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BOSONIZATION AND TWO-DIMENSIONAL ISING MODEL

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BOSONIZATION AND TWO-DIMENSIONAL ISING MODEL

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PLAN OF THE LECTURES

1. Two-dimensional Ising model

- TRANSFER MATRIX AND REDUCTION TO QUANTUM ISING CHAIN
- MAPPING ONTO MAJORANA FERMIONS. CONTINUUM LIMIT
- CRITICALITY: Z_2 CFT WITH $C = 1/2$. OPERATOR CONTENT

2. Abelian bosonization of two Ising models

- FREE MASSLESS DIRAC FERMION = TWO MASSLESS MAJORANAS
- ABELIAN BOSONIZATION OF THE DIRAC FERMION $\Rightarrow$ BOSONIZATION OF TWO ISING COPIES
- BOSONIZATION OF ALL ISING-MODEL OPERATORS

3. Applications

- HEISENBERG CHAIN IN THE CONTINUUM LIMIT. ABELIAN BOSONIZATION OF $SU(2)_1$ WZNW MODEL
- TWO-CHAIN ANTIFERROMAGNETIC $S=1/2$ LADDER: $SO(3)\times Z_2$ MODEL OF FOUR NONCRITICAL ISING SYSTEMS

1 Two-Dimensional Ising Model

STRONGLY ANISOTROPIC 2D ISING MODEL

Transfer matrix, τ-continuum limit

⇓

QUANTUM ISING CHAIN

Jordan-Wigner transformation

⇓

REAL (MAJORANA) FERMIONS ON 1D LATTICE

$|T - T_c|/T_c \ll 1$: *continuum limit*

⇓

QFT MODEL IN 1+1 DIMENSIONS: MASSIVE MAJORANA FERMION

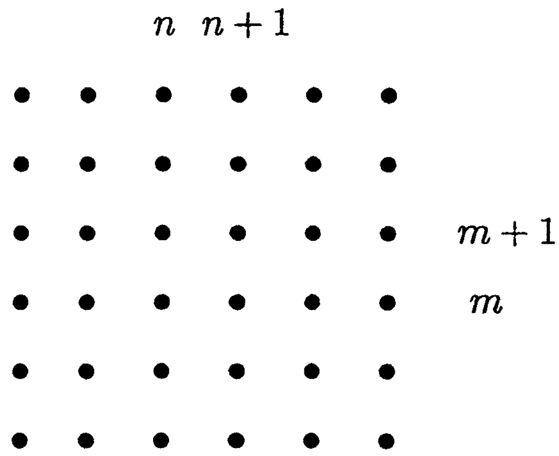
Criticality: massless limit

⇓

Z_2 CFT WITH CENTRAL CHARGE C = 1/2

1.1 τ-continuum limit and reduction to quantum Ising chain

Ising model on a square lattice with anisotropic n.n. couplings.



(1)

$$n = 1, 2, ..., N; \qquad m = 1, 2, ..., M \qquad (+ \text{ periodic boundary conditions})$$

$$\text{Ising variables:} \qquad \sigma_{nm} = \pm 1.$$

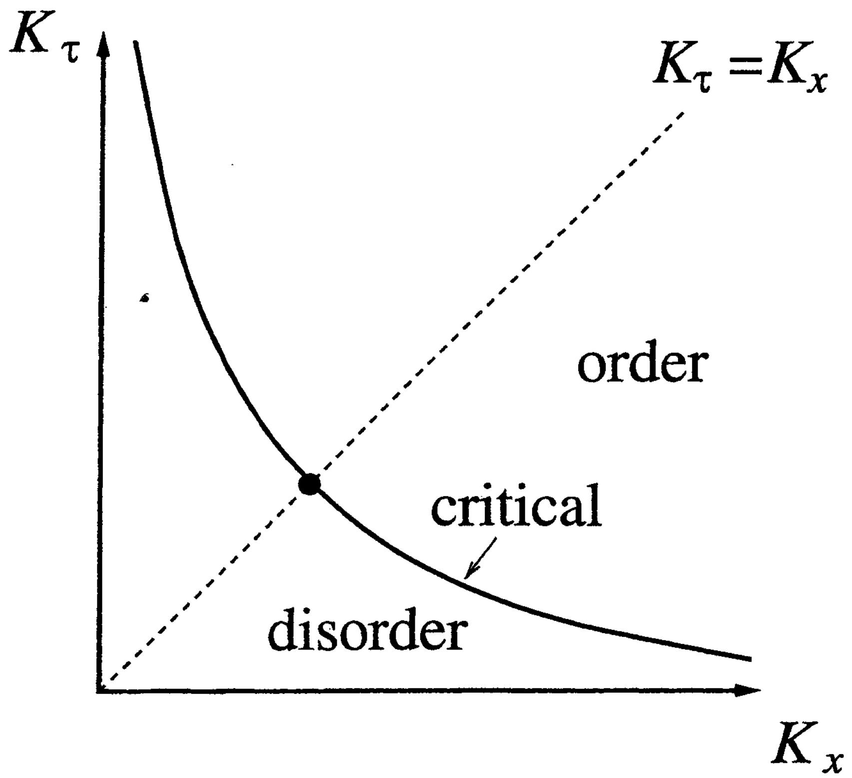
$$\text{Euclidean action} \quad = \quad \frac{\text{Energy}}{\text{Temperature}}$$

$$\begin{aligned} \mathcal{A} &= -\sum_{nm} \left(K_\tau \sigma_{nm} \sigma_{n,m+1} + K_x \sigma_{nm} \sigma_{n+1,m} \right) \\ Z &= \sum_{\{\sigma_{nm}\}} \exp\left(-\mathcal{A}[\sigma_{nm}]\right) \end{aligned}$$

Global Z_2 symmetry: $\quad \sigma_{nm} \to -\sigma_{nm}.$

Kramers-Wannier duality determines the critical curve:

$$\sinh 2K_x \sinh 2K_\tau = 1.$$



Transition point in the isotropic case:

$$K_c = J/T_c = \frac{1}{2} \ln \left(\sqrt{2} + 1\right).$$

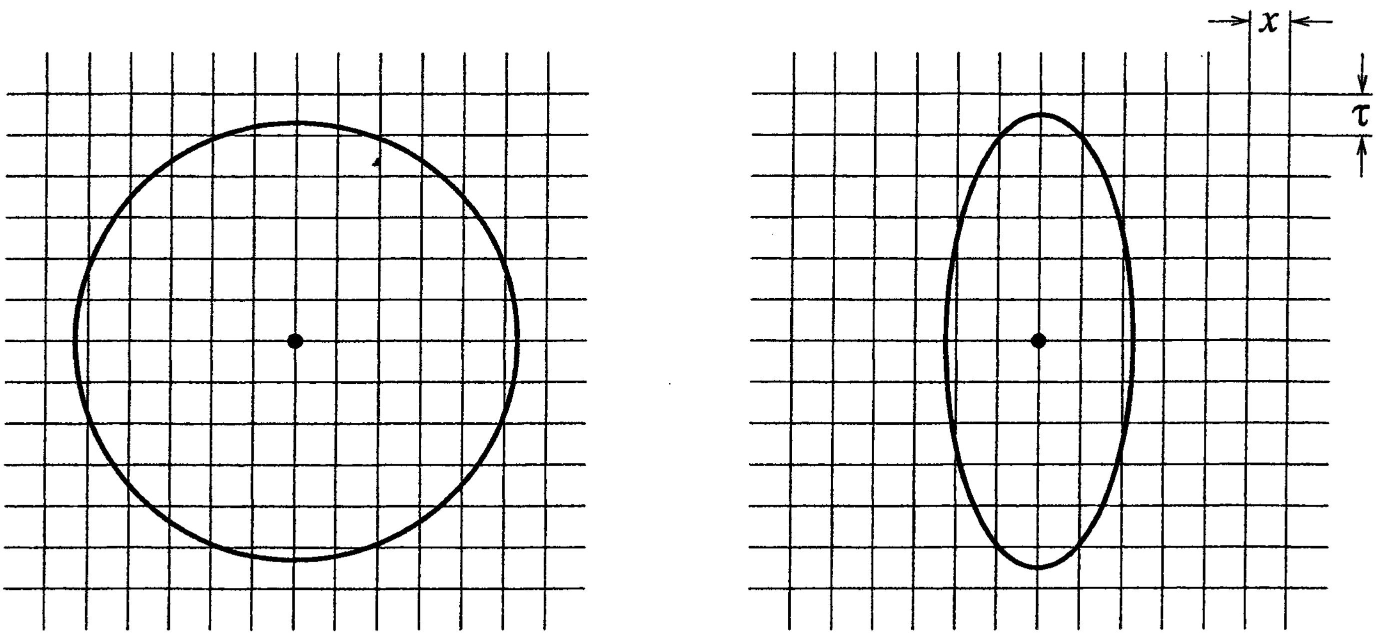
We will be dealing with a strongly anisotropic case:

$$K_\tau \gg 1, \quad K_x \ll 1.$$

Close to criticality

$$\underline{K_x \sim e^{-2K_\tau}.}$$

Suppose that T is close to T_c, so that the correlation length ξ_c is macroscopically large: $\xi_c/a \gg 1$. Consider the correlation function $\langle\sigma(\mathbf{r})\sigma(\mathbf{0})\rangle$ at distances $r \sim \xi_c$. In the isotropic case, $K_\tau = K_x$, the correlations are almost circular, whereas in the anisotropic case, $K_\tau \gg K_x$, they are ellipsoidal, strongly elongated in the τ-direction.

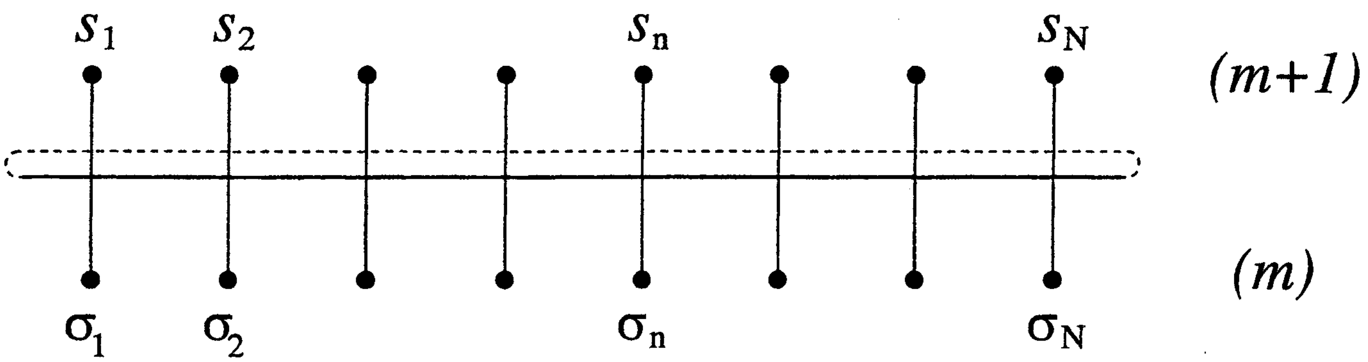


(A) (B)

To map (B) onto (A), squeeze the lattice in the τ-direction. This defines the so-called *τ-continuum limit* in which the coupling constants scale as follows:

$$K_x \propto \tau, \qquad e^{-2K_\tau} \propto \tau.$$

TRANSFER MATRIX $(2^N \times 2^N)$:



$$\{s_j\} = \{\sigma_{j,m+1}\}, \quad \{\sigma_j\} = \{\sigma_{j,m}\}$$

$$T_{m,m+1} = T(\{\sigma\},\{s\}) = \exp\left[-\frac{1}{2}K_\tau \sum_n (\sigma_n - s_n)^2 + \frac{1}{2}K_x \sum_n (\sigma_n\sigma_{n+1} + s_n s_{n+1})\right]$$

$$Z = \text{const } Tr\ \hat{T}^M.$$

Expected:

$$\hat{T} = 1 - \tau\hat{H} + O(\tau^2) = e^{-\tau\hat{H}+O(\tau^2)}$$

$\hat{H}$ – 1D quantum Hamiltonian

Two-row configurations:

$$\begin{array}{cccccccc} + & + & - & + & - & - & + & + \\ + & + & - & + & - & - & + & + \end{array} \qquad \{\sigma_n\} \equiv \{s_n\}: \quad K_x \sim \tau$$

$$\begin{array}{cccccccc} + & + & - & + & - & - & + & + \\ + & + & - & - & - & - & + & + \\ & & & \triangle & & & & \end{array} \qquad \text{1 spin flip}: \quad e^{-K_\tau} \sim \tau$$

$$\begin{array}{cccccccc} + & + & - & + & - & - & + & + \\ + & + & - & - & - & + & + & + \\ & & & \triangle & & \triangle & & \end{array} \qquad \text{2 spin flips}: \quad e^{-2K_\tau} \sim \tau^2 \ \text{(drop)}$$

etc

Parametrization:

$$\boxed{K_x = \tau, \qquad e^{-2K_\tau} = \lambda\tau}$$

QUANTUM ISING CHAIN = *Ising Chain in a Transverse Magnetic Field*:

$$\boxed{\hat{H} = -J \sum_{n=1}^{N} \left(\sigma_n^z \sigma_{n+1}^z + \lambda \sigma_n^x\right)}$$

1.2 Quantum Ising chain

Qualitative picture:

- $\lambda = 0$: classical 1D Ising chain. Long-range order (T=0), spontaneously broken Z_2:

$$\lim_{|n-m|\to\infty} \langle 0|\sigma_n^z \sigma_m^z|0\rangle = Q^2 = 1, \quad \langle 0|\sigma_n^z|0\rangle = \pm 1$$

$|\lambda| \ll 1$ – qualitatively the same result but with $Q^2 < 1$ (zero-point motion).

Small λ → LRO (spontaneously broken Z_2).

- $|\lambda| \to \infty$: decoupling – uncorrelated spins 1/2 in a magnetic field along x-axis.

$$\langle \tilde{0}|\sigma_n^z \sigma_m^z|\tilde{0}\rangle = 0.$$

At a large but finite λ

$$\lim_{|n-m|\to\infty} \langle \tilde{0}|\sigma_n^z \sigma_m^z|\tilde{0}\rangle \sim e^{-|n-m|/\xi}$$

Large λ → DISORDERED phase.

The passage from $|\lambda| \ll 1$ to $|\lambda| \gg 1$ requires a *quantum phase transition* at some $\lambda = \lambda_c$.

DUALITY TRANSFORMATION

$$
\begin{array}{ccccccc}
 & & n & & n+1 & & \\
\bullet & \star & \bullet & \star & \bullet & \star & \bullet \\
 & n-1/2 & & n+1/2 & & &
\end{array}
\tag{2}
$$

Dual spins: $\mu^{\alpha}_{n+1/2}$ $(\alpha = x, y, z)$.
Duality transformation:

$$
\boxed{\mu^{z}_{n+1/2} = \prod_{j=1}^{n} \sigma^{x}_{j}, \qquad \mu^{x}_{n+1/2} = \sigma^{z}_{n}\sigma^{z}_{n+1}}
$$

Inverse duality transformation:

$$
\boxed{\sigma^{z}_{n} = \prod_{j=0}^{n-1} \mu^{x}_{j+1/2}, \qquad \sigma^{x}_{n} = \mu^{z}_{n-1/2}\mu^{z}_{n+1/2}}
$$

$$
\mu^{z}_{n+1/2}\,|+\ +\ +\ +\ +\ +\ +\ +\,\rangle = |-\ -\ -\ -\ +\ +\ +\ +\,\rangle
$$

$$
\begin{array}{c} \triangle \\ (n+1/2) \end{array}
$$

$\mu^{z}_{n+1/2}$ creates a *kink*; hence - *disorder operator*.

Under duality transformation

$$\hat{H}[\sigma] \to \hat{H}[\mu] = -J\sum_n \left(\mu^x_{n+1/2} + \lambda\mu^z_{n-1/2}\mu^z_{n+1/2}\right).$$

$$\boxed{\hat{H}\left[\{\sigma\};\lambda\right] = \lambda\hat{H}\left[\{\mu\};1/\lambda\right]}$$

This is a quantum analog of Kramers-Wannier duality.

For each eigenvalue of $\hat{H}$

$$E(\lambda) = \lambda E(1/\lambda).$$

In particular, the mass gap satisfies

$$M(\lambda) = \lambda M(1/\lambda).$$

So if $M(\lambda) = 0$ at $\lambda = \lambda_c$, then $M(\lambda) = 0$ also at $\lambda = 1/\lambda_c$. Assuming that there exists only one critical point (this is know to be the case),

$$\boxed{\text{SELF-DUALITY} = \text{CRITICALITY:} \quad \lambda_c = 1}$$

Ordered phase	Disordered phase
$T < T_c: \quad \lambda < 1$	$T > T_c: \quad \lambda > 1$
$\langle\sigma^z\rangle \neq 0, \quad \langle\mu^z\rangle = 0$	$\langle\sigma^z\rangle = 0, \quad \langle\mu^z\rangle \neq 0$

Close to criticality

$$M(\lambda) \propto |\lambda - 1| \sim \frac{|T - T_c|}{T_c}.$$

1.3 Mapping onto Majorana fermions

Jordan-Wigner transformation for spinless *complex* fermions:

$$\sigma_n^x = 2a_n^\dagger a_n - 1$$
$$\sigma_n^z = (-1)^n \ \exp[\pm i\pi \sum_{j=1}^{n-1} a_j^\dagger a_j] \ (a_n^\dagger + a_n)$$

$$\{a_n, a_m^\dagger\} = \delta_{nm}, \qquad \{a_n, a_m\} = 0.$$

Tight-binding model:

$$\begin{aligned} \hat{H} &= -\sum_n \left(J\sigma_n^z \sigma_{n+1}^z + \Delta\sigma_n^x \right) \qquad (\Delta = \lambda J) \\ &\to \sum_n \left[J \underbrace{(a_n^\dagger - a_n)}\underbrace{(a_{n+1}^\dagger + a_{n+1})} - \Delta \underbrace{(a_n^\dagger - a_n)}\underbrace{(a_n^\dagger + a_n)} \right] \end{aligned}$$

Since only combinations $a_n^\dagger \pm a_n$ are present, introduce real, i.e. *Majorana* fermions:

$$\zeta_n = a_n^\dagger + a_n, \qquad \eta_n = -i(a_n^\dagger + a_n)$$
$$\{\zeta_n, \zeta_m\} = \{\eta_n, \eta_m\} = 2\delta_{nm}, \quad \{\zeta_n, \eta_m\} = 0$$

$$\zeta_n^\dagger = \zeta_n \quad \Rightarrow \quad \zeta_n = \frac{1}{\sqrt{N}} \sum_{k>0} \left(\zeta_k e^{ikn} + \zeta_k^\dagger e^{-ikn} \right)$$

ζ_k and $\zeta_k^\dagger = \zeta_{-k}$ represent independent modes only on the semiaxis $k > 0$.

Tight-binding Majorana model:

$$\hat{H} = \mathrm{i} \sum_n \left[J\eta_n(\zeta_{n+1} - \zeta_n) - (\Delta - J)\eta_n\zeta_n \right]$$

$|T - T_c|/T_c \ll 1 \quad \Rightarrow \quad |\Delta - J| \ll J \quad \Rightarrow$ continuum limit:

$$\begin{array}{rcl} a \to 0, \quad J, \Delta \to \infty, & & 2Ja = v, \quad 2(\Delta - J) = m \\ \zeta_n \to \sqrt{2a}\, \zeta(x), & & \eta_n \to \sqrt{2a}\, \eta(x) \\ \{\zeta(x), \zeta(x')\} = \{\eta(x), \eta(x')\} & = & \delta(x - x'), \quad \{\zeta(x), \eta(x')\} = 0 \end{array}$$

$$\hat{H} = \int \mathrm{d}x \, [\, \mathrm{i}v \, \eta(x)\partial_x\zeta(x) - \mathrm{i}m \, \eta(x)\zeta(x)]$$

Chiral rotation: $\xi_R = (\eta - \zeta)/\sqrt{2}$, $\xi_L = (\eta + \zeta)/\sqrt{2}$ $\rightarrow$ formally relativistic QFT of a massive Majorana fermion in 1+1 dimesions:

$$\begin{aligned} \mathcal{H}_M(x) &= \frac{\mathrm{i}v}{2} (\xi_L \partial_x \xi_L - \xi_R \partial_x \xi_R) - \mathrm{i}m \, \xi_R \xi_L \\ m &\sim (T - T_c)/T_c \end{aligned}$$

- Global Z_2 invariance: $\xi_R \to -\xi_R$, $\xi_L \to -\xi_L$.
- Duality transformation: $\xi_R \to -\xi_R$, $\xi_L \to \xi_L$ (or vice versa). Effectively $m \to -m$.

$$H_M = \sum_{k>0} \xi^\dagger(k)\,(kv\hat{\tau}_3 + m\hat{\tau}_2)\,\xi(k), \qquad \xi(k) = \begin{pmatrix} \xi_R(k) \\ \xi_L(k) \end{pmatrix}$$

Green function 2×2 matrix:

$$\hat{G}(k,\varepsilon) = -\frac{\mathrm{i}\varepsilon + kv\hat{\tau}_3 + m\hat{\tau}_2}{\varepsilon^2 + k^2v^2 + m^2}$$

Spectrum ($\mathrm{i}\varepsilon \to \omega + \mathrm{i}\delta$):

$$\underline{\omega^2 = k^2v^2 + m^2}$$

Immediate consequences:

- Correlation length:

$$\boxed{\xi_c \sim \frac{v}{|m|} \sim \frac{T_c}{|T - T_c|}}$$

- Specific heat: The mass dependence of the ground state energy (cf. condensation energy in a BCS superconductor):

$$\frac{1}{L}\left[\mathcal{E}_{\text{vac}}(m) - \mathcal{E}_{\text{vac}}(0)\right] \;\sim\; -\frac{m^2}{\Lambda}\ln\frac{\Lambda}{|m|}$$

(Λ = energy cutoff). Hence, the free energy density of a slightly noncritical 2D Ising model

$$\mathcal{F}(T) - \mathcal{F}(T_c) \sim -\frac{(T-T_c)^2}{T_c}\ln\frac{T_c}{|T-T_c|},$$

implying that

$$\boxed{C = -T\frac{\partial^2\mathcal{F}}{\partial T^2} \propto \ln\frac{T_c}{|T-T_c|}}$$

1.4 Criticality

$T \to T_c, \quad m \to 0$

Theory of a massless Majorana fermion   *in 1+1 (or 2 Euclidean) dimensions:*   *Minimal CFT with central charge $C = 1/2$.*

$\mathcal{R}^2_{x,\tau} \to \mathcal{C}_{z,\bar{z}}$:

$$z = \tau + \mathrm{i}x, \qquad \bar{z} = \tau - \mathrm{i}x \quad (v = 1)$$
$$\partial = \partial/\partial z, \qquad \bar{\partial} = \partial/\partial \bar{z}$$

Identify: $\xi \equiv \xi_L$, $\bar{\xi} \equiv \xi_R$. Euclidean action:

$$\boxed{\mathcal{A} = \int \mathrm{d}^2 z \left(\xi \bar{\partial} \xi + \bar{\xi} \partial \bar{\xi} \right)}$$

$$\delta A = 0 : \quad \bar{\partial}\xi = 0 \;\Rightarrow\; \xi = \xi(z) \;\text{(holomorphic)}$$
$$\partial\bar{\xi} = 0 \;\Rightarrow\; \bar{\xi} = \bar{\xi}(\bar{z}) \;\text{(antiholomorphic)}$$

i.e. in $1 + 1$ dimensions

$$\xi_L = \xi_L(x + t) \;\text{(left moving)}$$
$$\xi_R = \xi_R(x - t) \;\text{(right moving)}$$

Primary fields in CFT

For a primary field $f(z,\bar{z})$ with conformal dimensions h and $\bar{h}$ the two-point correlation function

$$\langle f(z_1,\bar{z}_1) f(z_2,\bar{z}_2)\rangle_{CFT} = \frac{\text{const}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}}$$

$$d = h + \bar{h} \qquad S = h - \bar{h}$$

(scaling dimension) (conformal spin)

- Chiral fermion fields $\xi, \bar{\xi}$:

$$\langle \xi(z_1)\xi(z_2)\rangle = \frac{1}{2\pi(z_1-z_2)}, \qquad \left(\frac{1}{2},0\right)$$

$$\langle \bar{\xi}(\bar{z}_1)\bar{\xi}(\bar{z}_2)\rangle = \frac{1}{2\pi(\bar{z}_1-\bar{z}_2)}, \qquad \left(0,\frac{1}{2}\right)$$

$$\underline{d = S = 1/2}$$

- Energy density $\varepsilon(z,\bar{z}) = \mathrm{i}\xi(z)\bar{\xi}(\bar{z})$

At $|T-T_c| \ll T_c$ $(|m| \ll \Lambda)$

$$\mathcal{A} = \mathcal{A}_{CFT} + m\int \mathrm{d}^2 z\; \varepsilon(z,\bar{z})$$

$$\langle \varepsilon(z_1,\bar{z}_1)\varepsilon(z_2,\bar{z}_2)\rangle \sim \frac{1}{|z_1-z_2|^2} \qquad \left(\frac{1}{2},\frac{1}{2}\right)$$

$\underline{d=1,\ \ S=0}$ (conformal scalar)

- ORDER/DISORDER OPERATORS $\sigma(z,\bar{z}), \mu(z,\bar{z})$

σ and μ are mutually nonlocal and each of these two fields is nonlocal in ξ. On the lattice

$$\sigma_n^z = \eta_n \prod_{j=1}^{n-1} (i\zeta_j\eta_j)\,, \quad \mu_{n+1/2}^z = \prod_{j=1}^{n} (i\zeta_j\eta_j)$$

$$\boxed{\begin{aligned} \zeta_n &= \sigma_n\mu_{n-1/2} = \mu_{n-1/2}\sigma_n \\ \eta_n &= i\sigma_n\mu_{n+1/2} = -i\mu_{n+1/2}\sigma_n \end{aligned}}$$

$\Downarrow$

FERMION = [ORDER PARAMETER]
× [DISORDER PARAMETER]

CFT proves (will be also shown below):

$$\sigma(z,\bar{z}),\ \mu(z,\bar{z}) : \quad \left(\frac{1}{16}, \frac{1}{16}\right) \quad \underline{d = 1/8, \quad S = 0.}$$

$$\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle = \langle\mu(z_1,\bar{z}_1)\mu(z_2,\bar{z}_2)\rangle \sim \frac{1}{|z_1 - z_2|^{1/4}}$$

2 Abelian bosonization of two Ising models

- Step 1: Start with a free massless Dirac fermion (no internal symmetry group):

$$\mathcal{L}_D(x) = \mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi,$$

$$\mathcal{H}_D(x) = -\mathrm{i}v\left[R^\dagger(x)\partial_x R(x) - L^\dagger(x)\partial_x L(x)\right], \quad \psi(x) = \begin{pmatrix} R(x) \\ L(x) \end{pmatrix}$$

Global U(1) symmetry $\rightarrow$ conserved current:

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \quad \partial_\mu j^\mu = 0.$$

Criticality with central charge $C_D = 1$.

Critical Ising model: $C_{\text{Ising}} = 1/2$, discrete (Z_2) symmetry $\rightarrow$ no Noether current (real fermions do not couple to electromagnetic field!).

For two identical Ising copies:
(i) $C = 1/2 + 1/2 = 1$,
(ii) U(1) symmetry is realized as O(2) rotations of the Majorana doublet.

$\Rightarrow$ Represent the Dirac fermion as two Majorana fermions.

$$\psi(x) = \Re e\ \psi(x) + \mathrm{i}\Im m\ \psi(x) = \frac{\xi^1(x) + \mathrm{i}\xi^2(x)}{\sqrt{2}}$$

$$\mathcal{H}_D = \mathcal{H}_M^1 + \mathcal{H}_M^2 = -(\mathrm{i}v/2)\left(\vec{\xi}_R\partial_x\vec{\xi}_R - \vec{\xi}_L\partial_x\vec{\xi}_L\right), \quad \vec{\xi} = (\xi^1, \xi^2)$$

$$\psi(x) \rightarrow e^{\mathrm{i}\alpha}\psi(x) \quad \Rightarrow \quad \begin{aligned} \xi^1(x) &\rightarrow \xi^1(x)\cos\alpha - \xi^2(x)\sin\alpha \\ \xi^2(x) &\rightarrow \xi^1(x)\sin\alpha + \xi^2(x)\cos\alpha \end{aligned}$$

• Step 2: Abelian bosonization of the Dirac field = bosonization of the two Majorana fields.

$$\left[\xi^1(x) + \mathrm{i}\xi^2(x)\right]_{R,L} = \frac{1}{\sqrt{\pi\alpha}} \exp\left[\pm \mathrm{i}\Phi_{R,L}(x)\right]$$

(i) $\alpha =$ short-distance cutoff of the *bosonic* theory.

(ii) $[\Phi_R, \Phi_L] = \mathrm{i}/4$ – to ensure anticommutation between the right and left components of the Fermi field.

$$j^\mu = (j^0, j^1)$$

$$j^0 = J_R + J_L, \qquad j^1 = J_R - J_L$$

Bosonization of chiral U(1) currents:

$$J_R = \ :R^\dagger R: \ = \mathrm{i}\xi_R^1\xi_R^2 = \frac{1}{\sqrt{\pi}}\partial_x\Phi_R$$

$$J_L = \ :L^\dagger L: \ = \mathrm{i}\xi_L^1\xi_L^2 = \frac{1}{\sqrt{\pi}}\partial_x\Phi_L$$

$$\Phi = \Phi_R + \Phi_L, \qquad \Theta = -\Phi_R + \Phi_L$$

(original field) (dual field)

$$j^0 = \frac{1}{\sqrt{\pi}}\partial_x\Phi, \qquad j^1 = -\frac{1}{\sqrt{\pi}}\partial_x\Theta$$

$$\partial_x\Phi = \mathrm{i}\sqrt{\pi}\left(\xi_R^1\xi_R^2 + \xi_L^1\xi_L^2\right), \quad \partial_x\Theta = \mathrm{i}\sqrt{\pi}\left(-\xi_R^1\xi_R^2 + \xi_L^1\xi_L^2\right)$$

- Step 3: Bosonization of fermionic mass bilinears.

$$R^\dagger L = -\frac{\mathrm{i}}{2\pi\alpha} e^{-\mathrm{i}\sqrt{4\pi}\Phi}, \quad R^\dagger L^\dagger = \frac{\mathrm{i}}{2\pi\alpha} e^{\mathrm{i}\sqrt{4\pi}\Theta}$$

$$\begin{aligned}
\mathrm{i}\pi\alpha\left(\xi_R^1\xi_L^1 + \xi_R^2\xi_L^2\right) &= \cos\sqrt{4\pi}\Phi,\\
\mathrm{i}\pi\alpha\left(\xi_R^1\xi_L^2 + \xi_L^1\xi_R^2\right) &= -\sin\sqrt{4\pi}\Phi,\\
\mathrm{i}\pi\alpha\left(\xi_R^1\xi_L^1 - \xi_R^2\xi_L^2\right) &= -\cos\sqrt{4\pi}\Theta,\\
\mathrm{i}\pi\alpha\left(\xi_R^1\xi_L^2 - \xi_L^1\xi_R^2\right) &= \sin\sqrt{4\pi}\Theta
\end{aligned}$$

Two copies of noncritical Ising models

≡ Free massive Dirac fermion

⇒ Sine-Gordon model with $\beta^2 = 4\pi$ (decoupling point)

$$\mathcal{H}_M[\vec{\xi}] = -(\mathrm{i}v/2)\left(\vec{\xi}_R\partial_R\vec{\xi}_R - \vec{\xi}_L\partial_R\vec{\xi}_L\right) - \mathrm{i}m\vec{\xi}_R\vec{\xi}_L$$

$$\Updownarrow$$

$$\mathcal{H}_{SG} = \frac{v}{2}\left[(\partial_x\Theta)^2 + (\partial_x\Phi)^2\right] - \frac{m}{\pi\alpha}\cos\sqrt{4\pi}\Phi$$

$$\partial_x\Theta(x) = \Pi(x), \quad [\Phi(x), \Pi(x')] = \mathrm{i}\delta(x - x').$$

- An example demonstrating the importance of the Ising model:

bosonic Hamiltonian:

$$\mathcal{H}_B = \frac{v}{2}\left[(\partial_x\Theta)^2 + (\partial_x\Phi)^2\right] - \frac{m_1}{\pi\alpha}\cos\sqrt{4\pi}\Phi - \frac{m_2}{\pi\alpha}\cos\sqrt{4\pi}\Theta$$

Both vertex perturbation to the Gaussian model have scaling dimension 1 and, hence, are strongly relevant $\rightarrow$ massive regime.

Is it always true? $m_1 = \pm m_2$ – self-duality points. Criticality?

Mapping onto two Majorana fields immediately solves the problem:

$$\mathcal{H}_B \rightarrow \sum_{j=1,2}\left[-\frac{\mathrm{i}v}{2}\left(\xi_R^j\partial_x\xi_R^j - \xi_L^j\partial_x\xi_L^j\right) - \mathrm{i}M_j\xi_R^j\xi_L^j\right]$$
$$M_1 = m_1 - m_2, \quad M_2 = m_1 + m_2$$

The spectrum consists of two *decoupled (!)* Majorana fermions with *different* masses. Ising criticality: $M_1 = 0$ or $M_2 = 0$.

Comment: equivalent representation – CDW and BCS-like pairings

$$\mathcal{H}_B \rightarrow -\mathrm{i}v\left(R^\dagger\partial_x R - L^\dagger\partial_x L\right) - \mathrm{i}m_1(R^\dagger L - h.c.) + \mathrm{i}m_2(R^\dagger L^\dagger - h.c.)$$

Chiral $\mathrm{U}(1)_R \times \mathrm{U}(1)_L$ symmetry fully broken: neither the particle number not the current conserved. Only $\mathrm{Z}_2 \times \mathrm{Z}_2$ left:

$$R \rightarrow -R, \quad L \rightarrow -L$$
$$R \rightarrow R^\dagger, \quad L \rightarrow L^\dagger \quad \text{(particle – hole symmetry)}$$

Hence Majorana fermions.

• Step 4: Bosonization of products of two Ising operators.

Consider two degenerate Ising models. At criticality 4 products

$$\sigma_1\sigma_2, \ \mu_1\mu_2, \ \sigma_1\mu_2, \ \mu_1\sigma_2$$

have the same scaling dimension $d = 1/8 + 1/8 = 1/4$. On the other hand, in the zero mass limit of the $\beta^2 = 4\pi$ sine-Gordon model, there are 4 vertex operators with the same dimension:

$$\cos\sqrt{\pi}\Phi, \ \ \sin\sqrt{\pi}\Phi, \ \ \cos\sqrt{\pi}\Theta, \ \ \sin\sqrt{\pi}\Theta.$$

There must be some correspondence between the two groups of 4 operators which should also hold at small deviations from criticality.

Heuristic derivation

$\beta^2 = 4\pi$ sine-Gordon model: At $m > 0$ (disordered Ising phase) the cosine potential has a degenerate set of minima at $(\Phi)_n = \sqrt{\pi}n$ $(n \in Z)$ implying that

$$\langle\cos\sqrt{\pi}\Phi\rangle \neq 0, \quad \langle\sin\sqrt{\pi}\Phi\rangle = 0,$$

and at the same time

$$\langle\sigma_1\rangle = \langle\sigma_2\rangle = 0, \quad \langle\mu_1\rangle = \langle\mu_2\rangle \neq 0.$$

At $m < 0$ (ordered Ising phase) $(\Phi)_n = \sqrt{\pi}(n + 1/2)$ implying that

$$\langle\cos\sqrt{\pi}\Phi\rangle = 0, \quad \langle\sin\sqrt{\pi}\Phi\rangle \neq 0,$$

with

$$\langle\sigma_1\rangle = \langle\sigma_2\rangle \neq 0 \quad \langle\mu_1\rangle = \langle\mu_2\rangle = 0.$$

Conclusion:

$$\boxed{\sigma_1\sigma_2 \sim \sin\sqrt{\pi}\Phi, \quad \mu_1\mu_2 \sim \cos\sqrt{\pi}\Phi}$$

Make a duality transformation in the sine-Gordon model:

$$\Phi \to \Theta : \quad \mathcal{H}_{SG} \to \frac{v}{2}\left[(\partial_x\Theta)^2 + (\partial_x\Phi)^2\right] - \frac{m}{\pi\alpha}\cos\sqrt{4\pi}\Theta$$

This corresponds to the duality transformation of the first Ising copy only:

$$\xi^1_R \to -\xi^1_R, \quad \xi^1_L \to \xi^j_L, \quad \xi^2_{R,L} \to \xi^2_{R,L}$$

implying that $\sigma_1 \leftrightarrow \mu_1$. So

$$\mu_1\sigma_2 \sim \sin\sqrt{\pi}\Theta, \quad \sigma_1\mu_2 \sim \cos\sqrt{\pi}\Theta$$

- CRITICAL ISING CORRELATORS FROM BOSONIZATION
  (*Zuber & Itzykson, 1977*)

$$\Gamma(\mathbf{r}) = \langle \sigma(\mathbf{r})\sigma(\mathbf{0})\rangle_c$$

$$K_{12}(\mathbf{r}) = \langle \sigma_1(\mathbf{r})\sigma_2(\mathbf{r})\sigma_1(\mathbf{0})\sigma_2(\mathbf{0})\rangle_c = \Gamma_1(\mathbf{r})\Gamma_2(\mathbf{r}) = \Gamma^2(\mathbf{r})$$

According to bosonization rules

$$\sigma_1(\mathbf{r})\sigma_2(\mathbf{r}) \sim \sin\sqrt{\pi}\Phi(\mathbf{r})$$

Φ is a *Gaussian* field:

$$\hat{H} = \frac{v}{2}\int dx \, \left[\Pi^2(x) + (\partial_x\Phi(x))^2\right] \quad \text{or} \quad \mathcal{A} = \frac{1}{2}\int d^2\mathbf{r}\left(\vec{\nabla}\Phi(\mathbf{r})\right)^2$$

2-point correlation function:

$$\langle\langle \Phi(\mathbf{r})\Phi(\mathbf{0})\rangle\rangle \equiv \langle \Phi(\mathbf{r})\Phi(\mathbf{0})\rangle - \langle\Phi\rangle^2 = -\frac{1}{2\pi}\ln\frac{|\mathbf{r}|}{\alpha}, \qquad (|\mathbf{r}| \gg \alpha)$$

Therefore, using the Baker-Hausdorff formula (Wick theorem for the Gaussian model)

$$\langle e^F\rangle = e^{\frac{1}{2}\langle F\rangle^2},$$

we obtain

$$\begin{aligned} K_{12}(\mathbf{r}) &= \langle \sin\sqrt{\pi}\Phi(\mathbf{r})\sin\sqrt{\pi}\Phi(\mathbf{0})\rangle \\ &= \frac{1}{2}\left[\langle\cos\sqrt{\pi}[\Phi(\mathbf{r}) - \Phi(\mathbf{0})]\rangle - \langle\cos\sqrt{\pi}[\Phi(\mathbf{r}) + \Phi(\mathbf{0})]\rangle\right] \\ &= \frac{1}{2}\Re e\,\langle e^{i\sqrt{\pi}[\Phi(\mathbf{r})-\Phi(\mathbf{0})]}\rangle = \frac{1}{2}\exp\left[-\frac{1}{2}\ln\frac{|\mathbf{r}|}{\alpha}\right] \\ &= \frac{1}{2}\left(\frac{\alpha}{|\mathbf{r}|}\right)^{1/2} \end{aligned} \tag{3}$$

Consequently, for a single critical Ising model

$$\langle \sigma(\mathbf{r})\sigma(\mathbf{0})\rangle_c \sim \frac{1}{|\mathbf{r}|^{1/4}}, \quad \Rightarrow \quad d_\sigma = 1/8$$

Similarly, $d_\mu = 1/8$

3 Applications

3.1 Background: Heisenberg spin-1/2 chain in the continuum limit

Isotropic [SU(2)-symmetric] S=1/2 antiferromagnetic Heisenberg chain:

$$H = J\sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1}, \quad J > 0,\ S = 1/2$$

Exactly solved by H. Bethe (1931). Known facts:

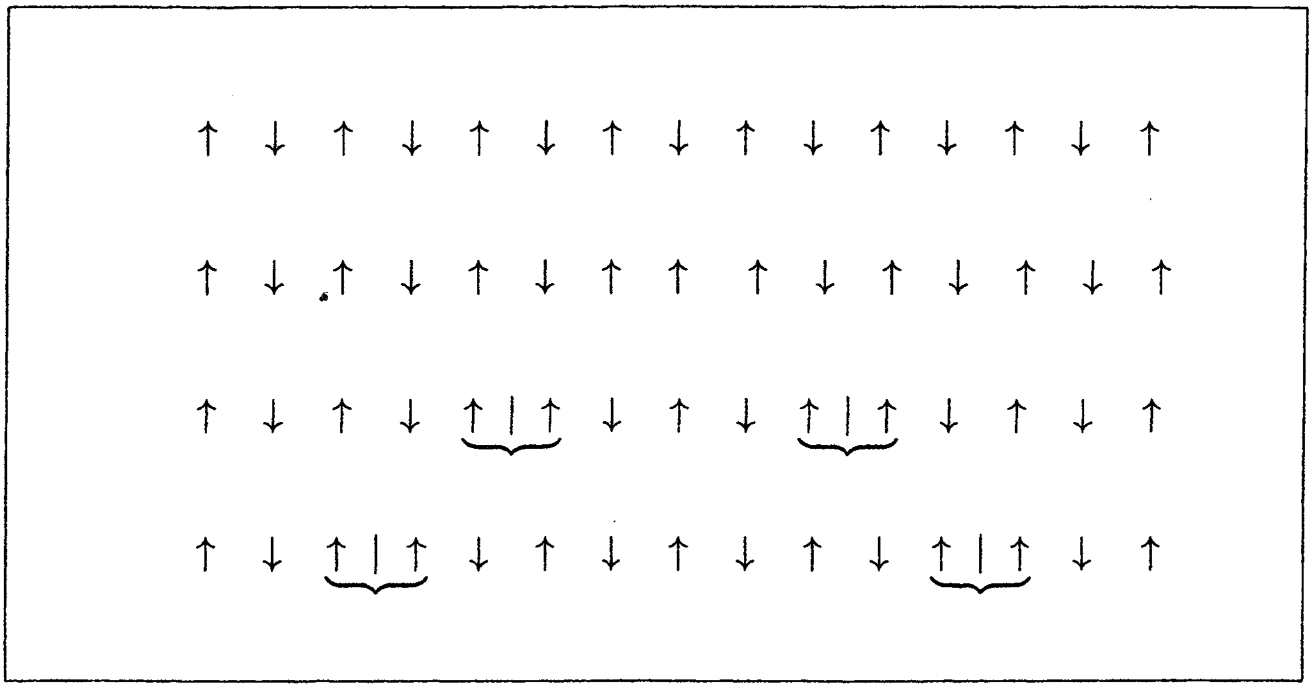
- no long-range order; $SU(2)_1$ criticality;
- elementary excitations - *spinons* - carry spin-1/2 and, in the long-wavelength limit, have a linear spectrum: $\omega_s(k) = v_s|k|$;
- spin-spin correlation functions follow power laws with *universal* critical exponents.

Anisotropic (XXZ) S=1/2 chain:

$$H_{\mathrm{XXZ}} = J\sum_n \left[\left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y\right) + \Delta S_n^z S_{n+1}^z\right]$$

- $\underline{-1 < \Delta \le 1}$ $\rightarrow$ no LRO, U(1) criticality, gapless spectrum, Δ-dependent critical exponents (Tomonaga-Luttinger liquid).

DECONFINED SPINONS IN A HEISENBERG CHAIN:



Continuum limit – two alternative routes

- *Luther and Peshel:*

XXZ model

$\Downarrow$ (Jordan – Wigner transformation)

Spinless interacting fermions on a 1D lattice

$\Downarrow$ (continuum limit)

Spinless Tomonaga – Luttinger liquid + Umklapp

$\Downarrow$ (Abelian bosonization)

Sine – Gordon model

$$\mathcal{H} = \frac{v}{2}\left[K\left(\partial_x\Theta\right)^2 + \frac{1}{K}\left(\partial_x\Phi\right)^2\right] - \frac{m_0}{\pi\alpha}\cos\sqrt{8\pi K}\Phi$$

$$1/K = 1 - (1/\pi)\arccos\Delta, \qquad m_0 \sim J\Delta$$

$|\Delta| < 1$: cosine perturbation irrelevant $\Rightarrow$ Gaussian model.

$\Delta = 1,\ K = 1$ (isotropic case): the model occurs at the SU(2)-symmetric weak-coupling separatrix of the Kosterlitz-Thouless phase diagram where the perturbation is *marginally* irrelevant.

- *Affleck; Haldane – Symmetry preserving fermionization:*

$$\mathbf{S}_n = \frac{1}{2}\psi^\dagger_{n\alpha}\vec{\sigma}_{\alpha\beta}\psi_{n\beta}$$

$$\psi \to \hat{U}\psi, \qquad \hat{U} \in SU(2)$$
$$\mathbf{S} \to \frac{1}{2}\psi^\dagger\hat{U}^\dagger\vec{\sigma}\hat{U}\psi = \mathcal{R}\mathbf{S}, \qquad \mathcal{R} \in SO(3)$$

$$\psi \to e^{i\gamma}\psi, \quad e^{i\gamma} \in U(1)$$
$$\mathbf{S} \to \mathbf{S}: \quad \text{charge U(1) redundant}$$

To kill unwanted charge excitations - *constraint*: exactly one particle per site.

$\Rightarrow$ large-U Hubbard model at 1/2 filling: mapping onto AF Heisenberg chain with $J \propto t^2/U$.

But - *no Mott transition in 1D Hubbard model:* the charge gap is generated at *any* $U > 0$.

$$U \ll t, \qquad m_c \propto \sqrt{Ut}\exp(-2\pi t/U)$$

$|E| \ll m_c$: only spin dynamics remains $\Rightarrow$ universal properties of the S=1/2 Heisenberg chain.

$$H = -t\sum_{i\sigma}(\psi^\dagger_{i\sigma}\psi_{i+1,\sigma} + h.c.) + U\sum_i n_{i\uparrow}n_{i\downarrow}, \quad 0 < U \ll t, \quad \sum_\sigma n_{is} = 1$$

$\Downarrow$ (non – Abelian bosonization)

$\mathcal{H}(x) = \mathcal{H}_c(x) + \mathcal{H}_s(x)$ – Charge – spin separation

Umklapp locks the charge and makes

$\mathcal{H}_c$ massive (Mott insulator).

$\Downarrow$ At low energies $|\omega| \ll m_{\text{charge}}$

CRITICAL SU(2)$_1$ WESS-ZUMINO-NOVIKOV-WITTEN (WZNW) MODEL with a marginally irrelevant perturbation (backscattering):

$$\mathcal{H}_s = \frac{2\pi v_s}{3}(: \mathbf{J}_R \cdot \mathbf{J}_R : + : \mathbf{J}_L \cdot \mathbf{J}_L :) - \gamma \mathbf{J}_R \cdot \mathbf{J}_L \quad (\gamma > 0)$$
$$C^{WZNW}_{SU(2)_1} = 1$$

Chiral vector currents $\mathbf{J}_{R,L}$ - generators of SU(2)$_{R,L}$

$$J^a_{R,L} = \frac{1}{2} : \psi^\dagger_{R,L;\alpha}\sigma^a_{\alpha\beta}\psi_{R,L;\beta} : \quad (a = x, y, z)$$

satisfy the Kac-Moody algebra:

$$[J^a_R(x), J^b_R(x')] = \mathrm{i}\epsilon^{abc}J^c_R(x)\delta(x - x') + \frac{k}{4\pi\mathrm{i}}\delta^{ab}\delta'(x - x')]$$
$$[J^a_L(x), J^b_L(x')] = \mathrm{i}\epsilon^{abc}J^c_L(x)\delta(x - x') - \frac{k}{4\pi\mathrm{i}}\delta^{ab}\delta'(x - x')]$$
$$[J^a_R(x), J^b_L(x')] = 0$$

with the level $k = 1$.

Smooth fields $(q \sim 0)$: $R^\dagger_\alpha \vec{\sigma}_{\alpha\beta} R_\beta, \quad L^\dagger_\alpha \vec{\sigma}_{\alpha\beta} L_\beta$

LOCAL SPIN DENSITY AND SPIN CURRENT:

$$\mathbf{J} = \mathbf{J}_R + \mathbf{J}_L, \quad \mathbf{j} = v_s \left(\mathbf{J}_R - \mathbf{J}_L\right)$$

Staggered fields $(q \sim \pi)$: $\langle R^\dagger_\alpha \vec{\sigma}_{\alpha\beta} L_\beta \rangle_{\text{charge}}, \quad \langle L^\dagger_\alpha \vec{\sigma}_{\alpha\beta} R_\beta \rangle_{\text{charge}}$

WZNW 2×2 matrix field:

$$\hat{g}(x) = \epsilon(x) + \mathrm{i} \sum_{a=1,2,3} n^a(x) \sigma^a \in SU(2)$$

$$\epsilon^2 + \mathbf{n}^2 = \text{const}$$

$$(h, \bar{h}) = (1/4, 1/4), \qquad d = 1/2, \; S = 0$$

DIMERIZATION OPERATOR AND STAGGERED MAGNETIZATION:

$$\epsilon(x) \sim Tr \; \hat{g}(x) \Leftarrow (-1)^n \mathbf{S}_n \cdot \mathbf{S}_{n+1},$$

$$n^a(x) \sim Tr \; [\sigma^a \hat{g}(x)] \Leftarrow (-1)^n \mathbf{S}_n$$

Local spin density of the S=1/2 Heisenberg chain:

$$\mathbf{S}(x) = \mathbf{J}(x) + (-1)^n \mathbf{n}(x)$$

ABELIAN BOSONIZATION OF $SU(2)_1$ WZNW MODEL

- Bosonize the Hubbard model using scalar fields $\varphi_{R,L;\alpha}$:

$$[R, L]_\alpha \simeq (2\pi\alpha)^{-1/2} \exp\left(\pm i\sqrt{4\pi}\varphi_{R,L;\alpha}\right)$$

- Charge and spin fields:

$$\Phi_c = \frac{\Phi_\uparrow + \Phi_\downarrow}{\sqrt{2}}, \qquad \Theta_c = \frac{\Theta_\uparrow + \Theta_\downarrow}{\sqrt{2}}$$
$$\Phi_s = \frac{\Phi_\uparrow - \Phi_\downarrow}{\sqrt{2}}, \qquad \Theta_s = \frac{\Theta_\uparrow - \Theta_\downarrow}{\sqrt{2}}$$

- Hamiltonian density: charge-spin separation:

$$\mathcal{H} = \mathcal{H}_c + \mathcal{H}_s$$

$$\begin{aligned} \mathcal{H}_c &= \frac{v_c}{2}\left[(\partial_x\Theta_c)^2 + (\partial_x\Phi_c)^2\right] \\ &+ \text{const } g\left[\partial_x\Phi_{cR}\partial_x\Phi_{cL} - \frac{1}{(2\pi\alpha)^2}\cos\sqrt{8\pi}\Phi_c\right] \\ &\quad (\text{Umklapp}: \ \text{marginally relevant perturbation}) \end{aligned}$$

$$\begin{aligned} \mathcal{H}_s &= \frac{v_s}{2}\left[(\partial_x\Theta_s)^2 + (\partial_x\Phi_s)^2\right] \\ &+ \text{const } g\left[\partial_x\Phi_{sR}\partial_x\Phi_{sL} - \frac{1}{(2\pi\alpha)^2}\cos\sqrt{8\pi}\Phi_s\right] \\ &\quad (\text{Backscattering}: \ \text{marginally irrelevant perturbation}) \end{aligned}$$

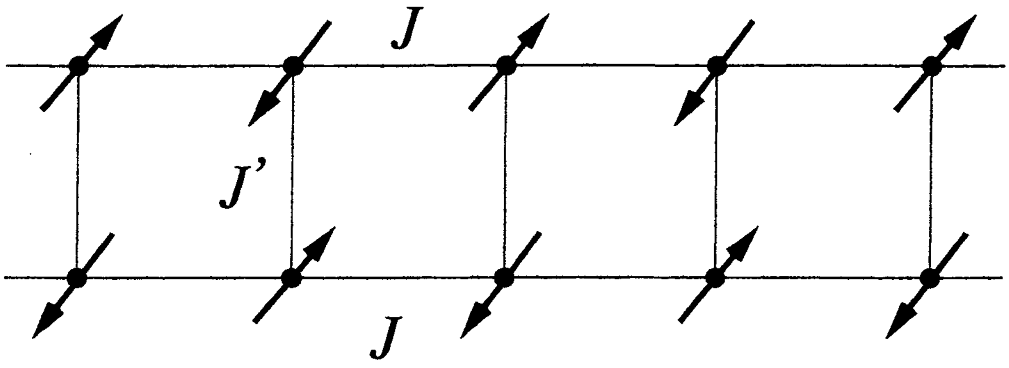
- Spin currents are expressed in terms of the spin fields only:

$$J^z_R = \frac{1}{\sqrt{2\pi}}\partial_x \Phi_{s,R}, \qquad J^z_L = \frac{1}{\sqrt{2\pi}}\partial_x \Phi_{s,L}$$
$$J^+_R = \frac{1}{2\pi\alpha}\exp(-\mathrm{i}\sqrt{8\pi}\Phi_{s,R}), \qquad J^+_L = \frac{1}{2\pi\alpha}\exp(\mathrm{i}\sqrt{8\pi}\Phi_{s,L})$$

- Staggered fields **n** and ϵ – charge needs to be locked:

$$n^z = -\frac{\lambda}{\pi\alpha} : \sin\sqrt{2\pi}\Phi_s :, \qquad n^\pm = \frac{\lambda}{\pi\alpha} : \exp\left(\pm\mathrm{i}\sqrt{2\pi}\Theta_s\right) :$$
$$\epsilon = \frac{\lambda}{\pi\alpha} : \cos\sqrt{2\pi}\Phi_s :$$
$$\lambda = \langle\cos\sqrt{2\pi}\Phi_c\rangle \neq 0 \qquad \text{(nonuniversal constant)}$$

3.2 Two-chain S=1/2 Heisenberg ladder



$$H = J \sum_{a=1,2} \sum_{n} \mathbf{S}_a(n) \cdot \mathbf{S}_a(n+1) + J' \sum_{n} \mathbf{S}_1(n) \cdot \mathbf{S}_2(n), \quad (J > 0)$$

$$|J'| \ll J \quad \Rightarrow \quad \text{continuum limit: } H \to \int \mathrm{d}x\, \mathcal{H}(x)$$

$$\begin{aligned} \mathcal{H} &= \frac{v_s}{2} \sum_{a=1,2} \left[(\partial_x \Theta_a)^2 + (\partial_x \Phi_a)^2 \right] + \mathcal{H}_{12} \\ \mathcal{H}_{12} &= J' a_0 \left[\underbrace{\mathbf{J}_1 \cdot \mathbf{J}_2}_{\text{marginal } (d=2)} + \underbrace{\mathbf{n}_1 \cdot \mathbf{n}_2}_{\text{relevant } (d=1)} \right] \end{aligned}$$

$$\Phi_\pm = \frac{\Phi_1 \pm \Phi_2}{\sqrt{2}}, \quad \Theta_\pm = \frac{\Theta_1 \pm \Theta_2}{\sqrt{2}}$$

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_+ + \mathcal{H}_- \\
\mathcal{H}_+ &= \frac{v_s}{2}\left[(\partial_x\Theta_+)^2 + (\partial_x\Phi_+)^2\right] - \frac{m}{\pi\alpha}\cos\sqrt{4\pi}\Phi_+ \\
\mathcal{H}_- &= \frac{v_s}{2}\left[(\partial_x\Theta_-)^2 + (\partial_x\Phi_-)^2\right] + \frac{m}{\pi\alpha}\cos\sqrt{4\pi}\Phi_- + \frac{2m}{\pi\alpha}\cos\sqrt{4\pi}\Theta_-
\end{aligned}$$

$(m = J'\lambda^2/2\pi)$ H.J. Schulz (1986)

$$\Phi_+ \Rightarrow \left(\xi^1, \xi^2\right), \quad \Phi_- \Rightarrow \left(\xi^3, \xi^4\right)$$

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_t[\vec{\xi}] + \mathcal{H}_s[\xi^4] + \mathcal{H}_{\text{marg}} \\
\mathcal{H}_t[\vec{\xi}] &= \sum_{a=1,2,3}\left[-\frac{iv_s}{2}\left(\xi^a_R\partial_x\xi^a_R - \xi^a_L\partial_x\xi^a_L\right) - im_t\xi^a_R\xi^a_L\right] \\
\mathcal{H}_s[\xi^4] &= -\frac{iv_s}{2}\left(\xi^4_R\partial_x\xi^4_R - \xi^4_L\partial_x\xi^4_L\right) - im_s\xi^4_R\xi^4_L
\end{aligned}$$

$m_t = m, \quad m_s = -3m$ $\underline{SO(3) \times Z_2}$

D. Shelton, A.M. Tsvelik & A.A. Nersesyan (1996)

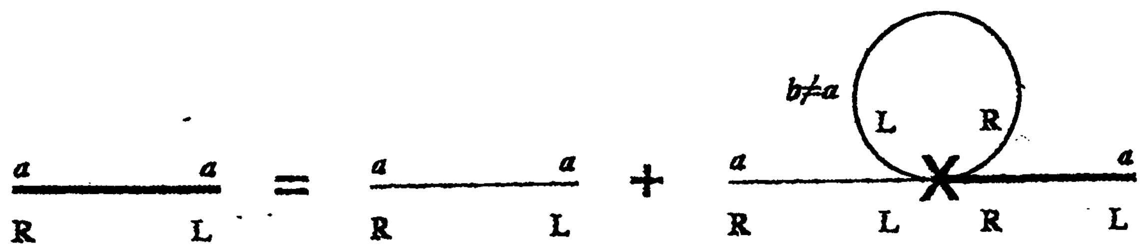
MARGINAL 4-FERMION INTERACTION

$$\mathcal{H}_{\text{marg}} = \frac{1}{2}\sum_{i\neq j} g_{ij}\left(\xi_R^i \xi_L^i\right)\left(\xi_R^j \xi_L^j\right),$$

$$g_{ii} = 0, \quad g_{ij} = g_{ji},$$
$$g_{12} = g_{23} = g_{31} = \frac{1}{2}J' a_0, \quad g_{14} = g_{24} = g_{34} = -\frac{1}{2}J' a_0.$$

$\Rightarrow$ weak mass renormalization:

$$\tilde{m}_i = m_i + \sum_{j(\neq i)} \frac{g_{ij}}{2\pi v_s} m_j \ln \frac{\Lambda}{|m_j|}$$



STAGGERED FIELDS OF THE SPIN LADDER

$$\mathbf{n}^+ = \mathbf{n}_1 \pm \mathbf{n}_2, \qquad \epsilon^\pm = \epsilon_1 \pm \epsilon_2$$

Abelian bosonization:

$$\begin{aligned}
n_x^+ &\sim \cos\sqrt{\pi}\Theta_+ \cos\sqrt{\pi}\Theta_-, & n_x^- &\sim \sin\sqrt{\pi}\Theta_+ \sin\sqrt{\pi}\Theta_- \\
n_y^+ &\sim \sin\sqrt{\pi}\Theta_+ \cos\sqrt{\pi}\Theta_-, & n_y^- &\sim \cos\sqrt{\pi}\Theta_+ \sin\sqrt{\pi}\Theta_- \\
n_z^+ &\sim \sin\sqrt{\pi}\Phi_+ \cos\sqrt{\pi}\Phi_-, & n_z^- &\sim \cos\sqrt{\pi}\Theta_+ \sin\sqrt{\pi}\Phi_- \\
\epsilon^+ &\sim \cos\sqrt{\pi}\Phi_+ \cos\sqrt{\pi}\Phi_-, & \epsilon^- &\sim \sin\sqrt{\pi}\Phi_+ \sin\sqrt{\pi}\Phi_-
\end{aligned}$$

Local representation in terms of the Ising order/disorder operators:

$$\begin{aligned}
\mathbf{n}^+ &\sim (\sigma_1\mu_2\sigma_3\mu_4, \quad \mu_1\sigma_2\sigma_3\mu_4, \quad \sigma_1\sigma_2\mu_3\mu_4) \\
\mathbf{n}^- &\sim (\mu_1\sigma_2\mu_3\sigma_4, \quad \sigma_1\mu_2\mu_3\sigma_4, \quad \mu_1\mu_2\sigma_3\sigma_4) \\
\epsilon^+ &\sim \mu_1\mu_2\mu_3\mu_4, \qquad \epsilon^- \sim \sigma_1\sigma_2\sigma_2\sigma_3
\end{aligned}$$

CRUCIAL:

For a standard ladder $m_t m_s < 0$

$J_\perp > 0:$ $m_t > 0$, Ising triplet $(1,2,3)$ disordered $\rightarrow$ T > T$_c$

$m_s < 0$, Ising singlet (4) ordered $\rightarrow$ T < T$_c$

$J_\perp < 0:$ *vice versa*

Correlation functions

Correlation functions in a noncritical Ising model ($T > T_c$):

$$\mathbf{r} = (\tau, x), \quad v \equiv 1 \qquad \text{Wu et al (1976)}$$

$$\langle \sigma(\mathbf{r})\sigma(\mathbf{0})\rangle \sim K_0(|m|r) \propto \frac{e^{-|m|r}}{\sqrt{|m|r}}, \quad (|m|r \gg 1)$$

$$\langle \mu(\mathbf{r})\mu(\mathbf{0})\rangle \sim 1 + O(e^{-2|m|r})$$

$T < T_c$: $\langle \sigma(\mathbf{r})\sigma(\mathbf{0})\rangle$ and $\langle \mu(\mathbf{r})\mu(\mathbf{0})\rangle$ interchanged.

$$\langle \mathbf{n}^+(\mathbf{r}) \cdot \mathbf{n}^+(\mathbf{0})\rangle \propto \frac{e^{-(2m_t+|m_s|)r}}{m_t\sqrt{|m_s|}r^{3/2}}, \qquad \langle \mathbf{n}^-(\mathbf{r}) \cdot \mathbf{n}^-(\mathbf{0})\rangle \propto \frac{e^{-m_t r}}{\sqrt{m_t r}}$$

$$\langle \epsilon^+(\mathbf{r})\epsilon^+(\mathbf{0})\rangle \propto \frac{e^{-|m_s|r}}{\sqrt{|m_s|r}}, \qquad \langle \epsilon^-(\mathbf{r})\epsilon^-(\mathbf{0})\rangle \propto \frac{e^{-3m_t r}}{(m_t r)^{3/2}}$$

$$K_0(mr) \Leftrightarrow \frac{1}{\omega^2 + q^2 + m^2}$$

$$\chi''(\pi - q, \omega) = \underbrace{Z\delta(\omega^2 - q^2 - m_t^2)}_{\text{massive S} = 1 \text{ magnon}} + \underbrace{\chi''_{\text{reg}}(\pi - q, \omega)}_{\text{incoherent background}\,:\, \omega \geq 3m_t}$$

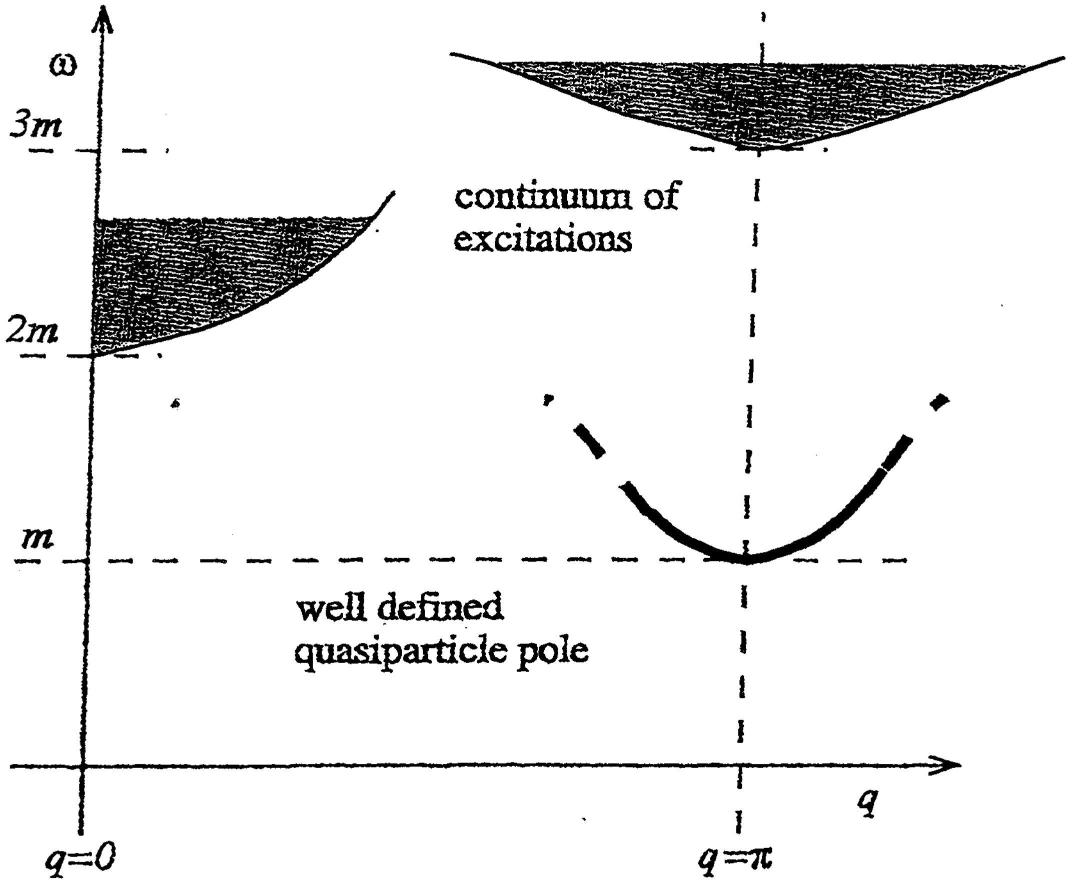
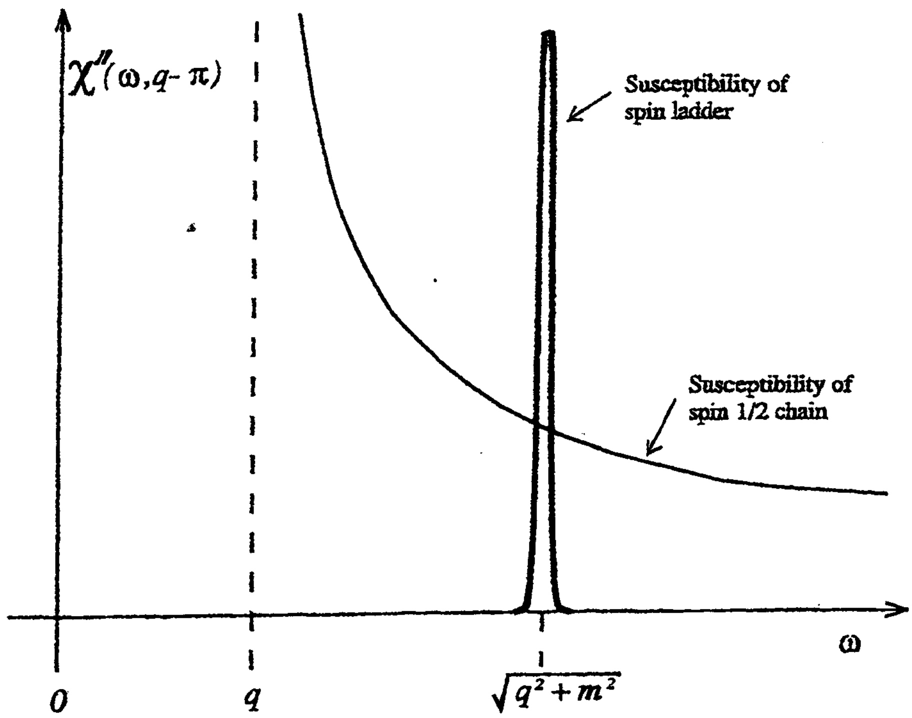


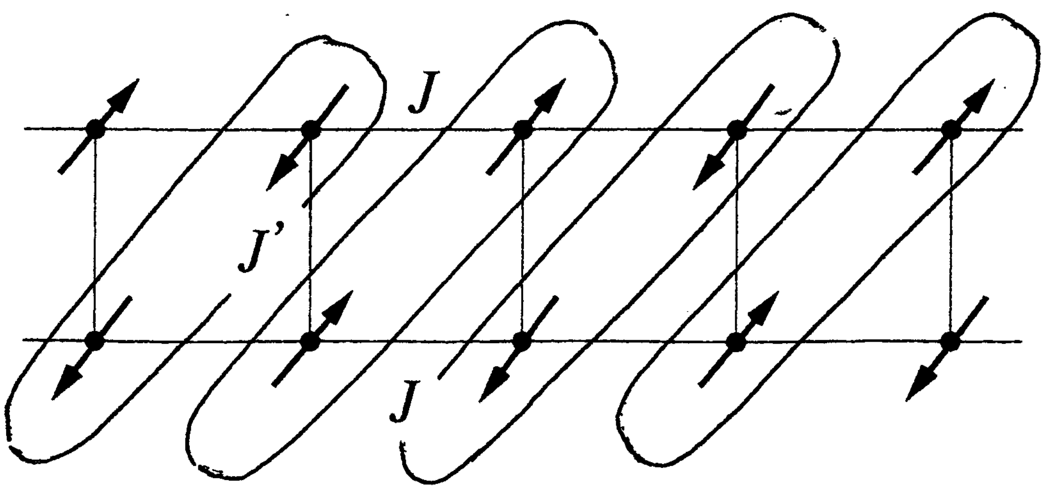
Fig. 21.3. The area of (ω, q) plane where the imaginary part of the dynamical magnetic susceptibility is finite.

- $J' = 0$: Gapless S=1/2 spinons of two decoupled Heisenberg chains $\Rightarrow$ broad continuum seen in $\chi''(\pi - q, \omega)$.
- $J' \neq 0$: Spinons confine to produce *massive coherent triplet excitations* – δ-peak in $\chi''(\pi - q, \omega)$.



- At energies $|\omega| < 2|m_t| + |m_s| \sim 5|m_t|$ $\Rightarrow$ effective *spin-1* Heisenberg chain with a small Haldane gap.

$J' > 0$: $\mathbf{N}_{S=1} = \mathbf{n}^-$



SPIN-1 HEISENBERG CHAIN WITH BIQUADRATIC EXCHANGE

$$H_{S=1}(\beta) = J \sum_n \left[\mathbf{S}_n \cdot \mathbf{S}_{n+1} - \beta \left(\mathbf{S}_n \cdot \mathbf{S}_{n+1} \right)^2 \right], \quad (J > 0,\ S = 1)$$

- $\beta = 0$: standard S=1 Heisenberg chain.
- $\beta = 1/3$: Valence Bond Solid (*Affleck, Kennedy, Lieb, Tasaki, 1988*)
- $\beta = 1$: Exactly integrable point (*Takhtajan; Babujan, 1982*)

$H(\beta = 1)$ – continuum limit $\Rightarrow$ level k = 2 SU(2) WZNW model:

$$\mathcal{H}^{WZNW}_{SU(2)_2} = \frac{\pi v}{2} (: \mathbf{I}_R \cdot \mathbf{I}_R : + : \mathbf{I}_L \cdot \mathbf{I}_L :) \quad \Leftrightarrow \quad H^0_M[\vec{\xi}] = -\frac{iv}{2} \sum_{a=1,2,3} (\xi^a_R \partial_x \xi^a_R - \xi^a_L \partial_x \xi^a_L)$$

$C^{WZNW}_{SU(2)_2} = 3/2 = 3 \times 1/2 \quad \rightarrow$ triplet of critical Ising models

$I^a_R = -\frac{i}{2} \epsilon^{abc} \xi^b_R \xi^c_R, \quad I^a_L = -\frac{i}{2} \epsilon^{abc} \xi^b_L \xi^c_L$ Zamilodchikov & Fattev (1986)

CFT: the mass term $im\vec{\xi}_R \cdot \vec{\xi}_L$ is the only relevant perturbation to $\mathcal{H}^{WZNW}_{SU(2)_2}$ allowed by all symmetries. O(3) model of massive Majorana fermions $\Rightarrow$ universal description of the S=1 Chain with a small Haldane mass.

$H_{S=1}(\beta)$ at $1 - \beta \ll 1$ continuum limit $\Rightarrow$

$$\mathcal{H}_M[\vec{\xi}] = -\frac{iv}{2} \left(\vec{\xi}_R \cdot \partial_x \vec{\xi}_R - \vec{\xi}_L \cdot \partial_x \vec{\xi}_L \right) - im\vec{\xi}_R \cdot \vec{\xi}_L$$

$$\mathbf{N}_{S=1} = \mathbf{n}^- |_{|m_s| \to \infty} = (\mu_1 \sigma_2 \mu_3,\ \sigma_1 \mu_2 \mu_3,\ \mu_1 \mu_2 \sigma_3)$$

Tsvelik (1990)