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PHENOMENOLOGY OF SUPERSYMMETRY

Lecture II

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Please note: These are preliminary notes intended for internal distribution only.

V. Supersymmetric theories of spin-½ fermions and their spin-0 boson superpartners

We have seen that the simplest supermultiplet consists of two real scalars (or equivalently, a complex scalar) and a Majorana fermion, all with mass m.

(corresponding to superspin $j=0$ if massive, or to superhelicity 0 if massless).

A Lagrangian that respects the supersymmetry algebra is given by:

$$\mathcal{L} = (\partial_\mu A)^*(\partial^\mu A) + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - \left|\frac{dW}{dA}\right|^2 - \frac{1}{2}\left[\frac{d^2W}{dA^2}\psi\psi + \left(\frac{d^2W}{dA^2}\right)^*\bar{\psi}\bar{\psi}\right]$$

where $W = W(A)$ is an arbitrary holomorphic function of A (i.e. W is an analytic function that depends on A but not A^*)

A = complex scalar
ψ = two-component Majorana fermion

If $W(A)$ is a cubic polynomial in A, then this is a renormalizable QFT called the Wess-Zumino model.

Simple example: $W = \frac{1}{2}mA^2$

result: a free theory of a complex scalar and a Majorana fermion, both of mass m.

a renormalizable example with interactions

$$W = \tfrac{1}{2} m A^2 + \tfrac{1}{3} g A^3 \qquad m, g \text{ real}$$

$$\mathcal{L} = (\partial_\mu A)^*(\partial^\mu A) + i\overline{\psi}\overline{\sigma}^\mu \partial_\mu \psi - \tfrac{1}{2} m(\psi\psi + \overline{\psi}\overline{\psi}) - m^2 A^* A$$
$$- g(A\psi\psi + A^*\overline{\psi}\overline{\psi}) - mg A^*A(A+A^*) - g^2(A^*A)^2$$

note that:

(i) $m_A = m_\psi$

(ii) $g_{A\psi\psi} = g$ while $g_{(A^*A)^2} = g^2$

Both (i) and (ii) are consequences of supersymmetry. Thus, supersymmetry relates different couplings of the theory.

Convert to four-component notation. Define $A = \frac{1}{\sqrt{2}}(S + iP)$. Then,

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu S)^2 + \tfrac{1}{2}(\partial_\mu P)^2 - \tfrac{1}{2} m^2 (S^2 + P^2)$$
$$+ \tfrac{i}{2}\overline{\Psi}_M \gamma^\mu \partial_\mu \Psi_M - \tfrac{1}{2} m \overline{\Psi}_M \Psi_M$$
$$- \frac{g}{\sqrt{2}}\left[S\overline{\Psi}_M \Psi_M - iP\overline{\Psi}_M \gamma_5 \Psi_M\right]$$
$$- \frac{mg}{\sqrt{2}} S(S^2 + P^2) - \tfrac{1}{4} g^2 (S^2 + P^2)^2.$$

Note that S is a scalar and P is a pseudoscalar.

[Had we chosen g complex, we would have generated CP-violating scalar-fermion interactions.]

The supersymmetric transformations are:

$$\delta_\xi A = \sqrt{2}\,\xi^\alpha \psi_\alpha$$
$$\delta_\xi \psi_\alpha = -i\sqrt{2}\,\sigma^\mu_{\alpha\dot\beta}\,\bar\xi^{\dot\beta}\,\partial_\mu A - \sqrt{2}\,\xi_\alpha \left(\frac{dW}{dA}\right)^*$$

where ξ is a constant (i.e. x-independent) anti-commuting infinitesimal parameter.

Note: by hermitian conjugation, one also obtains:

$$\delta_\xi A^* = \sqrt{2}\,\bar\xi_{\dot\alpha}\,\bar\psi^{\dot\alpha}$$
$$\delta_\xi \bar\psi_{\dot\alpha} = i\sqrt{2}\,\xi^\beta \sigma^\mu_{\beta\dot\alpha}\,\partial_\mu A^* - \sqrt{2}\,\bar\xi_{\dot\alpha}\,\frac{dW}{dA}$$

One can check that

$$\delta_\xi \mathcal{L} = \partial_\mu K^\mu$$ (exercise: derive K^μ explicitly)

which means that the action, $\int d^4x \mathcal{L}$, is invariant, if the field equations are satisfied.

Indeed, this is a symmetry. But is it supersymmetry?

First, consider ordinary space-time translations:

$$e^{ia_\mu P^\mu}\phi(x)\,e^{-ia_\mu P^\mu} = \phi(x+a)$$

for infinitesimal a,

$$i[P^\mu, \phi(x)] = \partial^\mu \phi(x)$$

Here, $\phi(x)$ is any generic field, either A or ψ.

Thus, for an infinitesimal space-time translation,

$$\delta_a \phi(x) \equiv \phi(x+a) - \phi(x)$$
$$\simeq a^\mu \partial_\mu \phi(x)$$
$$= i a^\mu [P_\mu, \phi(x)]$$

Likewise, if $\delta_\xi \phi(x)$ is a supersymmetric transformation, we expect:

$$\delta_\xi \phi(x) = i[\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha} \bar{Q}^{\dot\alpha}, \phi(x)]$$

Consider:

$$(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi(x) = \left[i(\eta Q + \bar\eta \bar Q), [i(\xi Q + \bar\xi \bar Q), \phi(x)] \right] - (\xi \leftrightarrow \eta)$$

$$= \left[[i(\eta Q + \bar\eta \bar Q), i(\xi Q + \bar\xi \bar Q)], \phi(x) \right]$$

using the Jacobi identity.

Finally, use the anti-commutation relations of the Q's. Since ξ and η are anti-commuting numbers, we have e.g.,

$$[\eta Q, \bar\xi \bar Q] = 2\eta^\alpha \sigma^\mu_{\alpha\dot\beta} \bar\xi^{\dot\beta} P_\mu$$

Thus, one ends up with:

$$[\delta_\eta, \delta_\xi]\phi(x) = 2(\xi \sigma^\mu \bar\eta - \eta \sigma^\mu \bar\xi)[P_\mu, \phi(x)]$$
$$= -2i(\xi \sigma^\mu \bar\eta - \eta \sigma^\mu \bar\xi)\partial_\mu \phi(x)$$

Let us test this result, using

$$\delta_\xi A = \sqrt{2}\,\xi^\alpha \psi_\alpha$$
$$\delta_\xi \psi_\alpha = -i\sqrt{2}\,\sigma^\mu_{\alpha\dot\beta}\,\bar\xi^{\dot\beta}\,\partial_\mu A - \sqrt{2}\,\xi_\alpha \left(\frac{dW}{dA}\right)^*$$

The result:

i) $[\delta_\eta \delta_\xi - \delta_\xi \delta_\eta]\,A(x) = -2i(\xi\sigma^\mu\bar\eta - \eta\sigma^\mu\bar\xi)\,\partial_\mu A$

ii) $[\delta_\eta \delta_\xi - \delta_\xi \delta_\eta]\,\psi_\alpha(x) = -2i(\xi\sigma^\mu\bar\eta - \eta\sigma^\mu\bar\xi)\,\partial_\mu \psi + R$

where $R=0$ if I impose the field equations satisfied by ψ.

We say that the supersymmetry algebra is <u>realized</u> <u>on-shell</u>, i.e. after imposition of the field equations.

<u>recall</u>: the action is invariant under δ_ξ only when the field equations are imposed.

The derivation of (ii) is non-trivial, and requires among other things the use of Fierz identities. These are a little simpler for two-component fermions as compared to four-component fermions. All such identities are based on:

$$\delta_\alpha{}^\beta\,\delta^{\dot\rho}{}_{\dot\gamma} = \tfrac{1}{2}\,\sigma^\mu_{\alpha\dot\gamma}\,\bar\sigma_\mu{}^{\dot\rho\beta}$$
$$\delta_\alpha{}^\beta\,\delta_\rho{}^\gamma = \tfrac{1}{2}\left[\delta_\alpha{}^\gamma\,\delta_\rho{}^\beta - \sigma^{\mu\nu}{}_\alpha{}^\gamma\,\sigma_{\mu\nu\rho}{}^\beta\right]$$
$$\delta^{\dot\beta}{}_{\dot\alpha}\,\delta^{\dot\rho}{}_{\dot\gamma} = \tfrac{1}{2}\left[\delta^{\dot\beta}{}_{\dot\gamma}\,\delta^{\dot\rho}{}_{\dot\alpha} - \bar\sigma^{\mu\nu\dot\beta}{}_{\dot\gamma}\,\bar\sigma_{\mu\nu}{}^{\dot\rho}{}_{\dot\alpha}\right]$$

which follow from the completeness of $\{I, \vec\sigma\}$ over the set of 2×2 matrices.

An alternative approach: Noether's theorem

By Noether's theorem, an invariance of the action implies the existence of a conserved current.

Given δ_ξ as defined above, we found that $\delta_\xi \mathcal{L} = \partial_\mu K^\mu$ for some combination of fields K^μ. Then, the Noether supercurrent corresponding to this invariance is:

$$\xi^\alpha J^\mu_\alpha + \bar{\xi}_{\dot\alpha} \bar{J}^{\mu\dot\alpha} = \sum_X \delta_\xi X \frac{\delta \mathcal{L}}{\delta(\partial_\mu X)} - K^\mu$$

where we sum over $X = A, \psi$. [exercise: evaluate J^μ_α explicitly in the Wess-Zumino model.]

Note that J^μ_α has both a vector and spinor index.

Noether's theorem states that, after imposing the field equations,

$$\partial_\mu J^\mu_\alpha = \partial_\mu \bar{J}^{\mu\dot\alpha} = 0$$

i.e. the supercurrent is conserved. The supercharges are defined in the usual way:

$$Q_\alpha = \int d^3x\, J^0_\alpha \quad , \qquad \bar{Q}^{\dot\alpha} = \int d^3x\, \bar{J}^{0\dot\alpha}$$

<u>exercise:</u> using the canonical commutation relations satisfied by the bosonic field A, and the canonical anti-commutation relations satisfied by ψ, show that:

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

where P^μ is expressed in terms of the quantum fields.*

*P^μ is the Noether charge of space-time translations

Auxiliary fields

The supersymmetric transformations shown are not optimal.
Note that:

(i) the transformations are non-linear if W is not quadratic in A

(ii) the supersymmetry algebra is realized only on-shell.

We avoid both (i) and (ii) by introducing an auxiliary complex scalar field F.

Consider the alternative Lagrangian:

$$\mathcal{L} = (\partial_\mu A)^*(\partial^\mu A) + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^*F$$
$$-F\frac{dW}{dA} - F^*\left(\frac{dW}{dA}\right)^* - \frac{1}{2}\left[\frac{d^2W}{dA^2}\psi\psi + \left(\frac{d^2W}{dA^2}\right)^*\bar{\psi}\bar{\psi}\right]$$

F is an auxiliary field since $\mathcal{L}$ depends on F but not $\partial_\mu F$.
So, F has trivial dynamics. The field equations for F and F^* are:

$$\frac{\partial\mathcal{L}}{\partial F} = 0 \quad , \quad \frac{\partial\mathcal{L}}{\partial F^*} = 0$$

That is,

$$F^* = \frac{dW}{dA} \quad , \quad F = \left(\frac{dW}{dA}\right)^*$$

Inserting these equations back into $\mathcal{L}$ returns us to the original Lagrangian. The theories are identical.

But, consider $\mathcal{L}$ before solving for F and F^*, and examine the following supersymmetric transformation:

$$\delta_\xi A = \sqrt{2}\,\xi^\alpha \psi_\alpha$$
$$\delta_\xi \psi_\alpha = -i\sqrt{2}\,\sigma^\mu_{\alpha\dot\beta}\,\bar\xi^{\dot\beta}\,\partial_\mu A - \sqrt{2}\,\xi_\alpha F$$
$$\delta_\xi F = -i\sqrt{2}\,\partial_\mu \psi^\alpha \sigma^\mu_{\alpha\dot\beta}\,\bar\xi^{\dot\beta}$$

One can now check the following results:

(i) $\delta_\xi \mathcal{L} = \partial_\mu \tilde{K}^\mu$ [exercise: find the explicit expression for $\tilde{K}^\mu$]

without imposing the field equations.

(ii) $[\delta_\eta \delta_\xi - \delta_\xi \delta_\eta]\, X(x) = -2i(\xi\sigma^\mu\bar\eta - \eta\sigma^\mu\bar\xi)\,\partial_\mu X$

for $X = A, \psi$, and F without imposing the field equations.

We say that the supersymmetry algebra is <u>realized off-shell</u>

Remarks:

1. Note the mass dimensions of the fields:
   $[\phi] = 1$, $[\psi] = \frac{3}{2}$, $[F] = 2$, which implies that $[\xi] = -\frac{1}{2}$.

2. Since $\delta_\xi F$ is a total divergence, $\int d^4x\, F$ is invariant under the supersymmetric transformation. This will be useful later.

note: $\delta_\xi F$ is a total divergence is a consequence of dimensional analysis. Since the supersymmetric transformation law is linear in the fields, $\delta_\xi F$ must involve ∂_μ since $[\partial_\mu] = 1$.

Counting degrees of freedom

On shell counting:

complex scalar A	2
Majorana spinor ψ	2

Before imposing the field equations, ψ_α $(\alpha = 1,2)$ has four degrees of freedom, since ψ is complex. [Equivalently, count ψ_α and $\bar{\psi}_{\dot\alpha}$ as four independent degrees of freedom].

The field equations are:

$$i\bar{\sigma}^\mu \partial_\mu \psi = \left(\frac{d^2W}{dA^2}\right)^* \bar{\psi}$$

and these relate ψ and $\bar{\psi}$, eliminating two of four degrees of freedom. [If $\frac{d^2W}{dA^2} = 0$, then $i\bar{\sigma}^\mu \partial_\mu \psi = 0$ is a relation between ψ_1 and ψ_2.]

Note: both A and ψ satisfy Klein-Gordon type field equations as well, but these do not affect the counting.

Off shell counting:

complex scalar A	2
Majorana spinor $\psi_\alpha, \bar{\psi}_{\dot\alpha}$	4
complex auxiliary field F	2

totals:

boson degrees of freedom $= 4$

fermion degrees of freedom $= 4$

In both cases, supersymmetry guarantees the equality of the number of boson and fermion degrees of freedom.

essons from the Wess-Zumino model

1. It is not clear how to build supersymmetric Lagrangians starting with a known supermultiplet of particles.

2. The supersymmetric transformation laws are not immediately obvious, even if a supersymmetric Lagrangian is given.

3. Checking that the supersymmetric transformations satisfy the supersymmetric algebra is quite laborious.

4. Checking that the action is invariant under the supersymmetric transformation is also tedious.

5. Off-shell supersymmetry increases the required number of fields, but leads to supersymmetry transformation laws that are linear in the fields.

6. $\int F(x)\, d^4x$ is invariant under the supersymmetry transformation. [a hint for (1) above?]

<u>ur goal</u>: develop a formalism in which step (1) is trivial and steps (2)–(4) are automatic.

Superfields and superspace

Recall that our quantum fields satisfy:

$$\phi(x) = e^{ix\cdot P}\phi(0)\,e^{-ix\cdot P}$$

where $\quad i[P^\mu, \phi(x)] = \partial^\mu\phi$

and $\quad \delta_a\phi(x) \equiv \phi(x+a) - \phi(x) = a^\mu\partial_\mu\phi(x) = ia^\mu[P_\mu, \phi(x)]$.

We then defined the supersymmetric transformation law in analogy with δ_a

$$\delta_\xi\phi(x) = i[\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha}, \phi(x)].$$

Here, $\phi(x)$ can be any field: A, ψ, F, etc.

It looks like Q_α and $\bar{Q}^{\dot\alpha}$ are generating a translation by a non-commuting infinitesimal ξ_α and $\bar{\xi}^{\dot\alpha}$. But what is being translated?

Let us extend space-time by introducing non-commuting co-ordinates:

$$\theta^\alpha, \ \bar{\theta}_{\dot\alpha} \qquad\qquad \alpha, \dot\alpha = 1, 2$$

i.e., $\{\theta^\alpha, \theta^\beta\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = \{\theta^\alpha, \bar\theta_{\dot\beta}\} = 0$.

Extend the translation operator $e^{ix\cdot P}$ to the super-translation operator:

$$G(x^\mu, \theta, \bar\theta) = e^{i(x\cdot P + \theta Q + \bar\theta\bar Q)}$$

Jargon: $(x^\mu; \theta^\alpha; \bar\theta_{\dot\alpha})$ is an 8-dimensional <u>superspace</u>.

Extend the field operator, $\phi(x) = e^{ix\cdot P}\phi(0)e^{-ix\cdot P}$
to the superfield:

$$\phi(x,\theta,\bar\theta) = G(x,\theta,\bar\theta)\,\phi(0,0,0)\,G^{-1}(x,\theta,\bar\theta)$$

<u>exercise:</u> Prove that:

$$G(y,\xi,\bar\xi)G(x,\theta,\bar\theta) = G(x+y+i(\xi\sigma\bar\theta-\theta\sigma\bar\xi),\ \xi+\theta,\ \bar\xi+\bar\theta)$$

note this non-trivial
extra space-time
translation

[Hint: use the BCH formula
$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}\ ]$$

Consider

$$G(y,\xi,\bar\xi)\phi(x,\theta,\bar\theta)G^{-1}(y,\xi,\bar\xi)$$

$$= G(y,\xi,\bar\xi)G(x,\theta,\bar\theta)\phi(0,0,0)[G(y,\xi,\bar\xi)G(x,\theta,\bar\theta)]^{-1}$$

$$= \phi(x+y+i(\xi\sigma\bar\theta-\theta\sigma\bar\xi),\ \xi+\theta,\ \bar\xi+\bar\theta)$$

For infinitesimal y, ξ and $\bar\xi$, we can write:

$$G(y,\xi,\bar\xi) \simeq 1 + i(y\cdot P + \xi Q + \bar\xi\bar Q)$$

and Taylor expand

$$\phi(x+y+i(\xi\sigma\bar\theta-\theta\sigma\bar\xi),\ \xi+\theta,\ \bar\xi+\bar\theta)$$
$$\simeq \phi(x,\theta,\bar\theta) + [y^\mu + i(\xi\sigma^\mu\bar\theta-\theta\sigma^\mu\bar\xi)]\partial_\mu\phi(x,\theta,\bar\theta)$$
$$+\left(\xi^\alpha\frac{\partial}{\partial\theta^\alpha} + \bar\xi_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\right)\phi(x,\theta,\bar\theta)$$

To first order, the end results are:

$$[\phi, P_\mu] = i\partial_\mu \phi$$
$$[\phi, \xi^\alpha Q_\alpha] = i\xi^\alpha\left(\frac{\partial}{\partial\theta^\alpha} + i(\sigma^\nu\bar\theta)_\alpha \partial_\nu\right)\phi$$
$$[\phi, \bar Q_{\dot\alpha}\bar\xi^{\dot\alpha}] = -i\left(\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\sigma^\nu)_{\dot\alpha}\partial_\nu\right)\bar\xi^{\dot\alpha}\phi$$

<u>Notation:</u> derivatives of Grassmann numbers

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}, \quad \partial^\alpha \equiv \frac{\partial}{\partial\theta_\alpha}, \quad \bar\partial_{\dot\alpha} \equiv \frac{\partial}{\partial\bar\theta^{\dot\alpha}}, \quad \bar\partial^{\dot\alpha} \equiv \frac{\partial}{\partial\bar\theta_{\dot\alpha}}$$
$$\partial_\alpha\theta^\beta = \delta_\alpha{}^\beta, \quad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}$$

<u>Warning:</u>

$$\partial_\alpha\theta_\beta = \varepsilon_{\beta\alpha} = -\varepsilon_{\alpha\beta}$$
$$\varepsilon^{\alpha\beta}\partial_\beta = -\partial^\alpha$$

Thus, we may represent P^μ, Q_α and $\bar Q_{\dot\alpha}$ by differential operators acting on superfields:

$$\hat P^\mu = i\partial^\mu$$
$$\hat Q_\alpha = i\partial_\alpha - (\sigma^\mu\bar\theta)_\alpha\partial_\mu$$
$$\hat{\bar Q}_{\dot\alpha} = -i\bar\partial_{\dot\alpha} + (\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$$

<u>exercise:</u> verify that when acting on a superfield,

$$\{\hat Q_\alpha, \hat{\bar Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}\hat P_\mu$$

Expanding the superfield in powers of $\theta, \bar{\theta}$:

Define $\quad \theta\theta \equiv \theta^\alpha \theta_\alpha$

$\qquad \bar{\theta}\bar{\theta} \equiv \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}$

Then, note that:

$$\theta^\alpha \theta^\beta = -\tfrac{1}{2}\varepsilon^{\alpha\beta}\theta\theta \qquad \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \tfrac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}$$

$$\theta_\alpha \theta_\beta = \tfrac{1}{2}\varepsilon_{\alpha\beta}\theta\theta \qquad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}$$

Clearly,

$$\theta^\alpha \theta^\beta \theta^\gamma = 0, \quad \text{etc.}$$

since the θ's anticommute imply that $(\theta^1)^2 = (\theta^2)^2 = 0$.

Three useful results

(i) $\quad (\theta\sigma^\mu\bar{\theta})\theta_\beta = -\tfrac{1}{2}\theta\theta(\sigma^\mu\bar{\theta})_\beta$

(ii) $\quad (\theta\sigma^\mu\bar{\theta})\bar{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\bar{\theta}\bar{\theta}(\theta\sigma^\mu)_{\dot{\beta}}$

iii) $\quad (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \tfrac{1}{2}g^{\mu\nu}(\theta\theta)(\bar{\theta}\bar{\theta})$

Exercise: prove these results [(iii) follows from the Fierz identities]

Thus, one can expand $\phi(x,\theta,\bar{\theta})$ as a Taylor series in θ and $\bar{\theta}$. The series is finite:

$$\phi(x,\theta,\bar{\theta}) = f(x) + \theta\zeta(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta\, m(x) + \bar{\theta}\bar{\theta}\, n(x)$$
$$+ \theta\sigma^\mu\bar{\theta}\, V_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\psi(x) + \theta\theta\bar{\theta}\bar{\theta}\, d(x)$$

where: f, m, n, V_μ, and d are commuting bosonic fields

ζ, χ, λ and ψ are anti-commuting fermionic fields.

How to compute the supersymmetric transformation law

For any superfield ϕ,

$$\delta_\xi \phi = i[\xi Q + \bar{\xi}\bar{Q}, \phi]$$

But we showed that

$$[\phi, \xi^\alpha Q_\alpha] = i\xi^\alpha \hat{Q}_\alpha \phi$$
$$[\phi, \bar{Q}_{\dot\alpha}\bar{\xi}^{\dot\alpha}] = i\hat{\bar{Q}}_{\dot\alpha}\bar{\xi}^{\dot\alpha}\phi$$

where $\hat{Q}_\alpha$, $\hat{\bar{Q}}_{\dot\alpha}$ are differential operators.

Thus,

$$\boxed{\delta_\xi \phi = -i(\xi\hat{Q} + \bar{\xi}\hat{\bar{Q}})\phi}$$

Insert the expansion

$$\phi(x,\theta,\bar\theta) = f(x) + \theta\zeta(x) + \bar\theta\bar\chi(x) + \ldots + \theta\theta\bar\theta\bar\theta d(x)$$

into the above equation and compare like terms in the $\theta, \bar\theta$ expansions. This yields the supersymmetric transformation law!

Number of degrees of freedom of $\phi(x,\theta,\bar\theta)$

$$\phi(x,\theta,\bar\theta) = f(x) + \theta\zeta(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x)$$
$$+ \theta\sigma^\mu\bar\theta V_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\psi(x) + \theta\theta\bar\theta\bar\theta d(x)$$

boson degrees of freedom

f, m, n, V_μ, d 16   8

fermion degrees of freedom

$\zeta, \chi, \lambda, \psi$ 16   8

$\uparrow$ if ϕ is complex $\uparrow$ if ϕ is real

Note: if ϕ is real, $f = f^*$, $d = d^*$, $n = m^*$, $V_\mu = V_\mu^*$,
$\zeta = \chi$, $\lambda = \psi$.

In either case, this is too many degrees of freedom to describe the supermultiplet of the Wess-Zumino model.

Conclusion: $\phi(x,\theta,\bar\theta)$ describes a reducible representation of supersymmetry. To describe an irreducible representation, one must impose supersymmetric constraints on ϕ.

Remark: the constraint $\phi = \phi^\dagger$ does lead to an irreducible representation. This is the off-shell supermultiplet with $j = \frac{1}{2}$. We will use this later when we study gauge fields.

Fermionic covariant derivatives

If $\phi(x,\theta,\bar\theta)$ is a superfield, it is easy to check that $\partial_\alpha \phi$ is not a superfield, since

$$\partial_\alpha(\delta_\xi \phi) \neq \delta_\xi(\partial_\alpha \phi)$$

[analogy with gauge theory: if ψ is a charged field transforming under a local gauge transformation, $\psi \to e^{i\Lambda(x)}\psi$, then $\partial_\mu \psi$ does not transform as ψ since $\partial_\mu(e^{i\Lambda(x)}\psi) \neq e^{i\Lambda(x)}\partial_\mu \psi$.]

The fermionic covariant derivative D is defined such that

$$D_\alpha(\delta_\xi \phi) = \delta_\xi(D_\alpha \phi)$$

$$\bar D_{\dot\alpha}(\delta_\xi \phi) = \delta_\xi(\bar D_{\dot\alpha}\phi)$$

Then, $D_\alpha \phi$ and $\bar D_{\dot\alpha}\phi$ both transform properly as superfields.

Using $\delta_\xi \phi = i[\xi Q + \bar\xi \bar Q, \phi]$, it follows that

$$D_\alpha(\xi Q + \bar\xi \bar Q) = (\xi Q + \bar\xi \bar Q) D_\alpha \qquad , \text{etc.}$$

Since ξ is anticommuting, and $\{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0$ it follows that:

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar Q_{\dot\beta}\} = \{\bar D_{\dot\alpha}, Q_\beta\} = \{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0.$$

Using the differential operator representation for $Q, \bar Q$ we can deduce an explicit realization of $D, \bar D$.

$$\boxed{\begin{aligned} D_\alpha &= \partial_\alpha - i(\sigma^\mu \bar\theta)_\alpha \partial_\mu \\ \bar D_{\dot\alpha} &= -\bar\partial_{\dot\alpha} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu \end{aligned}}$$

Further manipulation yields:

$$D^\alpha = -\partial^\alpha + i(\bar\theta\bar\sigma^\mu)^\alpha \partial_\mu$$

$$\bar D^{\dot\alpha} = \bar\partial^{\dot\alpha} - i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu$$

where by definition, $D^\alpha \equiv \varepsilon^{\alpha\beta} D_\beta$

$\bar D^{\dot\alpha} \equiv \varepsilon^{\dot\alpha\dot\beta} \bar D_{\dot\beta}$

[warning revisited: recall $\varepsilon^{\alpha\beta}\partial_\beta = -\partial^\alpha$, etc.]

Exercise: show that $\{D_\alpha, D_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$

i.e. the same relation satisfied by $\{\hat Q_\alpha, \hat{\bar Q}_{\dot\beta}\}$.

We are now in the position to impose an interesting constraint on ϕ:

$$\bar D_{\dot\alpha}\phi = 0$$

thereby reducing the number of degrees of freedom. Let's see what happens.

Chiral superfields

$$\bar{D}_{\dot\alpha}\phi = 0$$

$$\left(\frac{-\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\right)\phi(x,\theta,\bar\theta) = 0$$

solution:

$$\phi(x,\theta,\bar\theta) = \exp\left(-i\theta\sigma^\mu\bar\theta\,\partial_\mu\right)\phi(x,\theta)$$

where $\phi(x,\theta)$ is an arbitrary function of x,θ.

Expand $\phi(x,\theta)$ in a Taylor series in θ.

$$\phi(x,\theta) = A(x) + \sqrt{2}\,\theta\psi(x) - \theta\theta F(x)$$ [exact!]

(the $\sqrt{2}$ and minus sign are conventional)

Note that:

$$\exp(-i\theta\sigma^\mu\bar\theta\,\partial_\mu) = 1 - i\theta\sigma^\mu\bar\theta\,\partial_\mu - \tfrac{1}{4}\theta\theta\bar\theta\bar\theta\,\Box$$ [exact!]

[exercise: prove this]

$(\Box \equiv \partial_\mu\partial^\mu)$

Some more manipulation produces:

$$\boxed{\begin{aligned}\phi(x,\theta,\bar\theta) = {} & A(x) + \sqrt{2}\,\theta\psi(x) - \theta\theta F(x) - i\theta\sigma^\mu\bar\theta\,\partial_\mu A(x) \\ & + \frac{i}{\sqrt{2}}\theta\theta\left[\partial_\mu\psi(x)\sigma^\mu\bar\theta\right] - \frac{1}{4}\theta\theta\bar\theta\bar\theta\,\Box A(x)\end{aligned}}$$

The field content: A, ψ, F matches the off-shell fields of the superspin $j=0$ supermultiplet!

exercise: using $\delta_\xi \phi = -i(\xi\hat{Q} + \bar{\xi}\hat{\bar{Q}})\phi$, insert the chiral superfield $\phi(x,\theta,\bar\theta) = A + \sqrt{2}\,\theta\psi - \theta\theta F + \dots$ and derive the supersymmetric transformation laws $\delta_\xi A$, $\delta_\xi \psi$ and $\delta_\xi F$ written down previously.

It is useful to simplify the expression for the chiral superfield by modifying all operators according to:

$$\mathcal{O}_{new} = e^{i\theta\sigma^\mu\bar\theta\partial_\mu}\,\mathcal{O}\,e^{-i\theta\sigma^\mu\bar\theta\partial_\mu}$$

In the new "chiral representation",

$$\hat{Q}_\alpha = i\partial_\alpha$$
$$\hat{\bar{Q}}_{\dot\alpha} = -i\bar\partial_{\dot\alpha} + 2(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$$
$$D_\alpha = \partial_\alpha - 2i(\sigma^\mu\bar\theta)_\alpha\partial_\mu$$
$$\bar{D}_{\dot\alpha} = -\bar\partial_{\dot\alpha}$$

Now, the condition $\bar{D}_{\dot\alpha}\phi = 0$ simply means that ϕ is independent of $\bar\theta$. Let us call the chiral superfield

$$\phi_1(x,\theta) = A(x) + \sqrt{2}\,\theta\psi(x) - \theta\theta F(x)$$

in the chiral representation. Clearly,

$$\begin{aligned}\phi(x,\theta,\bar\theta) &= \exp(-i\theta\sigma^\mu\bar\theta\partial_\mu)\,\phi_1(x,\theta)\\ &= \phi_1(x - i\theta\sigma^\mu\bar\theta,\ \theta)\end{aligned}$$

Working in the chiral representation can simplify many calculations.

Anti-chiral superfields

$$D_\alpha \bar{\Phi} = 0$$

$$\bar{\Phi}(x,\theta,\bar{\theta}) = \exp(i\theta\sigma^\mu\bar{\theta}\partial_\mu)\,\bar{\phi}(x,\bar{\theta})$$

$$= A^*(x) + \sqrt{2}\,\bar{\theta}\bar{\Psi}(x) - \bar{\theta}\bar{\theta}\,F^*(x) + \dots$$

One can define the anti-chiral representation where

$$\hat{Q}_\alpha = i\partial_\alpha - 2i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu$$

$$\hat{\bar{Q}}_{\dot{\alpha}} = -i\bar{\partial}_{\dot{\alpha}}$$

$$D_\alpha = \partial_\alpha$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + 2i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu$$

and

$$\phi_2(x,\bar{\theta}) = A^*(x) + \sqrt{2}\,\bar{\theta}\bar{\Psi}(x) - \bar{\theta}\bar{\theta}\,F^*(x)$$

Then,

$$\phi(x,\theta,\bar{\theta}) = \exp(i\theta\sigma^\mu\bar{\theta}\partial_\mu)\,\phi_2(x,\bar{\theta})$$

$$= \phi_2(x + i\theta\sigma^\mu\bar{\theta},\, \bar{\theta})$$

Some jargon:

The F-component of a chiral superfield is the coefficient of the $\theta\theta$ term (with an extra minus sign). Sometimes I will write:

$$[\phi]_{\theta\theta} = -F.$$

Theorem: If ϕ is a chiral superfield then so is ϕ^n.
But $\phi^n \bar{\phi}^m$ is not a chiral superfield (n, m are positive integers).

proof: $\bar{D}_{\dot{\alpha}} \phi^n = n\phi^{n-1} \bar{D}_{\dot{\alpha}} \phi = 0.$

Theorem: F-terms transform as total derivatives.

recall: $\delta_\xi F = -i\sqrt{2}\, \partial_\mu \psi^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}}$

but now we identify

$$[\phi]_\theta = \sqrt{2}\, \psi(x)$$
$$[\phi]_{\theta\theta} = -F$$

Consequently,

$$\sum_{n \geq 1} [a_n \phi^n]_{\theta\theta} + h.c.$$

is a Lorentz scalar that transforms as a total divergence, and is thus a possible term in a supersymmetric Lagrangian.

exercise: multiply together two chiral multiplets. Do the computation in the chiral representation.

$$\phi_a(x) = A_a(x) + \sqrt{2}\,\theta\psi_a(x) - \theta\theta F_a(x)$$
$$\phi_b(x) = A_b(x) + \sqrt{2}\,\theta\psi_b(x) - \theta\theta F_b(x)$$

Show that:

$$[\phi_a \phi_b]_{\theta\theta} = -[A_a(x)F_b(x) + A_b(x)F_a(x) + \psi_a(x)\psi_b(x)]$$

Similarly,

$$[\phi_a\phi_b\phi_c]_{\theta\theta} = -[A_a A_b F_c + A_a A_c F_b + A_b A_c F_a + A_a \psi_b \psi_c + A_b \psi_a \psi_c + A_c \psi_a \psi_b]$$

The kinetic superfield

$$T\phi \equiv \tfrac{1}{4}\bar{D}^2\bar{\phi}$$

where $\bar{D}^2 \equiv \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}$, and $\bar{\phi}$ is anti-chiral.

Note that $\bar{D}_{\dot{\alpha}}(T\phi) = 0$ so that $T\phi$ is chiral.

An explicit computation gives [in the chiral representation]:

If $\phi = A + \sqrt{2}\,\theta\psi - \theta\theta F$, then

$$T\phi = F^* + i\sqrt{2}\,\theta\sigma^\mu\partial_\mu\bar{\psi} + \theta\theta\,\Box A^*$$

Let us compute:

$$[\phi T\phi]_F = -[\phi T\phi]_{\theta\theta}$$

$$[\phi^\dagger \phi]_F = -A\Box A^* + F^*F + i\psi\sigma^\mu\partial_\mu\bar{\psi}$$

$$= (\partial_\mu A)(\partial^\mu A^*) + F^*F + i\bar{\psi}\sigma^\mu\partial_\mu\psi$$

$$+ \text{ total divergence}$$

which we recognize as the kinetic energy of the Wess-Zumino model.

<u>Theorem</u>: For any chiral superfield,

$$[\phi]_F = \frac{1}{4}D^2\phi\Big|_{\theta=\bar{\theta}=0} = -\frac{1}{4}\partial^\alpha\partial_\alpha\phi\Big|_{\theta=\bar{\theta}=0}$$

since $D_\alpha = \partial_\alpha$ + θ-dependent terms.

Given any holomorphic function of the chiral superfield, $W(\phi)$, we can compute:

$$[W(\phi)]_F = -\frac{1}{4}\partial^\alpha\partial_\alpha W\Big|_{\theta=\bar{\theta}=0}$$

$$= -\frac{1}{4}\partial^\alpha\frac{\partial W}{\partial\phi}\partial_\alpha\phi\Big|_{\theta=\bar{\theta}=0} \qquad \text{by the chain rule}$$

$$= -\frac{1}{4}\left(\frac{\partial^2 W}{\partial\phi^2}\partial^\alpha\phi\,\partial_\alpha\phi\right) + \frac{\partial W}{\partial\phi}\frac{1}{4}D^2\phi\Big|_{\theta=\bar{\theta}=0}$$

Noting that:

$$\partial^\alpha\phi\,\partial_\alpha\phi\Big|_{\theta=\bar{\theta}=0} = -\varepsilon^{\alpha\beta}\partial_\beta\phi\,\partial_\alpha\phi\Big|_{\theta=\bar{\theta}=0} = -2\psi\cdot\psi$$

Conclusion:

$$[W(\phi)]_F = \frac{1}{2}\frac{d^2W}{dA^2}\psi\psi + \frac{dW}{dA}F$$

where $\frac{dW}{dA}$ means: replace ϕ with A in $W(\phi)$ and then compute the derivative.

The supersymmetric Lagrangian

$$\mathcal{L} = [\phi T \phi]_F - \left([W(\phi)]_F + h.c.\right)$$

$$= (\partial_\mu A)^*(\partial^\mu A) + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^*F$$
$$- F\frac{dW}{dA} - F^*\left(\frac{dW}{dA}\right)^* - \frac{1}{2}\left[\frac{d^2W}{dA^2}\psi\psi + \left(\frac{d^2W}{dA^2}\right)^*\bar{\psi}\bar{\psi}\right]$$

which recovers our previous result.

Proof that $\delta_\xi \mathcal{L} = \partial_\mu \tilde{K}^\mu$.

immediate! since $\mathcal{L}$ is the sum of F (and F^*) terms.

Some supersymmetric jargon: $W(\phi)$ is the superpotential. It is a holomorphic function of chiral superfields.

R-invariance

Recall $[R, Q_\alpha] = -Q_\alpha$

$[R, \bar{Q}_{\dot\alpha}] = \bar{Q}_{\dot\alpha}$

As a differential operator acting on superspace,

$$[\phi, R] = (\theta^\alpha \partial_\alpha - \bar{\theta}_{\dot\alpha} \bar{\partial}^{\dot\alpha} - 2n)\phi$$

$2n$ = chiral weight of the superfield ϕ. $n \in \mathbb{R}$.

i.e. $\hat{R} = \theta\partial - \bar{\theta}\bar{\partial} - 2n$

Under an R-transformation,

$$\delta_a \phi = ia[R, \phi] = -ia(\theta\partial - \bar{\theta}\bar{\partial} - 2n)\phi$$

Acting on a chiral superfield or anti-chiral superfield

$$\hat{R}\phi(x, \theta) = e^{2ina}\phi(x, e^{-ia}\theta)$$
$$\hat{R}\bar{\phi}(x, \bar{\theta}) = e^{-2ina}\bar{\phi}(x, e^{ia}\bar{\theta})$$

Infinitesimally,

$$\delta_a A = 2ina A$$
$$\delta_a \psi = (2n-1)ia\psi$$
$$\delta_a F = (2n-2)ia F$$

$2n$ = R-charge of the superfield ϕ.

Question: is $\delta_a \mathcal{L} = 0$?

exercise: prove that $[\phi T \phi]_F$ is R-invariant.

<u>Note</u>: the fully exponentiated form of the R-transformation is:

$$A \to e^{2ina} A$$
$$\psi \to e^{2i(n-\frac{1}{2})a} \psi$$
$$F \to e^{2i(n-1)a} F.$$

<u>Note</u>: $[W]_F$ is R-invariant only if W has R-charge, $2n$, equal to 2.

<u>proof</u>: this follows immediately from the transformation of F.

<u>example</u>: the Wess-Zumino model with

$$W(\phi) = \tfrac{1}{2} m \phi^2 + \tfrac{1}{3} g \phi^3$$

If $m=0$, $\mathcal{L}$ is R-invariant with the choice $n = \frac{1}{3}$.
If $g=0$, $\mathcal{L}$ is R-invariant with the choice $n = \frac{1}{2}$.
If both $m \neq 0$ and $g \neq 0$, then $\mathcal{L}$ is not R-invariant.

Finally, it should be noted that for a real superfield (not chiral or anti-chiral), the only possible choice for the R-charge is $n=0$.

$$\hat{R} \phi(x,\theta,\bar{\theta}) = \phi(x, e^{-ia}\theta, e^{ia}\bar{\theta})$$

if $\phi^\dagger = \phi$.

Grassmann integration

The rules:

$$\int d\theta^\alpha = \int d\bar{\theta}^{\dot\alpha} = 0$$

$$\int d\theta^\alpha\, \theta^\alpha = \int d\bar{\theta}^{\dot\alpha}\, \bar{\theta}^{\dot\alpha} = 1 \qquad \text{no sum over } \alpha, \dot\alpha.$$

definitions:

$$d^2\theta = -\tfrac{1}{4}\, d\theta^\alpha d\theta_\alpha$$

$$d^2\bar\theta = -\tfrac{1}{4}\, d\bar\theta_{\dot\alpha}\, d\bar\theta^{\dot\alpha}$$

$$d^4\theta = d^2\theta\, d^2\bar\theta$$

Then,

$$\int d^2\theta\, \theta\theta = \int d^2\bar\theta\, \bar\theta\bar\theta = \int d^4\theta\, \theta\theta\bar\theta\bar\theta = 1$$

and all other integrals equal zero.

It follows that for a chiral superfield,

$$\int d^2\theta\, \phi(x,\theta) = [\phi]_{\theta\theta} = -\tfrac{1}{4} D^2\phi\Big|_{\theta=\bar\theta=0}$$

while for an unconstrained superfield

$$\int d^4\theta\, \phi(x,\theta,\bar\theta) = [\phi]_{\theta\theta\bar\theta\bar\theta}$$

We can write a supersymmetric action as an integral over superspace. Simply note that:

$$[\bar{\Phi}\Phi]_{\theta\theta\bar{\theta}\bar{\theta}} = -[\phi T \phi]_{\theta\theta}$$

The Wess-Zumino action $S = \int d^4x \, \mathcal{L}$ can be written as:

$$S = \int d^4x \, d^4\theta \, \bar{\Phi}\Phi + \int d^4x \, d^2\theta \, W(\phi) + \int d^4x \, d^2\bar{\theta} \, W(\bar{\Phi}).$$

which is manifestly supersymmetric.

<u>Note:</u> We have already noted that F-terms transform as total divergences. For a general superfield

$$\phi = \dots + \theta\theta\bar{\theta}\bar{\theta} D$$

we can work out $\delta_\xi D$ and show that

$$[\phi]_D = [\phi]_{\theta\theta\bar{\theta}\bar{\theta}}$$

transforms as a total divergence. Thus, D-terms also make for suitable terms of a supersymmetric Lagrangian.

<u>Example:</u> $[\bar{\Phi}\Phi]_D = (\partial_\mu A)(\partial^\mu A^*) + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^*F$
$+ \text{total divergence}$

as expected.

Non-renormalization theorem

The superpotential is not renormalized to all orders in perturbation theory. Only the kinetic energy term is renormalized – this is wave function renormalization.

Hence, no quadratic divergences in the parameters of the superpotential. Mass renormalization is generated simply as a consequence of wave function renormalization, namely

$$m_B \phi_B^2 = m_R \phi_R^2$$

and $\phi_B = Z_\phi^{1/2} \phi_R$, which implies that

$$m_R = Z_\phi m_B .$$

Wave-function renormalization is logarithmic in the cutoff.

The original proof of this theorem demonstrated that all radiative corrections could be written in the form of an integral over $d^4\theta$. Thus, terms of the form

$$\int d^4x \, d^2\theta \, W(\phi)$$

are not affected.

V. Supersymmetric Gauge Theories

The real vector superfield satisfies

$$V(x,\theta,\bar\theta) = V^{+}(x,\theta,\bar\theta)$$

Expanding in $\theta,\bar\theta$:

$$\begin{aligned}V(x,\theta,\bar\theta) = {} & C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \tfrac{i}{2}\theta\theta[M(x)+iN(x)] \\ & -\tfrac{i}{2}\bar\theta\bar\theta[M(x)-iN(x)] + \theta\sigma^\mu\bar\theta V_\mu(x) \\ & + i\theta\theta\bar\theta[\bar\lambda(x) - \tfrac{i}{2}\bar\sigma^\mu\partial_\mu\chi(x)] \\ & - i\bar\theta\bar\theta\theta[\lambda(x) - \tfrac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)] \\ & + \tfrac{1}{2}\theta\theta\bar\theta\bar\theta[D(x) - \tfrac{1}{2}\Box C(x)]\end{aligned}$$

C, M, N, D and V_μ are real bosonic fields.
Factors of i and $\frac{1}{2}$ are convention.
The particular choices for coefficients of $\theta\theta\bar\theta$, $\bar\theta\bar\theta\theta$ and $\theta\theta\bar\theta\bar\theta$ is convenient for later purposes.

Define: $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$

degrees of freedom

8 bosonic : C, M, N, D, V_μ
8 fermionic : $\chi, \bar\chi, \lambda, \bar\lambda$

Field strength superfield [U(1) abelian gauge theory]

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$$

Note that $\bar{D}_{\dot\beta} W_\alpha = 0$, so that W_α is a spinor chiral superfield.

In the chiral representation, one can evaluate the θ-expansion of W_α. After much algebra:

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha F_{\mu\nu} - \theta\theta(\sigma^\mu\partial_\mu\bar\lambda)_\alpha$$

From here, we can work out the SUSY transformation law (SUSY = supersymmetry).

$$\delta_\xi \lambda = i\xi_\alpha D + \frac{1}{2}(\sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta \xi_\beta F_{\mu\nu}$$

$$\delta_\xi F_{\mu\nu} = i\partial_\mu(\xi\sigma_\nu\bar\lambda - \lambda\sigma_\nu\bar\xi) - i\partial_\nu(\xi\sigma_\mu\bar\lambda - \lambda\sigma_\mu\bar\xi)$$

$$\delta_\xi D = \partial_\mu(\xi\sigma^\mu\bar\lambda + \lambda\sigma^\mu\bar\xi)$$

Note: $\delta_\xi D$ is a total divergence, as expected.

Conclusion: $(\lambda, \bar\lambda, F_{\mu\nu}, D)$ is an irreducible supermultiplet.

Off-shell degrees of freedom:

4 fermionic degrees of freedom: $\lambda, \bar\lambda$
4 bosonic degrees of freedom: $D, F^{\mu\nu}$

Degrees of freedom of $F^{\mu\nu}$: an antisymmetric tensor has 6 d.o.f.'s. Since $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, it follows that $\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$. This is the Bianchi identity, which looks like 4 constraints ($\mu = 0,1,2,3$). But one of these is redundant since $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu F_{\rho\sigma} = 0$ automatically.

A supersymmetric Lagrangian:

Note that:

$$\frac{1}{4}[W^\alpha W_\alpha]_{\theta\theta} + \frac{1}{4}[\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}]_{\bar\theta\bar\theta}$$

$$= \frac{i}{2}(\lambda\sigma^\mu\partial_\mu\bar\lambda + \bar\lambda\bar\sigma^\mu\partial_\mu\lambda) + \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= i\bar\lambda\bar\sigma^\mu\partial_\mu\lambda + \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{total divergence}$$

This is the kinetic energy for a $U(1)$ gauge field V_μ with field strength $F_{\mu\nu}$ and for the gaugino λ, the fermionic superpartner of V_μ.

D has no dynamics; it is an auxiliary field.

Another form:

$$\mathcal{L} = \frac{1}{4}\int d^2\theta\, W^\alpha W_\alpha + \frac{1}{4}\int d^2\bar\theta\, \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}$$

In fact, both terms above are equal up to a total divergence. So, the action can be written as:

$$S = \frac{1}{2}\int d^4x\, d^2\theta\, W^\alpha W_\alpha$$

$$= \frac{1}{2}\int d^4x\, d^4\theta\, (D^\alpha V) W_\alpha$$

The latter form shows that this term does renormalize – this is wave function renormalization of V.

Gauge invariance

Let $\Lambda(x,\theta,\bar\theta)$ be a chiral superfield, i.e. $\bar D_{\dot\beta}\Lambda = 0$.

Then, $\bar\Lambda(x,\theta,\bar\theta) \equiv \Lambda^+(x,\theta,\bar\theta)$ is anti-chiral.

Consider:

$$V \longrightarrow V + i(\Lambda - \bar\Lambda).$$

Exercise: show that W_α is invariant under this transformation.

If we write:

$$\Lambda = A + \sqrt{2}\,\theta\psi - \theta\theta F - i\theta\sigma^\mu\bar\theta\,\partial_\mu A + \frac{i}{\sqrt{2}}\theta\theta(\partial_\mu\psi\sigma^\mu\bar\theta) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\,\Box A$$

we see that after some algebra that the components of V transform as follows:

$$C \rightarrow C + i(A - A^*)$$

$$\chi \rightarrow \chi + \frac{1}{\sqrt{2}}\psi$$

$$M + iN \rightarrow M + iN - 2F$$

$$V_\mu \rightarrow V_\mu + \partial_\mu(A + A^*)$$ $\leftarrow$ the usual $U(1)$-gauge transformation

$$\lambda \rightarrow \lambda$$

$$D \rightarrow D$$

Of course, $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is gauge invariant

$$F_{\mu\nu} \rightarrow F_{\mu\nu}$$

Wess-Zumino gauge

Choose the gauge functions A, ψ, F such that

$$C = \chi = M = N = 0$$

This is <u>not</u> a supersymmetric gauge choice;
SUSY transformations do not preserve this condition.
Nevertheless, it is very useful for computations.
In particular,

$$V_{wz}(x,\theta,\bar\theta) = \theta\sigma^\mu\bar\theta V_\mu(x) + i\theta\theta\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x).$$

Note that

$$V_{wz}^2(x,\theta,\bar\theta) = \frac{1}{2}\theta\theta\bar\theta\bar\theta V_\mu V^\mu$$

$$V_{wz}^n(x,\theta,\bar\theta) = 0 \qquad \text{for } n>2.$$

Note that:

$$e^{2gV_{wz}} = 1 + 2gV_{wz} + 2g^2V_{wz}^2 \qquad [\text{exact!}]$$

Gauge-invariant interactions

Let ϕ be a chiral superfield. Since it is complex, it can rotate under the gauge transformation by a phase rotation:

$$\phi \rightarrow e^{-2ig\Lambda}\phi$$

where Λ is a chiral superfield introduced earlier.

One can check that the kinetic energy term previously used:

$$\mathcal{L}_{KE} = \int d^4\theta\, \bar{\Phi}\phi$$

is not gauge invariant, since

$$\bar{\Phi} \longrightarrow \bar{\Phi}\, e^{2ig\bar{\Lambda}}$$

But,

$$\mathcal{L}_{KE} = \int d^4\theta\, \bar{\Phi}\, e^{2gV}\phi$$

is gauge invariant, under $V \rightarrow V + i(\Lambda - \bar{\Lambda})$.

In the Wess-Zumino gauge, this is a sensible polynomial interaction. In particular,

$$[\bar{\Phi}e^{2gV}\phi]_{\theta\theta\bar{\theta}\bar{\theta}} = FF^* + i\psi\sigma^\mu \bar{\mathcal{D}}_\mu\bar{\Psi} + (\mathcal{D}_\mu A)(\mathcal{D}^\mu A)^*$$
$$-i\sqrt{2}g(A\bar{\lambda}\bar{\Psi} - A^*\lambda\psi) + gAA^*D$$
$$+ \text{total divergence}$$

where $\mathcal{D}_\mu \equiv \partial_\mu + igV_\mu$ is the usual gauge-covariant derivative.

Supersymmetric U(1)-gauge theory

$$S = \frac{1}{2}\int d^4x\, d^4\theta\, (D^\alpha V) W_\alpha + \int d^4x\, d^4\theta\, \bar{\Phi} e^{2gV} \phi$$

$$= \int d^4x\, \mathcal{L}$$

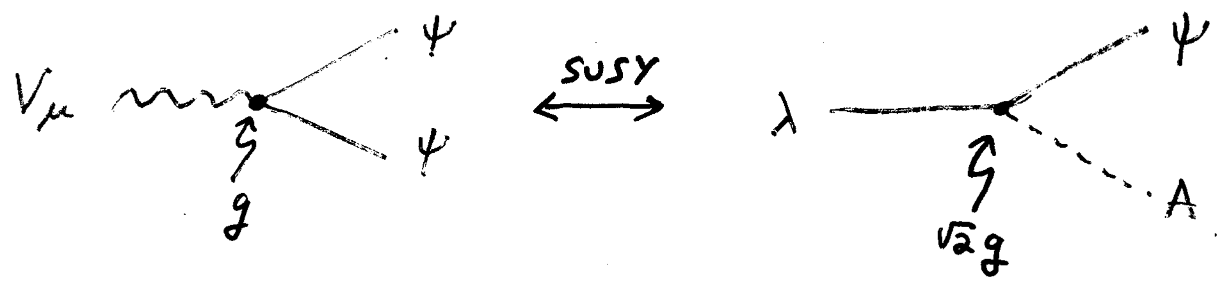
where:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\lambda}\bar{\sigma}^\mu \partial_\mu \lambda + \frac{1}{2} D^2 + F F^*$$

$$+ (\mathcal{D}_\mu A)(\mathcal{D}^\mu A)^* + i\bar{\psi}\bar{\sigma}^\mu \mathcal{D}_\mu \psi + g D A^* A$$

$$+ i\sqrt{2} g (A^* \lambda \psi - A \bar{\lambda} \bar{\psi})$$

this is a supersymmetric interaction of the scalar with its fermionic superpartner and the gaugino:



Let us eliminate the auxiliary fields by their field equations: $\frac{\partial \mathcal{L}}{\partial F} = 0$, $\frac{\partial \mathcal{L}}{\partial D} = 0$. Thus, $F = 0$ and

$$D = -g A A^*$$

The resulting Lagrangian is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\lambda}\bar{\sigma}^{\mu}\partial_{\mu}\lambda + (\mathcal{D}_{\mu}A)(\mathcal{D}^{\mu}A)^{*}$$
$$+ i\bar{\Psi}\bar{\sigma}^{\mu}\mathcal{D}_{\mu}\Psi + i\sqrt{2}g(A^{*}\lambda\Psi - A\bar{\lambda}\bar{\Psi})$$
$$- \frac{1}{2}g^{2}(A^{*}A)^{2}.$$

That is, there is a scalar potential of the form:

$$V_{scalar} = \frac{1}{2}g^{2}(A^{*}A)^{2}.$$

R-invariance

$$\hat{R}\,V(x,\theta,\bar{\theta}) = V(x, e^{-ia}\theta, e^{ia}\bar{\theta})$$

i.e. $n=0$. In the Wess-Zumino gauge,

$$V_{\mu} \rightarrow V_{\mu}$$
$$\lambda \rightarrow e^{ia}\lambda$$
$$D \rightarrow D$$

Thus, the R-invariance here is associated with the chiral symmetry of the massless gaugino.

On-shell degrees of freedom of the gauge supermultiplet

2 bosonic degrees of freedom (two helicities of V_{μ})
2 fermionic degrees of freedom (two helicities of λ)

The Fayet-Iliopoulos term

Another SUSY-invariant term can be added to $\mathcal{L}$:

$$\mathcal{L}_{FI} = 2\xi [V]_{\theta\theta\bar{\theta}\bar{\theta}} = \xi D + \text{total divergence}$$

The only modification to the previous work is that when one solves for the auxiliary fields,

$$D = -gAA^* - \xi$$

Thus,

$$V_{scalar} = \frac{1}{2}\left[gAA^* + \xi\right]^2$$

Note: in either case, we see that

$$V_{scalar} = \frac{1}{2} D^2$$

Generalization to many chiral superfields

With only one chiral superfield, it was not possible to add a superpotential $W(\phi)$ to the theory, since

$$\mathcal{L} = [W(\phi)]_{\theta\theta} + h.c.$$

is not gauge invariant. But, with more than one chiral superfield, gauge invariant superpotentials can be written.

If we now eliminate the corresponding F_i (i labels the chiral superfields):

$$\mathcal{L} = \sum_i F_i^* F_i - \frac{dW}{dA_i} F_i - \frac{dW}{dA_i^*} F_i^* + \ldots$$

we obtain

$$F_i^* = \frac{dW}{dA_i} \quad , \qquad F_i = \left(\frac{dW}{dA_i}\right)^*$$

which produces a term in the scalar potential:

$$V_{scalar} = -\sum_i \left| \frac{dW}{dA_i} \right|^2$$

$$\equiv F^* F$$

For the theory of charged chiral superfields, with

$$\Phi_i \to e^{-2ig q_i \Lambda} \Phi_i$$

q_i = U(1)-quantum number of the i^{th} superfield.

the total scalar potential is then given by:

$$\boxed{\begin{aligned} V_{scalar} &= \sum_i \left|\frac{dW}{dA_i}\right|^2 + \frac{1}{2}\left[\sum_i q_i g A_i^* A_i + \xi\right]^2 \\ &= \sum_i F_i^* F_i + \frac{1}{2} D^2 \end{aligned}}$$

In supersymmetric jargon, this is the sum of the F-term and the D-term contributions.

<u>Remark</u>: note that $V_{scalar} \geq 0$. In a supersymmetric ground state, the scalar fields may acquire vacuum expectation values such that

$$\langle V_{scalar} \rangle = 0.$$

example: SUSY-QED

superfield content:

ϕ_+ chiral superfield $q=+1$
ϕ_- chiral superfield $q=-1$
V real vector superfield

gauge invariant superpotential:

$$W(\phi_+, \phi_-) = m\phi_+\phi_-$$

This theory is R-invariant. Choose $R(\phi_\pm) = +1$, so that $R(W) = 2$ as required.

particle content

$$\phi_+ \to (A_+, \psi_+, F_+)$$
$$\phi_- \to (A_-, \psi_-, F_-)$$
$$V \to (\lambda, V_\mu, D)$$

Eliminate auxiliary fields F_+, F_-, D.
Combine ψ_+, ψ_- into a Dirac field. This is the electron (and positron). A_+, A_- are complex scalars called "selectrons", usually denoted by $\tilde{e}_L^\pm, \tilde{e}_R^\pm$ since they are super-partners of the left and right-handed electron.
V_μ is the photon and λ is a Majorana fermion (the "photino").

on-shell fermion d.o.f.'s		on-shell boson d.o.f.'s	
Dirac e^-, e^+	4	selectrons $\tilde{e}_L^\pm, \tilde{e}_R^\pm$	4
photino $\tilde{\gamma}$	2	photon	2

Non-abelian SUSY Yang-Mills theory

More complicated than the abelian theory, but with a similar structure. Briefly:

The chiral superfields are now multiplets under the gauge group, transforming as:

$$\phi_i \to (e^{-2ig\Lambda})_{ij}\, \phi_j$$

where $\Lambda = (\Lambda^a T^a)_{ij}$ and

$$[T_a, T_b] = i f_{abc} T_c$$

defines the non-abelian gauge group.

The combination

$$\bar{\phi}\, e^{2gV} \phi \qquad\qquad V \equiv V^a T^a$$

is gauge invariant if the gauge transformation satisfies:

$$e^{2gV'} = e^{-2ig\bar{\Lambda}}\, e^{2gV}\, e^{2ig\Lambda}$$

<u>exercise</u>: show that if the gauge group is abelian, the above result reduces to $V' = V + i(\Lambda - \bar{\Lambda})$.

Non-abelian field strength superfield

$$W_\alpha = -\frac{1}{8g}\bar{D}^2 e^{-2gV} D_\alpha e^{2gV}$$

$$\bar{W}_{\dot\alpha} = -\frac{1}{8g} D^2 e^{-2gV} \bar{D}_{\dot\alpha} e^{2gV}$$

Under the gauge transformation,

$$e^{2gV} \longrightarrow e^{-2ig\bar\Lambda} e^{2gV} e^{2ig\Lambda}$$

one can work out:

$$W_\alpha \longrightarrow e^{-2ig\Lambda} W_\alpha e^{2ig\Lambda}$$

Thus, $\mathrm{Tr}(W^\alpha W_\alpha)$ is gauge-invariant.

In the Wess-Zumino gauge,

$$W_\alpha^a = -i\lambda_\alpha^a + \theta_\alpha D^a - \frac{i}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha F_{\mu\nu}^a - \sigma^\mu(\mathcal{D}_\mu\bar\lambda^a)_\alpha\theta\theta$$

where a is the adjoint group index,

$$F_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a - g f_{abc} V_\mu^b V_\nu^c$$

$$\mathcal{D}_\mu\bar\lambda^a = \partial_\mu\bar\lambda^a - g f_{abc} V_\mu^b \bar\lambda^c$$

Note that the gaugino (like the gauge field) possesses non-abelian gauge charge.

Unlike the abelian case, W_α^a depends on the non-physical degrees of freedom (C, χ, M, N) in a general gauge choice.

A supersymmetric Lagrangian

$$\mathcal{L} = \frac{1}{2}\int d^2\theta \, \mathrm{tr}(W^\alpha W_\alpha) + h.c.$$

$$+ \int d^4\theta \, \bar{\Phi} e^{2gV} \Phi$$

$$+ \left[\int d^2\theta \, W(\Phi) + h.c. \right]$$

Note: by gauge invariance, no Fayet-Iliopoulos term is allowed since $[D^a]_{\theta\theta\bar{\theta}\bar{\theta}}$ carries a gauge index.

A gauge invariant superpotential requires

$$W(e^{-2ig\Lambda}\Phi) = W(\Phi)$$

Taking $\Lambda = \Lambda^a T^a$ infinitesimal, an equivalent condition is given by:

$$\frac{\partial W}{\partial \Phi_i} (T^a)_{ij} \Phi_j = 0 .$$

In components,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + i \bar{\lambda}^a \bar{\sigma}^\mu (\mathcal{D}_\mu \lambda)^a + \frac{1}{2} D^a D^a$$
$$+ F_i^* F_i + (\mathcal{D}_\mu A_i)(\mathcal{D}^\mu A_i)^* + i \bar{\psi}_i \bar{\sigma}^\mu (\mathcal{D}_\mu \psi)_i$$
$$+ g A_i^* T^a_{ij} A_j D^a - i\sqrt{2} g (\bar{\lambda}^a \bar{\psi}_i T^a_{ij} A_j - A_i^* T^a_{ij} \psi_j \lambda^a)$$
$$-\frac{1}{2} \frac{d^2 W}{dA_i dA_j} \psi_i \psi_j - \frac{1}{2} \left(\frac{d^2 W}{dA_i dA_j} \right)^* \bar{\psi}_i \bar{\psi}_j$$
$$- \frac{dW}{dA_i} F_i - \left(\frac{dW}{dA_i} \right)^* F_i^*$$

Eliminate the auxiliary fields:

$$F_i^* = \frac{dW}{dA_i}, \quad F_i = \left(\frac{dW}{dA_i} \right)^*, \quad D^a = -g A_i^* T^a_{ij} A_j$$

One obtains the following scalar potential:

$$\boxed{\begin{aligned} V_{scalar} &= \sum_i \left| \frac{dW}{dA_i} \right|^2 + \frac{1}{2} g^2 (A_i^* T^a_{ij} A_j)^2 \\ &= \sum_i F_i^* F_i + D^a D^a \end{aligned}}$$

Note: $V_{scalar} \geq 0$. In a supersymmetric ground state,

$$\langle V_{scalar} \rangle = 0.$$