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INTERACTION EFFECTS ON TRANSPORT COEFFICIENTS IN DISORDERED *d*-WAVE SUPERCONDUCTORS

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Motivations

• In a d-wave superconductor, like the cuprates, quasiparticles are gapless.

How does the theory of Anderson's localization work in such a case ?

• Since charge is not a conserved quantity in a superconductor, contrary to spin and energy, charge density fluctuations do not diffuse if disorder is present, while spin and thermal ones do.

How do we evaluate the quantum interference corrections to the quasiparticle charge conductivity ?

• Fermi liquid corrections to the quasiparticle transport coefficients are usually invoked to explain the doping and material dependence of properties like the penetration depth in the cuprates.

What is the interplay between localization and residual quasiparticle interaction ?

The Model

Nearest-neighbor tight-binding Hamiltonian with a pairing term of d-wave symmetry in a square lattice:

$$\begin{aligned}\mathcal{H} &= -\sum_{\langle ij\rangle}\sum_{\sigma} t_{ij}e^{i\phi_{ij}}c^{\dagger}_{i\sigma}c_{j\sigma} + H.c.\\ &+ \sum_{\langle ij\rangle}\Delta_{ij}\left(c^{\dagger}_{i\uparrow}c^{\dagger}_{j\downarrow} + c^{\dagger}_{j\uparrow}c^{\dagger}_{i\downarrow}\right) + H.c.\\ &+ \sum_{i\sigma}\left(\epsilon_i - \mu\right) c^{\dagger}_{i\sigma}c_{i\sigma}.\end{aligned}$$

- $t_{ij} = t_{ji} \in \mathcal{R}e$ are independent random variables, which are gaussian distributed with average t and width ut;
- $\phi_{ij} = -\phi_{ji}$ is a phase which, if finite, breaks time reversal;
- Δ_{ij} is the pair function with d-wave symmetry;
- ϵ_i is an on-site random energy.

If $\epsilon_i = 0$, no onsite disorder, and $\mu = 0$, half-filling, the Hamiltonian posseses a nesting property.

This situation is not completely unrealistic. Indeed, if the impurity scattering is close to the unitary limit, the on-site disorder effectively reduces to a random hopping.

Nesting property and chiral symmetry

If nesting occurs, the eigenfunctions at site $i = (n, m)$, in the Nambu spinor representation, have the property:

$$\phi_{-E}(n,m) = \hat{O}_{(\pi,\pi)}\phi_E(n,m) = (-1)^{n+m}\phi_E(n,m).$$

- The operator $\hat{O}_{(\pi,\pi)}$ shifts by (π, π) the momentum.

It follows that the two wavefunctions

$$\frac{1}{\sqrt{2}}\left[1 \pm (-1)^{n+m}\right]\phi_{E\to 0^+}(n,m),$$

are both eigenstates of zero energy, and are orthogonal, since they are defined on different sublattices: sublattice A ($n + m$ even), sublattice B ($n + m$ odd).

Therefore, at zero energy, which is the same as at the chemical potential, an additional symmetry is present (so called chiral symmetry), which makes each zero energy level at least twofold degenerate.

Due to the chiral symmetry present at $E = 0$ when nesting occurs, the localization properties of the wavefunctions at the chemical potential differ from those away, which are similar to those at the chemical potential but in the absence of nesting (on-site disorder present or $\mu \neq 0$).

Path integral formulation within replica trick method

We treat the disorder in the random hopping by means of the replica trick within a path integral representation.

We introduce the Grassmann spinors c_i and $\bar{c}_i$ with components $c_{i,\sigma,p,a}$ and $\bar{c}_{i,\sigma,p,a}$, $a = 1, \ldots, n$ is the replica index.

- $\sigma =\uparrow, \downarrow$ refers to the spin. σ_b $(b = x, y, z)$ are the Pauli matrices in spin space.
- $p = \pm E$ is the energy index. s_b $(b = 1, 2, 3)$ are the Pauli matrices in energy space.

The Nambu spinors are defined through

$$\Psi_i = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \bar{c}_i \\ i\sigma_y c_i \end{array} \right),$$

and $\bar{\Psi}_i = [c\Psi_i]^t$, with the charge conjugacy matrix $c = i\sigma_y \tau_1$.

- τ_b $(b = 1, 2, 3)$ are the Pauli matrices in the Nambu space.

Symmetries

The action reads

$$\begin{aligned} S &= \sum_{ij} \bar{\Psi}_i \left(-t_{ij} \mathrm{e}^{-i\phi_{ij}\tau_3} + i\Delta_{ij}\tau_2 s_1 - i\delta_{ij} E s_3\right) \Psi_j \\ &\equiv \sum_{ij} \bar{\Psi}_i \left(H_{ij} - i\delta_{ij} E s_3\right) \Psi_j \end{aligned}$$

- Magnetic impurities: $\delta S = \sum_i \bar{\Psi}_i \tau_3 \vec{\sigma} \cdot \vec{S}_i \Psi_i$.
- Magnetic field Zeeman splitting: $\delta S = \sum_i \bar{\Psi}_i \tau_3 \vec{\sigma} \cdot \vec{B} \Psi_i$.

We consider two different global unitary transformations, one for sublattice A and another for B:

$$\Psi_A = T_A \Psi_A, \quad \Psi_B = T_B \Psi_B.$$

We must impose for $E = 0$

$$c T_A^t c^t H_{AB} T_B = H_{AB}.$$

If $E \neq 0$, in addition also

$$c T_A^t c^t s_3 T_A = c T_B^t c^t s_3 T_B = s_3,$$

which implies $\boxed{T_A = T_B}$.

Symmetry groups

	$E=0$	$E \neq 0$
Yes Chiral, Yes $\hat{T}$	$\mathrm{U}(4n)\times\mathrm{U}(4n)$	$\mathrm{U}(4n)$
Yes Chiral, No $\hat{T}$	$\mathrm{U}(4n)$	$\mathrm{O}(4n)$
No Chiral, Yes $\hat{T}$	$\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$	$\mathrm{Sp}(2n)$
No Chiral, No $\hat{T}$	$\mathrm{Sp}(2n)$	$\mathrm{U}(2n)$
Magnetic field	$\mathrm{U}(2n)$	$\mathrm{U}(n)\times\mathrm{U}(n)$
Spin flip	$\mathrm{O}(2n)$	$\mathrm{U}(n)$

[See A. Altland and M.R. Zirnbauer, PRB **55**, 1142 (1997); T. Senthil, M.P.A. Fisher, L. Balents, C. Nayak, PRL **81**, 4704 (1998); T. Fukui, cond-mat/9905388]

Without disorder

In the absence of random hopping the action in momentum space reads

$$S_0 = \sum_k \bar{\Psi}_k \left(\epsilon_k + i\Delta_k \tau_2 s_1 - iEs_3\right) \Psi_k,$$

- $\epsilon_k = -2t\left(\cos k_x a + \cos k_y a\right) \equiv E_k \cos 2\theta_k;$
- $\Delta_k = 2\Delta\left(\cos k_x a - \cos k_y a\right) \equiv E_k \sin 2\theta_k,$

with

$$E_k = \sqrt{\epsilon_k^2 + \Delta_k^2}.$$

By the unitary transformation

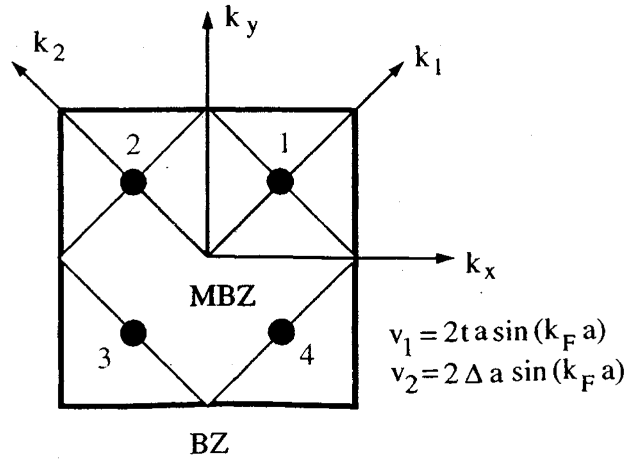
$$\Psi_k \to e^{i\theta_k \tau_2 s_1} \Psi_k,$$

$$S_0 = \sum_k \bar{\Psi}_k \left(E_k - iEs_3\right) \Psi_k.$$

The quasiparticle spectrum has four nodes at $(\pm k_F, \pm k_F)$. Close to these nodes

$$E_k^{1,3} \simeq \sqrt{v_1^2 k_1^2 + v_2^2 k_2^2},\ E_k^{2,4} \simeq \sqrt{v_2^2 k_1^2 + v_1^2 k_2^2},$$

show a Dirac-like behavior.



Average over disorder

Averaging over the random hopping leads to an additional term

$$S_{imp} = -2u^2t^2 \sum_{\langle ij \rangle} \left(\bar{\Psi}_i \Psi_j\right)^2 .$$

- $X_i^{\alpha\beta} = \Psi_i^{\alpha} \bar{\Psi}_i^{\beta}$,

$$S_{imp} = 2u^2t^2 \sum_{\langle ij \rangle} X_i^{\alpha\beta} X_j^{\beta\alpha} = \frac{1}{V} \sum_{q \in BZ} W_q Tr\left(X_q X_{-q}\right),$$

- $W_q = 2u^2t^2 \left(\cos q_x a + \cos q_y a\right)$.

If q is within the Magnetic BZ (MBZ), then $W_q = -W_{q+(\pi,\pi)} > 0$. We need to introduce two auxiliary fields

$$Q_{0q} = Q^{\dagger}_{0-q}, \quad Q_{3q} = Q^{\dagger}_{3-q},$$

to decouple the quartic term into

$$\begin{aligned} S_{imp} &= \frac{1}{V} \sum_{q \in MBZ} \frac{1}{4W_q} Tr\left[Q_{0q} Q_{0-q} + Q_{3q} Q_{3-q}\right] \\ &\quad - \frac{i}{V} \sum_{q \in MBZ} Tr\left[Q_{0q} X^t_{-q} + i Q_{3q} X^t_{-q-(\pi,\pi)}\right]. \end{aligned}$$

In the long wavelength limit, the auxiliary field in real space

$$Q_R = Q_{0R} + i(-1)^R Q_{3R},$$

is not hermitean and contains both a uniform and a staggered component.

Non linear σ-model. I

For smoothly varying Q_R, the integral over the Grassmann fields leads to

$$S = \int \frac{d^2R}{a^2}\frac{1}{4W_0}Tr\left[Q(R)Q(R)^\dagger\right] - \frac{1}{2}Tr\ln\left(iEs_3 - H + iQ\right).$$

1. We look for a saddle point $Q(R) = \Sigma s_3$. We find

$$1 = W_0\frac{1}{V}\sum_k \frac{1}{E_k^2+\Sigma^2}.$$

2. We project out longitudinal fluctuations. Apart from corrections of order u^2, this amounts to take

$$Q(R) = cT(R)^t c^t s_3 \Sigma T(R), \quad Q(R)Q^\dagger(R) = \Sigma^2.$$

$T(R)$ belongs to the coset group and describes the transverse modes.

The action of the transverse modes is then

$$\begin{aligned} S[Q] &= -\frac{1}{2}Tr\ln\left(iEs_3 - H + iQ\right) \\ &\simeq -\frac{1}{4}Tr\ln\left(iEs_3 - H + iQ\right)\left(-iEs_3 - H^\dagger - iQ^\dagger\right). \end{aligned}$$

The last equivalence holds since we expect an hermitean action.

Non linear σ-model. II

We need to calculate

$$D = (iEs_3 - H + iQ)\left(-iEs_3 - H^\dagger - iQ^\dagger\right)$$
$$= HH^\dagger + \Sigma^2 + E^2 + E\left(s_3Q^\dagger + Qs_3\right) + iHQ^\dagger - iQH^\dagger.$$

We notice that

- $$iH_{RR'}Q(R')^\dagger - iQ(R)H^\dagger_{RR'}$$
  $$= -iH_{RR'}\left[Q(R)^\dagger - Q(R')^\dagger\right]$$
  $$\simeq -iH_{RR'}\left(\vec{R} - \vec{R'}\right)\cdot\vec{\nabla}Q(R')^\dagger = \vec{J}\cdot\vec{\nabla}Q^\dagger,$$

where $\vec{J} = \partial_t R = -i[R, H]$ is the quasiparticle spin current,

- $HH^\dagger + \Sigma^2 \equiv G_0^{-1}$ is proportional to the identity.

By expanding in E and ∇Q we find, apart from constants,

$$-\frac{1}{4}Tr\ln D \simeq -\frac{E}{2}Tr\left(G_0Qs_3\right)$$
$$+\frac{1}{8}Tr\left(G_0\vec{J}\cdot\vec{\nabla}Q^\dagger G_0\vec{J}\cdot\vec{\nabla}Q^\dagger\right),$$

which is the desired action for the transverse modes and, as expected, has the form of a non-linear σ-model.

Non linear σ-model. III

Since

$$G_0(k) = \frac{1}{E_k^2 + \Sigma^2},$$

we can easily calculate the coefficients in the non-linear σ-model. We get

$$\begin{aligned} S_{NL\sigma M}[Q] &= \frac{\pi}{16\Sigma^2}\sigma \int d^2R\, Tr\left(\partial_\mu Q(R)^\dagger \alpha_{\mu\nu} \partial_\nu Q(R)\right) \\ &- E\frac{\pi N_0}{2\Sigma} \int d^2R\, Tr\left(Q(R) s_3\right), \end{aligned}$$

where the quasiparticle spin conductance

$$\begin{aligned} \sigma &= \frac{\Sigma^2}{\pi V} \sum_k \left(\vec{\nabla}\epsilon_k \cdot \vec{\nabla}\epsilon_k + \vec{\nabla}\Delta_k \cdot \vec{\nabla}\Delta_k\right) \left(\frac{1}{E_k^2 + \Sigma^2}\right)^2 \\ &\simeq \frac{1}{4\pi^2} \frac{v_1^2 + v_2^2}{v_1 v_2}, \end{aligned}$$

$\alpha_{\mu\nu} = \delta_{\mu\nu}$ for 4 nodes | $\alpha_{\mu\nu} = \delta_{\mu\nu}\frac{2v_\nu^2}{v_1^2+v_2^2}$ nodes 1 and 2

and v_1 and v_2 are the velocities parallel and perpendicular to the nodal direction.

$$N_0 = \frac{\Sigma}{\pi W_0},$$

is the density of states within the saddle point approximation.

Non linear σ-model. IV

In the presence of chiral symmetry, another term is found by integrating out the longitudinal modes

$$\delta S_{NL\sigma M} = \frac{\pi}{8 \cdot 16 \Sigma^4} \Pi \int dR \left| Tr \left[Q(R)^\dagger \vec{\nabla} Q(R) \right] \right|^2 ,$$

where Π is a parameter which controls the staggered density of states fluctuations.

It is important to notice that the logarithmic terms which appear upon integrating the gaussian propagator for 1 or 2 opposite nodes

$$\frac{1}{\sigma} \int d^2k \, \frac{1}{k_\mu \alpha_{\mu\nu} k_\nu} = 2\pi^2 v_1 v_2 \int dk_1 \, dk_2 \frac{1}{v_1^2 k_1^2 + v_2^2 k_2^2}$$
$$= 2\pi^2 \int dp_1 dp_2 \frac{1}{p_1^2 + p_2^2},$$

do not depend, within log-accuracy, on the velocity ratio ! We define the NLσM coupling constant

$$\boxed{g = \frac{1}{2\pi^2 \sigma} \text{ for 4 nodes}} \quad \boxed{g = \frac{v_1^2+v_2^2}{2 v_1 v_2} \frac{1}{2\pi^2 \sigma} \text{ for 1 or 2 nodes}}$$

Up to terms of order u^2, for 1 or 2 opposite nodes $\boxed{g = 1}$.

We can use the standard Wilson-Polyakov renormalization group to derive the β-functions for the coupling constant g and for the density of states.

β-functions

When chiral symmetry is present, we have to consider another running parameter

$$c = \frac{1}{2\pi^2\Pi}, \quad \beta_c = -\beta_g \frac{c^2}{ng^2},$$

and

$$\Gamma = \frac{g}{c + ng}.$$

	β_g	β_N
Yes Chiral, Yes $\hat{T}$	$8ng^2$	$(\Gamma/4 - 8n)g$
Yes Chiral, No $\hat{T}$	$4ng^2$	$(-1 + \Gamma/4 - 4n)g$
No Chiral, Yes $\hat{T}$	$2(2n+1)g^2$	$(-1 + 4n)g$
No Chiral, No $\hat{T}$	$(2n+1)g^2$	$(-1 + 2n)g$
Magnetic field	$2ng^2$	$-2ng$
Spin flip	$(n-1)g^2$	$(1-n)g$

- Delocalized phases with diverging DOS
- Localized phases with vanishing DOS
- Universality class of the IQHE.

$g = 1$?

The above RG equations describe the flow of the running parameters for small enough g. However, in the 1 or 2 node case $g = 1$ up to corrections of order u^2 !

As showed by Nersesyan, Tsvelik and Wenger, Nucl. Phys. B **438**, 561 (1995), this model can be mapped onto a 1+1 dimensional model of interacting electrons. They showed that the density of state maps onto a mass term, which, if finite, would break the chiral symmetry between right and left moving fermions. As it is known, this operator has dimension $\alpha = 1$. Keeping into account the electron-electron interaction generated by disorder, NT&W showed that the dimension gets the standard Luttinger liquid corrections, namely $1 - \alpha \propto u^2$. Since we have neglected terms of order u^2, which are due to the longitudinal modes, it is not at all surprising that we get, in the case they considered (no lattice chiral symmetry), an exponent exactly equal to 1 !

However, thanks to their analysis, we know that longitudinal fluctuations reduce the value of g below 1, so that we can still give a sense to the above RG equations.

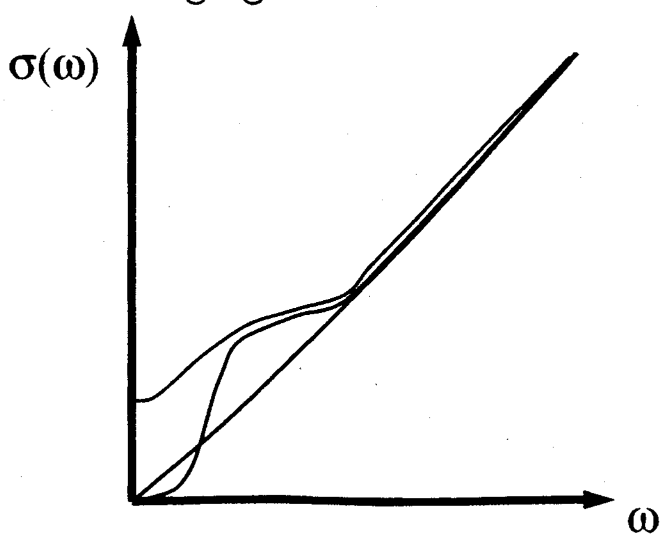
Some comments

$$\begin{aligned}\sigma &= \frac{\Sigma^2}{\pi V}\sum_k \left(\vec{\nabla}\epsilon_k\cdot\vec{\nabla}\epsilon_k + \vec{\nabla}\Delta_k\cdot\vec{\nabla}\Delta_k\right)\left(\frac{1}{E_k^2+\Sigma^2}\right)^2 \\ &\simeq \frac{1}{4\pi^2}\frac{v_1^2+v_2^2}{v_1 v_2},\end{aligned}$$

If $\Sigma = 0$, namely $N_0 = 0$, then $\sigma = 0$. This is the well known fact that

$$\lim_{\omega=0}\lim_{q=0} \mathcal{I}m\chi_{JJ}(q,\omega) = 0$$

for a pure system. Indeed, in an ideal metal, the conductivity displays only the Drude peak which derives from the diamagnetic term of the current. However, in our case, σ is the spin conductance which, at zero magnetic field, has no diamagnetic term. Therefore delocalized quasiparticles with zero DOS $\Longrightarrow \sigma = 0$, localized qp with zero DOS $\Longrightarrow \sigma = 0$, and delocalized qp with diverging DOS $\Longrightarrow \sigma = const.$.



Charge conductance

The d-wave gap function is a mass term for the charge-density diffusive modes. Charge conductance is still defined through the coefficient of the gradient square term in the action of the charge modes.

- Therefore we have re-derived the non-linear σ-model with

$$Q \rightarrow c^t T_c^t c Q T_c,$$

where

$$T_c = \mathrm{e}^{W_c},$$

describes the charge modes which are gapless in the absence of superconductivity;

- expanded the action at second order in W_c;
- finally we have integrated out the massless modes at the gaussian level.

Within the Drude approximation, the charge conductance

$$\sigma_c = \frac{\Sigma^2}{\pi V} \sum_k \vec{\nabla}\epsilon_k \cdot \vec{\nabla}\epsilon_k \left(\frac{1}{E_k^2 + \Sigma^2}\right)^2 \simeq \frac{1}{4\pi^2}\frac{v_1}{v_2}.$$

The one loop corrections to σ_c are those of σ if chiral symmetry is not present, including both spin flip and magnetic field cases. When chiral symmetry holds, we find that

$$\delta\sigma_c = -\sigma_c 2g \ln s.$$

Namely, while quasiparticle spin is delocalized, charge is localized.

Interaction effects

Since we have already set up all the machinery of the non-linear σ-model, we can easily take into account residual quasiparticle interaction along the same line of the Finkelstein approach. [See A.M. Finkel'stein, Z. Phys. B **56**, 189 (1984).]

Since Coulomb interaction spoils nesting symmetry already at the Hartree-Fock level, unless the density is not really close to half-filling, we can assume that chiral symmetry is absent. [Work is in progrees for the case in which chiral symmetry is present.]

Within the Landau Fermi liquid theory, we introduce three scattering amplitudes between the quasiparticles:

- U_s $\mathrm{p-h}$ singlet channel,
- U_t $\mathrm{p-h}$ triplet channel,
- U_c $\mathrm{p-p}$ Cooper channel.

The projection of the p-h singlet channel onto the diffusive modes is zero, since charge is not diffusive. For the same reason, only the p-p τ_1 channel contributes, which corresponds to fluctuations of an is-component to the pair order parameter [see Khveshenko, Yashenkin and Gornyi, PRL **86**, 4668 (2001)].

Interaction effects

The one loop corrections to σ and to the DOS can be easily derived and we find

$$\frac{\delta\sigma}{\sigma} = \frac{\delta N_0}{N_0} = 2gN_0\left(\frac{3}{2}U_t + \frac{1}{2}U_c\right)\ln s.$$

If time-reversal symmetry is broken, the U_c contribution drops out. A Zeeman terms removes 2/3 of the triplet contribution, while spin-flip scattering removes the whole correction.

Therefore, interaction acts as a delocalizing mechanism, and enhances the DOS. Moreover, spin fluctuations are enforced, which suggests that the system may have a quite efficient magnetic response.

Wess-Zumino-Witten term

The original Hamiltonian is not invariant under $\Delta \to -\Delta$. On the contrary, the effective NLσM is invariant. Did we miss some terms ? Since, at $E = 0$, $\Delta \to -\Delta$ implies $H \to H^\dagger$, the missing term should arise from

$$S_\Gamma[Q] = -\frac{1}{4} Tr \ln\left(-H + iQ\right) + \frac{1}{4} Tr \ln\left(-H^\dagger + iQ\right).$$

We can easily evaluate its variation along a massless path

- $\delta Q Q + Q \delta Q = 0$, no chiral symmetry.

$$\delta S_\Gamma = -\frac{i}{4} Tr\left(G_H \delta Q\right) + \frac{i}{4} Tr \ln\left(G_{H^\dagger} \delta Q\right),$$

where

$$G_H = \left(-H + iQ\right)^{-1}, \quad G_{H^\dagger} = \left(-H^\dagger + iQ\right)^{-1}.$$

We notice that

$$\left(-H^\dagger \mp iQ\right)\left(-H \pm iQ\right) \simeq G_0^{-1} \mp \vec{\nabla} Q \cdot \vec{J},$$

hence

$$\begin{aligned} G_H &= \left[1 - G_0 \vec{\nabla} Q \cdot \vec{J}\right]^{-1} G_0 \left(-H^\dagger - iQ\right) \\ G_{H^\dagger} &= \left(-H - iQ\right) G_0 \left[1 + \vec{\nabla} Q \cdot \vec{J} G_0\right]^{-1}. \end{aligned}$$

At leading order in the gradient expansion, we find

$$\delta S_\Gamma \simeq \frac{1}{2} Tr\left[\delta Q Q \partial_\mu Q \partial_\nu Q G_0 J_\mu G_0 J_\nu^\dagger G_0\right].$$

Since

$$J_\mu J_\nu^\dagger = \frac{1}{2}\delta_{\mu\nu}\left(\vec{\partial}\epsilon\cdot\vec{\partial}\epsilon + \vec{\partial}\Delta\cdot\vec{\partial}\Delta\right) - i\tau_2 s_1 \epsilon_{\mu\nu} E\vec{\partial}E\times\vec{\partial}\theta,$$

$$\delta S_\Gamma = i\Gamma \int d^2R\; Tr\left[\tau_2 s_1 \delta Q Q \partial_\mu Q \epsilon_{\mu\nu}\partial_\nu Q\right],$$

where

$$\Gamma = \frac{1}{2V}\sum_k \frac{E_k\vec{\partial}E_k\times\vec{\partial}\theta_k}{\left(E_k^2+\Sigma^2\right)^3} = \frac{k}{16\pi}\frac{1}{\Sigma^4}.$$

Here k counts the number of vortices minus antivortices in the Brillouin zone. Introducing a coordinate which parametrizes the massless path, we find the standard expression

$$S_k = i\frac{k}{12\pi}\frac{1}{4\Sigma^6}\int d^3R\; \epsilon_{\alpha\beta\gamma} Tr\left[\tau_2 s_1 Q\partial_\alpha Q Q\partial_\alpha Q Q\partial_\alpha Q\right].$$

In the time reversal invariant case, the massless modes belong to Sp$(2n)$, while Q is a $8n \times 8n$ matrix. Hence, if we write the above expression in terms of the massless modes, we expect a factor four to appear. This would imply that the effective action corresponds to a Wess-Zumino-Witten model Sp$(2n)_k$.