = (M^{-1})^{T\,\alpha}{}_\beta\, \xi^\beta$$

$$= [i\sigma^2 M (i\sigma^2)^T]^\alpha{}_\beta\, \xi^\beta$$

which motivates the definition:

$$\varepsilon^{\alpha\beta} \equiv i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

i.e. $\varepsilon^{12} = -\varepsilon^{21} = 1$. Then,

$$\boxed{\xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta}$$

The matrices M and $(M^{-1})^T$ are related by similarity transformation, or equivalently by a change of basis; hence the corresponding representations are equivalent.

This is similar to the well-known result in $SU(2)$ that the $\underset{\sim}{2}$ and $\underset{\sim}{2}^*$ representations are equivalent.

Either ξ_α or ξ^α is a good candidate for the $(\frac{1}{2},0)$ representation.

For the $(0,\frac{1}{2})$ representation, introduce the "dotted" spinor indices:

$$\bar{\eta}^{\dot{\alpha}} \longrightarrow \bar{\eta}^{\dot{\alpha}\,\prime} = (M^{-1})^{\dagger\,\dot{\alpha}}{}_{\dot{\beta}}\,\bar{\eta}^{\dot{\beta}} \qquad \dot{\alpha},\dot{\beta} = \dot{1},\dot{2}$$

where

$$(M^{-1})^{\dagger} \simeq I - \frac{i\vec{\theta}\cdot\vec{\sigma}}{2} + \frac{\vec{\beta}\cdot\vec{\sigma}}{2}\,.$$

An equivalent description is via the <u>conjugate</u> representation M^{*}:

$$\bar{\eta}_{\dot{\alpha}} \longrightarrow \bar{\eta}_{\dot{\alpha}}' = (M^{*})_{\dot{\alpha}}{}^{\dot{\beta}}\,\bar{\eta}_{\dot{\beta}}$$

where

$$\boxed{\bar{\eta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\,\bar{\eta}_{\dot{\beta}}}$$

and:

$$\epsilon^{\dot{\alpha}\dot{\beta}} = i\sigma^{2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{\dot{\alpha}\dot{\beta}} = -i\sigma^{2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So, $\epsilon^{\dot{\alpha}\dot{\beta}} = \epsilon^{\alpha\beta}$, etc.

Note that $\bar{\eta}_{\dot{\alpha}}$ and η_{α}^{*} have the same transformation law, so we may equate them:

$$\bar{\eta}_{\dot{\alpha}} = \eta_{\alpha}^{*}$$

Similarly,

$$\bar{\eta}^{\dot{\alpha}} = \eta^{\alpha\,*}.$$

Thus, ξ_α and $\bar{\eta}^{\dot\alpha}$ are the fundamental building blocks for constructing spin-½ quantum fields. To construct a field theory, we need to be able to construct Lorentz invariant scalar combinations of ξ and $\bar{\eta}$ in order to construct the Lagrangian.

A basic property of the Lorentz invariant matrix M is that:

$$\varepsilon^{\alpha\beta} M_\alpha{}^\rho M_\beta{}^\sigma \equiv \varepsilon^{\rho\sigma} \det M$$
$$= \varepsilon^{\rho\sigma}$$

It then follows that under $\xi_\alpha \to M_\alpha{}^\beta \xi_\beta$

$$\chi_\alpha \to M_\alpha{}^\beta \chi_\beta$$

$$\chi\xi \equiv \chi^\alpha \xi_\alpha = \varepsilon^{\alpha\beta}\chi_\beta \xi_\alpha$$

is invariant under Lorentz transformations. Similarly,

$$\bar\chi\bar\xi \equiv \bar\chi_{\dot\alpha}\bar\xi^{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta}\bar\xi^{\dot\alpha}$$

is invariant.

Note carefully the placement of the indices.

<u>Notes</u>:

1. $\chi\xi = \xi\chi$, using $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$ and anti-commuting properties of the two-component fermion field

2. $\bar\chi\bar\xi = \bar\xi\bar\chi$

3. $(\chi\xi)^\dagger = (\chi^\alpha\xi_\alpha)^\dagger = \bar\xi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\xi\bar\chi = \bar\chi\bar\xi$

Hermitian conjugation reverses the order

Conclusion:

$$\chi\xi + \bar{\chi}\bar{\xi}$$

is Lorentz invariant and Hermitian. This is a candidate for a term in the Lagrangian.

We still need a candidate for a kinetic energy term.
Introduce:

$$\sigma^\mu = (I;\ \vec{\sigma})$$

$$\bar{\sigma}^\mu = (I,\ -\vec{\sigma})$$

Note that:

$$p_\mu \sigma^\mu = p_0 I - \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_0 + p_3 \end{pmatrix}$$

is a Hermian 2x2 matrix. So is $M p_\mu \sigma^\mu M^\dagger$. Thus, there exists a p'_μ such that:

$$\boxed{p'_\mu \sigma^\mu = M p_\mu \sigma^\mu M^\dagger}$$

Exercise: Using $\det(p_\mu\sigma^\mu) = p_0^2 - |\vec{p}|^2$ and $\det M = 1$, show that $p_0'^2 - |\vec{p}\,'| = p_0^2 - |\vec{p}|^2$ and conclude that $p_\mu \to p'_\mu$ under the Lorentz transformation M.

The spinor index structure of the boxed equation above is:

$$p'_\mu \sigma^\mu_{\alpha\dot{\alpha}} = M_\alpha{}^\beta (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} p_\mu \sigma^\mu_{\beta\dot{\beta}}$$

Thus, we have deduced the spinor index structure of σ^μ:

$$\sigma^\mu_{\alpha\dot\alpha}$$

which immediately allows one to construct another Lorentz invariant quantity:

$$i\chi^\alpha \sigma^\mu_{\alpha\dot\alpha}\, \partial_\mu \bar\chi^{\dot\alpha} \equiv i\chi\sigma^\mu\partial_\mu\bar\chi$$

The factor of i is inserted since $\frac{i}{2}\chi\sigma^\mu\overleftrightarrow{\partial}_\mu\bar\chi$ (which differs from $i\chi\sigma^\mu\partial_\mu\bar\chi$ by a total divergence) is hermitian and thus a candidate for a kinetic energy term in the Lagrangian.

<u>exercise</u>: Show that $(\chi\sigma^\mu\bar\xi)^\dagger = \xi\sigma^\mu\bar\chi$.

Similarly, the index structure of $\bar\sigma^\mu$ is:

$$\bar\sigma^{\mu\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}$$

<u>exercise</u>: Show that $\chi\sigma^\mu\bar\xi = -\bar\xi\bar\sigma^\mu\chi$.

That is, $\bar\sigma^\mu$ does not lead to an independent Lorentz invariant quantity.

<u>exercise</u>: Show that:

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2g^{\mu\nu}$$
$$\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = 2g^{\mu\nu}$$

Lorentz transformations in two component notation

$$\sigma^{\mu\nu}{}_\alpha{}^\beta = \frac{1}{4}\left(\sigma^\mu_{\alpha\dot\alpha}\,\bar\sigma^{\nu\dot\alpha\beta} - \sigma^\nu_{\alpha\dot\alpha}\,\bar\sigma^{\mu\dot\alpha\beta}\right)$$

$$\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta} = \frac{1}{4}\left(\bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu{}_{\alpha\dot\beta} - \bar\sigma^{\nu\dot\alpha\alpha}\sigma^\mu{}_{\alpha\dot\beta}\right)$$

Explicitly,

$$\sigma^{ij} = -i\varepsilon^{ijk}\frac{1}{2}\sigma^k = \bar\sigma^{ij}$$

$$\sigma^{i0} = -\sigma^{0i} = \frac{1}{2}\sigma^i = -\bar\sigma^{i0} = \bar\sigma^{0i}$$

note:

$\varepsilon_{\mu\nu\alpha\beta}\,\sigma^{\alpha\beta} = -2i\sigma_{\mu\nu}$

$\varepsilon_{\mu\nu\alpha\beta}\,\bar\sigma^{\alpha\beta} = 2i\bar\sigma_{\mu\nu}$

where $\varepsilon_{0123} = +1$.

Comparing with

$$\exp -\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu} \simeq I - i\vec\theta\cdot\frac{\vec\sigma}{2} - \vec\beta\cdot\frac{\vec\sigma}{2}$$

$\theta^i = \frac{1}{2}\varepsilon^{ijk}\theta_{jk}$

$\beta^i = \theta^{i0} = -\theta^{0i}$

we deduce that

$$\boxed{S^{\mu\nu} = i\sigma^{\mu\nu}}$$ for the $(\frac{1}{2},0)$ representation

Similarly,

$$\boxed{S^{\mu\nu} = i\bar\sigma^{\mu\nu}}$$ for the $(0,\frac{1}{2})$ representation.

Exercise: Show that:

$$\chi\sigma^{\mu\nu}\xi \equiv \chi^\alpha\sigma^{\mu\nu}{}_\alpha{}^\beta\xi_\beta = -\xi^\alpha\sigma^{\mu\nu}{}_\alpha{}^\beta\chi_\beta \equiv -\xi\sigma^{\mu\nu}\chi$$

$$\bar\chi\bar\sigma^{\mu\nu}\bar\xi \equiv \bar\chi_{\dot\alpha}\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta}\bar\xi^{\dot\beta} = -\bar\xi_{\dot\alpha}\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta}\bar\chi^{\dot\beta} \equiv -\bar\xi\bar\sigma^{\mu\nu}\bar\chi$$

$$(\chi\sigma^{\mu\nu}\xi)^\dagger = \bar\chi\bar\sigma^{\mu\nu}\bar\xi$$

Four-component notation

$$\psi = \begin{pmatrix} \xi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}$$

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_{\mu\alpha\dot{\beta}} \\ \bar{\sigma}_\mu^{\dot{\alpha}\beta} & 0 \end{pmatrix}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = 2i \begin{pmatrix} \sigma^{\mu\nu}{}_\alpha{}^\beta & 0 \\ 0 & \bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}$$

Note: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ $\qquad$ $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

Projection operators

$$P_L = \frac{1}{2}(1-\gamma_5) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_R = \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

$$\psi_L \equiv P_L\psi = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}$$

$$\psi_R \equiv P_R\psi = \begin{pmatrix} 0 \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}$$

Dirac adjoint

$$\psi^\dagger = (\bar{\xi}_{\dot\alpha} \;\; \eta^\alpha)$$

Introduce the matrix A

$$A = \begin{pmatrix} 0 & \delta^{\dot\alpha}{}_{\dot\beta} \\ \delta_\alpha{}^\beta & 0 \end{pmatrix} \qquad\qquad A\gamma^\mu A^{-1} = \gamma^{\mu\dagger}$$

and the Dirac adjoint

$$\overline{\Psi} = \psi^\dagger A = (\eta^\beta \;\; \bar{\xi}_{\dot\beta})$$

remark: numerically, $A = \gamma^0$, although each has a different spinor-index structure.

Charge conjugation matrix

$$C = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix} \qquad\qquad C^{-1}\gamma^\mu C = -\gamma^{\mu T}$$

The charge conjugated four-component spinor is:

$$\psi^c = C\overline{\Psi}^T = C(\psi^\dagger A)^T = \begin{pmatrix} \eta_\alpha \\ \bar{\xi}^{\dot\alpha} \end{pmatrix}$$

remark: numerically, $C = i\gamma^0\gamma^2$, although each has a different spinor index structure.

TRANSLATION TABLE

$$\psi_1 = \begin{pmatrix} \xi_1 \\ \overline{\eta_1} \end{pmatrix} \qquad \psi_2 = \begin{pmatrix} \xi_2 \\ \overline{\eta_2} \end{pmatrix}$$

$$\boxed{\begin{aligned} \overline{\Psi}_1 P_L \Psi_2 &= \eta_1 \xi_2 \\ \overline{\Psi}_1 P_R \Psi_2 &= \overline{\eta}_2 \overline{\xi}_1 \end{aligned}}$$

$$\overline{\Psi^c_1} P_L \Psi_2 = \xi_1 \xi_2$$

$$\overline{\Psi}_1 P_R \Psi_2^c = \overline{\xi}_1 \overline{\xi}_2$$

$$\boxed{\overline{\Psi}_1 \gamma^\mu P_L \Psi_2 = \overline{\xi}_1 \overline{\sigma}^\mu \xi_2}$$

$$\overline{\Psi^c_1} \gamma^\mu P_R \Psi_2^c = -\overline{\xi}_2 \overline{\sigma}^\mu \xi_1$$

$$\boxed{\begin{aligned} \tfrac{i}{2} \overline{\Psi}_1 \sigma^{\mu\nu} P_L \Psi_2 &= \eta_1 \sigma^{\mu\nu} \xi_2 \\ \tfrac{i}{2} \overline{\Psi}_1 \sigma^{\mu\nu} P_R \Psi_2 &= \overline{\xi}_1 \overline{\sigma}^{\mu\nu} \overline{\eta}_2 \end{aligned}}$$

$$\overline{\Psi^c_1} P_L \Psi_2^c = \xi_1 \eta_2$$

$$\overline{\Psi^c_1} P_R \Psi_2^c = \overline{\xi}_2 \overline{\eta}_2$$

$$\overline{\Psi}_1 P_L \Psi_2^c = \eta_1 \eta_2$$

$$\overline{\Psi^c_1} P_R \Psi_2 = \overline{\eta}_1 \overline{\eta}_2$$

$$\overline{\Psi^c_1} \gamma^\mu P_L \Psi_2^c = \overline{\eta}_1 \overline{\sigma}^\mu \eta_2$$

$$\boxed{\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 = -\overline{\eta}_2 \overline{\sigma}^\mu \eta_1}$$

$$-\tfrac{i}{2} \Psi_1^c \sigma^{\mu\nu} P_L \Psi_2^c = \xi_1 \sigma^{\mu\nu} \eta_2$$

$$-\tfrac{i}{2} \Psi_1^c \sigma^{\mu\nu} P_R \Psi_2^c = \overline{\eta}_1 \overline{\sigma}^{\mu\nu} \overline{\xi}_2$$

It follows that:

$$\overline{\Psi}_1 \Psi_2 = \eta_1 \xi_2 + \overline{\eta}_2 \overline{\xi}_1$$

$$\overline{\Psi}_1 \gamma_5 \Psi_2 = -\eta_1 \xi_2 + \overline{\eta}_2 \overline{\xi}_1$$

$$\overline{\Psi}_1 \gamma^\mu \Psi_2 = \overline{\xi}_1 \overline{\sigma}^\mu \xi_2 - \overline{\eta}_2 \overline{\sigma}^\mu \eta_1$$

$$\overline{\Psi}_1 \gamma^\mu \gamma_5 \Psi_2 = -\overline{\xi}_1 \overline{\sigma}^\mu \xi_2 - \overline{\eta}_2 \overline{\sigma}^\mu \eta_1$$

$$-\tfrac{i}{2} \overline{\Psi}_1 \sigma^{\mu\nu} \Psi_2 = \eta_1 \sigma^{\mu\nu} \xi_2 + \overline{\xi}_1 \overline{\sigma}^{\mu\nu} \overline{\eta}_2$$

Note:
If $\Psi_1 = \Psi_2 = \Psi_M$
then

$$\overline{\Psi}_M \gamma^\mu \Psi_M = 0$$

$$\overline{\Psi}_M \sigma^{\mu\nu} \Psi_M = 0$$

Four-component Majorana spinor

Set $\eta = \xi$. Then,

$$\Psi_M = \begin{pmatrix} \xi_\alpha \\ \bar{\xi}^{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \xi \\ i\sigma^2 \xi^* \end{pmatrix}$$

One can check that $\Psi_M^c = \Psi_M$

Majorana field theory

$$\mathcal{L} = i\bar{\Psi}\bar{\sigma}^\mu \partial_\mu \Psi - \tfrac{1}{2} m (\Psi\Psi + \bar{\Psi}\bar{\Psi})$$

$$= \tfrac{i}{2}\bar{\Psi}\bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} \Psi - \tfrac{1}{2} m (\Psi\Psi + \bar{\Psi}\bar{\Psi}) + \text{total divergence}$$

where Ψ is a two-component fermion.

note: $\bar{\Psi}\overleftrightarrow{\partial_\mu}\Psi \equiv \bar{\Psi}(\partial_\mu\Psi) - (\partial_\mu\bar{\Psi})\Psi$

Translating to four-component notation: $\Psi_M \equiv \begin{pmatrix} \Psi \\ \bar{\Psi} \end{pmatrix}$

$$\bar{\Psi}_M \gamma^\mu \partial_\mu \Psi_M = \bar{\Psi}\bar{\sigma}^\mu \partial_\mu \Psi - (\partial_\mu \bar{\Psi})\bar{\sigma}^\mu \Psi$$

$$= \bar{\Psi}\bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} \Psi$$

so that:

$$\mathcal{L} = \tfrac{i}{2}\bar{\Psi}_M \gamma^\mu \partial_\mu \Psi_M - \tfrac{1}{2} m \bar{\Psi}_M \Psi_M$$

A Dirac fermion is equivalent to two mass-degenerate Majorana fermions.

Start with:

$$\mathcal{L} = i\left(\overline{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 + \overline{\psi}_2 \bar{\sigma}^\mu \partial_\mu \psi_2\right) - \tfrac{1}{2} m_{ij}\, \psi_i \psi_j - \tfrac{1}{2} m_{ij}^* \,\overline{\psi}_i \overline{\psi}_j$$

where $m_{ij} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$

and diagonalize m_{ij}. The corresponding eigenvalues are $\pm m$. Let:

$$\psi_a = \frac{\psi_1 + \psi_2}{\sqrt{2}} \qquad\qquad \psi_1 = \frac{\psi_a + i\psi_b}{\sqrt{2}}$$

or

$$i\psi_b = \frac{\psi_1 - \psi_2}{\sqrt{2}} \qquad\qquad \psi_2 = \frac{\psi_a - i\psi_b}{\sqrt{2}}$$

the factor of i is inserted here so that both two-component fermions have positive mass.

Then,

$$\mathcal{L} = i\left(\overline{\psi}_a \bar{\sigma}^\mu \partial_\mu \psi_a + \overline{\psi}_b \bar{\sigma}^\mu \partial_\mu \psi_b\right) - \tfrac{1}{2} m\left(\psi_a \psi_a + \overline{\psi}_a \overline{\psi}_a + \psi_b \psi_b + \overline{\psi}_b \overline{\psi}_b\right)$$

corresponding to two mass-degenerate two-component spinors.

Dirac field theory

$$\mathcal{L} = i(\bar{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 + \bar{\psi}_2 \bar{\sigma}^\mu \partial_\mu \psi_2) - \frac{1}{2} m_{ij} \psi_i \psi_j - \frac{1}{2} m_{ij}^* \bar{\psi}_i \bar{\psi}_j$$

where $m_{ij} \equiv \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$

and ψ_1, ψ_2 are both two-component spinors.

The Dirac spinor is

$$\psi_D = \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix}$$

Note that:

$$\begin{aligned} \bar{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 + \bar{\psi}_2 \bar{\sigma}^\mu \partial_\mu \psi_2 &= \bar{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 - (\partial_\mu \bar{\psi}_2) \bar{\sigma}^\mu \psi_2 \\ &\quad + \text{total divergence} \\ &= \bar{\psi}_D \gamma^\mu \partial_\mu \psi_D + \text{total divergence} \end{aligned}$$

and

$$\psi_1 \psi_2 + \bar{\psi}_1 \bar{\psi}_2 = \bar{\psi}_D \psi_D$$

Thus,

$$\mathcal{L}_D = i \bar{\psi}_D \gamma^\mu \partial_\mu \psi_D - m \bar{\psi}_D \psi_D$$

The see-saw mechanism

$$\mathcal{L} = i(\bar{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 + \bar{\psi}_2 \bar{\sigma}^\mu \partial_\mu \psi_2) - \frac{1}{2} m_{ij} \psi_i \psi_j - \frac{1}{2} m_{ij}^* \bar{\psi}_i \bar{\psi}_j$$

where $m_{ij} = \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix}$ and $m_D \ll M$.

Eigenvalues of m_{ij}: $\frac{1}{2} M \left[1 \pm \sqrt{1 + \frac{4 m_D^2}{M^2}} \right]$

i.e. for $m_D \ll M$, the two mass eigenvalues are M, $-\frac{m_D^2}{M}$.

Eigenstates:

$$i\psi_a \simeq \psi_1 - \frac{m_D}{M} \psi_2$$

$$\psi_b \simeq \psi_2 + \frac{m_D}{M} \psi_1$$

Then,

$$\frac{1}{2} m_D (\psi_1 \psi_2 + \psi_2 \psi_1) + \frac{1}{2} M \psi_2 \psi_2 + h.c.$$

$$\simeq \frac{1}{2} \left[\frac{m_D^2}{M} \psi_a \psi_a + M \psi_b \psi_b + h.c. \right] + O\left(\frac{m_D^3}{M^2} \right)$$

which corresponds to a theory of two Majorana fermions, one very light and one very heavy (the seesaw).

III. Supersymmetry — extending Poincaré symmetry

Initially, attempts were made to extend Poincaré symmetry by combining space-time and internal symmetries in a non-trivial way. Coleman and Mandula proved that these attempts would always fail. They assumed all symmetry generators satisfy commutation relations.

The breakthrough was achieved by considering a new class of "fermionic" generators that satisfy anti-commutation relations. Haag, Lopuszanski and Sohnius discovered which fermionic symmetry generators were allowed and proved they must transform either as $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$, i.e. spin-½ generators.

The above result implies that:

$$[Q_\alpha, J^{\mu\nu}] = i\sigma^{\mu\nu}{}_\alpha{}^\beta Q_\beta$$

where Q_α is a $(\frac{1}{2},0)$ symmetry operator. In addition,

$$[Q_\alpha, P^\mu] = 0$$

since Q_α is translationally invariant (no explicit x-dependence).

Similarly, $\overline{Q}^{\dot\alpha}$ is a $(0,\frac{1}{2})$ symmetry operator, so

$$[\overline{Q}^{\dot\alpha}, J^{\mu\nu}] = i\overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}\overline{Q}^{\dot\beta}$$

$$[\overline{Q}^{\dot\alpha}, P^\mu] = 0$$

The Q's satisfy anti-commutation relations. Given the Lorentz properties of the Q's, there is little freedom left. For example, $\{Q_\alpha, \overline{Q}_{\dot\beta}\}$ transforms as $(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$ which is a four-vector.

$$\boxed{\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu}$$

where the 2 is conventional (and can be achieved by re-scaling the Q's). The other anti-commutators are:

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

Exercise: In 4-component notation, define $Q_M = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix}$. Show that:

$$[Q_M, J^{\mu\nu}] = \tfrac{1}{2}\sigma^{\mu\nu} Q_M$$
$$\{Q_M, \bar{Q}_M\} = 2\gamma^\mu P_\mu \qquad \text{where } \bar{Q}_M = Q_M^\dagger A.$$

Theorem:

The vanishing of the vacuum energy is a necessary and sufficient condition for the existence of a unique supersymmetric vacuum.

proof: multiply $\{Q_\alpha, \bar{Q}_{\dot\beta}\}$ by $\bar\sigma^{\nu\dot\beta\alpha}$ and use $\mathrm{tr}\,\sigma^\mu\bar\sigma^\nu = 2g^{\mu\nu}$ to obtain:

$$\bar\sigma^{\mu\dot\beta\alpha}\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 4P^\mu$$

For $\mu = 0$, this reads:

$$4P^0 = Q_1Q_1^* + Q_1^*Q_1 + Q_2Q_2^* + Q_2^*Q_2$$

Thus, if $|0\rangle$ is the vacuum state, then

$$\langle 0|P^0|0\rangle \geq 0$$

and

$$\langle 0|P^0|0\rangle = 0 \iff Q_\alpha|0\rangle = 0.$$

if this condition is satisfied then the vacuum is supersymmetric

Recall the Casimir operators of the Poincaré algebra, P^2 and W^2.
Note that:

$$[P^2, Q_\alpha] = [P^2, \overline{Q}_{\dot{\alpha}}] = 0$$

but,

$$[W^2, Q_\alpha] \neq 0$$
$$[W^2, \overline{Q}_{\dot{\alpha}}] \neq 0$$

Thus, the irreducible representations of the supersymmetry algebra will contain different spins. If $|B\rangle$ is a boson and $|F\rangle$ is a fermion, then,

$$Q_\alpha |B\rangle = |F\rangle \qquad \text{(schematic)}$$
$$Q_\alpha |F\rangle = |B\rangle$$

where $|F\rangle$ and $|B\rangle$ differ by half a unit of spin, since Q_α is a $(\frac{1}{2}, 0)$ symmetry operator.

<u>Definition:</u> $(-1)^F$ is an operator defined such that

$$(-1)^F |B\rangle = +|B\rangle$$
$$(-1)^F |F\rangle = -|F\rangle$$

<u>Theorem:</u>

(i) $Q_\alpha (-1)^F = -(-1)^F Q_\alpha$

(ii) $\operatorname{tr} (-1)^F = 0$ (for fixed non-zero P_μ)

<u>consequence of (ii)</u>: supersymmetric multiplets contain equal numbers of bosonic and fermionic degrees of freedom

proof of (ii): evaluate $\operatorname{tr}[(-1)^F \{Q_\alpha, \overline{Q}_{\dot{\beta}}\}]$ in two ways.

First way - use the anticommutation relation for the Q's.

Second way - expand the anticommutator and manipulate the expression using (i) until you get zero.

R-invariance

The supersymmetry algebra can be extended slightly by noting that Q_α is inherently complex. The supersymmetric algebra is unchanged under:

$$Q_\alpha \rightarrow e^{-i\gamma} Q_\alpha$$
$$\overline{Q}_{\dot\alpha} \rightarrow e^{i\gamma} \overline{Q}_{\dot\alpha}$$

where γ is a real number. Thus, introduce a new symmetry generator R such that:

$$e^{i\gamma R} Q_\alpha e^{-i\gamma R} = e^{-i\gamma} Q_\alpha$$
$$e^{i\gamma R} \overline{Q}_{\dot\alpha} e^{-i\gamma R} = e^{i\gamma} \overline{Q}_{\dot\alpha}$$

Taking γ infinitesimal,

$$[R, Q_\alpha] = -Q_\alpha$$
$$[R, \overline{Q}_{\dot\alpha}] = \overline{Q}_{\dot\alpha}$$

We may choose $[R, P^\mu] = [R, J^{\mu\nu}] = 0$ to extend the supersymmetry algebra.

Casimir operators of the supersymmetric algebra

P^2 remains a Casimir operator since it commutes with all the generators. This means that <u>all the states that make up a supersymmetric multiplet have the same mass</u>.

W^2 is no longer a Casimir operator. We need a supersymmetric generalization of the Pauli-Lubanski vector W_μ.

definition:

$$B_\mu \equiv W_\mu - \frac{1}{8}\sigma_\mu^{\dot{\alpha}\beta}[Q_\beta, \overline{Q}_{\dot{\alpha}}]$$

The factor of $\frac{1}{8}$ was chosen so that

$$[B_\mu, Q_\alpha] = -\frac{1}{2}P_\mu Q_\alpha$$
$$[B_\mu, \overline{Q}_{\dot{\alpha}}] = \frac{1}{2}P_\mu \overline{Q}_{\dot{\alpha}}$$

note:

$$[B_\mu, B_\nu] = -i\varepsilon_{\mu\nu\rho\sigma}B^\rho P^\sigma$$
$$[B_\mu, P_\nu] = 0$$

Next, we define:

$$C_{\mu\nu} \equiv B_\mu P_\nu - B_\nu P_\mu$$

It is easy to check that

$$[C_{\mu\nu}, Q_\alpha] = [C_{\mu\nu}, \overline{Q}_{\dot{\alpha}}] = [C_{\mu\nu}, P_\sigma] = 0.$$

Since $C_{\mu\nu}C^{\mu\nu}$ is Lorentz invariant, it follows that

$$[C_{\mu\nu}C^{\mu\nu}, J^{\alpha\beta}] = 0.$$

Hence, $C_{\mu\nu}C^{\mu\nu}$ is the second Casimir operator of the supersymmetry algebra.

We can write:

$$C_{\mu\nu}C^{\mu\nu} = 2[B^\mu B_\mu P^2 - (B^\mu P_\mu)^2]$$

Case 1: $P^2 > 0$.

In a frame where $P^\mu = (m; \vec{0})$,

$$C^{\mu\nu} C_{\mu\nu} = 2m^2 (B^\mu B_\mu - B_0^2) = -2m^2 |\vec{B}|^2$$

Moreover, in this frame:

$$[B^i, B^j] = i m \varepsilon^{ijk} B^k$$

Define $m J^k = B^k$. Then the J^k satisfy angular momentum commutation relations and:

$$C^{\mu\nu} C_{\mu\nu} = -2m^4 \vec{J}^2$$

has eigenvalues $-2m^4 j(j+1)$, where $j = 0, \frac{1}{2}, 1, \dots$ is the "superspin". The irreducible representations are thus $|m, j\rangle$.

Using $\vec{w} = m \vec{S}$,

$$J^k = S^k - \frac{1}{8m} \sigma^{k\dot{\alpha}\beta} [Q_\beta, \bar{Q}_{\dot{\alpha}}]$$

In the rest frame, $\{Q_1, \bar{Q}_{\dot{1}}\} = \{Q_2, \bar{Q}_{\dot{2}}\} = 2m$ and all other anti-commutators vanish. Thus, the state $|\Omega\rangle \equiv Q_1 Q_2 |m, j, j_3\rangle$ satisfies the condition $Q_\alpha |\Omega\rangle = 0$.

The only non-vanishing states then are:

$$|\Omega\rangle, \quad \bar{Q}^{\dot{1}} |\Omega\rangle, \quad \bar{Q}^{\dot{2}} |\Omega\rangle \quad \text{and} \quad \bar{Q}^{\dot{1}} \bar{Q}^{\dot{2}} |\Omega\rangle$$

We can compute the spin of these states by working out the eigenvalues with respect to S^3 and $\vec{S}^2$.

$$|m, j\rangle \; \begin{cases} j_3 = j \\ j_3 = j-1 \\ \vdots \\ j_3 = -j \end{cases} \xrightarrow{\text{fixed } j_3} \begin{array}{ll} S_3 = j_3 & |\Omega\rangle \\ S_3 = j_3 + \frac{1}{2} & \bar{Q}^{\dot{1}} |\Omega\rangle \\ S_3 = j_3 - \frac{1}{2} & \bar{Q}^{\dot{2}} |\Omega\rangle \\ S_3 = j_3 & \bar{Q}^{\dot{1}} \bar{Q}^{\dot{2}} |\Omega\rangle \end{array}$$

example: $j=0$

possible values of s_3:

$$s_3 = 0,\ +\tfrac{1}{2},\ -\tfrac{1}{2},\ 0$$

corresponds to two real scalars* and one Majorana fermion.

scalar degrees of freedom = 2 [*equivalently, one complex scalar]

fermion degrees of freedom = 2

exercise: show that $j=\frac{1}{2}$ corresponds to a real vector field, a real scalar field and two Majorana fermions (or equivalently one Dirac fermion).

Case 2: $P^2=0$

One can show that for $P^2=0$,

$$L_\mu \equiv W_\mu - \frac{1}{16}\sigma_\mu^{\dot\alpha\beta}[Q_\beta, \bar{Q}_{\dot\alpha}]$$

satisfies:
$$P^\mu L_\mu = 0$$
$$[L_\mu, L_\nu] = -i\varepsilon_{\mu\nu\alpha\beta}L^\alpha P^\beta$$

(the same relations satisfied by W_μ). Thus, for $P^2=0$, L_μ is proportional to P_μ.

$$L_\mu = (K+\tfrac{1}{4})P_\mu \qquad K = \text{super-helicity}$$

If $Q_{\dot\alpha}|\Omega\rangle$ as before, a simple computation yields:

$$W_\mu|\Omega\rangle = (K+\tfrac{1}{2})P_\mu|\Omega\rangle$$

However, for $P^2=0$, only two states survive. The massless supermultiplet consists of particles of helicity K and $K+\frac{1}{2}$. We must add the corresponding anti-particles (CPT-conjugates), which yields states of helicity $-K$ and $-(K+\frac{1}{2})$.

Examples:

(i) $K=0$

helicities $0, \frac{1}{2}$ $\oplus$ helicities $0, -\frac{1}{2}$

corresponding to a massless complex scalar and a massless Majorana fermion

(ii) $K=\frac{1}{2}$

helicities $\frac{1}{2}, 1$ $\oplus$ helicities $-\frac{1}{2}, -1$

corresponding to a massless Majorana fermion and a massless real vector field.

Further generalizations

Extended supersymmetry introduces N spin-½ $(\frac{1}{2},0)$ generators, Q^A_α $(A=1,\ldots,N)$ and N $(0,\frac{1}{2})$ generators $\overline{Q}_{\dot\alpha A}$. The algebra involving the Q's is more complicated.

In these lectures, we will not make this additional generalization. The theories discussed here are based on $N=1$ supersymmetry.

The main reason for this choice is due to the fact that $N=1$ theories can describe chiral fermions (as seen in nature). For $N>1$, all left-handed fermions have right-handed partners, so the construction of realistic theories of this type are much more difficult.