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INTRODUCTION TO ABELIAN BOSONIZATION

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INTRODUCTION TO ABELIAN BOSONIZATION

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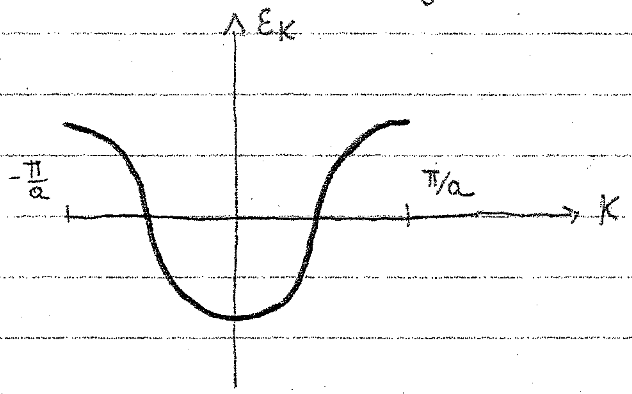
1. Bosonization of interacting spinless fermions
2. Bosonization of the Heisenberg model through the Jordan-Wigner transformation
3. Hubbard model

Interacting spinless fermions

Let us consider a one dimensional model of spinless fermions described by a non interacting Hamiltonian of the form

$$H_0 = \sum_k \varepsilon_k \, c_k^+ c_k \qquad (1)$$

with a band dispersion of the form



The momentum has quantized values which depend on the boundary conditions. If the chain has L sites, then

$$k = \frac{2\pi n}{La}, \quad n \in \left[-\frac{L}{2}, \frac{L}{2}\right] \quad \text{for periodic boundary conditions}$$

$$k = \frac{\pi}{La}(2n+1), \quad n \in \left[-\frac{L}{2}, \frac{L}{2}\right] \quad \text{for antiperiodic boundary conditions}$$

Our scope is to study model (1) in the presence of a generic interaction

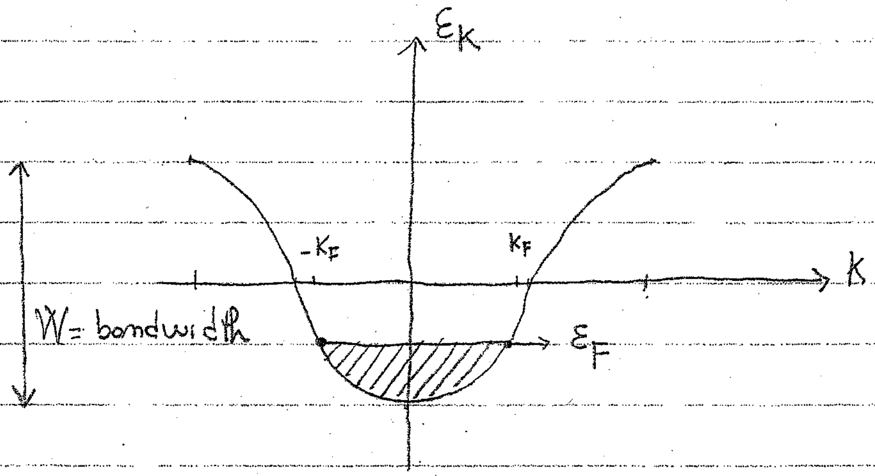
$$H_{int} = \frac{1}{2L} \sum_q V(q)\, \rho(q)\, \rho(-q) \qquad (2)$$

where $\rho(q) = \sum_k c_k^+ c_{k+q}$ is the Fourier transform of the density.

We will adopt an approach which is substantially similar to what it is usually done to study quantum fluctuations in models which develop, at the mean field level, an order parameter.

Let us start by the non interacting Hamiltonian (1).
If N fermions are present, the ground state is obtained by occupying the momentum eigenstates up to a Fermi momentum such that

$$\sum_{K=-K_F}^{K_F} = N \simeq L\,\frac{K_F}{\pi} \quad \text{(if } N \text{ and } L \text{ are large)}$$



This ground state (Fermi sea) has a kind of order parameter, namely the average of the occupation number in momentum space

$$\langle c_K^\dagger c_K \rangle \equiv \langle n_K \rangle = \Theta(K_F - |K|)$$

If we assume that the average value of the band energy

$$\frac{1}{N} \langle H_0 \rangle \simeq \epsilon_F$$

is much bigger than the interaction energy between two electrons at the typical distance, i.e. $V(q)$ at $q \sim K_F$, then the only degrees of freedom which are affected by interaction are those around each Fermi point, $\pm K_F$. We define as Right moving fermions those with positive momentum (c_{RK}) and Left moving fermions those with $K<0$, (c_{LK}). Analogously the Right and Left moving densities are defined through

$$\rho_R(q) = \sum_K c^+_{RK} c_{RK+q} \qquad \rho_L(q) = \sum_K c^+_{LK} c_{LK+q}$$

Since the interaction is weak compared to ϵ_F, the Fourier components of the densities which are affected by interaction are those with

$$|q| \ll K_F$$

In this limit, they are also well defined the Right and Left mover densities in real space $\rho_R(x)$ and $\rho_L(x)$, since they describe the distribution functions of electrons with momentum $K \sim \pm K_F$ which vary in space over distances $\Delta x \sim |1/q|$ such that, from $\Delta x \Delta K \sim 1$

$$\Delta K \sim \frac{1}{\Delta x} \sim |q| \ll K_F$$

The unperturbed ground state is identified by the "order parameters"

$$\langle n_{RK} \rangle = \langle c^{+}_{RK} c_{RK} \rangle = \theta(K_F - K) \qquad K>0$$

$$\langle n_{LK} \rangle = \langle c^{+}_{LK} c_{LK} \rangle = \theta(K + K_F) \qquad K<0$$

Let us analyse the role of interaction in the same spirit as one studies the role of quantum fluctuations on models which develop an order parameter at the mean field level.

The idea is to derive the equations of motion by making the approximation that, whenever a commutator of two operators has a component proportional to the order parameter, we substitute that with its average value on the ground state.

Let us show how this procedure works in the well known spin wave theory.

• Spin wave theory.

At each site R there is a spin with components $S_x(R), S_y(R), S_z(R)$. The classical ground state has an order parameter

$$\langle S_z(R) \rangle = M(R)$$

Then, within spin wave approximation

$$[S_x(R), S_y(R')] = i\,\delta_{R,R'}\, S_z(R) \simeq i\,\delta_{R,R'}\, M(R)$$

$$[S_{x,y}(R), S_z(R')] = [S_{x,y}(R), S_z(R')] = \mp i\,\delta_{R,R'}\, S_{y,x}(R)$$

$\downarrow$ as if it is equivalent to

$$-\frac{1}{2M(R')}\left(S_x^2(R') + S_y^2(R')\right) + \frac{1}{2} + M(R')$$

Let us apply this idea to our case.

Let us start by calculating

$$[\rho_R(q), \rho_R(q')] = \sum_{kp} [c^+_{Rk} c_{Rk+q}, c^+_{Rp} c_{Rp+q'}] =$$

$$= \sum_{kp} \delta_{p,k+q} c^+_{Rk} c_{Rp+q'} - \delta_{k,p+q'} c^+_{Rp} c_{Rk+q}$$

By a change of variables, it is equal to

$$\sum_{kp} \delta_{p,k+q} c^+_{Rk} c_{Rk+q+q'} - \delta_{p,k+q'} c^+_{Rk} c_{Rk+q+q'}$$

If the sums run for all k's, the above term would be zero. However, since all the fermions which are involved should be Right movers, with positive momentum, hence the first δ-function implies $k+q>0$ while the second $k+q'>0$, then one finds

$$[\rho_R(q), \rho_R(q')] = \sum_{k=\mathrm{Max}(0,-q,-q-q')}^{\mathrm{Max}(0,-q',-q-q')} + \sum_{k=\min(\frac{\pi}{a},\frac{\pi}{a}-q',\frac{\pi}{a}-q-q')}^{\min(\frac{\pi}{a},\frac{\pi}{a}-q,\frac{\pi}{a}-q-q')} c^+_{Rk} c_{Rk+q+q'}$$

Since all q's are small, for $q \neq -q'$, this operator involves excitations very far from k_F, either close to $k\sim 0$ or $k\sim\frac{\pi}{a}$. As we expect the ground state occupation number not to be modified for $k\sim 0$ and $k\sim\frac{\pi}{a}$, we can safely assume the commutator to be zero. On the contrary, for $q=-q'$, the commutator becomes related to the order parameter.

Namely

$$\left[\rho_R(q), \rho_R(-q)\right] = \sum_{K=\mathrm{Max}(0,-q)}^{\mathrm{Max}(0,q)} + \sum_{K=\min\left(\frac{\pi}{a}, \frac{\pi}{a}+q\right)}^{\min\left(\frac{\pi}{a}, \frac{\pi}{a}-q\right)} n_{RK}$$

In the spirit of what we previously said, we approximate this operator with its average value on the Fermi sea. Namely

$$\left[\rho_R(q), \rho_R(-q)\right] = \left\langle \left[\rho_R(q), \rho_R(-q)\right] \right\rangle$$

$$= \sum_{K=\mathrm{Max}(0,-q)}^{\mathrm{Max}(0,q)} \theta(K_F - K) = \frac{qL}{2\pi}$$

Therefore

$$\left[\rho_R(q), \rho_R(q')\right] = \delta_{q,-q'} \frac{qL}{2\pi} \qquad (2)$$

Analogously for the Left movers one finds

$$\left[\rho_L(q), \rho_L(q')\right] = -\delta_{q,-q'} \frac{qL}{2\pi} \qquad (3)$$

In real space

$$\left[\rho_R(x), \rho_R(y)\right] = \frac{1}{L^2}\sum_{qq'} e^{iqx} e^{iq'y} \left[\rho_R(q), \rho_R(q')\right] e^{-\alpha|q|}$$

$$= \frac{1}{2\pi L}\sum_q q\, e^{iq(x-y)-\alpha|q|} = -\frac{i}{2\pi}\frac{\partial}{\partial x}\frac{1}{L}\sum_q e^{iq(x-y)-\alpha|q|}$$

$$= -\frac{i}{2\pi}\frac{\partial}{\partial x}\delta_\alpha(x-y)$$

where α is a cut off [Recall that $|q| \ll K_F$ was a starting condition] as well as

$$\left[\rho_L(x), \rho_L(y)\right] = \frac{i}{2\pi}\frac{\partial}{\partial x}\delta(x-y) \qquad (4)$$

$$\left[\rho_R(q), H_0\right] = \sum_{k,p>0} \varepsilon_p \left[c^+_{Rk} c_{Rk+q}, c^+_{Rp} c_{Rp}\right]$$

$$= \sum_{k,p>0} \varepsilon_p \left(\delta_{p,k+q}\, c^+_{Rk} c_{Rp} - \delta_{k,p}\, c^+_{Rp} c_{Rk+q}\right)$$

$$= \sum \left(\varepsilon_{k+q} - \varepsilon_k\right) c^+_{Rk} c_{Rk+q} \simeq \underbrace{v_F}_{\left.\frac{\partial \epsilon_k}{\partial k}\right|_{k=k_F}} \cdot q\, \rho_R(q)$$

Since $\left.\dfrac{\partial \epsilon_k}{\partial k}\right|_{k=-k_F} = -v_F$

$$[\rho_L(q), H_0] = -v_F \cdot q\, \rho_L(q)$$

Given the above commutation relations, we can write

$$H_0 = \frac{\pi}{L} v_F \sum_q \rho_R(q)\rho_R(-q) + \rho_L(q)\rho_L(-q)$$

$$= \pi v_F \int dx\, \rho_R(x)\rho_R(x) + \rho_L(x)\rho_L(x)$$

Moreover, the continuity equation for R and L movers imply

$$i\dot{\rho}_R(q) = [\rho_R(q), H_0] = v_F q\, \rho_R(q) = q\, J_R(q)$$

$$i\dot{\rho}_L(q) = [\rho_L(q), H_0] = -v_F q\, \rho_L(q) = q\, J_L(q)$$

Hence

$$J_R(q) = v_F \rho_R(q) \qquad (5)$$

$$J_L(q) = -v_F \rho_L(q) \qquad (6)$$

are the current operators

The commutation relations (2) and (3) provide a bosonic representation of the long wavelength density fluctuations. We define, for $q > 0$,

$$b_{Rq} = -i\sqrt{\frac{2\pi}{qL}}\,\rho_R(q) \qquad b^+_{Rq} = i\sqrt{\frac{2\pi}{qL}}\,\rho_R(-q) \qquad (4)$$

as well as

$$b_{Lq} = i\sqrt{\frac{2\pi}{qL}}\,\rho_L(-q) \qquad b^+_{Lq} = -i\sqrt{\frac{2\pi}{qL}}\,\rho_L(q) \qquad (5)$$

which satisfy, for $a, b = R, L$

$$[\,b_{aq}\,,\,b^+_{bq'}\,] = \delta_{ab}\,\delta_{qq'}$$

$$[\,b_{aq}\,,\,b_{bq'}\,] = 0$$

Notice that, for $q=0$, $\rho_R(q=0)$ and $\rho_L(q=0)$ correspond to the densities of right and left moving fermions, $\frac{N_R}{L}$ and $\frac{N_L}{L}$, respectively, where $N_{R,L}$ are number operators.

Therefore we can write

$$\rho_R(x) = \frac{i}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\left(e^{iqx}\,b_{Rq} - e^{-iqx}\,b^+_{Rq}\right) + \frac{N_R}{L} \qquad (6)$$

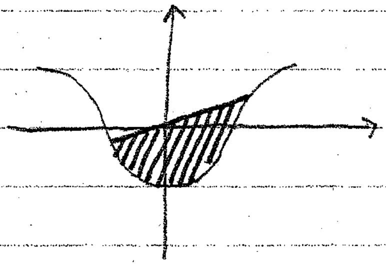
$$\rho_L(x) = -\frac{i}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\left(e^{-iqx}\,b_{Lq} - e^{iqx}\,b^+_{Lq}\right) + \frac{N_L}{L} \qquad (7)$$

We can also introduce fields which are conjugate of $\rho_{R,L}$ — Indeed the fields

$$\phi_R(x) = \frac{1}{L}\sum_{q>0}\sqrt{\frac{2\pi L}{q}}\left(e^{iqx} b_{Rq} + e^{-iqx} b^+_{Rq}\right) + K_{FR}\,x - \Theta_R + \frac{\pi}{2} N_L \quad (8)$$

$$\phi_L(x) = \frac{1}{L}\sum_{q>0}\sqrt{\frac{2\pi L}{q}}\left(e^{-iqx} b_{Lq} + e^{iqx} b^+_{Lq}\right) + K_{FL}\,x + \Theta_L + \frac{\pi}{2} N_R \quad (9)$$

where $K_{FR} = 2\pi \frac{N_R}{L}$, $K_{FL} = 2\pi \frac{N_L}{L}$ are the right and left Fermi momenta, corresponding to a situation.



satisfy

$$\begin{aligned} \left[\phi_R(x), \rho_R(y)\right] &= -i\,\delta(x-y) \qquad & \frac{1}{2\pi}\nabla\phi_R(x) &= \rho_R(x) \\ \left[\phi_L(x), \rho_L(y)\right] &= i\,\delta(x-y) \qquad & \frac{1}{2\pi}\nabla\phi_L(x) &= \rho_L(x) \end{aligned} \quad (10)$$

if $[\Theta_R, N_R] = [\Theta_L, N_L] = i$. The last term in (8) and (9) implies that

$$\left[\phi_R(x), \phi_L(y)\right] = -i\pi \qquad (11)$$

which will turn useful later

The θ-operators, which are conjugate of $N_{R,L}$, are needed because the sum over q does not include $q=0$.

Since ϕ_R and ϕ_L are conjugate of the densities, one can easily construct operators which create or destroy particles. Indeed

$e^{-i\phi_R(x)}$ creates a Right fermion in x since

$$e^{i\phi_R(x)}\, \rho_R(y)\, e^{-i\phi_R(x)} = \rho_R(y) + i\left[\phi_R(x), \rho_R(y)\right]$$

$$= \rho_R(y) + \delta(x-y)$$

and analogously $e^{i\phi_L(x)}$. In turns, these operators can be used to construct Fermi fields. It is customary to define shifted ϕ's. Given a reference state with

$$\langle N_R\rangle = \langle N_L\rangle = \frac{N}{2} \qquad K^{\circ}_{FR} = K^{\circ}_{FL} = \pi\frac{N}{L} \equiv K_F$$

we define new ϕ_R and ϕ_L like in (8) and (9), by substituting the number operators by

$$N_{R,L} \rightarrow \Delta N_{R,L} = N_{R,L} - \frac{N}{2}$$

implying that $\rho_{R,L}(x) = \frac{1}{2\pi}\nabla\phi_{R,L} + \frac{N}{2L}$

Then, the operators

$$\Psi_R(X) = \frac{1}{\sqrt{2\pi\alpha}} \, e^{iK_F X + i\phi_R(X)} \qquad (12)$$

$$\Psi_L(X) = \frac{1}{\sqrt{2\pi\alpha}} \, e^{-iK_F X - i\phi_L(X)} \qquad (13)$$

behave like Fermi fields which annihilate a right or left fermion at position X. To show it, we recall that, given two operators A and B both commuting with $[A,B]$, then

$$e^{A} e^{B} = e^{B} e^{A} e^{-[B,A]}$$

$$e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]} = e^{B} e^{A} e^{-\frac{1}{2}[B,A]}$$

Therefore

$$\Psi_R(X)\,\Psi_L(y) = \frac{1}{2\pi\alpha} \, e^{iK_F(X-y)} \, e^{i\phi_R(X)} \, e^{-i\phi_L(y)}$$

$$= \frac{1}{2\pi\alpha} \, e^{iK_F(X-y)} \, e^{-i\phi_L(y)} \, e^{i\phi_R(X)} \, e^{-\overset{i\pi}{[\phi_L,\phi_R]}} = -\Psi_L(y)\,\Psi_R(X)$$

Moreover

$$\Psi_R(x)\Psi_R^{\dagger}(y) = \frac{1}{2\pi\alpha} e^{iK_F(x-y)} e^{i\phi_R(x)} e^{-i\phi_R(y)} =$$

$$= \frac{1}{2\pi\alpha} e^{iK_F(x-y)} e^{i(\phi_R(x)-\phi_R(y))} e^{+\frac{1}{2}[\phi_R(x),\phi_R(y)]}$$

$$[\phi_R(x), \phi_R(y)] = \frac{2\pi}{L}\sum_{q>0}\frac{1}{q}\left(e^{iq(x-y)} - e^{-iq(x-y)}\right) + \frac{2\pi}{L} i(x-y)$$

$$= \frac{2\pi}{L}\sum_q \frac{1}{q} e^{iq(x-y)} e^{-\alpha|q|} = 2i\tan^{-1}\frac{(x-y)}{\alpha} \simeq i\pi\,\mathrm{sign}(x-y) \qquad (14)$$

Analogously

$$[\phi_L(x), \phi_L(y)] = -2i\tan^{-1}\frac{(x-y)}{\alpha} \simeq -i\pi\,\mathrm{sign}(x-y) \qquad (15)$$

Hence

$$\Psi_R(x)\Psi_R^{\dagger}(y) = \frac{1}{2\pi\alpha} e^{iK_F(x-y)} e^{i(\phi_R(x)-\phi_R(y))} e^{+i\frac{\pi}{2}\mathrm{sign}(x-y)} \qquad (16)$$

while

$$\Psi_R^{\dagger}(y)\Psi_R(x) = \frac{1}{2\pi\alpha} e^{iK_F(x-y)} e^{i(\phi_R(x)-\phi_R(y))} e^{+i\frac{\pi}{2}\mathrm{sign}(y-x)} \qquad (17)$$

Hence, for $|x-y| \gg \alpha$,

$$\psi_R(x)\psi_R^\dagger(y) + \psi_R^\dagger(y)\psi_R(x) = 0 = \{\psi_R(x), \psi_R^\dagger(y)\}$$

The same holds for ψ_L

$$\{\psi_L(x), \psi_L^\dagger(y)\} = 0 \qquad |x-y| \gg \alpha.$$

What does it happen when we send $x \to y$? We can not simply take $\phi_R(x) - \phi_R(y) \to 0$ in (16) and (17), since the exponential operator is singular. We first have to normal order it and then take the limit $x \to y$.

$$\phi_R(x) - \phi_R(y) \simeq \frac{1}{L}\sum_{q>0}\sqrt{\frac{2\pi L}{q}}\left(e^{-iqx} - e^{-iqy}\right) b^+_{Rq}$$
$$+ \frac{1}{L}\sum_{q>0}\sqrt{\frac{2\pi L}{q}}\left(e^{iqx} - e^{iqy}\right) b_{Rq} = \phi^{(+)} + \phi^{(-)}$$

$$e^{i(\phi^{(+)} + \phi^{(-)})} = e^{i\phi^{(+)}}\, e^{i\phi^{(-)}}\, e^{\frac{1}{2}[\phi^{(+)}, \phi^{(-)}]}$$

$$\frac{1}{2}[\phi^{(+)}, \phi^{(-)}] = -\frac{1}{2L}\, 2\pi \sum_{q>0}\frac{1}{q}\left(e^{-iqx} - e^{-iqy}\right)\left(e^{iqx} - e^{iqy}\right) e^{-\alpha q}$$

$$= -\frac{\pi}{L}\sum_{q>0}\frac{1}{q}\, e^{-\alpha q}\left(2 - 2\cos q(x-y)\right) = \frac{1}{2}\ln\frac{\alpha^2}{\alpha^2 + (x-y)^2} \qquad (18)$$

[The same result is valid for ϕ_L]

By adding the phase factor $\frac{1}{2}\left[\phi_{R/L}(x), \phi_{R/L}(y)\right] = \pm i \tan^{-1}\frac{x-y}{\alpha}$, we have

$$\frac{1}{2}\ln\frac{\alpha^2}{\alpha^2+(x-y)^2} + i\tan^{-1}\frac{x-y}{\alpha} = \ln\frac{\alpha}{\alpha - i(x-y)}$$

in $\psi_R(x)\psi_R^+(y)$ $\left[\psi_L^+(y)\,\psi_L(x)\right]$

$$\frac{1}{2}\ln\frac{\alpha^2}{\alpha^2+(x-y)^2} - i\tan^{-1}\frac{x-y}{\alpha} = \ln\frac{\alpha}{\alpha + i(x-y)}$$

in $\psi_R^+(y)\psi_R(x)$ $\left[\psi_L(x)\,\psi_L^+(y)\right]$

$$\left\{\psi_{R/L}(x), \psi_{R/L}^+(y)\right\} \simeq \frac{1}{2\pi\alpha}\, e^{\pm i K_F (x-y)} : e^{\pm i\left(\phi_{R/L}(x) - \phi_{R/L}(y)\right)} :$$

$$\left[\frac{\alpha}{\alpha \mp i(x-y)} + \frac{\alpha}{\alpha \pm i(x-y)}\right] \qquad (13)$$

$$= e^{\pm i K_F(x-y)} : e^{\pm i\left(\phi_R(x) - \phi_R(y)\right)} : \underbrace{\frac{1}{\pi}\frac{\alpha}{\alpha^2+(x-y)^2}}_{\delta_\alpha(x-y)} \xrightarrow[\alpha\to 0]{} \delta(x-y)$$

Therefore the anticommutator gives a δ-function regularized by α. This was predictable since $\psi^+_{R,L}(x)$ does not really create a fermion at x, but in a wave packet around x of width α.

The previous calculation also shows that a particular care has to be taken when we consider the product of two bosonized operators at the same point. The correct way to do the product is

- given two operators $A(x)$ and $B(x)$ then

$$A(x)\,B(x) = \lim_{x\to y} \Big(A(x)\,B(y)\Big)\Big|_{\frac{|x-y|}{\alpha} \gg 1}$$

Example: How to recover $\psi^+_R(x)\psi_R(x) = \rho_R(x)$

Instead of $\{\psi_R(x), \psi^+_R(y)\}$, let us calculate $\psi^+_R(x)\psi_R(y)$. We find, see Eq (18),

$$\psi^+_R(x)\psi_R(y) \simeq \frac{1}{2\pi\alpha}\, e^{-iK_F(x-y)} : e^{-i(\phi_R(x)-\phi_R(y))} : \frac{\alpha}{\alpha - i(x-y)}$$

$$\simeq \frac{1}{2\pi}\,\frac{1}{\alpha - i(x-y)} : 1 - iK_F(x-y) - i\,\nabla\phi_R(x)\,(x-y) + \cdots :$$

Hence

$$\psi^+_R(x)\,\psi_R(x) = \lim_{x\to y}\Big(\psi^+_R(x)\,\psi_R(y)\Big)_{\frac{|x-y|}{\alpha}\gg 1} = \frac{1}{2\pi}\nabla\phi_R(x) + \frac{K_F}{2\pi} \equiv \rho_R(x)$$

The equations of motion for ρ_R and ρ_L imply that

$$b_{Rq}(t) = b_{Rq}\, e^{-i v_F q t} \qquad b_{Lq}(t) = b_{Lq}\, e^{-i v_F q t}$$

hence

$$\phi_R(x,t) = \phi_R(x - v_F t)$$

$$\phi_L(x,t) = \phi_L(x + v_F t)$$

Therefore, analogously to (18), we find the following correlation functions

$$\langle \psi_{R/L}(x,t)\, \psi^{+}_{R/L}(0,0) \rangle = \frac{1}{2\pi}\, e^{\pm i K_F x} \frac{1}{\alpha \mp i(x \mp v_F t)}$$

$$\langle \psi^{+}_{R/L}(0,0)\, \psi_{R/L}(x,t) \rangle = \frac{1}{2\pi}\, e^{\pm i K_F x} \frac{1}{\alpha \pm i(x \mp v_F t)}$$

so that

$$G_{R/L}(x,t) = -i \langle T\left(\psi_{R/L}(x t)\, \psi^{+}_{R/L}(0,0) \right) \rangle$$

$$= \frac{1}{2\pi}\, e^{\pm i K_F x} \frac{1}{\pm x - v_F t + i\alpha\, \mathrm{sign}(t)}$$

$$\langle \rho_{R/L}(x,t)\, \rho_{R/L}(0) \rangle = \frac{1}{4\pi^2} \langle \nabla\phi_{R/L}(x,t)\, \nabla\phi_{R/L}(0) \rangle$$

$$= \frac{1}{2\pi} \frac{1}{\left[\alpha \mp i(x \mp v_F t)\right]^2}$$

For future utility, let us calculate

$$G_{\beta_R \beta_L}(x,t) = \langle e^{i\beta_R \phi_R(x,t) + i\beta_L \phi_L(x,t)}\, e^{-i\beta_R \phi_R(0) - i\beta_L \phi_L(0)} \rangle$$

Since $\left[\phi_R(x),\ \phi_L(y) - \phi_L(z)\right] = 0$ then

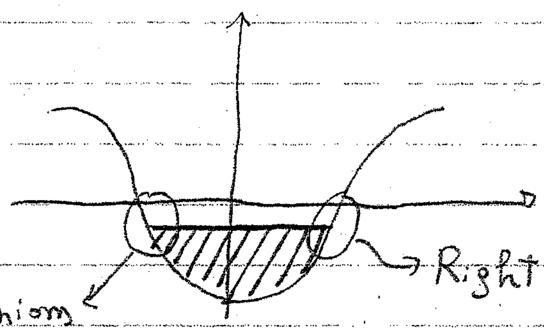
$$G_{\beta_R \beta_L}(x,t) = \langle e^{i\beta_R \phi_R(x,t)}\, e^{-i\beta_R \phi_R(0)} \rangle \langle e^{i\beta_L \phi_L(x,t)}\, e^{-i\beta_L \phi_L(0)} \rangle$$

$$= \left[\frac{\alpha}{\alpha - i(x - v_F t)}\right]^{\beta_R^2} \left[\frac{\alpha}{\alpha + i(x + v_F t)}\right]^{\beta_L^2}$$

Let us now switch on interaction

$$H_{int} = \frac{1}{2L} \sum_q V(q)\, \rho(q)\rho(-q)$$

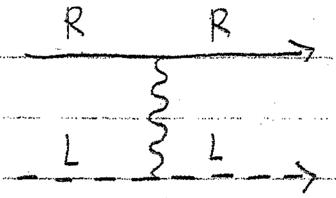
Since we are assuming a weak coupling approach, we keep of all the scattering amplitudes generated by $V(q)$, only those which act among fermions close to the Fermi momenta.



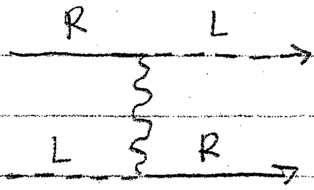
Left fermions ----→

Right fermions : ⟶

For generic filling $\frac{N}{L}$, there are two such scattering processes

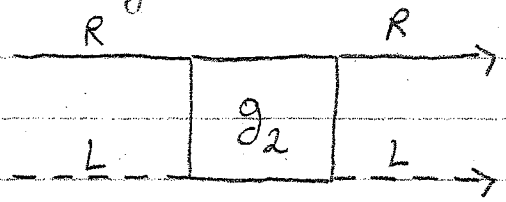


$V(q) \sim V(q=0)$



$V(q) \sim V(q=2K_F)$

Shortly



$$g_2 = V(0) - V(2K_F)$$

and

$$H_{int} = \frac{g_2}{L} \sum_q \rho_R(q)\rho_L(-q)$$

Therefore, the low energy Hamiltonian is

$$H = H_0 + H_{int} = \frac{\pi v_F}{L} \sum_q \rho_R(q)\rho_R(-q) + \rho_L(q)\rho_L(-q)$$
$$+ \frac{g_2}{L} \sum_q \rho_R(q)\rho_L(-q) \qquad (20)$$

Written in terms of the b_{Rq} and b_{Lq}

$$H = \frac{v_F}{L} \sum_{q>0} q \left(b^+_{Rq} b_{Rq} + b^+_{Lq} b_{Lq} \right)$$
$$+ \frac{g_2}{2\pi L} \sum_{q>0} q \left(b_{Rq} b_{Lq} + b^+_{Lq} b^+_{Rq} \right)$$

is a bilinear form which can easily be diagonalized. However we will adopt a different approach, which is more useful for the following

Let us introduce two new fields

$$\phi(x) = \frac{1}{\sqrt{4\pi}} \left(\phi_R(x) + \phi_L(x) \right)$$

$$\Theta(x) = \frac{1}{\sqrt{4\pi}} \left(\phi_L(x) - \phi_R(x) \right) \qquad (21)$$

as well as $\Pi(x) = \nabla \Theta(x)$

$$[\phi(x), \Theta(y)] = \frac{1}{4\pi} \Big\{ -[\phi_R(x), \phi_R(y)] + [\phi_L(x), \phi_L(y)]$$
$$+ [\phi_R(x), \phi_L(y)] - [\phi_L(x), \phi_R(y)] \Big\}$$

$$= \frac{1}{4\pi} \left\{ -4i \tan^{-1} \frac{(x-y)}{\alpha} - 2\pi i \right\}$$

$$\simeq \frac{1}{4\pi} \left\{ -2\pi i \, \mathrm{sign}(x-y) - 2\pi i \right\} = -i \, \Theta(x-y) \qquad (22)$$

Heavyside step function

Therefore

$$[\phi(x), \Pi(y)] = i \, \delta(x-y) \qquad (23)$$

are conjugate variables.

From (21) we have

$$\phi_R(x) = \sqrt{\pi}\left(\phi(x) - \Theta(x)\right)$$
$$\phi_L(x) = \sqrt{\pi}\left(\phi(x) + \Theta(x)\right)$$

Since $\rho_{R/L} = \frac{1}{2\pi}\nabla\phi_{R,L}$

$$H = \frac{v_F}{4\pi}\int dx\ \nabla\phi_R\nabla\phi_R + \nabla\phi_L\nabla\phi_L$$
$$+ \frac{g_2}{4\pi^2}\int dx\ \nabla\phi_R\nabla\phi_L$$

$$= \frac{1}{2}v_F\int dx\ \nabla\phi\nabla\phi + \Pi\Pi \quad + \frac{g_2}{4\pi}\int dx\ \nabla\phi\nabla\phi - \Pi\Pi$$

$$= \frac{1}{2}v_F\int dx\ \left(1 + \frac{g_2}{2\pi v_F}\right)\nabla\phi\nabla\phi + \left(1 - \frac{g_2}{2\pi v_F}\right)\Pi\Pi$$

$$= \frac{1}{2}v_F\sqrt{1 - \left(\frac{g_2}{2\pi v_F}\right)^2}\int dx\ \frac{1}{K}\nabla\phi\nabla\phi + K\,\Pi\Pi \qquad (24)$$

where $K = \sqrt{\dfrac{1 - \frac{g_2}{2\pi v_F}}{1 + \frac{g_2}{2\pi v_F}}}$. Notice that the density and the current are given by

$$\rho(x) = \rho_R(x) + \rho_L(x) = \frac{1}{\sqrt{\pi}}\nabla\phi(x)$$
$$J(x) = v_F\left(\rho_R(x) - \rho_L(x)\right) = -\frac{1}{\sqrt{\pi}}\nabla\Theta(x)$$

The canonical transformation

$$\phi \to \sqrt{K}\,\phi$$
$$\Pi \to \frac{1}{\sqrt{K}}\,\Pi \qquad (25)$$

makes (24) diagonal

$$H = \frac{1}{2} v \int dx \; \nabla\phi\nabla\phi + \Pi\Pi \qquad \left[v = v_F \sqrt{1-\left(\frac{g_2}{2\pi v_F}\right)^2} \right]$$

This Hamiltonian describes the normal modes of the low energy particle-hole excitations (in Landau theory the zero sounds, which exhaust in one dimension the particle-hole spectrum). The big advantage of Bosonization is that we can also calculate single particle correlations.

$$\phi_{R/L}(x) \to \sqrt{\pi}\left(\sqrt{K}\,\phi \mp \frac{1}{\sqrt{K}}\,\Theta\right)$$

$$= \frac{1}{2}\sqrt{K}\left(\phi_R + \phi_L\right) \mp \frac{1}{2\sqrt{K}}\left(\phi_L - \phi_R\right)$$

$$= \frac{1}{2}\left(\sqrt{K} \pm \frac{1}{\sqrt{K}}\right)\phi_R + \frac{1}{2}\left(\sqrt{K} \mp \frac{1}{\sqrt{K}}\right)\phi_L$$

the new fields which diagonalize H

Therefore

$$\langle \psi_{R/L}(xt)\,\psi^{+}_{R/L}(0)\rangle = \frac{1}{2\pi\alpha}\left[\frac{\alpha}{\alpha - i(x-vt)}\right]^{\frac{1}{4}\left(\sqrt{K}\pm\frac{1}{\sqrt{K}}\right)^2}\left[\frac{\alpha}{\alpha + i(x+vt)}\right]^{\frac{1}{4}\left(\sqrt{K}\mp\frac{1}{\sqrt{K}}\right)^2}$$

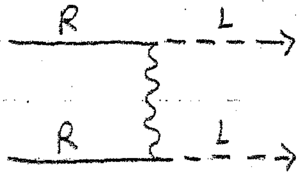
$$= \frac{1}{2\pi\alpha}\,\frac{\alpha}{\alpha \mp i(x\mp vt)}\,\frac{\alpha \mp i(x\mp vt)}{\alpha}\left[\frac{\alpha}{\alpha - i(x-vt)}\right]^{\frac{1}{4}\left(K+\frac{1}{K}\pm 2\right)}\left[\frac{\alpha}{\alpha + i(x+vt)}\right]^{\frac{1}{4}\left(K+\frac{1}{K}\mp 2\right)}$$

$$= \frac{1}{2\pi}\,\frac{1}{\alpha \mp i(x\mp vt)}\left[\frac{\alpha^2}{(\alpha - i(x-vt))(\alpha + i(x+vt))}\right]^{\frac{1}{4}\left(K+\frac{1}{K}-2\right)}$$

$$\frac{1}{4}\left(K+\frac{1}{K}-2\right) \to \frac{1}{4K}(K-1)^2$$

The single particle Green's functions acquire an anomalous exponent $\frac{1}{2K}(K-1)^2$

Let us now consider the half filling case $\frac{N}{L} = \frac{1}{2}$. In this case $K_F = \pi \frac{N}{L} = \frac{\pi}{2}$, and an additional scattering amplitude is allowed close to the Fermi points, namely



$\sim V(2K_F) = V(\pi)$ (Umklapp scattering)

In real space it corresponds to an interaction term $V(x-y) = V(\pi)(-1)^{x-y} f(|x-y|)$, where $f(r)$ is some short range function such that $\int dr\, f(r) = 1$. Hence

$$\text{Umklapp} = V(\pi) \int dx dy\, (-1)^{x-y} f(|x-y|) \left[\psi_R^+(x) \psi_R^+(y) \psi_L(y) \psi_L(x) + H.c. \right]$$

$$= \frac{V(\pi)}{(2\pi\alpha)^2} \int dx dy\, f(|x-y|) \left[e^{-\frac{1}{2}[\phi_R(x),\phi_R(y)]} e^{-\frac{1}{2}[\phi_L(y),\phi_L(x)]} e^{-i(\phi_R(x)+\phi_R(y)+\phi_L(x)+\phi_L(y))} + H.c. \right]$$

$$\simeq -V(\pi) \frac{2}{(2\pi\alpha)^2} \int dx \cos\sqrt{16\pi}\,\phi(x) \equiv \frac{2g_3}{(2\pi\alpha)^2} \int dx \cos\sqrt{16\pi}\,\phi(x)$$

with $g_3 = -V(\pi)$. After the canonical transformation (25)

$$H_{\text{Umklapp}} = \frac{2g_3}{(2\pi\alpha)^2} \int dx \cos\sqrt{16\pi K}\,\phi(x)$$

and the Hamiltonian reads

$$H = \frac{1}{2} v \int dx \quad \nabla\phi\nabla\phi + \Pi\Pi \quad + \frac{1}{v} \frac{4 g_3}{(2\pi\alpha)^2} \cos\sqrt{16\pi K}\, \phi(x) \qquad (26)$$

which is a sine-Gordon model. The dimension of the UmKlapp is $4K$, to be compared with the dimension of $\nabla\phi\nabla\phi + \Pi\Pi$, which is 2. Therefore the UmKlapp gets marginal at $K=\frac{1}{2}$ and relevant for $K<\frac{1}{2}$. When it is relevant, the UmKlapp locks the field ϕ to a value which minimizes $g_3 \cos\sqrt{16\pi K}\,\phi$, and a gap opens in the spectrum.

- How to calculate dimensions.

Let us normal order $\cos\sqrt{16\pi}\,\phi(x)$. We get

$$\cos\sqrt{16\pi K}\,\phi(x) = \; :\cos\sqrt{16\pi K}\,\phi(x): \, \exp\left[-4K \frac{2\pi}{L} \sum_{q>0} \frac{1}{q} e^{-\alpha q}\right]$$

$q>0 \Rightarrow q \geq \frac{2\pi}{L}$

$$\simeq -4K \int_{\frac{2\pi}{L}}^{\frac{1}{\alpha}} \frac{dq}{q} = -4K \ln \frac{L}{2\pi\alpha}$$

and

$$\frac{4 g_3}{v} \frac{1}{(2\pi\alpha)^2} \cos\sqrt{16\pi K}\,\phi = \frac{4 g_3}{v} (2\pi\alpha)^{4K-2} \left(\frac{1}{L}\right)^{4K} :\cos\sqrt{16\pi K}\,\phi:$$

The relevance / irrelevance is controlled by the behavior under the limit $\alpha \to 0$.

Bosonization of the Heisenberg model

Let us consider a 1d Heisenberg model

$$H = J \sum_{i=1}^{L} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \qquad (1)$$

We can map this model onto a spinless fermion model via the so called Jordan-Wigner transformation.
We write

$$S_i^z = \frac{1}{2} - c_i^+ c_i = \frac{1}{2} - n_i \qquad (2)$$

where $n_i = 0,1$ is the spinless fermion number at site i.
Therefore $S_i^z = \frac{1}{2}$ corresponds to an empty site, and $S_i^z = -\frac{1}{2}$ to an occupied site. Let us show that

$$S_i^+ = e^{i\pi \sum_{j=1}^{i-1} n_j} c_i \qquad S_i^- = c_i^+ e^{-i\pi \sum_{j=1}^{i-1} n_j} \qquad (3)$$

are good representation for spin operators. Let us define

$$\xi_i = e^{i\pi \sum_{j=1}^{i-1} n_j}$$

the string operator. Since $\xi_i^2 = 1$, we easily check that

$$[S_i^+, S_i^-] = [c_i, c_i^+] = 1 - 2n_i = 2S_i^z$$

$$[S_i^+, S_i^z] = -\xi_i [c_i, n_i] = -S_i^+$$

Moreover, since for $i > j$

$$\xi_i c_j = -c_j \xi_i$$

while $\xi_i c_j = c_j \xi_i$ for $i \le j$, then

$$[S_i^+, S_j^{+(-)}] = c_i \xi_i \xi_j c_j^{(+)} - c_j^{(+)} \xi_j \xi_i c_i = 0$$

as it should be. Let us rewrite (1) using (2) and (3)

$$S_i^x S_{i+1}^x + S_i^y S_{i+1}^y = \frac{1}{2}\left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+\right) =$$

$$= \frac{1}{2}\left(c_i \xi_i \xi_{i+1} c_{i+1}^+ + c_i^+ \xi_i \xi_{i+1} c_{i+1}\right) =$$

$$= \frac{1}{2}\left(c_i e^{i\pi n_i} c_{i+1}^+ + c_i^+ e^{i\pi n_i} c_{i+1}\right) = \frac{1}{2}\left(c_{i+1}^+ c_i + c_i^+ c_{i+1}\right)$$

$$S_i^z S_{i+1}^z = \left(\frac{1}{2} - n_i\right)\left(\frac{1}{2} - n_{i+1}\right) = \frac{1}{4} + n_i n_{i+1} - \frac{1}{2} n_i - \frac{1}{2} n_{i+1}$$

Hence

$$H = \frac{J}{2} \sum_i c_i^+ c_{i+1} + \text{H.c.} + J\Delta \sum_i \frac{1}{4} + n_i n_{i+1} - \frac{1}{2} n_i - \frac{1}{2} n_{i+1} \qquad (4)$$

In the case of an open chain, i.e. site 1 not coupled to L, the last two terms are

$$-\frac{J\Delta}{2} \sum_i n_i + n_{i+1} = \frac{J\Delta}{2}\left(n_1 + n_L\right) - J\Delta \sum_i n_i$$

Since $\sum_i n_i = N$, the total number of fermions, is a conserved quantity, this term introduces a scattering potential at the edge sites 1 and L. It is therefore better to work with a closed chain, by adding the bond

$$\frac{J}{2}\left(S_1^+ S_L^- + S_1^- S_L^+\right) + J\Delta S_1^z S_L^z \qquad (5)$$

We notice that

$$S_1^+ S_L^- = c_1 e^{i\pi \sum_{j=1}^{L-1} n_j} c_L^+ = c_1 c_L^+ e^{i\pi \sum_{j=1}^{L-1} n_j} =$$

$$= c_1 c_L^+ e^{i\pi \sum_{j=1}^{L} n_j} e^{i\pi n_L} = (-1)^N c_1 c_L^+ e^{i\pi n_L} = (-1)^N c_1 c_L^+$$

Hence

$$(5) = \frac{J}{2}(-1)^N \left(c_1 c_L^+ + H.c. \right) + J\Delta\left(\frac{1}{4} + n_1 n_L - \frac{1}{2} n_1 - \frac{1}{2} n_L \right)$$

We further apply the transformation

$$c_i \longrightarrow (-1)^i c_i \qquad (6)$$

such that (1) + (5) read (apart from constant terms)

$$H = -\frac{J}{2} \sum_i \left(c_i^+ c_{i+1} + H.c. \right) + J\Delta \sum_i n_i n_{i+1}$$

$$+ \frac{J}{2}(-1)^{N+L} \left(c_1^+ c_L + c_L^+ c_1 \right) + J\Delta\, n_1 n_L \qquad (7)$$

The factor $(-1)^{N+L}$ makes the ground state non degenerate, as it should be for bosonic models. For simplicity, let us take $N+L=$ odd. In such a case, (7) is equivalent to the spinless fermion Hamiltonian

$$H = -t \sum_i c_i^+ c_{i+1} + H.c. + V \sum n_i n_{i+1}$$

$$= \sum_k \varepsilon_k c_k^+ c_k + \frac{1}{L} \sum_q V \cos qa\, \rho_q \rho_{-q} \qquad (8)$$

with $t = \frac{J}{2}$, $V = J\Delta$ and $\varepsilon_K = -2t\cos Ka$, a being the lattice spacing. The Hamiltonian (8) can be analysed within bosonization. In this specific case

$$K_F = \frac{\pi}{a}\frac{N}{L} = \frac{\pi}{2a} - \frac{\pi}{La} S^z_{TOTAL}$$

$$v_F = 2ta \sin K_F a$$

$$g_2 = 2V\left(1 - \cos 2K_F a\right)$$

At half filling $g_3 = -2V\cos 2K_F a = +2V$ is the Umklapp.
As we showed, the Umklapp is irrelevant/marginal/relevant if

$$K \gtreqless 1/2$$

What does $K = 1/2$ correspond to? Let us consider the bosonized expression of the spin operators.
As before, the spin operators will be related to phase field operators which are assumed to be slowly varying, namely varying on distances $\Delta x \gg \frac{1}{K_F} \sim a$. In this approximation, a continuum limit $x = i \times a$ = continuous variable is well defined, and

$$S_i^+ \simeq S_x^+ \simeq (-1)^{x/a} e^{i\pi\int^x dy\, \rho(y)} \left(\Psi_R(x) + \Psi_L(x)\right)$$

where $(-1)^{x/a}$ comes from (6), and

$$\rho(y) = \rho_R(y) + \rho_L(y) = \frac{1}{2\pi}\nabla\phi_R(y) + \frac{1}{2\pi}\nabla\phi_L(y) + \frac{N}{L}$$

More explicitly

$$S^+_x \simeq \frac{(-1)^{x/a}}{\sqrt{2\pi\alpha}} e^{\frac{i}{2}\left(\phi_R(x)+\phi_L(x)\right)} e^{iK_F x}\left[e^{iK_F x + i\phi_R(x)} + e^{-iK_F x - i\phi_L(x)}\right]$$

$$\simeq \frac{(-1)^{x/a}}{\sqrt{2\pi\alpha}} e^{\frac{i}{2}\left(\phi_R(x)+\phi_L(x)\right)} - \frac{i}{\sqrt{2\pi\alpha}} e^{\frac{i}{2}\left(3\phi_R(x)+\phi_L(x)\right)}$$

$$= \frac{(-1)^{x/a}}{\sqrt{2\pi\alpha}} e^{-i\sqrt{\pi}\,\Theta(x)} - \frac{i}{\sqrt{2\pi\alpha}} e^{i\sqrt{\pi}\left(2\phi(x)-\Theta(x)\right)}$$

$$= \frac{(-1)^{x/a}}{\sqrt{2\pi\alpha}} e^{-i\sqrt{\frac{\pi}{K}}\,\Theta(x)} - \frac{i}{\sqrt{2\pi\alpha}} e^{i\sqrt{\pi}\left(2\sqrt{K}\phi(x)-\frac{1}{\sqrt{K}}\Theta(x)\right)}$$

Therefore

$$\langle S^+_x S^-_y \rangle \sim (-1)^{\frac{x-y}{a}} \left(\frac{1}{|x-y|}\right)^{\frac{1}{2K}} \qquad (S)$$

On the other hand

$$S^z_i \sim S^z_x \simeq \Psi^+(x)\Psi(x) = \Psi^+_R(x)\Psi_R(x) + \Psi^+_L(x)\Psi_L(x)$$

$$+ \Psi^+_R(x)\Psi_L(x) + \Psi^+_L(x)\Psi_R(x) =$$

$$= \frac{1}{\sqrt{\pi}}\nabla\phi(x) + \frac{1}{2\pi\alpha}\left[e^{-2iK_F x} e^{-i\phi_R(x)} e^{-i\phi_L(x)} + H.c.\right]$$

$$= \frac{1}{\sqrt{\pi}}\nabla\phi(x) + \frac{1}{2\pi\alpha}\left[i e^{2iK_F x} e^{-i\sqrt{4\pi}\,\phi(x)} + H.c.\right] =$$

$$= \frac{1}{\sqrt{\pi}} \nabla\phi(x) + \frac{1}{2\pi\alpha} \left[i e^{-2ik_F x} e^{-i\sqrt{4\pi K}\,\phi(x)} + H.c. \right]$$

If $S^z_{TOT} = 0$, which includes the ground state for spin isotropic Hamiltonians, then $N = \frac{L}{2}$, i.e. $k_F = \frac{\pi}{2}$, hence the leading contribution to S^z_x is

$$S^z_x \sim \frac{1}{2\pi\alpha} (-1)^{x/a} \left[i e^{-i\sqrt{4\pi K}\,\phi(x)} + H.c. \right]$$

Therefore

$$\langle S^z_x S^z_y \rangle \sim (-1)^{\frac{x-y}{a}} \left(\frac{1}{|x-y|} \right)^{2K} \qquad (10)$$

If $\Delta = 1$, spin isotropy is a symmetry of (1), hence the ground state should have $S^z_{TOT} = 0$ and (10) equal to (9), which implies

$$2K = \frac{1}{2K} \implies K = \frac{1}{2}$$

Therefore for $\Delta < 1$, $K > 1/2$ [$K = 1$ for $\Delta = 0$!] and Umklapp is irrelevant. The spin excitations are gapless and the correlation functions decay as power laws. If $\Delta > 1$ $K < \frac{1}{2}$, Umklapp is relevant. A gap opens and $\sqrt{16\pi}\,\phi$ is locked to values π mod (2π) [see Eq. (23) for $g_3 > 0$]. In this case

$$\langle S^z_x S^z_y \rangle \sim (-1)^{\frac{x-y}{a}} \text{ constant}$$

which corresponds to a Néel state.
What does it happen if $K = \frac{1}{2}$, namely at the isotropic point?

In this case Umklapp is marginal, and from (8) $V = 2t$. This implies a strong coupling limit which is outside the limit of applicability of bosonization of model (8). We must resort to better controlled models which represent (1)

The Hubbard model

The Hubbard model is described by the Hamiltonian

$$H = -t \sum_{i\sigma} c^{+}_{i\sigma} c_{i\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \qquad (1)$$

The interaction term represents an on site Coulomb repulsion which prevents two electrons, with opposite spin, to be on the same site. If the number of electrons N is equal to the number of sites L and $U \gg t$, the Hubbard model is insulating, each site being occupied by an electron which does not move, since that would cost energy $U \gg t$. Therefore, at energy $\ll U$, charge degrees of freedom are frozen, and only the spin degrees of freedom matter. Indeed (1) can be mapped onto an Heisenberg antiferromagnet, clearly spin isotropic, with $J = \frac{4t^2}{U}$.

However, at half filling, the model (1) is insulating not only for $U \gg t$, but for any value of $U > 0$. This happens because of the nesting property of the Fermi surface together with the filling being commensurate. Therefore charge degrees of freedom are frozen for any U, and since there is no phase transition as a function of U, we expect the spin degrees of freedom to still be described by an Heisenberg model. Therefore we have the possibility to access the spin isotropic point of the Heisenberg model in the weak coupling, $U \ll t$, regime of the Hubbard model (1)

Let us bosonize (1). We have to introduce four fields $\phi_{R\uparrow}$, $\phi_{L\uparrow}$, $\phi_{R\downarrow}$ and $\phi_{L\downarrow}$. Moreover, to preserve the proper anticommutation relations among the Fermi fields we must impose that

$$[\phi_p(x), \phi_q(y)] = \pm i\pi \quad \text{for } p \neq q \qquad p,q = R\uparrow, L\uparrow, R\downarrow, L\downarrow$$

We write ($\sigma = \uparrow, \downarrow$)

$$\phi_{R\sigma}(x) = \frac{1}{L} \sum_{q>0} \sqrt{\frac{2\pi L}{q}} \left(e^{iqx} b_{R\sigma q} + e^{-iqx} b^+_{R\sigma q} \right) + K_{FR\sigma} x - \theta_{R\sigma} + (\cdots)_{R\sigma}$$

$$\phi_{L\sigma}(x) = \frac{1}{2} \sum_{q>0} \sqrt{\frac{2\pi L}{q}} \left(e^{-iqx} b_{L\sigma q} + e^{iqx} b^+_{L\sigma q} \right) + K_{FL\sigma} x + \theta_{L\sigma} + (\cdots)_{L\sigma}$$

where

$$(\cdots)_{R\uparrow} = \frac{\pi}{2} N_{L\uparrow} + \frac{\pi}{2} N_{R\downarrow} + \frac{\pi}{2} N_{L\downarrow}$$

$$(\cdots)_{L\uparrow} = \frac{\pi}{2} N_{R\uparrow} - \frac{\pi}{2} N_{R\downarrow} - \frac{\pi}{2} N_{L\downarrow}$$

$$(\cdots)_{R\downarrow} = -\frac{\pi}{2} N_{R\uparrow} - \frac{\pi}{2} N_{L\uparrow} + \frac{\pi}{2} N_{L\downarrow}$$

$$(\cdots)_{L\downarrow} = +\frac{\pi}{2} N_{R\uparrow} + \frac{\pi}{2} N_{L\uparrow} + \frac{\pi}{2} N_{R\downarrow}$$

This choice implies that

$$[\phi_{R\uparrow}, \phi_{L\uparrow}] = [\phi_{R\downarrow}, \phi_{L\downarrow}] = -i\pi$$

$$[\phi_{R\uparrow}, \phi_{R\downarrow}] = [\phi_{L\uparrow}, \phi_{L\downarrow}] = i\pi$$

$$[\phi_{R\uparrow}, \phi_{L\downarrow}] = [\phi_{L\uparrow}, \phi_{R\downarrow}] = -i\pi$$

The advantage of this choice is that, if we introduce

$$\phi_\sigma = \frac{1}{\sqrt{4\pi}}\left(\phi_{R\sigma} + \phi_{L\sigma}\right) \qquad \Theta_\sigma = \frac{1}{\sqrt{4\pi}}\left(\phi_{L\sigma} - \phi_{R\sigma}\right)$$

then $[\phi_\uparrow, \phi_\downarrow] = [\phi_\uparrow, \Theta_\downarrow] = [\Theta_\uparrow, \phi_\downarrow] = 0$ while $[\Theta_\uparrow, \Theta_\downarrow] = i$.

We introduce the charge and spin combinations

$$\phi_{c/s} = \frac{1}{\sqrt{2}}\left(\phi_\uparrow \pm \phi_\downarrow\right), \qquad \Theta_{c/s} = \frac{1}{\sqrt{2}}\left(\Theta_\uparrow \pm \Theta_\downarrow\right) \qquad (2)$$

as well as $\Pi_{c/s} = \nabla \Theta_{c/s}$, which satisfy, thanks to our choice,

$$[\phi_c, \phi_s] = [\Pi_c, \Pi_s] = [\phi_c, \Pi_s] = [\Pi_c, \phi_s] = 0$$

and $[\phi_{c/s}(x), \Pi_{c/s}(y)] = i\,\delta(x-y)$.

In terms of these fields, the non interacting part of the Hamiltonian can be written as

$$H_0 = \frac{v_F}{2} \sum_{q=c,s} \int dx \; \nabla\phi_q \nabla\phi_q + \Pi_q \Pi_q$$

with $v_F = 2ta\sin k_F a$. Let us bosonize the interaction

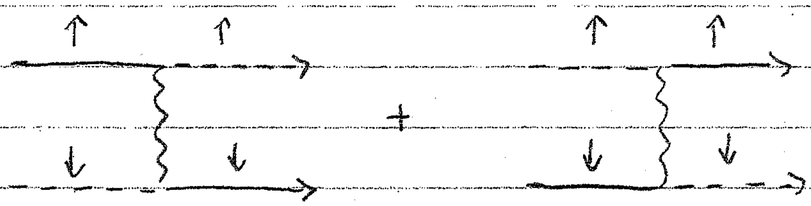
$$H_{int} = U \sum_i n_{i\uparrow} n_{i\downarrow} = \frac{U}{L} \sum_q \rho_{q\uparrow} \rho_{-q\downarrow}$$

The term with $q \sim 0$ is

$$V_a \int dx \left(\frac{1}{2\pi}\right)^2 \left(\nabla\phi_{R\uparrow} + \nabla\phi_{L\uparrow}\right)\left(\nabla\phi_{R\downarrow} + \nabla\phi_{L\downarrow}\right)$$

$$= \frac{V_a}{2\pi} \int dx \quad \nabla\phi_c \nabla\phi_c - \nabla\phi_s \nabla\phi_s \qquad (4)$$

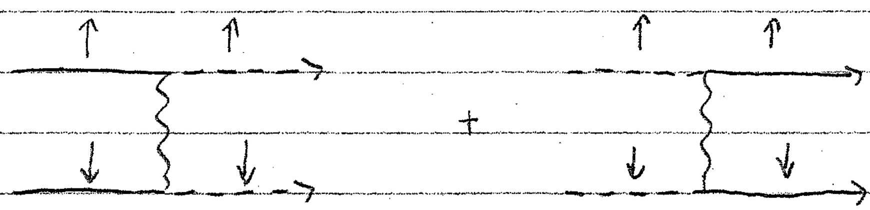
There are two processes at $q \sim 2K_F$. The first is the so called backscattering



By bosonization this term reads

$$\frac{2V_a}{(2\pi\alpha)^2} \int dx \quad \cos\sqrt{8\pi}\,\phi_s(x) \qquad (5)$$

The other process is the Umklapp



which reads

$$-\frac{2V_a}{(2\pi\alpha)^2} \int dx \quad \cos\left(\sqrt{8\pi}\,\phi_c(x) + 4K_F x\right) \qquad (6)$$

At half filling, $K_F = \frac{\pi}{2a}$, and $4K_F x = 2\pi \frac{x}{a}$ can be dropped out of (6) since $\frac{x}{a}$ = integer.

The full bosonized Hamiltonian at half filling is therefore

$$H = \frac{v_F}{2} \sum_{p=c,s} \int dx \; \nabla\phi_p \nabla\phi_p + \Pi_p \Pi_p$$

$$+ \frac{g_c}{2\pi} \int dx \; \nabla\phi_c \nabla\phi_c + \frac{g_s}{2\pi} \int dx \; \nabla\phi_s \nabla\phi_s \qquad (7)$$

$$+ \frac{2 g_1}{(2\pi\alpha)^2} \int dx \; \cos\sqrt{8\pi}\, \phi_s - \frac{2 g_3}{(2\pi\alpha)} \int dx \; \cos\sqrt{8\pi}\, \phi_c$$

where $g_c = Va$ $g_s = -Va$ $g_1 = g_3 = Va$. Both charge and spin fields are described by a Sine-Gordon Hamiltonian with marginal $\cos\sqrt{8\pi}\,\phi$. Since spin isotropy holds, the sine Gordon model in the spin sector has to be equivalent to the Heisenberg model with $\Delta = 1$, which corresponds in the spinless fermion model to a strong coupling point while in this case to a weak coupling one. The bilinear terms can be diagonalized by a canonical transformation

$$\phi \to \sqrt{K}\,\phi \qquad \Pi \to \frac{1}{\sqrt{K}}\,\Pi$$

where

$$K_c = \frac{1}{\sqrt{1+\frac{g_c}{\pi v_F}}} \simeq 1 - \frac{g_c}{2\pi v_F} = 1 - \frac{V_a}{2\pi v_F} < 1$$

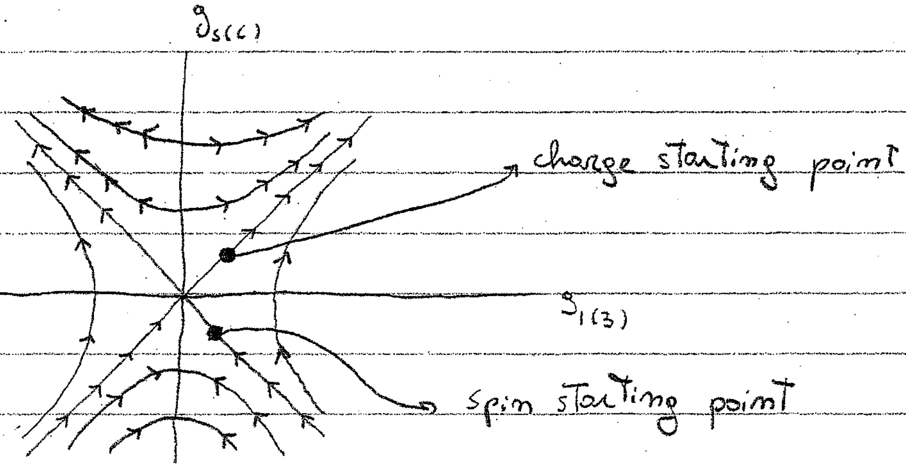
$$K_s = \frac{1}{\sqrt{1+\frac{g_s}{\pi v_F}}} \simeq 1 - \frac{g_s}{2\pi v_F} = 1 + \frac{V_a}{2\pi v_F} > 1$$

$$v_c = v_F \sqrt{1+\frac{g_c}{\pi v_F}} \qquad v_s = v_F \sqrt{1+\frac{g_s}{\pi v_F}}$$

Since all interaction terms in (7) are marginal, one can do a Renormalization group analysis, which leads to the following equations

$$\dot{g}_s = \frac{1}{\pi v_s} g_1^2 \qquad \dot{g}_1 = \frac{1}{\pi v_s} g_1 g_s$$

$$\dot{g}_c = \frac{1}{\pi v_c} g_3^2 \qquad \dot{g}_3 = \frac{1}{\pi v_s} g_3 g_c$$



Given the starting values, one finds that the Umklapp is a marginally relevant coupling. Therefore a charge gap opens and the field ϕ_c is locked to $0 \mod\left(\frac{2\pi}{\sqrt{8\pi}}\right)$. As expected, the model is insulating for any $U>0$.

On the contrary, the backscattering is marginally irrelevant. The fixed point corresponds to $g_s^* = g_1^* = 0$, hence $K_s^* = 1$. Namely the spin fixed point Hamiltonian reduces simply to

$$H_s^* = \frac{v_s}{2}\int dx \; \nabla\phi_s \nabla\phi_s + \Pi_s \Pi_s$$

which describes gapless spin excitations. This analysis gives us also the opportunity to derive the expression of the spin operators directly at the isotropic point.

$$S^+(x) = \psi^+_\uparrow(x)\psi_\downarrow(x) = \psi^+_{R\uparrow}(x)\psi_{R\downarrow}(x) + \psi^+_{L\uparrow}(x)\psi_{L\downarrow}(x)$$

$$+ \psi^+_{R\uparrow}(x)\psi_{L\downarrow}(x) + \psi^+_{L\uparrow}(x)\psi_{R\downarrow}(x).$$

$$\psi^+_{R\uparrow}(x)\psi_{R\downarrow}(x) = \frac{1}{2\pi\alpha} e^{-i\phi_{R\uparrow}} e^{i\phi_{R\downarrow}} = \frac{1}{2\pi\alpha} e^{-i(\phi_{R\uparrow}-\phi_{R\downarrow})} e^{\frac{1}{2}[\phi_{R\uparrow},\phi_{R\downarrow}]}$$

$$= \frac{i}{2\pi\alpha} e^{-i\sqrt{2\pi}(\phi_s-\theta_s)}$$

$$\psi^+_{L\uparrow}\psi_{L\downarrow} = +\frac{i}{2\pi\alpha} e^{i\sqrt{2\pi}(\phi_s+\theta_s)}$$

$$\psi^+_{R\uparrow}(x)\,\psi_{L\downarrow}(x) = \frac{(-1)^x}{2\pi\alpha}\, e^{-i\phi_{R\uparrow}}\, e^{-i\phi_{L\downarrow}}$$

$$= \frac{i}{2\pi\alpha}(-1)^x\, e^{-i(\phi_{R\uparrow}+\phi_{L\downarrow})} = +\frac{i}{2\pi\alpha}(-1)^x\, e^{-i\sqrt{2\pi}\,\phi_c}\, e^{i\sqrt{2\pi}\,\theta_s}$$

Since the charge field is locked to $\phi_c = n\frac{\pi}{\sqrt{2\pi}}$, then

$$\psi^+_{R\uparrow}\psi_{L\downarrow} = +\frac{i}{2\pi\alpha}(-1)^x\, C\, e^{i\sqrt{2\pi}\,\theta_s}, \qquad C = \langle e^{-i\sqrt{2\pi}\phi_c}\rangle$$

Analogously $\psi^+_{L\uparrow}\psi_{R\downarrow} = \frac{i}{2\pi\alpha}(-1)^x C\, e^{i\sqrt{2\pi}\,\theta_s}$, so that

$$S^+(x) = \frac{i}{2\pi\alpha}\, e^{-i\sqrt{2\pi}(\phi_s-\theta_s)} + \frac{i}{2\pi\alpha}\, e^{i\sqrt{2\pi}(\phi_s+\theta_s)}$$

$$+ \frac{i}{\pi\alpha}(-1)^x C\, e^{i\sqrt{2\pi}\,\theta_s}$$

Finally

$$S^z(x) = \frac{1}{2}\left(\psi^+_\uparrow\psi_\uparrow - \psi^+_\downarrow\psi_\downarrow\right) = \frac{1}{2}\left(\psi^+_{R\uparrow}\psi_{R\uparrow} + \psi^+_{L\uparrow}\psi_{L\uparrow} - \psi^+_{R\downarrow}\psi_{R\downarrow} - \psi^+_{L\downarrow}\psi_{L\downarrow}\right)$$

$$+\frac{1}{2}\left(\psi^+_{R\uparrow}\psi_{L\uparrow} - \psi^+_{R\downarrow}\psi_{L\downarrow} + h.c.\right) = \frac{1}{\sqrt{2\pi}}\nabla\phi_s + \frac{1}{\pi\alpha}(-1)^x \cos\sqrt{2\pi}\,\phi_c\, \sin\sqrt{2\pi}\,\phi_s$$

$$\simeq \frac{1}{\sqrt{2\pi}}\nabla\phi_s + \frac{C}{\pi\alpha}(-1)^x \sin\sqrt{2\pi}\,\phi_s$$

There is another operator which plays an important role in many spin chain models, namely the dimerization. In the fermionic language it corresponds to

$$\epsilon_i = (-1)^i \sum_\sigma c^+_{i\sigma} c_{i+1\sigma} + h.c.$$

Upon bosonization

$$\epsilon(x) = (-1)^x \sum_\sigma \left[\psi^+_{R\sigma}(x)\, \psi_{L\sigma}(x+a) + \psi^+_{L\sigma}(x)\, \psi_{R\sigma}(x+a) \right]$$

$$\doteq \frac{1}{2\pi\alpha} \left[i e^{i\phi_{L\uparrow}} e^{i\phi_{R\uparrow}} + i e^{i\phi_{L\downarrow}} e^{i\phi_{R\downarrow}} + h.c. \right]$$

$$\doteq + \frac{2}{\pi\alpha} \cos\sqrt{2\pi}\,\phi_c \cos\sqrt{2\pi}\,\phi_s \simeq + \frac{2C}{\pi\alpha} \cos\sqrt{2\pi}\,\phi_s$$